"Ueber optische Strahensysteme," J. f. d. reine und angewandte Math. **97** (1884), 172-176.

On optical ray systems

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Let a first and second medium be separated by a surface whose equation is:

$$f_{12}(x_{12}, y_{12}, z_{12}) = 0, \tag{1}$$

where x, y, z mean the rectangular coordinates of a point of the surface.

Let the wave surface of the first medium be given by the equation:

$$f_1(\rho_1 \xi_1, \rho_1 \eta_1, \rho_1 \zeta_1) = 0, \tag{2}$$

and the second one by the equation:

$$f_2(\rho_2 \xi_2, \rho_2 \eta_2, \rho_2 \zeta_2) = 0. \tag{3}$$

In these expressions, ρ_1 and ρ_2 mean the guiding rays, and ξ_1, η_1, ζ_1 and ξ_2, η_2, ζ_2 mean the cosines of their directions, such that with the assumption of a certain unit of time, ρ_1 and ρ_2 will give the velocity that light has in their directions in the medium in question directly.

A light ray that starts from the point (x_{01}, y_{01}, z_{01}) in the first medium with the direction (ξ_1, η_1, ζ_1) meets the separation surface at the point (x_{02}, y_{02}, z_{02}), and one draws the segment r_1 between them. The refracted ray then has the direction (ξ_2, η_2, ζ_2), and one draws the segment r_2 in the second medium up to the point (x_{23}, y_{23}, z_{23}). One will then have the equations:

$$x_{12} = x_{01} + r_1 \xi_1, \qquad y_{12} = y_{01} + r_1 \eta_1, \qquad z_{12} = z_{01} + r_1 \zeta_1, \tag{4}$$

$$x_{23} = x_{12} + r_2 \xi_2, \qquad y_{23} = y_{12} + r_2 \eta_2, \qquad z_{23} = z_{12} + r_2 \zeta_2. \tag{5}$$

It follows from the principle of fastest arrival time that:

$$\left\{\begin{array}{c} \dfrac{A_1}{\sum A_1\rho_1\xi_1} - \dfrac{A_2}{\sum A_2\rho_2\xi_2} = \mu A_{12}, \quad \dfrac{B_1}{\sum A_1\rho_1\xi_1} - \dfrac{B_2}{\sum A_2\rho_2\xi_2} = \mu B_{12}, \\ \dfrac{C_1}{\sum A_1\rho_1\xi_1} - \dfrac{C_2}{\sum A_2\rho_2\xi_2} = \mu C_{12}. \end{array}\right. \tag{6}$$

Now and later, the Σ sign means that the summands that are written down are added to two other summands that have the same meaning for the z and z axes that the term written down has for the x axis. M is an undetermined factor, and the following equations are valid for the quantities A, B, C:

$$(7)\qquad \sum A_1\, d(\rho_1\, \xi_1) = 0, \qquad \sum A_2\, d(\rho_2\, \xi_2) = 0, \qquad \sum A_{12}\, dx_{12} = 0.$$

Equations (6) are also true for reflections, except that the wave surface in the first medium is also to be regarded as the wave surface for the second medium. If the quantities x_{01}, y_{01}, z_{01}, ξ_1, η_1, ζ_1 are functions of two independent variables u, v, by which, a ray system is defined in the first medium, then one can calculate the determining data of the second ray system that arises from this as functions of u, v with the help of equations (1)-(7). The derivation of these functions yields that the ray in the second system that belongs to a pair of values (u, v) will correspond to the ray in the first system that belongs to the same pair of values (u, v) in such a way that it arises from this refraction.

If one adds equations (6), multiplied by dx_{12}, dy_{12}, dz_{12}, in turn, then brings the second term on the left-hand side to the right-hand side, and considers equations (7), then one will obtain:

$$(8)\qquad \frac{\sum A_1 dx_{12}}{\sum A_1 \rho_1\, \xi_1} = \frac{\sum A_1 dx_{12}}{\sum A_2 \rho_2\, \xi_2}\,.$$

It follows from equations (4) that:

$$(9)\qquad \sum A_1\, dx_{12} = \sum A_1\, dx_{01} + \sum A_1\, d(r_1\, \xi_1).$$

Now, one has:

$$\sum A_1\, d(r_1\, \xi_1) = \sum A_1\, d\left(\frac{r_1}{\rho_1}\cdot \rho_1\, \xi_1\right) = d\left(\frac{r_1}{\rho_1}\right)\sum A_1\, \rho_1\, \xi_1 + \frac{r_1}{\rho_1}\sum A_1\, d(\rho_1\, \xi_1)\ ,$$

or, due to the first of equations (7):

$$\sum A_1\, d(r_1\, \xi_1) = d\left(\frac{r_1}{\rho_1}\right)\cdot \sum A_1\, \rho_1\, \xi_1\ .$$

Therefore, equation (9) can be written:

$$(10)\qquad \frac{\sum A_1 dx_{12}}{\sum A_1 \rho_1 \xi_1} = \frac{\sum A_1 dx_{01}}{\sum A_1 \rho_1 \xi_1} + d\left(\frac{r_1}{\rho_1}\right).$$

With the use of (10), (8) becomes:

$$(11)\qquad \frac{\sum A_1 dx_{01}}{\sum A_1 \rho_1 \xi_1} + d\left(\frac{r_1}{\rho_1}\right) = \frac{\sum A_2 dx_{12}}{\sum A_2 \rho_2 \xi_2}\,.$$

If the surface (x_{23}, y_{23}, z_{23}) is the separating surface between the second medium and a third one then one will obtain an equation that is similar to equation (11) between the determining data of the second ray system and the third one that arises from it by refraction, and so forth.

If n media are present, and the rays of the n^{th} ray system in the n^{th} medium draw the segment r_n to the surface ($x_{n,\,n+1}$, $y_{n,\,n+1}$, $z_{n,\,n+1}$), where r_n can be chosen to be an arbitrary function of u, v, then n equations of the form of equation (11) will be true. If one adds them then one will obtain:

$$\frac{\sum A_1 dx_{01}}{\sum A_1 \rho_1 \xi_1} + \sum_{s=1}^{n} d\left(\frac{r_s}{\rho_s}\right) = \frac{\sum A_n dx_{n,n+1}}{\sum A_n \rho_n \xi_n}. \tag{12}$$

This equation expresses the relations that exist between two ray systems, one of which emerges from the other one by the various reflections and refractions in media with arbitrary wave surfaces. It will also still be valid when n becomes infinitely large – i.e., when the light goes through inhomogeneous media. One can then think of every inhomogeneous medium as decomposed into infinitely small strips, and regard each of these components as a homogeneous medium.

If the first ray system consists of rays that start from a luminous point, and one takes that point to be the starting point of the rays then one will have $dx_{01} = dy_{01} = dz_{01} = 0$, and equation (12) will read:

$$\frac{\sum A_n dx_{n,n+1}}{\sum A_n \rho_n \xi_n} = d\left(\sum_{s=1}^{n} \frac{r_s}{\rho_s}\right). \tag{13}$$

If one then determines r_n such that the right-hand side of this equation vanishes, so:

$$\sum_{s=1}^{n} \frac{r_s}{\rho_s} = c, \tag{14}$$

where c means the integration constant, then one will get:

$$\sum A_n \, dx_{n,n+1} = 0. \tag{15}$$

However, r_s / ρ_s gives the time that it takes a light ray to traverse the s^{th} medium, and therefore the surfaces ($x_{n,\,n+1}$, $y_{n,\,n+1}$, $z_{n,\,n+1}$), which are determined by equation (14), are obtained from their definitions in such a way that all of the rays of the ray system that emanate from luminous points at the same time will meet each of these surfaces at the same time, or that a light motion has propagated from a luminous point in the last medium at one point in time for an arbitrary time duration to a certain surface that is defined by equation (14). Therefore, this surface will be given the name of "wave surface," in the *Kirchhoff-Helmholtz* sense. However, in order to avoid confusion with the surface that is referred to as the "wave surface of a medium," I would like to call the surfaces that are defined by equation (14) *surfaces of equal arrival time*, where "equal" is taken to have the sense of "simultaneous."

Now, (15) says that the cosines of the directions of the normals to these surfaces are proportional to A_n , B_n , C_n , so they are equal to the cosines of the directions of the normals to the wave surface in the n^{th} medium. We thus have the theorem:

I. *For optical ray systems, the rays are inclined with respect to the surfaces of equal arrival time in all directions in the same way as the guiding rays of the wave surface of the ray system (which are parallel to them) are inclined with respect to it.*

This immediately yields the following theorem:

II. *For optical ray systems whose wave surface is a sphere, the rays of the system are normal to the surface of equal arrival time.*

III. *The necessary and sufficient condition for a ray system to be optically representable in an isotropic medium is that its rays be normals to a surface.*

Equation (13) will always be fulfilled then. The first of the aforementioned theorems is an analogue of the *Malus-Dupin* theorem for optical ray systems in media with arbitrary wave surfaces. The third one is an extension of the same theorem, insofar as the media that light must traverse before it comes back to an isotropic medium can be not just isotropic or crystalline, but media with completely arbitrary wave surfaces.

As a result of equation (13), $\frac{\sum A_n dx_{n,n+1}}{\sum A_n \rho_n \xi_n}$ will be the complete differential of a function of u, v when the n^{th} ray system is representable as an optical one, and this condition is also sufficient, since when it is fulfilled, a refracting surface can always be determined in such a way that all of the conditions will be satisfied. As I showed in my dissertation (*), this agrees with the content of the theorem that *Kummer* (**) stated without proof, and which reads:

Any infinitely-thin optical ray bundle inside of a homogeneous, transparent medium has the property that its two focal planes will cut out two curves from the wave surface of light that belongs to this medium,(whose center can be chosen to lie on the axis of the ray bundle) that will intersect in conjugate directions. Any ray bundle that has this property is also actually optically representable.

We understand the term "conjugate directions" of the wave surface to mean the directions of two conjugate diameters that are attached to the point of the wave surface in question to two infinitely-small *Dupin* conic sections – viz., the indicatrix – which will be ellipses or hyperbolas according to whether the surface is convex-concave or concave-convex at that place, resp.

(*) "Ueber die Beziehnungen zwischen zwei allgemeinen Strahlensystemen, von welchen das eine durch die verschiedensten Reflexionen und Brechungen in Medien mit beliebiger Wellenfläche aus dem anderen hervorgegangen ist, und die hieraus für optisch darstellbare Strahlensysteme sich ergebenden Folgerungen," Inaugural-Dissertation, Berlin, 1883.

(**) Monatsberichte der Königlichen Preuss. Akademie der Wissenschaften zu Berlin, from the year 1860, pp. 470.