"Quaternionen, Semivectoren, und Spinoren," Zeit. Phys. **95** (1935), 337-354.

Quaternions, semi-vectors, and spinors

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(Received on 23 April 1935)

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The representation of Lorentz transformations by quaternions allows one to develop the theory of semi-vectors and spinors very simply. The decomposition of semi-vectors of the first (second, resp.) kind into two spinors of the first (second, resp.) kind then corresponds precisely to the decomposition of quaternions into two right-invariant (left-invariant, resp.) subalgebras. This decomposition is mediated by idempotent quaternions and depends upon the choice of two complex parameters that establish the two of the ∞^1 invariant planes that generate the minimal cone in the space of semi-vectors of the first (second, resp.) kind.

The connection between the theory of spinors that was developed by van der Waerden [1]) and the theory of semi-vectors that was introduced by Einstein and Mayer [2]) has been the subject of numerous investigations [3]). The connection between quaternions and semi-vectors has also been treated already [4]). Here, it shall be shown the entire theory of *semi-vectors and spinors* can be developed in a unified and elementary way from the theory of quaternions by the addition of the idempotent quaternions.

In § 1, the *Lorentz transformations* will be discussed in general. § 2 presents the essential concepts of the *theory of quaternions*, from which the *idempotent quaternions* will be introduced in § 3, and the *decomposition of quaternions* into two left-invariant (right-invariant, resp.) subalgebras that they make possible will be developed. § 4, which is linked with § 2, treats the well-known *representation* of quaternions by matrices. In § 5, the results that were found in § 3 will be carried over to the representation and developed further. § 6 presents the *geometric interpretation* of these results. In § 7, the *representation of Lorentz transformations* by quaternions will be treated, and § 8 discusses the theory of *semi-vectors and spinors* that follows from it.

In this paper, we restrict ourselves to the group of *proper* Lorentz transformations, and in a later publication we would first next to go into the transformation properties of semi-vectors and spinors under the transformations of the complete Lorentz group from the standpoint that is chosen here.

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[1]) Cf., O. Laporte and G. Uhlenbeck, Phys. Rev. **37** (1931), 1380.

[2]) A. Einstein und W. Mayer, Berl. Ber. (1932), pp. 552.

[3]) J. A. Schouten, Zeit. Phys. **84** (1933), 92; V. Bargmann, Helv. Phys. Acta **7** (1934), 57; J. Ullmo, Journ. de phys. **5** (1934), 230.

[4]) W. Scherrer, Comment. Math. Helv. **7** (1935), 141.

§ 1. Generalities on Lorentz transformations.

In our investigations, we will always denote the spatial coordinates by x_1, x_2, x_3 and in place of the time t, we shall employ the Minkowski variable $x_4 = ict$ (c = speed of light). This has the advantages that Lorentz transformations (LT) will be orthogonal transformations of these four variables, and that the difference between covariant and contravariant quantities goes away.

The translations $\mathfrak{L}$ of the full Lorentz group divide into two classes: The class of proper LT's, $\mathfrak{L}_0$ – i.e., the orthogonal transformations of the variables x_1, x_2, x_3, x_4 whose transformation matrix has the determinant + 1 – and the class of improper LT's, $\mathfrak{L}_1$, which can each be composed of a reflection $\mathfrak{s}$ in the spatial coordinates:

$$\mathfrak{s}: \qquad x_1^s = -x_1, \qquad x_2^s = -x_2, \qquad x_3^s = -x_3, \qquad x_1^s = x_4 ,$$

and a proper LT. The improper LT's, in contrast to the proper LT's, do not define a group, since the composition of two improper LT's always yields a proper LT.

In the sequel, we will regard the LT as vector maps in four-dimensional space R_4 . By $\{x\}$ ($[x]$, resp.), we would like this to mean the matrix with one column (one row, resp.) that contains the components x_1, x_2, x_3, x_4 of the vector $\mathbf{x}$ in the column (row, resp.). The vector $\mathbf{x}$ will be associated with the vector $\mathbf{x}'$ under a LT. For a transition from one to another not-necessarily-orthogonal coordinate system, we would like to denote the components of $\mathbf{x}$ by x_i^*.

Upper-case Latin symbols will denote square matrices in what follows. From now on, $\bar{A}$ means the matrix whose elements are complex conjugates of those of the matrix A, $A^\times$ means the *transposed* matrix that arises from A by reflection in the main diagonal, $\tilde{A}$ is the matrix that is *adjoint* (i.e., Hermitian conjugate) to A, A^{-1} is the matrix that is reciprocal to A, and $|A|$ is the determinant of the matrix A.

We can then write the LT in the form:

$$\{x'\} = L\,\{x\} \tag{1}$$

or in the form:

$$[x'] = [x]\,L^{-1},$$

where:

$$L^\times = L^{-1}, \tag{2}$$

since the LT is orthogonal. One thus has:

$$|L| = \pm 1, \tag{3}$$

where one has + or – according to whether we are dealing with a proper or improper LT, respectively. The condition (2) and the one (3) that follows from it are not the only ones that the matrix of a (proper or improper) LT $\mathfrak{L}$ is subject to. One must still observe the reality properties. Namely, the matrix that corresponds to an LT $\mathfrak{L}$ must be *real* in the coordinate system x_1, x_2, x_3, $x_0 = ct$: If we introduce these coordinates by means of:

$$\{x^*\} = U\,\{x\},$$

where [1]):

$$U = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & -i \end{pmatrix},$$

then the matrix of the LT $\mathfrak{L}$ that is referred to these coordinates:

$$L^* = U\,L\,U^{-1}$$

must be real; i.e., one must have:

$$U\,L\,U^{-1} = \bar{U}\,\bar{L}\,\bar{U}^{-1}.$$

Since:

$$\bar{U} = U^{-1}, \qquad U^2 = (U^2)^{-1} = G,$$

where:

$$-\,G = \begin{pmatrix} -1 & & & \\ & -1 & & \\ & & -1 & \\ & & & +1 \end{pmatrix}$$

is the matrix that represents the reflection $\mathfrak{s}$, this leads to the equation:

$$\bar{L} = G\,L\,G, \tag{4}$$

which expresses the fact that the matrix elements of L that stand in the fourth row and the fourth column are pure imaginary, except for l_{44} [2]). We must further require of an LT $\mathfrak{L}$ that it should not invert the time direction, which leads to the condition:

$$l_{44} > 0. \tag{5}$$

Equations (2), (4), and (5) already encompass all of the conditions to be imposed upon L. Equation (4) can also amount to a statement about the reflection $\mathfrak{s}$: Namely, it expresses the fact that the reflection $\mathfrak{s}$ that is represented by G commutes with an LT $\mathfrak{L}$ when and only when $\mathfrak{L}$ degenerates into merely a transformation of the spatial coordinates.

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[1]) For diagonal matrices, we omit the zeroes that lie outside of the diagonal.

[2]) l_{ik} means the element of the matrix L that appears in the i^{th} column and the k^{th} row.

However, in the present paper, we shall restrict ourselves to the proper LT $\mathfrak{L}_0$, so in addition to (2), (4), and (5), we further assume that:

$$| L | = 1.$$

§ 2. Quaternions.

We understand a *quaternion* **a** to mean a four-component hypercomplex number [1]):

$$\mathbf{a} = a_1\,\mathbf{e}_1 + a_2\,\mathbf{e}_2 + a_3\,\mathbf{e}_3 + a_4\,\mathbf{e}_4\,,$$

whose *components* a_1, a_2, a_3, a_4 can be arbitrary complex numbers, and whose *basis quantities* (i.e., *unit quaternions*) $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$, $\mathbf{e}_4$ obey the following multiplication rules:

$$\left.\begin{aligned} \mathbf{e}_1^2 = \mathbf{e}_2^2 = \mathbf{e}_3^2 = -\mathbf{e}_0,\ \mathbf{e}_i\mathbf{e}_4 = \mathbf{e}_4\mathbf{e}_i = \mathbf{e}_i \quad (i = 1, 2, 3, 4), \\ \mathbf{e}_2\mathbf{e}_3 = -\mathbf{e}_3\mathbf{e}_2 = \mathbf{e}_1,\ \mathbf{e}_3\mathbf{e}_1 = -\mathbf{e}_1\mathbf{e}_3 = \mathbf{e}_2,\ \mathbf{e}_1\mathbf{e}_2 = -\mathbf{e}_2\mathbf{e}_1 = \mathbf{e}_3. \end{aligned}\right\} \tag{6}$$

$\mathbf{e}_4$ is therefore the principal unit of the quaternions. We understand the sum of two quaternions **a** and **b** to mean the quaternion **c** whose components c_i are given by $a_i + b_i$. A quaternion will be multiplied by a number when all of the components of this quaternion are multiplied by this number. The product of two quaternions will be defined by (6) and the distributive laws:

$$\left.\begin{aligned} \mathbf{a}(\mathbf{b}+\mathbf{c}) = \mathbf{ab}+\mathbf{ac}, \\ (\mathbf{a}+\mathbf{b})\mathbf{c} = \mathbf{ac}+\mathbf{bc}. \end{aligned}\right\} \tag{7}$$

The validity of the associative law follows from (6) and (7):

$$\mathbf{a}\,(\mathbf{b}\,\mathbf{c}) = (\mathbf{a}\,\mathbf{b})\,\mathbf{c}.$$

On the other hand, from (6), the commutative law is no longer true for the multiplication of two quaternions.

The *conjugate* quaternion $\breve{\mathbf{a}}$ to the quaternion **a** is defined by:

$$\breve{\mathbf{a}} = -\,a_1\,\mathbf{e}_1 - a_2\,\mathbf{e}_2 - a_3\,\mathbf{e}_3 + a_4\,\mathbf{e}_4\,.$$

By the term *adjoint* quaternion $\tilde{\mathbf{a}}$ to the quaternion **a**, we would like this to mean the quaternion:

$$\tilde{\mathbf{a}} = -\,\bar{a}_1\,\mathbf{e}_1 - \bar{a}_2\,\mathbf{e}_2 - \bar{a}_3\,\mathbf{e}_3 + \bar{a}_4\,\mathbf{e}_4$$

whose components are complex conjugate to those of $\breve{\mathbf{a}}$. We would like to call a quaternion for which $\mathbf{a} = \tilde{\mathbf{a}}$ *self-adjoint.*

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[1]) Cf., e.g., H. Rothe, Enc. Math. Wiss. III, sec. 11, pp. 1300.

We would like to denote the quaternion whose components are complex conjugate to those of **a** by $\bar{\mathbf{a}}$. One then has:

$$\tilde{\mathbf{a}} = \bar{\breve{\mathbf{a}}} = \breve{\bar{\mathbf{a}}}.$$

We will call a quaternion for which $\mathbf{a} = \bar{\mathbf{a}}$ *real*.

We understand the *norm* $\mathcal{N}a$ of a quaternion **a** to mean the (generally complex) number:

$$\mathcal{N}a = a_1^2 + a_2^2 + a_3^2 + a_4^2 .$$

If the norm of a quaternion **a** equals:

$$\mathcal{N}a = 1$$

then we call **a** a *unit quaternion*. If:

$$\mathcal{N}a = 0$$

for a quaternion **a** then we call **a** a *null quaternion* [1]).

The reciprocal quaternion $\mathbf{a}^{-1}$ to a quaternion **a**, for which one would have:

$$\mathbf{a}^{-1}\mathbf{a} = \mathbf{a}\,\mathbf{a}^{-1} = \mathbf{e}_4,$$

is defined when and only when $\mathcal{N}a \neq 0$. One then has:

$$\mathbf{a}^{-1} = \frac{1}{\mathcal{N}a}\breve{\mathbf{a}} .$$

The norm of the product of several quaternions is equal to the product of their norms. Furthermore, one has the equation:

$$\widetilde{\mathbf{abc}} = \tilde{\mathbf{c}}\tilde{\mathbf{b}}\tilde{\mathbf{a}} . \tag{8}$$

Likewise, the conjugate quaternion to the product of several quaternions is the product of the conjugate quaternions in the opposite sequence.

§ 3. Idempotent quaternions.

We would now like to introduce a special class of quaternions: the *idempotent* quaternions, which have the property that they coincide with their squares. An idempotent quaternion **ε** is then defined by the requirement that [2]):

$$\boldsymbol{\varepsilon}\,\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}.$$

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[1]) We distinguish this from the quaternion 0, for which $a_1 = a_2 = a_3 = a_4 = 0$.

[2]) Cf., e.g., H. Weyl, *Theory of Groups and Quantum Mechanics*, London, 1931, chap. III, § 13, chap. V, § 3.

With the exception of the principal unit $\mathbf{e}_4$ and the quaternion 0, a quaternion $\boldsymbol{\varepsilon}$:

$$\boldsymbol{\varepsilon} = \varepsilon_1\,\mathbf{e}_1 + \varepsilon_2\,\mathbf{e}_2 + \varepsilon_3\,\mathbf{e}_3 + \varepsilon_4\,\mathbf{e}_4$$

fulfills this condition if and only if:

$$\mathcal{N}\boldsymbol{\varepsilon} = 0, \qquad \varepsilon_4 = \tfrac{1}{2},$$

so:

$$a_1^2 + a_2^2 + a_3^2 = -\tfrac{1}{4}, \quad \varepsilon_4 = \tfrac{1}{2}. \tag{9}$$

In the sequel, we will always understand $\boldsymbol{\varepsilon}$ to mean an idempotent quantity that is different from 0 and $\mathbf{e}_4$, thus a null quaternion for which (9) is true. Due to (9), along with $\boldsymbol{\varepsilon}$, $\breve{\boldsymbol{\varepsilon}}$, $\bar{\boldsymbol{\varepsilon}}$, and $\tilde{\boldsymbol{\varepsilon}}$ are also idempotent. In particular, one has:

$$\boldsymbol{\varepsilon}\,\breve{\boldsymbol{\varepsilon}} = \breve{\boldsymbol{\varepsilon}}\,\boldsymbol{\varepsilon} = 0 \tag{10}$$

and

$$\boldsymbol{\varepsilon} + \breve{\boldsymbol{\varepsilon}} = \mathbf{e}_4\,. \tag{11}$$

One calls two idempotent quantities *independent* when (10) is true. We can characterize $\boldsymbol{\varepsilon}$ as a *primitive idempotent* quaternion, as compared to $\mathbf{e}_4$, since $\boldsymbol{\varepsilon}$, in contrast to $\mathbf{e}_4$, may no longer be decomposed into the sum of idempotents and non-zero quaternions. One easily proves by means of (11) that $\breve{\boldsymbol{\varepsilon}}$ is the only primitive quaternion that is independent of $\boldsymbol{\varepsilon}$.

Equation (11) allows one to define the following decomposition for an arbitrary quaternion $\mathbf{a}$:

$$\mathbf{a} = \mathbf{a}\,\mathbf{e}_4 = \mathbf{a}\,(\boldsymbol{\varepsilon} + \breve{\boldsymbol{\varepsilon}}) = \mathbf{a}\,\boldsymbol{\varepsilon} + \mathbf{a}\,\breve{\boldsymbol{\varepsilon}} = \underset{1}{\mathbf{a}} + \underset{2}{\mathbf{a}}\,, \tag{12}$$

where:

$$\underset{1}{\mathbf{a}} = \mathbf{a}\,\boldsymbol{\varepsilon}, \qquad \underset{2}{\mathbf{a}} = \mathbf{a}\,\breve{\boldsymbol{\varepsilon}}. \tag{12a}$$

The quaternions $\underset{1}{\mathbf{a}}$ and $\underset{2}{\mathbf{a}}$ then fulfill the equations:

$$\underset{1}{\mathbf{a}} = \underset{1}{\mathbf{a}}\,\boldsymbol{\varepsilon}, \qquad \underset{2}{\mathbf{a}} = \underset{2}{\mathbf{a}}\,\breve{\boldsymbol{\varepsilon}}.$$

The totality of quaternions $\underset{1}{\mathbf{x}}$ for which one has (for a fixed $\boldsymbol{\varepsilon}$):

$$\underset{1}{\mathbf{x}}\,\boldsymbol{\varepsilon} = \underset{1}{\mathbf{x}} \tag{13}$$

defines a left-invariant subalgebra $P_1(\boldsymbol{\varepsilon})$ inside of the algebra of quaternions [1]). Namely, along with $\underset{1}{\mathbf{a}}$ and $\underset{1}{\mathbf{b}}$:

[1]) Also called a *left ideal*; cf., e.g., B. L. van der Waerden, *Moderne Algebra*, Berlin, 1930.

$$\underset{1}{\mathbf{a}} + \underset{1}{\mathbf{b}} = \underset{1}{\mathbf{a}}\,\boldsymbol{\varepsilon} + \underset{1}{\mathbf{b}}\,\boldsymbol{\varepsilon} = (\underset{1}{\mathbf{a}} + \underset{1}{\mathbf{b}})\,\boldsymbol{\varepsilon} \tag{14}$$

also belongs to this totality, and furthermore, along with $\underset{1}{\mathbf{a}}$, $c\underset{1}{\mathbf{a}}$ (c, an arbitrary number) also belongs, and along with $\underset{1}{\mathbf{a}}$, one also has:

$$\mathbf{z}\underset{1}{\mathbf{a}} = \mathbf{z}\underset{1}{\mathbf{a}}\,\boldsymbol{\varepsilon}, \tag{15}$$

if $\mathbf{z}$ is an entirely arbitrary quaternion. All of the quaternions that belong to $P_1(\boldsymbol{\varepsilon})$ can also be characterized by the requirement:

$$\underset{1}{\mathbf{x}}\,\breve{\boldsymbol{\varepsilon}} = 0, \tag{16}$$

which is equivalent to (13), since in the event that the last equation is true, one also has:

$$\underset{1}{\mathbf{x}} = \underset{1}{\mathbf{x}}\,\mathbf{e}_4 = \underset{1}{\mathbf{x}}\,(\boldsymbol{\varepsilon} + \breve{\boldsymbol{\varepsilon}}) = \underset{1}{\mathbf{x}}\,\boldsymbol{\varepsilon} + \underset{1}{\mathbf{x}}\,\breve{\boldsymbol{\varepsilon}} = \underset{1}{\mathbf{x}}\,\boldsymbol{\varepsilon}\,.$$

Likewise, the totality of all quaternions $\underset{2}{\mathbf{x}}$ for which:

$$\underset{2}{\mathbf{x}}\,\breve{\boldsymbol{\varepsilon}} = \underset{2}{\mathbf{x}}$$

defines a left-invariant subalgebra $P_2(\boldsymbol{\varepsilon})$, for which equations that are analogous to (14), (15), (16) are true.

Since, according to (12), any quaternion $\mathbf{a}$ (for a given $\boldsymbol{\varepsilon}$) can be uniquely decomposed into a quaternion that belongs to $P_1(\boldsymbol{\varepsilon})$ and one that belongs to $P_2(\boldsymbol{\varepsilon})$, the algebra of quaternions is the sum of the two left-invariant subalgebras $P_1(\boldsymbol{\varepsilon})$ and $P_2(\boldsymbol{\varepsilon})$. According to (13) [(16), resp.], $\boldsymbol{\varepsilon}$ ($\breve{\boldsymbol{\varepsilon}}$, resp.) plays the role of right unity in the subalgebra $P_1(\boldsymbol{\varepsilon})$ [$P_2(\boldsymbol{\varepsilon})$, resp.].

By means of an idempotent quantity $\boldsymbol{\varepsilon}$, we can also perform yet another decomposition of an arbitrary quaternion $\mathbf{a}$, namely:

$$\mathbf{a} = \mathbf{e}_4\,\mathbf{a} = (\boldsymbol{\varepsilon} + \breve{\boldsymbol{\varepsilon}})\,\mathbf{a} = \boldsymbol{\varepsilon}\,\mathbf{a} + \breve{\boldsymbol{\varepsilon}}\,\mathbf{a} = \overset{1}{\mathbf{a}} + \overset{2}{\mathbf{a}}, \tag{17}$$

where:

$$\overset{1}{\mathbf{a}} = \boldsymbol{\varepsilon}\,\mathbf{a}, \qquad \overset{2}{\mathbf{a}} = \breve{\boldsymbol{\varepsilon}}\,\mathbf{a}.$$

$\overset{1}{\mathbf{a}}$ and $\overset{2}{\mathbf{a}}$ now fulfill the equations:

$$\overset{1}{\mathbf{a}} = \boldsymbol{\varepsilon}\overset{1}{\mathbf{a}}, \qquad \overset{2}{\mathbf{a}} = \breve{\boldsymbol{\varepsilon}}\,\overset{2}{\mathbf{a}}.$$

The totality of quantities $\overset{1}{\mathbf{x}}$ for which one has:

$$\boldsymbol{\varepsilon}\,\overset{1}{\mathbf{x}} = \overset{1}{\mathbf{x}},$$

defines a right-invariant subalgebra $R_1(\boldsymbol{\varepsilon})$, one likewise shows that along with $\overset{1}{\mathbf{a}}$ and $\overset{1}{\mathbf{b}}$, their sum also belongs to $R_1(\boldsymbol{\varepsilon})$, and along with $\overset{1}{\mathbf{a}}$, $c\,\overset{1}{\mathbf{a}}$ also belongs. Furthermore, if $\mathbf{z}$ is an arbitrary quaternion then along with $\overset{1}{\mathbf{a}}$, $\overset{1}{\mathbf{a}}\,\mathbf{z}$ also belongs to $R_1(\boldsymbol{\varepsilon})$. A second right-invariant subalgebra $R_2(\boldsymbol{\varepsilon})$ will be defined by the totality of quaternions $\overset{2}{\mathbf{x}}$ for which one has:

$$\breve{\boldsymbol{\varepsilon}}\,\overset{2}{\mathbf{x}} = \overset{2}{\mathbf{x}}.$$

According to (17), the algebra of quaternions is then once more the sum of $R_1(\boldsymbol{\varepsilon})$ and $R_2(\boldsymbol{\varepsilon})$.

Any quaternion that belongs to a right-invariant or left-invariant subalgebra is naturally a *null quaternion*, since $\boldsymbol{\varepsilon}$ is a null quaternion.

§ 4. Representation of the quaternions by two-rowed matrices.

As is known, the quaternions may be represented by two-rowed matrices in an invertible way; i.e., any quaternion may be associated with a two-rowed matrix in such a way that the sum (product, resp.) of two quaternions corresponds to the sum (product, resp.) of the associated matrices, and conversely.

The basis quantities $\mathbf{e}_i$ will thus be associated with the following matrices [1]):

$$\mathbf{e}_1 \to E_1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \qquad \mathbf{e}_2 \to E_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad \mathbf{e}_3 \to E_3 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \qquad \mathbf{e}_4 \to E_4 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

The quaternion:

$$\mathbf{a} = a_1\,\mathbf{e}_1 + a_2\,\mathbf{e}_2 + a_3\,\mathbf{e}_3 + a_4\,\mathbf{e}_4$$

then corresponds to the matrix:

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} a_4 + ia_3 & -a_2 + ia_1 \\ a_2 + ia_1 & a_4 - ia_3 \end{pmatrix}. \tag{18}$$

Conversely, an arbitrary matrix:

[1]) All other possible representations in terms of two-rowed matrices come from this one by similarity transformations; in the representation employed, the basis quantities will be represented by unitary matrices.

$$B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$$

corresponds uniquely to a quaternion:

$$\mathbf{b} = b_1\,\mathbf{e}_1 + b_2\,\mathbf{e}_2 + b_3\,\mathbf{e}_3 + b_4\,\mathbf{e}_4\,,$$

where:

$$b_1 = -\frac{i}{2}(b_{12} + b_{21}), \qquad b_2 = -\frac{1}{2}(b_{12} - b_{21}),$$
$$b_3 = -\frac{i}{2}(b_{11} - b_{22}), \qquad b_4 = +\frac{1}{2}(b_{11} + b_{22}).$$

This representation has the following properties:

1. The norm of the quaternion **a** is equal to the determinant of its associated matrix.
2. The spur (i.e., trace) of the matrix A, $\mathcal{SP}\,A$ is equal to $2a_4$.
3. If the matrix A is associated with the quaternion **a** then the adjoint quaternion $\tilde{\mathbf{a}}$ is associated with the matrix $\tilde{A}$ that is the Hermitian conjugate of A.

§ 5. Decomposition of quaternions.

From the theorems above, it follows that a matrix $\mathcal{E}$ represents a *primitive idempotent* quaternion $\boldsymbol{\varepsilon}$ if and only if:

$$|\,\mathcal{E}\,| = 0, \qquad \mathcal{SP}\,\mathcal{E} = 1. \tag{19}$$

The matrix representation allows one to give very simply relations that the components of a quaternion must satisfy if that quaternion is to belong to one of the two left-invariant (right-invariant, resp.) subalgebras that are determined by $\boldsymbol{\varepsilon}$. Equation (16) reads, in matrix form [1]):

$$\underset{1}{X}\,\breve{\mathcal{E}} = 0,$$

and leads to the following equations for the elements of $\underset{1}{X}$:

$$\frac{\underset{1}{x_{12}}}{\underset{1}{x_{11}}} = \frac{\underset{1}{x_{22}}}{\underset{1}{x_{21}}} = -\frac{\varepsilon_{22}}{\varepsilon_{21}}\,. \tag{20}$$

Likewise, we get for the quaternions $\underset{2}{\mathbf{x}}$ that belong to a $P_2(\boldsymbol{\varepsilon})$:

[1]) In which we understand $\breve{\mathcal{E}}$ to mean the matrix that is associated with $\breve{\boldsymbol{\varepsilon}}$.

$$\frac{\underset{2}{x}_{12}}{\underset{2}{x}_{11}} = \frac{\underset{2}{x}_{22}}{\underset{2}{x}_{21}} = -\frac{\varepsilon_{21}}{\varepsilon_{11}}. \tag{21}$$

One gets similar equations for the components of a quaternion $\overset{1}{\mathbf{x}}$ that belongs to an $R_1(\boldsymbol{\varepsilon})$:

$$\frac{\overset{1}{\mathbf{x}}_{21}}{\overset{1}{\mathbf{x}}_{11}} = \frac{\overset{1}{\mathbf{x}}_{22}}{\overset{1}{\mathbf{x}}_{12}} = \frac{\varepsilon_{21}}{\varepsilon_{11}} \tag{22}$$

and finally for the quaternions $\overset{2}{\mathbf{x}}$ that belong to $R_2(\boldsymbol{\varepsilon})$:

$$\frac{\overset{2}{\mathbf{x}}_{21}}{\overset{2}{\mathbf{x}}_{11}} = \frac{\overset{2}{\mathbf{x}}_{22}}{\overset{2}{\mathbf{x}}_{12}} = -\frac{\varepsilon_{21}}{\varepsilon_{22}}. \tag{23}$$

The idempotent quaternion $\boldsymbol{\varepsilon}$ will already be determined uniquely by the two ratios that appear in (20) and (21):

$$\frac{\varepsilon_{22}}{\varepsilon_{21}} = \frac{\varepsilon_4 - i\varepsilon_3}{\varepsilon_2 + i\varepsilon_1}, \qquad -\frac{\varepsilon_{11}}{\varepsilon_{21}} = -\frac{\varepsilon_4 + i\varepsilon_3}{\varepsilon_2 + i\varepsilon_1} = \chi,$$

where the two numbers φ and χ must be different from each other [1]). If we regard φ and χ as the parameters of $\boldsymbol{\varepsilon}$ then this yields the components of $\boldsymbol{\varepsilon}$:

$$\varepsilon_1 = -\frac{i}{2}\frac{1-\varphi\chi}{\varphi-\chi}, \qquad \varepsilon_2 = \frac{1}{2}\frac{1+\varphi\chi}{\varphi-\chi}, \qquad \varepsilon_3 = \frac{i}{2}\frac{\varphi+\chi}{\varphi-\chi}, \qquad e_4 = \frac{1}{2}. \tag{24}$$

Equations (20) and (21) express the fact that in the matrix $\underset{1}{X}$ ($\underset{2}{X}$, resp.) that is associated with the quaternion $\underset{1}{\mathbf{x}}$ ($\underset{2}{\mathbf{x}}$, resp.) the second *column* is equal to φ (χ, resp.) times the first *column*. By contrast, equations (22) and (23) express the fact that in the matrices $\overset{1}{X}$ ($\overset{2}{X}$, resp.) that are associated with the quaternions $\overset{1}{\mathbf{x}}$ ($\overset{2}{\mathbf{x}}$, resp.) the first *row* equals $-\chi$ ($-\varphi$, resp.) times the second *row*.

The decomposition of the quaternion $\mathbf{a}$ into two quaternions $\underset{1}{\mathbf{a}}$ and $\underset{2}{\mathbf{a}}$ that belong to two left-invariant subalgebras then corresponds to the following decomposition of the matrix A that is associated with them:

[1]) In fact, if one had $\varphi = \chi$ then it would follow that $\varepsilon_{11} = -\varepsilon_{22}$, which is impossible, according to (19).

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} \alpha_1 & \varphi\alpha_1 \\ \alpha_2 & \varphi\alpha_2 \end{pmatrix} + \begin{pmatrix} \alpha_3 & \chi\alpha_3 \\ \alpha_4 & \chi\alpha_4 \end{pmatrix} = \underset{1}{A} + \underset{2}{A}, \tag{25}$$

which yields:

$$\left.\begin{aligned}
\alpha_1 &= \frac{a_{12} - \chi a_{11}}{\varphi - \chi} = \frac{ia_1 - a_2 - \chi(ia_3 + a_1)}{\varphi - \chi}, \\
\alpha_2 &= \frac{-\chi a_{21} + a_{22}}{\varphi - \chi} = \frac{-\chi(ia_1 + a_2) - ia_3 + a_4}{\varphi - \chi}, \\
\alpha_3 &= \frac{-a_{12} + \varphi a_{11}}{\varphi - \chi} = \frac{ia_1 + a_2 + \varphi(ia_3 + a_4)}{\varphi - \chi}, \\
\alpha_4 &= \frac{\varphi a_{21} - a_{22}}{\varphi - \chi} = \frac{\varphi(ia_1 + a_2) + ia_3 + a_4}{\varphi - \chi}.
\end{aligned}\right\} \tag{26}$$

In more concise matrix form, this may be written:

$$\{\alpha\} = S\,\{a\}, \tag{27}$$

where:

$$S = \frac{1}{\varphi - \chi}\begin{pmatrix} i & -1 & -i\chi & -\chi \\ -i\chi & -\chi & -i & 1 \\ -i & 1 & i\varphi & \varphi \\ i\varphi & \varphi & i & -1 \end{pmatrix}. \tag{27a}$$

For a quaternion $\mathbf{a}\,\boldsymbol{\varepsilon} = \mathbf{a}$, one then has, from (25):

$$\alpha_3 = 0, \qquad \alpha_4 = 0, \tag{28}$$

and for a quaternion $\mathbf{a}\,\breve{\boldsymbol{\varepsilon}} = \mathbf{a}$, one has:

$$\alpha_1 = 0, \qquad \alpha_2 = 0. \tag{28}$$

The decomposition of a quaternion **a** into the quaternions $\overset{1}{\mathbf{a}}$ and $\overset{2}{\mathbf{a}}$ that belong to $R_1(\boldsymbol{\varepsilon})$ and $R_2(\boldsymbol{\varepsilon})$ then corresponds to the following decomposition of A:

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} -\chi\beta_1 & -\chi\beta_2 \\ \beta_1 & \beta_2 \end{pmatrix} + \begin{pmatrix} -\varphi\beta_3 & -\varphi\beta_4 \\ \beta_3 & \beta_4 \end{pmatrix} = \overset{1}{A} + \overset{2}{A}, \tag{30}$$

from which, one gets, in a similar way:

$$\{\beta\} = T\,\{a\}, \tag{31}$$

where:

$$T = \frac{1}{\varphi - \chi}\begin{pmatrix} i\varphi & \varphi & i & 1 \\ i & -1 & -i\varphi & \varphi \\ -i\chi & -\chi & -i & -1 \\ -i & 1 & i\chi & -\chi \end{pmatrix}. \tag{31a}$$

For a quaternion $\boldsymbol{\varepsilon}$ **a** = **a**, one has, from (30):

$$\beta_3 = 0, \qquad \beta_4 = 0, \tag{32}$$

and for a quaternion $\breve{\boldsymbol{\varepsilon}}$ **a** = **a**, one has:

$$\beta_1 = 0, \qquad \beta_2 = 0. \tag{32}$$

§ 6. Geometric interpretation of the decomposition.

The results obtained by be expressed simply when we resort to a geometric manner of expression. To that end, we consider the quaternion **a** to be a vector in a four-dimensional space with $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$, $\mathbf{e}_4$ as its basis vectors. We can consider the product:

$$\mathbf{a}' = \mathbf{b}\ \mathbf{a} \tag{34}$$

to be a map in this space by which any quaternion **a** can be associated with another quaternion **a′**. A second group of maps $\mathbf{a} \to \mathbf{a}''$ can be induced by the product:

$$\mathbf{a}'' = \mathbf{a}\ \mathbf{b}, \tag{35}$$

and finally, the maps $\mathbf{a} \to \mathbf{a}'''$:

$$\mathbf{a}''' = \mathbf{b}\ \mathbf{a}\ \mathbf{c}$$

define a third group.

The totality of null quaternions is a three dimensional structure in this space: the *minimal cone,* whose equation is:

$$a_1^2 + a_2^2 + a_3^2 + a_4^2 = 0.$$

However, another structure remains invariant under the maps (34) [(35), resp.], to which we would now like to turn.

The totality of quaternions for which (13) is true obviously lies in a *plane*, whose equation is (28), and which is already determined uniquely by the parameter φ from (26). This plane remains *invariant* under all maps (34), since, from (15), the quaternion **a′** lies in that plane. Since the parameter φ admits no restriction, there is a simply infinite family of planes that remain invariant under the maps (34), and which all lie on the minimal cone; we can call each such plane *left-invariant.* We also see that, conversely, equation (28) has only one solution φ for every null quaternion; therefore, one and only one left-invariant plane goes through each point of the minimal cone [1]).

Any primitive idempotent quaternion $\boldsymbol{\varepsilon}$ now defines *two* (not necessarily distinct) planes $\mathfrak{p}_1$ and $\mathfrak{p}_2$, whose equations are given by (28) [(29), resp.]. Therefore, $\mathfrak{p}_1$ depends upon only the parameter φ, while $\mathfrak{p}_2$ depends upon only the parameter χ. All left-invariant subalgebras $P_1(\boldsymbol{\varepsilon})$ lie in the plane $\mathfrak{p}_1$, and all of the quaternions that belong to the left-invariant subalgebra $P_2(\boldsymbol{\varepsilon})$ lie in the plane $\mathfrak{p}_2$. Any quaternion can be decomposed into the sum of two quaternions that lie in distinct left-invariant planes.

Entirely analogous considerations may be applied to the group of maps (35). There is a family of *right-invariant* [i.e., invariant under the maps (35)] planes that lie on the minimal cone. One and only one right-invariant plane goes through each point of the minimal cone [2]). Any primitive idempotent quaternion defines *two right-invariant planes* $\mathfrak{r}_1$ and $\mathfrak{r}_2$, to which the right-invariant subalgebras $R_1(\boldsymbol{\varepsilon})$ and $R_2(\boldsymbol{\varepsilon})$ belong. Therefore, $\mathfrak{r}_1$ is determined uniquely by the parameter χ according to (32) and $\mathfrak{r}_2$, by the parameter φ according to (33).

The decomposition of quaternions into two left-invariant (right-invariant, resp.) subalgebras is therefore equivalent to the decomposition of the space that we are considering into two irreducible subspaces that invariant under the maps (34) [(35), resp.].

We thus return to the maps (34) and introduce new basis vectors $\boldsymbol{l}_i$, such that α_1, α_2, α_3, α_4 become the components of $\mathbf{a}$ in the new coordinates. The basis vectors $\boldsymbol{l}_i$ are then determined uniquely by the requirement that:

$$\sum_i a_i \mathbf{e}_i = \sum_i \alpha_i \boldsymbol{l}_i \,. \tag{36}$$

$\boldsymbol{l}_1$, $\boldsymbol{l}_2$ lie in the left-invariant plane that is determined by φ and $\boldsymbol{l}_3$, $\boldsymbol{l}_4$ lie in the left-invariant plane that is determined by χ, and indeed, from (31), (31a), and (36), this yields:

$$\left.\begin{aligned} \boldsymbol{l}_1 &= \tfrac{1}{2}[-\varphi(\mathbf{e}_2 + i\mathbf{e}_1) + \mathbf{e}_4 - i\mathbf{e}_3], \\ \boldsymbol{l}_2 &= \tfrac{1}{2}[\mathbf{e}_3 - i\mathbf{e}_1 + \varphi(\mathbf{e}_4 + i\mathbf{e}_3)]. \end{aligned}\right\} \tag{37}$$

$\boldsymbol{l}_3$ and $\boldsymbol{l}_4$ are obtained from $\boldsymbol{l}_1$ and $\boldsymbol{l}_2$ when one replaces φ with χ.

A special transformation law under the maps (34) is true for the components α_i . The equation:

$$\mathbf{a}' \,\boldsymbol{\varepsilon} = \mathbf{b}\,\mathbf{a}\,\boldsymbol{\varepsilon}$$

reads, in matrix form:

$$\begin{pmatrix} \alpha_1' & \varphi\alpha_1' \\ \alpha_2' & \varphi\alpha_2' \end{pmatrix} = B \begin{pmatrix} \alpha_1 & \varphi\alpha_1 \\ \alpha_2 & \varphi\alpha_2 \end{pmatrix}. \tag{38}$$

[1]) Except for the point 0, 0, 0, 0, which is common to all invariant planes.
[2]) Naturally, this plane is always distinct from the left-invariant plane that goes through this point.

(38) is obviously equivalent to:

$$\begin{Bmatrix} \alpha_1' \\ \alpha_2' \end{Bmatrix} = B \begin{Bmatrix} \alpha_1 \\ \alpha_2 \end{Bmatrix}. \tag{39}$$

Likewise, one has:

$$\begin{Bmatrix} \alpha_3' \\ \alpha_4' \end{Bmatrix} = B \begin{Bmatrix} \alpha_3 \\ \alpha_4 \end{Bmatrix}.$$

For the maps (35), we introduce new basis vectors $\mathbf{r}_i$, such that β_1, β_2, β_3, β_4 become the components of **a** in the new coordinate system. From (31) and (31a), it then follows that:

$$\left.\begin{aligned} \mathbf{r}_1 &= \tfrac{1}{2}[\mathbf{e}_2 - i\mathbf{e}_1 - \chi(\mathbf{e}_4 - i\mathbf{e}_3)], \\ \mathbf{r}_2 &= \tfrac{1}{2}[\chi(\mathbf{e}_3 + i\mathbf{e}_1) + (\mathbf{e}_4 + i\mathbf{e}_3)]. \end{aligned}\right\} \tag{40}$$

One then gets $\mathbf{r}_3$ and $\mathbf{r}_4$ when one replaces χ with φ in $\mathbf{r}_1$ and $\mathbf{r}_2$.

For the map:

$$\mathbf{a}'' = \mathbf{a}\,\mathbf{b},$$

we get the following transformation law for the β_i from (30):

$$\begin{gathered} \begin{Bmatrix} \beta_1'' \\ \beta_2'' \end{Bmatrix} = B \begin{Bmatrix} \beta_1 \\ \beta_2 \end{Bmatrix}, \\ \\ \begin{Bmatrix} \beta_3'' \\ \beta_4'' \end{Bmatrix} = B \begin{Bmatrix} \beta_3 \\ \beta_4 \end{Bmatrix}. \end{gathered} \tag{41}$$

§ 7. Representation of Lorentz transformations by quaternions.

It is well-known that the LT's can be represented by quaternions [1]). We would like to briefly derive this fact. The four-vector $\hat{x}$ may regarded as the quaternion:

$$\mathbf{x} = x_1\,\mathbf{e}_1 + x_2\,\mathbf{e}_2 + x_3\,\mathbf{e}_3 + x_4\,\mathbf{e}_4\,;$$

since x_1, x_2, x_3 are real and x_4 is imaginary the quaternion $i\mathbf{x}$ is self-adjoint. Furthermore, if $\bar{\mathbf{q}}_1$ and $\mathbf{q}_2$ are two arbitrary unit quaternions then:

$$\mathbf{y} = \mathbf{q}_1\,\mathbf{x}\,\mathbf{q}_2 \tag{42}$$

represents an orthogonal four-dimensional transformation, since:

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[1]) F. Klein, Phys. Zeit. **12** (1911), 17; L. Silberstein, *Theory of Relativity*, London, 1924.

$$\mathcal{N}y = \mathcal{N}x.$$

However, in order for (42) to represent a LT, i**x**, along with i**y**, must be self-adjoint, so, from equation (8), one must have:

$$\mathbf{q}_1\,\mathbf{x}\,\mathbf{q}_2 = \tilde{\mathbf{q}}_2\mathbf{x}\tilde{\mathbf{q}}_1\,.$$

One easily confirms that this requirement leads to the equation:

$$\mathbf{q}_2 = \pm\,\tilde{\mathbf{q}}_1\,.$$

Thus, it is only with the use of the + sign that the requirement (5) is fulfilled. For an LT, one then has:

$$\mathbf{x}' = \mathbf{q}\,\mathbf{x}\,\tilde{\mathbf{q}}\,. \tag{43}$$

This transformation is always a proper LT. The quaternion **q** will therefore be determined uniquely by the LT $\mathfrak{L}_0$, up to sign.

We can likewise write down equation (43) in matrix form when we associate the unit quaternion **q** with the unimodular matrix Q and the quaternion **x** with the matrix X. One then has:

$$X' = Q\,X\,\tilde{Q}\,. \tag{44}$$

The element x'_{ik} of X' can be calculated by the rules of matrix multiplication to be:

$$x'_{ik} = \sum_{r=1}^{2}\sum_{s=1}^{2} q_{ir}\bar{q}_{ks}x_{rs}\,, \tag{45}$$

from which, according to (18):

$$x_{11} = x_4 + ix_3\,, \qquad x_{12} = -\,x_2 + ix_1\,, \qquad x_{21} = x_2 + ix_1, \qquad x_{22} = x_4 - ix_3\,.$$

We see that linear combinations of the coordinates (namely, the x'_{ik}) transform under proper LT's like tensorial objects of second rank on a two-dimensional (complex) manifold. The question of what the tensors of first rank would be on this manifold likewise leads to the spinors. Therefore, in the next paragraph we will arrive at the spinors in a natural way.

It is remarkable that by means of the representation (43) of LT's one gets all of the conditions that were posed in § 1 with no further assumptions. In particular, if **q** is a real unit quaternion – so Q is a unitary matrix – then (44) represents a similarity transformation under which the spur of X – thus, x_4 – also remains invariant; (43) then represents a spatial rotation in this case.

§ 8. Semi-vectors and spinors.

Equation (43) suggests the introduction of two types of quaternions, which transform under this LT according to the following schema:

$$\underline{\mathbf{u}}' = \underline{\mathbf{u}}\,\tilde{\mathbf{q}}\,, \tag{46}$$

$$\underline{\underline{\mathbf{v}}}' = \mathbf{q}\,\underline{\underline{\mathbf{v}}}\,, \tag{47}$$

where we put one (two, resp.) bars under a quaternion according to whether it transforms according to (46) [(47), resp.]. We will call a quaternion that transforms according to (46) [(47), resp.] a *semi-quaternion of the first (second, resp.) kind.*

We have essentially treated the transformations (46) and (47) already in § 6, and can thus briefly summarize them. Since $\mathbf{q}$ is a unit quaternion, the transformations (46) and (47) are orthogonal. The *norm* of a semi-quaternion of the first, as well as the second, kind is then an invariant under LT's, and furthermore, for two semi-quaternions $\underline{\mathbf{u}}$, $\underline{\mathbf{v}}$ of the first kind *the quaternion:*

$$\mathbf{w} = \underline{\mathbf{u}}\;\underline{\breve{\mathbf{v}}} \tag{48}$$

is an invariant under LT's, and for two *semi-quaternions* $\underline{\underline{\mathbf{u}}}$, $\underline{\underline{\mathbf{v}}}$ *of the second kind the quaternion:*

$$\mathbf{s} = \underline{\underline{\breve{\mathbf{u}}}}\;\underline{\underline{\mathbf{v}}} \tag{49}$$

is an invariant. Moreover, the quaternion that is *adjoint* to a semi-quaternion of the first kind is a semi-quaternion of the second kind, and vice versa; thus, if:

$$\mathbf{u}' = \mathbf{u}\;\tilde{\mathbf{q}}$$

then, from (8), one has:

$$\tilde{\mathbf{u}}' = \mathbf{q}\;\tilde{\mathbf{u}}\,.$$

The product of a semi-quaternion of the second kind with a semi-quaternion of the first kind:

$$\mathbf{t} = \underline{\underline{\mathbf{v}}}\;\underline{\mathbf{u}} \qquad \text{(observe the order of the terms!)}$$

transforms like the position quaternion $\mathbf{x}$, so it defines a four-vector.

We can *identify* the space of semi-quaternions of the first (second, resp.) kind with the space that we introduced in § 6 if we admit *only* the transformations (34) in it and restrict them with the demand of orthogonality, in addition. If a semi-quaternion in this space were regarded as a vector then we would call it a *semi-vector* of the first (second, resp.) kind. In the *space of semi-vectors* of the first (second, resp.) kind that we now consider, from § 6, *there is a family of* ∞^1 *invariant planes that generate the minimal cone.* The right-invariant planes of § 6 are now the invariant planes in the space of semi-vectors of the first kind and the left-invariant planes are the invariant planes in the space of semi-vectors of the second kind.

We can decompose each semi-vector of the first (second, resp.) kind into two *special semi-vectors* of the first (second, resp.) kind, *each of which lies in an invariant plane.* This decomposition may be accomplished simply with the help of a primitive idempotent quaternion $\boldsymbol{\varepsilon}$ by the formula:

$$\underline{\mathbf{u}} = \boldsymbol{\varepsilon}\underline{\mathbf{u}} + \breve{\boldsymbol{\varepsilon}}\,\underline{\mathbf{u}}$$

for a semi-vector of the first kind and by the formula:

$$\underline{\underline{\mathbf{v}}} = \underline{\underline{\mathbf{v}}}\boldsymbol{\varepsilon} + \underline{\underline{\mathbf{v}}}\,\breve{\boldsymbol{\varepsilon}}$$

for the semi-vectors of the second kind.

The semi-vectors of the first (second, resp.) kind that lie in an invariant plane will be called spinors of the first (second, resp.) kind; they thus have only two independent components. Each semi-vector of the first (second, resp.) kind can be decomposed into two spinors of the first (second, resp.) kind; however, this decomposition *depends upon the choice of two complex parameters that determine the planes in which these spinors lie.*

The transformation properties of spinors of the first (second, resp.) kind become especially simple when we introduce new basis vectors that depend on the choice of invariant plane. If we introduce them in the same way as we did in § 6 by means of (40) [(37), resp.] and we denote the components of spinors of the first (second, resp.) kind in these new coordinates by ξ_1, ξ_2 (η_1, η_2, resp.) then we get from equation (41) for the spinors of the first kind:

$$\begin{Bmatrix} \xi_1' \\ \xi_2' \end{Bmatrix} = \bar{Q} \begin{Bmatrix} \xi_1 \\ \xi_2 \end{Bmatrix},$$

and from equation (39), we get for the spinors of the second kind:

$$\begin{Bmatrix} \eta_1' \\ \eta_2' \end{Bmatrix} = Q \begin{Bmatrix} \eta_1 \\ \eta_2 \end{Bmatrix},$$

where Q is the matrix that represents the quaternion $\mathbf{q}$. The spinors of the second kind thus transform as complex conjugates to the spinors of the first kind.

If one defines the spinors by starting with equation (45) in the previous paragraph then the individual character of the invariant planes in which these spinors lie is lost; by the introduction of basis vectors that are adapted to the invariant plane, the transformation is indeed equal to $\bar{Q}$ (Q, resp.), so it is independent of the parameter that determines the plane. Naturally, this parameter itself remains invariant under a proper LT, due to the invariance of the plane that it determines.

We further remark that for two semi-vectors of the first (second, resp.) kind that lie in an invariant plane, the quaternion:

$$\mathbf{w} = \underline{\mathbf{u}}\,\underline{\breve{\mathbf{v}}} = \underline{\mathbf{u}}\,\boldsymbol{\varepsilon}\,\breve{\boldsymbol{\varepsilon}}\,\underline{\breve{\mathbf{v}}} = 0$$

which is invariant from (48), (the quaternion:

$$\mathbf{s} = \underline{\underline{\breve{\mathbf{u}}}}\;\underline{\underline{\mathbf{v}}} = \underline{\underline{\mathbf{u}}}\;\breve{\boldsymbol{\varepsilon}}\;\boldsymbol{\varepsilon}\;\underline{\underline{\mathbf{v}}} = 0$$

which is invariant from (49), resp.) must always vanish, so, in particular, the scalar product of two semi-vectors of the first (second, resp.) kind that lie in the same invariant plane must always vanish [1]).

Many times, it can be advantageous to carry out the decomposition of semi-vectors in such a way that both of the invariant planes $\mathfrak{r}_1$, $\mathfrak{r}_2$ ($\mathfrak{p}_1$, $\mathfrak{p}_2$, resp.) into which the space of semi-vectors of the first (second, resp.) kind gets split are *unitarily orthogonal* to each other – i.e., such that for each arbitrary semi-vector of the first (second, resp.) kind $\overset{1}{\mathbf{u}}$ ($\underset{1}{\mathbf{v}}$, resp.) that lies in $\mathfrak{r}_1$ ($\mathfrak{p}_1$, resp.) and for each arbitrary semi-vector of the first (second, resp.) kind $\overset{2}{\mathbf{u}}$ ($\underset{2}{\mathbf{v}}$, resp.) that lies in $\mathfrak{r}_2$ ($\mathfrak{p}_2$, resp.) the equation:

$$\overset{1}{u}_1\,\overline{\overset{2}{u}_1} + \overset{1}{u}_2\,\overline{\overset{2}{u}_2} + \overset{1}{u}_3\,\overline{\overset{2}{u}_3} + \overset{1}{u}_4\,\overline{\overset{2}{u}_4} = 0, \tag{50}$$

(the equation:

$$\underset{1}{v_1}\,\overline{\underset{2}{v_1}} + \underset{1}{v_2}\,\overline{\underset{2}{v_2}} + \underset{1}{v_3}\,\overline{\underset{2}{v_3}} + \underset{1}{v_4}\,\overline{\underset{2}{v_4}} = 0, \tag{51}$$

resp.) is true. Since:

$$\overset{1}{\mathbf{u}} = \boldsymbol{\varepsilon}\overset{1}{\mathbf{u}}, \qquad \overset{2}{\mathbf{u}} = \breve{\boldsymbol{\varepsilon}}\,\overset{2}{\mathbf{u}}, \qquad (\underset{1}{\mathbf{v}} = \underset{1}{\mathbf{v}}\boldsymbol{\varepsilon}, \qquad \underset{2}{\mathbf{v}} = \underset{2}{\mathbf{v}}\,\breve{\boldsymbol{\varepsilon}}, \text{ resp.}),$$

(50) [(51), resp.] may be fulfilled if and only if:

$$\overline{\boldsymbol{\varepsilon}}\;\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}\;\overline{\boldsymbol{\varepsilon}} = 0. \tag{52}$$

However, since $\boldsymbol{\varepsilon}$ is always idempotent, like $\overline{\boldsymbol{\varepsilon}}$, and the only primitive idempotent quaternion for which (52) is true is $\breve{\boldsymbol{\varepsilon}}$, one must have:

$$\overline{\boldsymbol{\varepsilon}} = \breve{\boldsymbol{\varepsilon}},$$

so $\boldsymbol{\varepsilon}$ must be self-adjoint. Since ε_1, ε_2, ε_3 are then pure imaginary, it follows from equation (24) that:

$$\varphi\,\overline{\chi} = -1.$$

We then get the following simple formulas for the components of $\boldsymbol{\varepsilon}$:

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[1]) The fourth component of the product $\mathbf{u}\,\breve{\mathbf{v}}$ is, in fact, the scalar product of the semi-vectors $\mathbf{u}$ and $\mathbf{v}$.

$$\varepsilon_1 = -\frac{i}{2}\frac{\varphi+\bar{\varphi}}{1+\varphi\bar{\varphi}}, \qquad \varepsilon_2 = -\frac{i}{2}\frac{\varphi-\bar{\varphi}}{1+\varphi\bar{\varphi}}, \qquad \varepsilon_3 = -\frac{i}{2}\frac{1-\varphi\bar{\varphi}}{1+\varphi\bar{\varphi}}, \qquad \varepsilon_4 = \frac{1}{2}.$$

In summary, we can say: The decomposition of semi-vectors of the first (second, resp.) kind into two spinors of the first (second, resp.) kind corresponds precisely to the decomposition of quaternions into two right-invariant (left-invariant, resp.) subalgebras. Each of the ∞^1 invariant planes in the space of semi-vectors of the first (second, resp.) kind can be regarded as the planes of spinors of the first (second, resp.) kind in it. The great simplicity of all the formulas when one expresses them in terms of quaternions shows that the latter are suitable tools for the description of semi-vectors and spinors.

At this point, let me cordially thank the *Fundusz Kultury Narodowej* for making it possible for me to do this work by the granting of a stipend.

Remark in correction. In the meantime, I have found that one can exhibit a wave equation very simply by means of quaternions that is invariant under the full group of Lorentz transformations. It reads:

$$\left.\begin{aligned} \mathbf{d}\,\bar{\boldsymbol{\psi}}\,\mathbf{a} &= \boldsymbol{\varphi}\,\mathbf{b}, \\ \breve{\mathbf{d}}\,\boldsymbol{\varphi}\,\mathbf{a} &= \breve{\boldsymbol{\psi}}\,\mathbf{b}, \end{aligned}\right\} \tag{53}$$

where $\boldsymbol{\psi}$ is a semi-vector of the first kind and $\boldsymbol{\varphi}$ is a semi-vector of the second kind. $\mathbf{d}$ means the quaternion whose components d_i are given by:

$$d_i = \frac{\partial}{\partial x_i} - i\,\varepsilon\,\Phi_i\,,$$

since Φ_i are thus the components of the four-potential. $\mathbf{a}$ and $\mathbf{b}$ are arbitrary constant quaternions that are subject to only the requirement that:

$$\mathbf{a} = -\tilde{\mathbf{a}}\,, \qquad \mathbf{b} = -\tilde{\mathbf{b}}\,.$$

The four-current:

$$\mathbf{I} = \tilde{\boldsymbol{\psi}}\,\breve{\mathbf{a}}\,\boldsymbol{\psi} + \boldsymbol{\varphi}\,\mathbf{a}\,\tilde{\boldsymbol{\varphi}}$$

is then divergence-free.

The system of equations (53) proves to be essentially identical to that of Einstein and Mayer [1]), which is know to explain the existence of two elementary particles with different masses. However, in quaternion form, (53) makes the discussion of this system of equations much simpler, as will be shown in a paper that will appear shortly in this journal.

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[1]) Proc. Kon. Ak. v. Wet. Amsterdam **36** (1933), 497.