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STATICS.

ON THE EQUILIBRIUM OF A FLEXIBLE CURVE ON A CURVED SURFACE

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PROBLEM: *What is the curve of double curvature that is affected by a ponderous, homogeneous, perfectly-flexible, and inextensible filament of well-defined length that is fixed at its two extremities by two points on an given, but arbitrary, curved surface that is subject to the action of gravity on that surface when one assumes that the surface exerts no friction? What is the tension in the filament at any of the points along it, and what is the normal pressure that it exerts on the curved surface at that point?*

SOLUTION: Let $S = 0$ be the equation of the proposed surface in terms of x, y, z when it is referred to three rectangular axes that are chosen in such a way that the z-axis is vertical. One lets α, β, γ denote the three angles that the normal at the point (x, y, z) makes with the respective coordinates. Upon setting:

$$\frac{dS}{dx} = P, \qquad \frac{dS}{dy} = Q, \qquad \frac{dS}{dz} = R, \tag{1}$$

$$\frac{1}{\sqrt{P^2 + Q^2 + R^2}} = V, \tag{2}$$

to abbreviate, one will have:

$$\cos\alpha = PV, \quad \cos\beta = QV, \quad \cos\gamma = RV, \tag{3}$$

and the differential equation for the surface S will be:

$$P\,dx + Q\,dy + R\,dz = 0. \tag{4}$$

Upon taking the sum of the products of equations (3) with P, Q, R, respectively, one will find that:

$$P\cos\alpha + Q\cos\beta + R\cos\gamma = (P^2 + Q^2 + R^2)\,V = \frac{1}{V}. \tag{5}$$

In these various formulas, the sign of V will vary in such a way that the flexible curve will lie on one side of the surface S or the other. One must determine it for each problem by considering the particular case, just as one determines the constants in integral calculus.

If we consider an arbitrary arc of the flexible curve and replace the tensions at the extremities of that arc with equivalent forces that are tangent to that curve at those points then we must define an equilibrium system in which equilibrium will still persist when we suppose that this portion suddenly becomes inflexible without changing its curvature.

That portion of the flexible curve exerts a pressure on the surface S at each point of that curve that one can replace with an equal and opposite force that is normal to that surface, but one must nonetheless remove it from the entire part that pertains to that arc of the flexible curve that is kept in equilibrium by the action of all those forces and will thus become a free system of invariable form in which the forces must consequently have a zero resultant. The sums of their components parallel to the three axes must then be zero separately, which will lead us to three equations in the givens and the unknowns of the problem. We shall therefore address the study of those equations.

Take the arc of the flexible curve that goes from its lowest point, at which its tangent is horizontal, up to any one of its points (x, y, z). Its tension at the lowest point can be associated with a weight, so we suppose that if one broke the curve at that location then if one were to maintain equilibrium, it would be necessary to add a vertical prolongation of a length a of the same density that passes through an infinitely-small fixed pulley. Now take the unit of weight to be the weight of a unit length of that curve, so we can say that the tension at its lowest point is a. Furthermore, since the x and y axes are simply subject to being rectangular in the horizontal xy-plane and can have arbitrary directions in that plane, moreover, we can make them turn in such a way that the tangent at the lowest point of the curve, and consequently the tension a at that point, will be parallel to the x-axis. As for the tension at the point (x, y, z) that points along the tangent at that point, we represent it by T. When we let s denote the arc of the curve that is found between the lowest point and the latter point, the components of that tension parallel to the three axes will be:

$$T\frac{dx}{ds},\quad T\frac{dy}{ds},\quad T\frac{dz}{ds},$$

resp.

Let N be the normal pressure that is exerted by the flexible curve on the surface S at the point (x, y, z). That pressure will be a function of the three coordinates of that point and will vary with it. However, along the entire extent of an element ds, one can regard it as constant and proportional to the extent of that element for which it will thus be expressed by $N\,ds$, and since it is normal to the surface S, its components parallel to the axes will be:

$$N\,ds\cdot\cos\alpha,\quad N\,ds\cdot\cos\beta,\quad N\,ds\cdot\cos\gamma,$$

respectively. We must then replace the sum of the components parallel to the axes of the pressure of the entire arc from the lowest point, which is the origin of S, with the integrals:

$$-\int N\cos\alpha\,ds\ ,\qquad -\int N\cos\beta\,ds\ ,\qquad -\int N\cos\gamma\,ds\ ,$$

resp.

Finally, to the components parallel to the z-axis, one adds the weight of the curve, taken negatively, and from the preceding conventions on the choice of units of weight and length, that must be expressed by $-s$.

If we then express the idea that the sums of the components parallel to the three axes are separately zero and observe that the tension a must have the opposite sign to the component of T parallel to the x-axis, we will have:

$$T\frac{dx}{ds} - \int N\cos\alpha\, ds - a = 0,$$

$$T\frac{dy}{ds} - \int N\cos\beta\, ds = 0,$$

$$T\frac{dz}{ds} - \int N\cos\gamma\, ds - s = 0\,.$$

One can infer the values of the two unknowns T and N from those three equations, along with an equation in x, y, z that is independent of those unknowns, and which will be the equation of a surface that cuts the surface S along the required curve. Upon differentiating them, they will become:

$$\left.\begin{aligned} dT\cdot\frac{dx}{ds} + Td\,\frac{dx}{ds} - N\cos\alpha\, ds = 0,\\ dT\cdot\frac{dy}{ds} + Td\,\frac{dy}{ds} - N\cos\beta\, ds = 0,\\ dT\cdot\frac{dz}{ds} + Td\,\frac{dz}{ds} - N\cos\gamma\, ds = 0.\end{aligned}\right\} \tag{6}$$

Since one has:

$$dx^2 + dy^2 + dz^2 = ds^2, \tag{7}$$

it will then follow upon considering s to be the independent variable that one has:

$$dx\cdot d\,\frac{dx}{ds} + dy\cdot d\,\frac{dy}{ds} + dz\cdot d\,\frac{dz}{ds} = 0. \tag{8}$$

In addition, since the normal to the surface S at (x, y, z) is perpendicular to the tangent to the curve at that point, one will have:

$$\frac{dx}{ds}\cos\alpha + \frac{dy}{ds}\cos\beta + \frac{dz}{ds}\cos\gamma = 0. \tag{9}$$

Finally, when equation (7) is divided by ds, that will give:

$$\frac{dx}{ds}dx + \frac{dy}{ds}dy + \frac{dz}{ds}dz = 0. \tag{10}$$

Having said that, if one takes the sum of the products of equations (6) with *dx*, *dy*, *dz*, respectively, while keeping the relations (8), (9), (10) in mind, then when one divides by *ds* that will give:

$$dT = dz,$$

so

$$T = z + A, \tag{11}$$

in which *A* is an arbitrary constant. Hence, the tension *T* at the point (*x*, *y*, *z*) is entirely independent of the length of the arc of the curve between that point to the lowest point of the curve, as well as the surface on which it is located. That tension depends upon only the vertical distance between those two points; i.e., the distance between the horizontal planes that contain them, respectively.

If one lets *c* denote the vertical coordinate of the lowest point of the curve then it will be necessary that $z = c$ must pertain to $T = a$, which will give:

$$a = c + A. \tag{12}$$

Hence, one will see that if one arranges the arbitrary *xy*-plane in such a way that one has $c = a$, it will result that $A = 0$, and as a result $T = z$; i.e., if one replaces the tension at the lowest point by a prolongation of the curve to a sufficient length that it can pass through an infinitely-small, frictionless pulley and hang vertically then one can replace the tension at another arbitrary point of that curve with a prolongation of the same nature that must then terminate below with the former on the same horizontal plane. Hence, one can conclude, more generally, that if one replaces the tensions at the two extremities of an arbitrary arc of the curve with prolongations of that curve of a sufficient length to pass through infinitely-small pulleys and hang vertically then those two prolongations must terminate below in the same horizontal plane (*). Therefore, it will further result that the difference between the tensions at two arbitrary points of the curves will be constantly equal to the weight of a portion of that curve whose length is equal to the vertical distance between those two points.

In order to preserve our results in full generality, we keep *A*, and upon replacing *T* and *dT* with their values $z + A$ and *dz*, resp., in equations (6) and dividing by *ds*, those equations will become:

$$\left.\begin{array}{l} \dfrac{dz}{ds}\dfrac{dx}{ds} + (z+A)\dfrac{d^2x}{ds^2} - N\cos\alpha = 0, \\ \dfrac{dz}{ds}\dfrac{dy}{ds} + (z+A)\dfrac{d^2y}{ds^2} - N\cos\beta = 0, \\ \dfrac{dz}{ds}\dfrac{dz}{ds} + (z+A)\dfrac{d^2z}{ds^2} - N\cos\gamma = 1. \end{array}\right\} \tag{13}$$

(*) As one sees, that is a generalization of what established in vol. XIX, pp. 347.

In order to easily infer the value of the normal pressure N at the point (x, y, z) from those equation, we take the sum of their products with P, Q, R, respectively, and when we recall the relations (4) and (5), that will give:

$$(z+A)\left(P\frac{d^2x}{ds^2}+Q\frac{d^2y}{ds^2}+R\frac{d^2z}{ds^2}\right)-\frac{N}{V}=R\,;$$

hence, one will infer that:

$$N = V\left\{(z+A)\left(P\frac{d^2x}{ds^2}+Q\frac{d^2y}{ds^2}+R\frac{d^2z}{ds^2}\right)-R\right\},$$

or rather (2):

$$N = \frac{(z+A)\left(P\frac{d^2x}{ds^2}+Q\frac{d^2y}{ds^2}+R\frac{d^2z}{ds^2}\right)-R}{\sqrt{P^2+Q^2+R^2}}. \tag{14}$$

Upon substituting that value in the last of equations (13), along with the value (3) of $\cos\gamma$, we will get:

$$(z+A)\left(P\frac{d^2x}{ds^2}+Q\frac{d^2y}{ds^2}\right) = (P^2+Q^2)\left\{(z+A)\frac{d^2z}{ds^2}-1\right\}+(P^2+Q^2+R^2)\left(\frac{dz}{ds}\right)^2 \tag{15}$$

for the second-order differential equation of a surface that must cut the surface S along the desired curve.

As a first application of these general formulas, suppose that the curve lies on an inclined plane, and in order to make the tangent at the lowest point be parallel to the x-axis, as those formulas require, make the inclined plane pass through that axis, and let ω be the angle that it makes with the xz-plane. Its equation will be:

$$y\cos\omega = z\sin\omega. \tag{16}$$

Here, we will have:

$$S = y\cos\omega - z\sin\omega,$$

$$P = \frac{dS}{dx} = 0, \qquad Q = \frac{dS}{dy} = \cos\omega, \qquad R = \frac{dS}{dz} = -\sin\omega,$$

so

$$P^2+Q^2 = \cos^2\omega, \qquad P^2+Q^2+R^2 = 1\,.$$

Upon substituting those values in formulas (14) and (15), they will become:

$$N = (z+a)\left(\frac{d^2y}{ds^2}\cos\omega-\frac{d^2z}{ds^2}\sin\omega\right)+\sin\omega, \tag{17}$$

$$(z + a)\left\{\frac{d^2y}{ds^2}\sin\omega+\frac{d^2z}{ds^2}\cos\omega\right\}\cos\omega+\left(\frac{dz}{ds}\right)^2 = \cos^2\omega, \tag{18}$$

resp. However, after two differentiations, equation (16) will give:

$$\frac{dy}{ds}=\frac{dz}{ds}\cdot\frac{\sin\omega}{\cos\omega}, \qquad \frac{d^2y}{ds^2}=\frac{d^2z}{ds^2}\cdot\frac{\sin\omega}{\cos\omega}. \tag{19}$$

If one substitutes those values in equations (17) and (18), they will become:

$$N = \sin\omega, \tag{20}$$

$$(z + A)\frac{d^2z}{dx^2}+\left(\frac{dz}{dx}\right)^2 = \cos^2\omega, \tag{21}$$

resp. Equation (20) tells us that the pressure on the inclined plane is constant at all points of the curve, and that for a unit length of that curve, it will be equal to the unit of weight multiplied by the cosine of the inclination of that plane above the horizon.

Equation (21) amounts to:

$$\frac{d\left\{(z+A)\dfrac{dz}{dx}\right\}}{ds} = \cos^2\omega,$$

which will give:

$$(z + A)\frac{dz}{dx} = s\cos^2\omega + B\cos\omega,$$

upon integrating, in which B is an arbitrary constant. Upon multiplying by 2 ds and integrating once more, one will infer from this that:

$$(z + A)^2 = s^2\cos^2\omega + 2Bs\cos\omega + C,$$

in which C is a new constant.

One infers from the last equation that:

$$s\cos\omega = -B \pm\sqrt{(z+A)^2-(C-B^2)}\ ;$$

hence, upon differentiating:

$$ds\cos\omega = \pm\frac{(z+A)\,dz}{\sqrt{(z+A)^2-(C-B^2)}},$$

and upon squaring:

$$ds^2 \cos^2 \omega; \quad \text{i.e.,} \quad (dx^2 + dy^2 + dz^2) \cos^2 \omega = \frac{(z+A)^2 dz^2}{(z+A)^2 - (C-B^2)},$$

or rather, upon replacing $dy^2 \cos^2 \omega$ with $dz^2 \sin^2 \omega$ using (19):

$$dx^2 \cos^2 \omega + dz^2 = \frac{(z+A)^2 dz^2}{(z+A)^2 - (C-B^2)},$$

which will give:

$$dx \cos \omega = \sqrt{C-B^2} \frac{dz}{\sqrt{(z+A)^2 - (C-B^2)}}.$$

That is then the differential equation of the projection of the curve onto the xz-plane.

If one desires that the x-axis should be tangent to the lowest point then it will be necessary for $z = 0$ to refer to $dz / dx = 0$, which will give $C - B^2 = A^2$, by means of which, the equation will become:

$$dx \cos \omega = \frac{A\,dz}{\sqrt{(z+A)^2 - A^2}}.$$

However, from (12), one will have $c = 0$, so $A = a$, which will give:

$$dx \cos \omega = \frac{a\,dz}{\sqrt{(z+a)^2 - a^2}}.$$

Upon integrating once more, that will give:

$$e^{(x \cos \omega)/a} = \frac{z + a + \sqrt{(z+a)^2 - a^2}}{D},$$

in which D is a new constant. If one desires, in addition, that the lowest point should be the origin of the coordinates then it will be necessary that x and z should be zero at the same time, which will give $D = a$, and as a result:

$$a \cdot e^{-(x \cos \omega)/a} = z + a + \sqrt{(z+a)^2 - a^2},$$

which is an equation that will finally give:

$$2\,(z + a) = a\{e^{(x \cos \omega)/a} + e^{-(x \cos \omega)/a}\}, \tag{22}$$

when it is solved with respect to $z + a$, which is then the original equation of the projection of the curve onto the vertical xz-plane. If one supposes that $\omega = 0$, so $\cos \omega = 1$, then it will become:

$$2\,(z + a) = a\,(e^{x/a} + e^{-x/a})\,;$$

i.e., the equation of the ordinary catenary, as it must be (*).

As a second application, suppose that the curve is situated on an arbitrary cylindrical surface that has its rectilinear elements vertical, and consequently parallel to the z-axis. The z-coordinate will not enter into S then, in such a way that one will have:

$$R = \frac{dS}{dz} = 0,$$

so when one divides equations (14) and (15) by $P^2 + Q^2$ they will then become:

$$N = \frac{(z+A)\left(P\dfrac{d^2x}{ds^2} + Q\dfrac{d^2y}{ds^2}\right)}{\sqrt{P^2+Q^2}}, \tag{23}$$

$$(z + A)\frac{d^2z}{ds^2} + \left(\frac{dz}{ds}\right)^2 = 1. \tag{24}$$

The last equation is nothing by equation (21) in which one has set $\cos\omega = 1$ and changed x into s. If, as we have done, we then place the lowest point at the origin of the coordinates then, from (22), it will give:

$$2\,(z + a) = a\,(e^{s/a} + e^{-s/a})\,. \tag{25}$$

However, if one develops the cylindrical surface on a vertical plane then z and s will be the rectangular coordinates of the curve. Hence, the development of a curve that is situated on a cylindrical surface with vertical rectilinear elements will be an ordinary curve, which should have been easy to predict.

As for the normal pressure (23) that is exerted by the curve on the cylindrical surface at each point, one senses that it must differ according to the nature of that surface. In order to give an example of the manner of calculating it in each case, suppose that one is dealing with a cylinder of revolution of radius equal to r that has the z-axis for its axis; one will have:

$$x^2 + y^2 = r^2, \tag{26}$$

(*) One can arrive at the differential equation of the catenary on an arbitrary inclined plane directly by considering the fact that gravity along an inclined plane is nothing but gravity in a vertical plane multiplied by the sine of the inclination of the plane. One then concludes the differential equation of the catenary on an arbitrary curved surface by considering that at each of its points, that curve will be found to be situated in the tangent plane to the curved surface at that point. One can also obtain the equation of the catenary that is situated on a curved surface by proposing to trace out a curve on that surface such that the distance from its center of gravity to the xy-plane will be the least possible, which presents an interesting application of the method of variations.

so

$$S = x^2 + y^2 - r^2,$$

$$P = \frac{dS}{dx} = 2x, \quad Q = \frac{dS}{dy} = 2y, \quad P^2 + Q^2 = 4\,(x^2 + y^2) = 4r^2\,.$$

Upon substituting this in (23) and observing that one has $A = a$ here, that will give:

$$r\,N = (z + a)\left(x\frac{d^2x}{ds^2} + y\frac{d^2y}{ds^2} \right). \tag{27}$$

However, after two successive differentiations, equation (26) will give:

$$x\frac{dx}{ds} + y\frac{dy}{ds} = 0,$$

$$x\frac{d^2x}{ds^2} + y\frac{d^2y}{ds^2} = -\left(\frac{dx}{ds}\right)^2 - \left(\frac{dy}{ds}\right)^2 = \left(\frac{dz}{ds}\right)^2 - 1.$$

Upon substituting that in (27), one will then have:

$$r\,N = (z + a)\left\{\left(\frac{dz}{ds}\right)^2 - 1\right\};$$

on the other hand, equation (25) will give:

$$\frac{dz}{ds} = \tfrac{1}{2}(e^{s/a} - e^{-s/a}),$$

so

$$\left(\frac{dz}{ds}\right)^2 = \tfrac{1}{4}(e^{2s/a} - e^{-2s/a} - 2)\,,$$

and

$$\left(\frac{dz}{ds}\right)^2 - 1 = \tfrac{1}{4}(e^{2s/a} - e^{-2s/a} + 2) = \left(\frac{e^{s/a} + e^{-s/a}}{2}\right)^2 = \left(\frac{z + a}{a}\right)^2.$$

Upon substituting, that will give:

$$N = \frac{(z+a)^3}{a^2 r};$$

i.e., if one draws another plane below the xy-plane that is also horizontal and is at a distance that is equal to the length of the arc of the curve whose weight brings equilibrium to the tension at the lowest point then the pressures will grow like the cubes of the altitudes of the various points of the curve above that plane. All other things being equal, they will grow larger as the radius of the cylinder grows smaller.

As a third application, suppose that the curve sits on a conical surface of revolution that has the z-axis for its axis, the origin for its summit, and its generating angle equal to ω; that is the problem that was posed on page 87 of vol. XVIII of the *Annales*. The equation of the cone will then be:

$$(x^2 + y^2)\cos^2\omega = z^2 \sin^2\omega, \tag{28}$$

in such a way that one will have:

$$S = (x^2 + y^2)\cos^2\omega - z^2\sin^2\omega,$$

so

$$P = \frac{dS}{dx} = 2x\cos^2\omega, \qquad Q = \frac{dS}{dy} = 2y\cos^2\omega, \qquad R = \frac{dS}{dz} = -2z\sin^2\omega,$$

$$P^2 + Q^2 = 4(x^2 + y^2)\cos^4\omega, \quad P^2 + Q^2 + R^2 = 4\{(x^2 + y^2)\cos^4\omega + z^2\sin^4\omega\},$$

and when one substitutes those values in formulas (14) and (15), they will become:

$$N = \frac{(z+A)\left\{\left(x\frac{d^2x}{ds^2} + y\frac{d^2y}{ds^2}\right)\cos^2\omega - z\frac{d^2z}{ds^2}\sin^2\omega\right\} + z\sin^2\omega}{\sqrt{(x^2+y^2)\cos^4\omega + z^2\sin^4\omega}},$$

$$z(z+A)\sin^2\omega\cos^2\omega + (x^2+y^2)\left\{(z+A)\frac{d^2z}{ds^2} - 1\right\}\cos^4\omega$$
$$+ \left\{(x^2+y^2)\cos^4\omega + z^2\sin^4\omega\right\}\left(\frac{dz}{ds}\right)^2 = 0.$$

Upon next replacing $(x^2 + y^2)\cos^2\omega$ with its value $z^2\sin^2\omega$ in these two equations, they will become:

$$Nz\sin\omega = (z+A)\left\{\left(x\frac{d^2x}{ds^2} + y\frac{d^2y}{ds^2}\right)\cos^2\omega - z\frac{d^2z}{ds^2}\sin^2\omega\right\} + \sin^2\omega, \tag{29}$$

$$(z+A)\left(x\frac{d^2x}{ds^2} + y\frac{d^2y}{ds^2} + z\frac{d^2z}{ds^2}\right)\cos^2\omega + z\left(\frac{dz}{ds}\right)^2 = z\cos^2\omega. \tag{30}$$

However, after two differentiations, one will infer from equation (28) that:

$$\left(x\frac{dx}{ds}+y\frac{dy}{ds}\right)\cos^2\omega = z\frac{dz}{ds}\sin^2\omega,$$

$$\left(x\frac{d^2x}{ds^2}+y\frac{d^2y}{ds^2}\right)\cos^2\omega = z\frac{d^2z}{ds^2}\sin^2\omega - \left\{\left(\frac{dx}{ds}\right)^2+\left(\frac{dy}{ds}\right)^2\right\}\cos^2\omega+\left(\frac{dz}{ds}\right)^2\sin^2\omega,$$

or rather:

$$\left(x\frac{d^2x}{ds^2}+y\frac{d^2y}{ds^2}\right)\cos^2\omega = z\frac{d^2z}{ds^2}\sin^2\omega - \left\{1-\left(\frac{dz}{ds}\right)^2\right\}\cos^2\omega+\left(\frac{dz}{ds}\right)^2\sin^2\omega,$$

or finally:

$$\left(x\frac{d^2x}{ds^2}+y\frac{d^2y}{ds^2}\right)\cos^2\omega = z\frac{d^2z}{ds^2}\sin^2\omega + \left(\frac{dz}{ds}\right)^2 - \cos^2\omega.$$

Substituting that in equations (29) and (30) will give:

$$Nz\sin\omega = (z+A)\left\{\left(\frac{dz}{ds}\right)^2 - \cos^2\omega\right\} + z\sin^2\omega, \tag{31}$$

$$z\,(z+A)\frac{d^2z}{ds^2}+(2z+A)\left\{\left(\frac{dz}{ds}\right)^2 - \cos^2\omega\right\} = 0. \tag{32}$$

One can put the latter into the form:

$$(2z+A)\sqrt{\cos^2\omega-\left(\frac{dz}{ds}\right)^2} - z\,(z+A)\frac{\dfrac{d^2z}{ds^2}}{\sqrt{\cos^2\omega-\left(\dfrac{dz}{ds}\right)^2}} = 0,$$

or, upon multiplying by dz:

$$\sqrt{\cos^2\omega-\left(\frac{dz}{ds}\right)^2}\,dz\,(z+A) + z\,(z+A)\;d\sqrt{\cos^2\omega-\left(\frac{dz}{ds}\right)^2} = 0,$$

or rather:

$$d\left\{z\,(z+A)\sqrt{\cos^2\omega-\left(\frac{dz}{ds}\right)^2}\right\} = 0,$$

and upon integrating, that will give:

$$z\,(z + A)\,\sqrt{\cos^2\omega - \left(\frac{dz}{ds}\right)^2} = E,$$

in which E is an arbitrary constant. Upon letting c denote the vertical coordinate of the lowest point of the curve, as usual, it will be necessary that $z = c$ must refer to $dz / ds = 0$, which will give:

$$c\,(c + A)\cos\omega = E,$$

or rather (12):

$$E = a\,c\cos\omega,$$

in such a way that the integral will become:

$$z\,(z + A)\sqrt{\cos^2\omega - \left(\frac{dz}{ds}\right)^2} = a\,c\cos\omega. \tag{33}$$

One can infer that:

$$\left(\frac{dz}{ds}\right)^2 - \cos^2\omega = -\frac{a^2c^2\cos^2\omega}{z^2(z+A)^2}, \tag{34}$$

which is a value that will give:

$$N = \frac{z^3(z+A)\sin^2\omega - a^2c^2\cos^2\omega}{z^3(z+A)\sin\omega}, \tag{35}$$

when it is substituted in formula (31), and this is a formula to which we shall return shortly.

Since the right-hand side of equation (33) is constant and its left-hand side is the product of three factors, the first two of which increase with z, it will follow that the third factor must become smaller and smaller as z becomes larger and that it must be precisely zero when z is infinite, so one must then have:

$$\frac{dz}{ds} = \pm\cos\omega,$$

which can obviously be true only to the extent that the curve coincides with a generator of the cone. Hence, the asymptotes of the conical curve will be two generators of the surface on which it is found.

In the case of z infinite, formula (35) will give simply $N = \sin\omega$, and that must indeed be the case, since the infinite branches of the curve are the same as in the case of a rectilinear curve that lies on an inclined plane along the direction of its greatest slope.

In the case where one has $A = 0$ or $c = a$, as in (12), formula (35) will become:

$$N = \frac{z^4\sin^2\omega - a^4\cos^2\omega}{z^4\sin\omega};$$

the pressure will then be zero when one has:

$$z^2 \sin \omega = a^2 \cos \omega, \qquad \text{in which} \qquad z = \frac{a}{\sqrt{\tan \omega}}.$$

That value of z will be greater than a – i.e., greater than the distance from the lowest point of the curve to the summit of the cone – whenever the generator angle ω is less than half a right angle, so the lower part of the curve will leave the cone in order to become an ordinary catenary. The extreme tensions of that new catenary will be (11):

$$T = \frac{a}{\sqrt{\tan \omega}}.$$

Equation (34) will give:

$$\left(\frac{ds}{dz}\right)^2 \qquad \text{or} \qquad \frac{dx^2 + dy^2}{dz^2} + 1 = \frac{z^3 (z + A)^2}{\{z^2 (z + A)^2 - a^2 c^2\} \cos^2 \omega},$$

and consequently:

$$dx^2 + dy^2 = \frac{z^2 (z + A)^2 \sin^2 \omega + a^2 c^2 \cos^2 \omega}{\{z^2 (z + A)^2 - a^2 c^2\} \cos^2 \omega} dz^2,$$

or even better:

$$z^2 (z + A)^2 \{(dx^2 + dy^2) \cos^2 \omega - dz^2 \sin^2 \omega\} = a^2 c^2 (dx^2 + dy^2 + dz^2) \cos^2 \omega;$$

however, one infers from equation (28) that:

$$z = \sqrt{x^2 + y^2} \frac{\cos \omega}{\sin \omega}, \qquad dz = \frac{x\,dx + y\,dy}{z\sqrt{x^2 + y^2}} \cdot \frac{\cos \omega}{\sin \omega},$$

$$z + A = \frac{A \sin \omega + \sqrt{x^2 + y^2} \cos \omega}{\sin \omega}.$$

Upon substitution, that will give:

$$(x^2 + y^2)\{A \sin \omega + \sqrt{x^2 + y^2} \cos \omega\}^2 \{(x^2 + y^2)(dx^2 + dy^2) - (x\,dx + y\,dy)^2\}$$

$$= a^2 c^2 \{(x^2 + y^2)(dx^2 + dy^2) \sin^2 \omega - (x\,dx + y\,dy)^2 \cos^2 \omega\} \sin^2 \omega,$$

and that is the differential equation of the projection of the curve onto the xy-plane.

If we set:

$$x = r \sin \theta, \qquad y = r \cos \theta,$$

in order to pass to polar coordinates then it will result that:

$$dx = dr \sin\theta + r\, d\theta \cos\theta, \qquad dy = dr \cos\theta - r\, d\theta \sin\theta,$$

$$x^2 + y^2 = r^2, \qquad dx^2 + dy^2 = dr^2 + r^2\, d\theta^2, \qquad x\, dx + y\, dy = r\, dr,$$

which will give:

$$r^4 (A \sin\omega + r \cos\omega)^2\, d\theta^2 \cos^2\omega = a^2 c^2 (dr^2 + r^2\, d\theta^2 \sin^2\omega) \sin^2\omega,$$

upon substitution, which is an equation from which one can infer that:

$$d\theta = \frac{ac\, dr \sin\omega}{r\sqrt{(A\sin\omega + r\cos\omega)^2 r^2 \cos^2\omega - a^2c^2\sin^4\omega}}.$$

It does not seem that this value is generally integrable in finite form.

As a consequence, we confine ourselves to considering the case in which $A = 0$ and $c = a$; it will then become:

$$d\theta = \frac{a^2\, dr \sin\omega}{r\sqrt{r^4\cos^4\omega - a^4\sin^4\omega}},$$

and upon integration, that will give:

$$2\,(\theta + F)\sin\omega = \arccos\frac{a^2\sin^2\omega}{r^2\cos^2\omega},$$

in which F is the arbitrary constant. In order to determine it, one must remark that θ is zero for the projection of the lowest point on the xy-plane and that one must then have:

$$r = a\frac{\sin\omega}{\cos\omega},$$

which will give $F = 0$, in such a way that one has simply:

$$2\theta\sin\omega = \arccos\frac{a^2\sin^2\omega}{r^2\cos^2\omega}, \tag{36}$$

for the required polar equation, which will give:

$$\theta = \pm\frac{\omega}{4\sin\omega}$$

for r infinite; that is then one-half the angle that is subtended between the projections of the asymptotes of the curve in this case.

In this case, it will be easy to learn the nature of the curve that is described by the flexible curve on the development of the cone. Let R be the radius vector of that curve, whose projection is r, and let Θ be the angle of the development of the cone that pertains to the angle θ, so one will have:

$$r = \frac{R}{\sin\omega}, \qquad \theta = \frac{\Theta}{\sin\omega}.$$

If one then substitutes that in equation (35) then it will become:

$$2\Theta = \arccos \frac{a^2 \sin^4 \omega}{R^2 \cos^2 \omega},$$

or rather:

$$a^2 \sin^4 \omega = R^2 \cos^2 \omega \cos 2\Theta = R^2 (\cos^2 \Theta - \sin^2 \Theta) \cos \omega.$$

If one sets:

$$R \sin \Theta = X, \quad R \cos \Theta = Y,$$

in order to have rectangular coordinates then, upon substitution, it will become:

$$(Y^2 - X^2) \cos^2 \omega = a^2 \sin^4 \omega,$$

which will be the equation of an equilateral hyperbole no matter what the generating angle of the cone is, moreover.

As a last application, suppose that the curve is located on a sphere that is given by the equation:

$$x^2 + y^2 + z^2 = r^2, \tag{37}$$

and after two differentiations, that will give:

$$x\,dx + y\,dy + z\,dz = 0, \tag{38}$$

$$x\frac{d^2x}{ds^2} + y\frac{d^2y}{ds^2} + z\frac{d^2z}{ds^2} + 1 = 0, \tag{39}$$

and as a result:

$$P = x, \quad Q = y, \quad R = z, \quad P^2 + Q^2 = x^2 + y^2 = r^2 - z^2,$$

$$P^2 + Q^2 + R^2 = r^2.$$

With the aid of those various results, formulas (14) and (15) will become:

$$N = -\frac{2z + A}{r}, \tag{40}$$

$$r^2\,(z + A)\,\frac{d^2z}{ds^2} + r^2\left(\frac{dz}{ds}\right)^2 + z\,(2z + A) = r^2. \tag{41}$$

Formula (39) shows that, all other things being equal, the pressure is inversely proportional to the radius of the sphere. In the particular case of $A = 0$ or $c = a$, one will have $N = -\,2z\,/\,r$, in such a way that the pressure will be proportional to the elevation at each point of the curve above the plane of the horizontal great circle. It will have its maximum at the pole of the circle, at which it will be equal to 2; i.e., to twice the weight per unit length of the curve.

In that same special case, equation (41) will become simply:

$$z\,\frac{d^2z}{ds^2} + \left(\frac{dz}{ds}\right)^2 = \frac{r^2 - 2z}{r^2},$$

or rather:

$$r^2 d\left\{\frac{d(r^2 - 2z^2)}{ds}\right\}^2 + 4d\,(r^2 - 2z^2)^2 = 0,$$

which will give:

$$r^2\left\{\frac{d(r^2 - 2z^2)}{ds}\right\}^2 + 4\,(r^2 - 2z^2)^2 = G,$$

after a first integration. One determines the constant G by observing that when $dz\,/\,ds$ or $\frac{d(r^2 - 2z^2)}{ds}$ vanishes, one must have $z = c$, which will give:

$$4\,(r^2 - 2c^2)^2 = G,$$

and as a result:

$$r^2\left\{\frac{d(r^2 - 2z^2)}{ds}\right\}^2 = 4\{(r^2 - 2c^2)^2 - (r^2 - 2z^2)^2\}.$$

Upon taking the square root, one will get:

$$\frac{2\,ds}{r} = \frac{-\,d(r^2 - 2z^2)}{\sqrt{(r^2 - 2c^2)^2 - (r^2 - 2z^2)^2}},$$

and upon integrating, that will give:

$$\frac{2\,s}{r} = \arccos\frac{r^2 - 2z^2}{r^2 - 2c^2}.$$

There is no point in adding a constant here, since $s = 0$ when $z = c$, as it must be.

At the point for which $z = r / \sqrt{2}$, one will have $2s / r = \omega / 2$, so $s = \pi r / 4$. In the particular case in question, that will then be the length of the portion of the curve that is found between the point in question down to the lowest point. Hence, it will follow that the total arc length of the curve that is found between two points that are situated in that way is equal to precisely one-fourth of the circumference of a great circle.

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