Excerpted from L. Boltzmann, *Vorlesungen über die Prinzipe der Mechanik*, II Teil, Johann Ambrosius Barth, Leipzig, 1904, chap. IV, pp. 162-212.

IV. Analogies with physical, especially heat-theoretic, theorems.

§ 42. Analogue of supplied heat.

The special character of the equations of thermodynamics is qualified by the fact that the increment in supplied heat is not a complete differential expression. However, the differential δE of the supplied energy will always be a complete differential expression, as long as we consider scleronomic systems, as in the previous paragraphs. Thus, as long as we restrict ourselves to the consideration of such systems and place the differential of the supplied heat on a parallel with the increment δE in total energy, no analogy will exist already in this most important context.

On that basis, *Clausius* has already considered systems in which there are distant forces whose laws of action change in time, such that parameters that are very slowly-varying in time enter into the force function V in place of certain constants, and the mechanical system that is considered is *rheonomic*.

If one thinks of, e.g., a piston that seals a warm gas from the action of repulsive normal forces that are external to the gas molecules, and which suddenly assume enormous values in close proximity to the outer surface, then one can think of a slow withdrawal of the piston under the action of a slow change in the force function of those forces. *Clausius* also treated central motions for which the law of action of the central force contained parameters that varied in time analogously.

However, a complication in the calculations entered into this *Clausius* picture of the variability of the law of action for the forces of nature. One always added an additive, arbitrary constant to the force function V of the these forces, which we can think of as being determined by the fact that for a certain position of all material points of the system (viz., the zero level of the potential) one will have $V = 0$. For scleronomic systems, it is entirely irrelevant which position one chooses for it. By contrast, if the law of action of the force changes in time then the work that is required to go from one zero position to another will also change. The absolute value of V then changes in various ways, according to whether one chooses one zero position or the other one, and in order to determine it completely, one must state which special position one has chosen to be the zero position.

It is always best to choose that position to be the one in which all material points are so far from each other and from all remaining points that act upon them that no perceptible force acts upon any of them. In physical cases, this choice of zero position will always be possible. It is only for laws of force that are constructed by mathematical abstraction – e.g., when the force that acts between two material points is directly proportional to the distance between them – that this choice of zero position can become impossible.

The *Clausius* assumption that the law of action of the forces that act between the material points changes in time indeed gives a complete analogy with the thermodynamic equations; we observe nothing in nature itself that would suggest that the law of action of certain natural forces would vary in time. Indeed, physics research would even cease completely if we did not know that the laws of nature that we had found up to now would still be correct for later times. Therefore, under the *Clausius* assumption, one can arrive at an unambiguous definition of an extremely fluctuating energy balance only under more or less arbitrary assumptions, and one is advised to replace the assumption of the variability of the law of action of the forces with the assumption that (v) material points interact with the n material points that define the system considered. The former points shall be completely immobile during the unvaried motion; during the variation of the motion, however, they will change their position extremely slowly. The aforementioned computational difficulty then drops away.

It will not be the energy that is supplied by all of the n points (which has been regarded as analogous to the heat that is supplied), but the work that is done on the n points as a result of their motion under the influence of the forces that act upon them that are due to the ν points, that corresponds to the external work that is supplied to a body, and merely the remaining supplied energy that corresponds to the supplied heat, such that the differential of the energy component that corresponds to the supplied heat is not a complete differential, while the differential of the total supplied energy still is.

The position of the n material points shall be determined by s coordinates (viz., the rapidly-varying ones), but those of the v points shall be determined by g coordinates (viz., the slowly-varying parameters).

One thus obtains (e.g., in the following way) a good picture of the irreversible changes of state of a gas that is sealed by a piston. The molecular motion and internal atomic motion of the gas molecule can correspond to the rapid motion of the n material points that are being considered. The molecules of the piston, whose thermal motion we can ignore without changing the problem substantially, can correspond to the v material points that move only under a variation of the state of the gas (and indeed extremely slowly as long as its changes of state are irreversible).

§ 43. Concept of cyclic motions and the motions that they relate to.

One now deals once more with ascribing properties to the rapid motion of the n material points that would serve as characteristic properties of a true picture of thermal motion as much as possible.

The problem of giving a brief, systematic summary of all basic types of mechanical systems that could be used for that purpose will thus be complicated by the fact that many of them have some features in common with other ones, and the various authors sometimes regard one feature and sometimes the other as the most essential one, which also makes the terminology fluctuate. In the summary that I seek here, I will then strive to attain either completeness or the greatest possible overview of the classification, and I will also be forced to deviate somewhat in the terminology, sometimes of one author and sometimes of another.

If we accept the basic picture of the mechanical theory of heat then one of the striking properties of thermal energy consists of the fact that in a warm body one indeed continually finds the most animated motion in the smallest particles, but we nevertheless discern no variation in its externally-visible and perceptible state, while we clearly distinguish how the state of a body continually changes in time when it moves.

We also find this property in other domains of physics. We do not see the slightest temporal variation in an electric current of unvarying intensity, in whose vicinity one finds magnets or masses of iron at rest, except what is due to the driving battery, and yet *Maxwell* explained its properties by the hypothesis that the essence of the electric current consists of a violent motion whose arena is partially inside of the current conductor and partially in the surrounding ether.

We must then look at mechanical models that possess similar properties. An example of such a model is given by a rotating rigid body that is absolutely symmetric around its axis that exhibits no motion besides a rapid rotation around that axis. Another example is given by an irrotational current in an absolutely homogeneous, incompressible, inviscid fluid in a channel that returns to itself and has absolutely rigid walls. We refer to such motions as *cyclic.*

Special cyclic systems have already been employed in mechanics and the theory of heat, especially by *Rankine*. *Maxwell* first treated general cyclic systems and employed them in order to explain electromagnetic and electrodynamic phenomena. Its application to the theory of heat in general form was made by *Rankine*, and we have *Helmholtz* to thank for the further development of the basic equations for it, which had already been presented by *Maxwell*, as well as for the foundations of the now-conventional terminology.

Cyclic systems, in the strictest sense (in what follows, we would like to call them *true cycles*), are ones in which arbitrary motions are indeed found, and thus, such that when any massive particle leaves a location in space, a completely identical one will always immediately replace it that has the same parallel velocity that the first particle had at that location in space. A coordinate is called a *true cyclic* coordinate when the system performs a motion such that the specified coordinate changes while the remaining coordinates remain constant.

The molecular motions that represent heat in the mechanical theory of heat are not strictly cyclic according to the picture of that theory. It is only for a large number of moving molecules that as soon as a molecule leaves a certain state of motion, it always soon arrives in the neighborhood of another molecule in a very similar state of motion, such that we perceive no change externally. For that reason, one must extend the concept of the true cyclic system: The characteristic of the true cyclic system consists of the fact that all of its properties depend, not upon the absolute values of the true cyclic coordinates, but merely on their rates of change. The value of a cyclic coordinate that is not differentiated with respect to time can thus enter into either the expression for the *vis viva* or into the expression for the forces that act upon the system or into the functions that express conditions. In generalizing the concept of true cyclic coordinates, we would like to follow *Hertz's* lead in casually referring to any coordinate whose value, which is not differentiated with respect to time, enters into all of the expressions as a *cyclic* coordinate.

When either internal or external forces act upon a system, as *Hertz* assumed of all systems, the rectangular coordinates themselves will be cyclic, in the event that no condition equations are present.

Systems whose motion is periodic have a certain relationship to cyclic systems when they thus incessantly repeat precisely the same state of motion in the same sequence in the course of a long span of time, and we would like to briefly call them *periodic systems*. When the periodically moving masses only play a subordinate role, the periodic systems can have almost all properties of the cyclic ones, although they are distinguished from true cyclic systems only by, e.g., that fact that they contain rotating gears, pistons that go back and forth, or moving masses that oscillate in other ways.

Helmholtz went further and considered systems that are subject to merely the condition that not only the sum of the kinetic and potential energy, but each of those energies individually, always remains constant. *Clausius* defined an even more general concept when he referred to a motion for which the value of any of the rectangular coordinates or any of the velocity components of a material point in the coordinate directions never increases beyond all bounds when the motion takes place for an arbitrarily long time interval as a *stationary* motion, while I would rather prefer to use the word "finite." If, in addition, the motion is indeed not periodic in the sense that all material points simultaneously return to precisely their old position, speed, and direction in the course of a finite time and then begin the same motion all over again, but a regularity exists in the motion such that the temporal mean of the *vis viva*, one velocity component, the value of any rectangular coordinate of any material point, or the total force function V, etc., hastens to a fixed limit when one allows the time interval over which that mean is taken to exceed all limits in an arbitrary way without varying the motion, then we would like to call such a motion a *mesic* one.

§ 44. Special examples.

Before we go into the calculations, we would like to clarify what we said with some examples.

The *first example* is the one that has been already been mentioned several times that comes from the kinetic theory of gases or fluids that can form droplets. The system will be defined by n material points that, from the intuitions of the mechanical theory of heat, move just like the molecules of a gas that follows the *van der Waals* law or a fluid that can form droplets in a cylindrical vessel with rigid walls that is closed from above by a completely dense, frictionless piston. The raising of the *vis viva* (perhaps by molecular impacts that come from anywhere outside) corresponds to the heat supplied by the raising of the temperature, whereas the work that is done against the internal forces that act between the n material points corresponds to the internal work. The s variables are the quantities that are necessary for the determination of the position of the n material points.

The piston always moves only slowly, such that its pressure is always nearly equal to the opposing pressure of the material points between which equilibrium in the *vis viva* should also nearly exist. The g variables determined the position of the piston. The work that is done by the force that the piston exerts on the material points is the external work. It is equal to the work that is done on the piston by the forces that act upon it from the outside. For a sufficiently large number of molecules, this system is not a true cycle, but it is isokinetic, finite, and mesic, but still not periodic.

Second example: central motion model. As we already did at the conclusion of paragraph 41, we similarly consider the central motion of a single material point. However, let us take precautions that the two constants λ and a, which determine the law according to which the central force acts, should vary slowly during the central motion.

In place of the *Clausius* assumption of a direct variability in the laws of nature, we would like to think of the variability of λ and a as coming about by ordinary mechanical means. If we first treat the central motion of a planet around the Sun then we can perhaps arrange that masses (e.g., meteor stones) always plunge into the Sun from the outside, such that its mass, and therefore also its force of attraction to the planets, will also increase in time. If one would like to construct a closed process that is analogous to the *Carnot* circular process then one would have to, e.g., first plunge masses into the Sun. That would produce external work. The *vis viva* of the central motion, which corresponds to the thermal energy of the warm body, would then be reduced. The same masses must then be once more taken from the Sun out to an infinite distance. Less work would need to be done by that than was gained before by the plunging, since indeed the planet is now further away and exerts less attraction. Finally, the energy of the orbital motion of the planet must again be brought to the old place by a corresponding supply of energy, and we assume that the form, position, and velocities of motion are again the same at the conclusion of the process as they were at the start. Since the path is always closed here, complete analogies with the second law already exist. If $\bar{T}$ is the mean *vis viva* of the planet in its orbital motion, and δQ is the energy that one must supply for the purpose of raising the *vis viva* then it will not be δQ, but $\delta Q / \bar{T}$, that is a complete differential, as long as the mass of the Sun always increases and decreases so slowly that the increase or decrease during a planetary orbit can be considered to be small and uniform in time. One can verify that by performing the calculations in detail, although the introduction of masses into the Sun is, after all, still a somewhat inconvenient process for the sake of calculations.

A device that is indeed somewhat abstract, but mechanically much clearer, and which illustrates all conceivable cases in the greatest generality is the following one: A very small, completely smooth ball of mass m moves on a smooth, horizontal plane. Let a flexible, massless string of unvarying length be fixed upon it that goes through a hole in the plane and then hangs down vertically and carries a massless, magnetic pole A that moves without friction in a vertical tube at its end. Vertically beneath it, one finds an extremely short magnet that can rotate around a horizontal axis whose very close poles shall be called B and C.

One can now slowly supply *vis viva* to the small ball m during the central motion by means of small impacts (this corresponds to the supply of heat) and also slowly rotate the magnet (which corresponds to the motion of the piston). One can thus vary the state slowly and also return to the old state of motion in another way when one, e.g., first rotates the short magnet by less agitated motion of the small ball m, then supplies *vis viva*, then slowly returns to the old position by a violent motion of the short magnet, and then again removes just as much *vis viva* until the *vis viva* arrives at the old value, and thus also changes the direction of motion in precisely the way that will ultimately put it once more on the same path with the same position of the magnet. The variables that determine the position of the mass m in the plane are the ones that we previously called the s variables, while the g variables reduce to a single one, namely, the angle of rotation of the magnet.

There are two ways that one can arrange that the force that acts upon the magnet from the outside does not vary periodically during the unvaried motion, but only needs to be slowly-varying in time, in the event that the motion is varied: First of all, one assumes that the orbital period of the mass m is very short and that the moment of inertia of the magnet relative to its rotational axis is enormous, such that it makes only a very small rotation during the motion of the mass m from perihelion to aphelion. Secondly, one imagines infinitely many equally-constructed masses m on the plane, instead of a single one, and which have all possible phases of the same central motion simultaneously, and without mutually perturbing each other, move independently of each other and all of then will be affixed to the magnet in the same way and by means of the same device that was described above. In that way, one can convert the system into an isokinetic one (in the *Helmholtz* sense) and likewise also a true cyclic one, when in fact all of these masses already continuously cover the entire surface that they sweep out in the course of time under central motion at the initial time in a suitable way. Thus, the knowledge of a single cyclic variable is then by no means sufficient for the determination of the position of any of the massive particles that take part in the central motion, along with the slowly-varying coordinates that determine its position or that of the magnet, but two more variables are required for that (e.g., two rectangular coordinates in the plane, or the path length and the direction of motion at a given distance from the force center).

If one would like to arrange that the central motion of each mass results from *Newton's* law of gravitation then one could apply no ordinary magnets, but rather the pole A must be attracted to the closer pole B by a force that varies in proportion to the first power of the distance, but be repelled by the distant pole C with an equal force. Then, it is once more not δQ, but $\delta Q / \bar{T}$, that will be a complete differential. If one has two magnets at the same time, one of which exhibits the behavior that was depicted above, while the other one obeys the same law as an ordinary magnet, and each of them can rotate independently of the other one then one will obtain a central motion under which the central force will obey the law that was mentioned at the end of paragraph 41, and the two constants λ and a can be varied slowly independently of each other.

$\delta Q / \bar{T}$ is then also not a complete a differential, and one sees that $\delta Q / \bar{T}$ is not a complete differential for all isokinetic systems, as well as not all pure cyclic ones. In regard to the rigorous calculation of all examples, I refer to Wien Sitz. Ber. II, **92**, pp. 853, Oct. 1885, Exn. Rep. d. Physik **22**, pp. 135.

Third example: A mass rotates rapidly around an axis, and its distance from the axis is the slowly-varying parameter. This is an instructive example of a cyclic system in the broader sense, according to *Hertz's* terminology, that is not a true cycle. For the sake of brevity, in what follows, it shall always be referred to as the *centrifugal model*. On the beautiful analogies that this simple mechanical device exhibits with *Carnot's* theorem and the behavior of complete gases, cf., my *Vorlesungen über Maxwells Theorie der*

Elektrizität und des Lichtes, volume 1, lecture 2. Another device is described in the same book (lectures 4 and 6), for which two mutually-independent cyclic motions are possible.

Fourth example: The fluid flow model. An inviscid, incompressible fluid flows irrotationally in a channel that returns to itself. The form of the channel and also its cross-section can vary slowly at different places without that varying the total cavity of the channel. This system is a true cycle.

§ 45. Either periodicity or a cyclic character of the motion is assumed.

We would now like to carry out the calculations in the greatest generality that is possible. We think of n material points (the small balls m of the central motion model) whose position is determined by s generalized coordinates. σ conditions that are completely unvarying in time can possibly exist between the latter, which therefore must also remain true for all varied motions of them.

Later, we will assume that the motion of these n material points is either cyclic or has been converted into one. However, for the time being, we would like to leave them entirely general.

These n points define the mechanical system considered. In addition, three types of material points shall be in effect.

1. Along with the n material points, (n') other ones shall be in interaction, and the latter ones shall always keep the same position in space, and therefore, with no loss of generality, they can also be added to the n points. One can find a fixed, unvarying mass – e.g., in the central motion model, at the place where the string that exerts an arbitrary central force on the small ball m goes through the hole. Such masses can also have a distribution in the path plane of the small ball that is fixed in some other way.

2. In addition to the n material points, there are v other ones in interaction, whose position, which is determined by g generalized coordinates (slowly-varying variables or parameters), is often entirely unvarying and often once more extremely slowly-varying. The v points will be regarded as external to the system in question. They correspond to the magnet in the central motion model.

3. (N) other material points are present that shall act merely upon the v points, but not upon the n points at all, and therefore shall remain completely outside of the system considered. The forces that they exert upon the first points must be in complete equilibrium with the forces that the n points exert upon the v points, as long as the latter remain completely at rest, while the state must differ from equilibrium only slightly when the v move slowly. In the central motion model, these are the forces that must preserve equilibrium with the forces that the pole A exerts upon the magnet.

Let T be the *vis viva* of the n points, F, the force function of all of their interactions with each other and the action of the forces that arise from the n' points. We call the work that is done by all of these forces the *internal work*, and we shall call the work that is done by the forces that are exerted by the v points on the n points (i.e., the external forces) the *external work*. The external force on a coordinate p_h of the n points – i.e., the one that is exerted upon it by means of the interaction of the n and v points – shall be denoted by $\mathfrak{P}_h$. The forces that act between the n and the v points shall have a force function, in any case, that shall be denoted by Ω; by contrast, the total force function $F + \Omega$ of all forces that act between the n, n', and v points shall be called V. As long as one prefers to say nothing at all about the v points, the external influence on the n points is determined simply by the forces $\mathfrak{P}_h$ that have the force function Ω, but which then include parameters that are slowly-varying in time.

First, the n material points shall move during the time interval $t_1 - t_0$, with unvarying positions of the v points, which we would like to refer to as the unvaried motion. The analogues of the mean values of T, V, etc., during that time interval, as calculated by formula (236), shall be denoted by $\bar{T}$, $\bar{V}$.

We next compare the unvaried motion to another motion that happens in a way that differs from the first one by only infinitely little, begins at the same time t_0 as the unvaried motion, and ends at a time $t_1 + \delta t_1$ that differs from t_1 by only infinitely little.

Any state A of the unvaried motion that takes place at any time t will always correspond to the state B of the varied motion that takes place at the same time t. The *vis viva* T of the n points in the state B shall be greater than that in state A by δT.

The values of the s coordinates of all n points will, in any case, be somewhat different in state B than they were in state A. We denote the increases in F, Ω, and V that come about as a result of this situation by δF, $\delta\Omega$, and δV, resp. Furthermore, if:

$$P_h = -\frac{\partial F}{\partial p_h} + \mathfrak{P}_h \tag{240}$$

is the total generalized force that acts upon a coordinate p_h of the system of n points then one will have:

$$\delta F + \delta\Omega = \delta V = -\sum_{h=1}^{s} P_h\,\delta p_h\,. \tag{241}$$

δV is then precisely the quantity that was denoted in that way in the foregoing, and represents the total amount of energy that was supplied to the system of n points that will be performed against all of the forces that act upon the n points. Furthermore, since δT represents the increase in the *vis viva* of that point, the n points must be supplied with the total energy:

$$\delta E = \delta T + \delta V = \delta J_n + \delta\Omega = \delta J_{n,v} - \delta_v\Omega \tag{242}$$

in some way in order to take the state A of the system to the state B. The forces that produce this supply of energy, and to which we would now like to restrict the name of *additional forces*, are entirely new forces that are completely different from all of the ones that act during the unvaried motion. We make this energy δE analogous to the heat δQ that supplied to a body, while we make $J_n = T + V$ (or also, if we would prefer, $J_{n,v} =$

$T + V$), along with the total internal energy of a warm body, analogous to the *vis viva* of the molecular motion and the heat that is produced by doing internal work, resp.

§ 46. The extended system.

The increases in the force functions Ω and V, which arise from the fact that the v material points have somewhat different positions in the state B than they do in state A, shall be denoted by $\delta_v\Omega$ and $\delta_v V$, resp., such that the total increase that the quantities V and Ω experience under the transition from state A to state B is:

(243) $$\delta_{\text{tot.}} V = \delta V + \delta_v V, \qquad \delta_{\text{tot.}}\Omega = \delta\Omega + \delta_v\Omega,$$

and the expression:

(244) $$\delta J_{n,v} = \delta T + \delta V + \delta_v V$$

represents the total increase in the total energy $J_{n,v} = T + V$ of the extended system of $n + v$ points.

One can then say: If the transition from the unvaried state to the varied one takes place at precisely the time t in such a way that precisely the state A goes to state B then the system of n points will be supplied with the energy δE from the additional forces, which supplies the v points with the energy $-\delta\Omega$ by its influence, such that its internal energy will increase by $\delta J_n = \delta T + \delta F$, or it will employ the part $\delta T + \delta F$ of the total energy δE that is supplied by the additional forces towards increasing the proper energy, but the part $\delta\Omega$ will go to the performing of external work.

Ω is the force function of the interaction of the n and the v points. Since we are dealing with merely a mechanical picture of certain natural phenomena here, it is completely arbitrary which point one adds to the system considered and which ones that one regards as external. One analogy with the properties of warm bodies, in particular, can emerge in one case, and another analogy in another case. One can then also count the v points as belonging to the system considered, such that:

$$V = F + \Omega$$

is then the potential energy, and:

$$J_{n,v} = T + V = T + F + \Omega$$

is the total energy of the total system considered. One would then once more make $\delta T + \delta V$ analogous to the supplied heat, but now $\delta_v V = \delta_v\Omega$ is analogous to the supplied energy that takes the form of external work to the system, $-\delta_v\Omega$ is heat that is expended upon doing external work.

If the forces that have the force function Ω are not ordinary forces at a distance, but have the value zero for a certain distance from the points between which the forces act, but immediately increase to infinity for a somewhat larger or smaller distance, then Ω will be constant, other than that, so:

$$\delta\Omega + \delta_v\Omega = 0,$$

and both viewpoints will come down to the same thing. This condition is fulfilled by the centrifugal model with no further assumptions. In the picture of a gas that is closed by a piston, one can think of it as being fulfilled in any case with no essential modification of the problem, since even in that case, only a vanishing amount of energy will be employed for the variation of the force function of the forces that act between gas molecules and that of the piston, so the variation of Ω will not come under consideration.

We have let $\mathfrak{P}_h$ denote the forces that the v points exert upon the n points (e.g., the piston on the gas). In the case of v points at rest, the N points exert similarly-denoted forces on the v points (the hand or the weight of the load on the outer side of a slightly-moving piston that closes the gas). The n exert equal and oppositely-denoted forces on the v, or the v on the N points. The same statement is true for an extremely slow motion of the v points in which the entire process becomes nearly irreversible (e.g., the irreversible extension of gases), at least to a high degree of approximation.

Therefore, $-\delta_v V$ is also the work that is performed by means of the motion of the v points against the forces that the N points exert upon the v points and imply that the latter must remain at rest for unvaried motion, but for varied motion, they will move only exceedingly slowly. Namely, the latter forces are completely the same as the forces $\mathfrak{P}_h$.

§ 47. Application of the principle of least work.

δE, δT, and δV now have the same meanings as in formula (223), § 36. V generally contains, in addition to the coordinates of the n, also those of the v points, and the expressions that included in the latter play the role of slowly-varying parameters in the expression for the force function of the n points, as long as the v points move slowly; as long as the motion remains unvaried, this will never happen by itself. During the unvaried motion, V is then a function of the coordinates of the n material points that does not contain time explicitly, and that alone was assumed at the beginning of § 36. The effects that come about for infinitely small motions of the v points will be counted with the additional forces there. Equation (223) is then once more true here, namely:

$$2\delta\int_{t_0}^{t_1} T\,dt = \int_{t_0}^{t_1} \delta E\,dt + \sum_{h=1}^{s}(q_h^1\,\delta p_h^1 - q_h^0\,\delta p_h^0).$$

Up to now, we have consider the unvaried motion during the time interval $t_1 - t_0$, and completely independent of that, the varied motion during the time $t_1 + \delta t_1 - t_0$.

We would now like to concern ourselves with consideration of the transition from one motion to the other. Thus, it is initially entirely irrelevant to which of the n points the additional forces have supplied energy at the time moments in the course of the entire time interval $t_1 - t_0$. By contrast, it is not irrelevant whether the total shift in the v points happens at a single moment that lies between t_0 and t_1 or is composed of several shifts, and when this (these, resp.) shifts take place in the course of the time interval $t_1 - t_0$. When the total displacement of the v points is the same, the value of the increase $\delta E - \delta\Omega = \delta(T + F) = \delta J_n$ in the total energy of the n points, and likewise, the value of the increase $\delta J_{n,v} = \delta T + \delta F + \delta_{\text{tot.}}\Omega = \delta T + \delta_{\text{tot.}} V$ in the total *vis viva* and the force function of the $n + v$ at the end of the motion, and thus at the time $t_1 + \delta t_1$, will naturally be independent of when that displacement took place. By contrast, let the total external work, whether we define it by $\delta\Omega$ or by $-\delta_v\Omega$, depend upon when the displacement of the v points took place, and likewise, the energy $\delta T + \delta V$ that the additional force must supply in total to the n points during the time interval $t_1 - t_0$. The latter is then equal to $\delta T + \delta_{\text{tot.}} V - \delta_v\Omega = \delta T + \delta F + \delta\Omega$.

Now, the v points, in particular, should move uniformly during the entire time interval $t_1 - t_0$ from the position that they have for the unvaried motion to the position that they have for the varied motion, such that they have the initial position at time t_0, but arrive at the latter at exactly the time t_1. Thus, whereas the motion of the n points can happen with greater or lesser velocity, the v points shall move infinitely slow, but completely uniform, such that they can cover only very small paths during the finite time interval $t_1 - t_0$, which is why we have called the quantities that express their influence the slowly-varying parameters.

If we then let $\delta_v\Omega$ denote the work that must be performed by the v points against the forces that arise from the force function Ω when the entire displacement of the v points happened at the moment in time t then the work that is performed by the gradual displacement of the v points during the time dt that we consider now will be equal to:

$$\frac{\delta_v\Omega\, dt}{(t_1 - t_0)},$$

in which $\delta_v\Omega$ is different for the various phases of the motion, and is therefore a function of t. The total work that is performed by the v points against the forces that arise from the force function Ω during the time interval $t_1 - t_0$ will then be:

$$\delta_v\Omega_1 = \frac{1}{t_1 - t_0}\int_{t_0}^{t_1} \delta_v\Omega\, dt \tag{245}$$

in the case of the gradual displacement of the v points. If suitable additional forces act thereby that, together with the displacement of v points, take the unvaried motion to the varied motion, such that the material points at the time t_0 move in such a way that the unvaried motion corresponds to that time, as opposed to how they move at the time $t_1 + \delta t_1$ in precisely such a way that would befit the time $t_1 + \delta t_1$ in the varied motion (the time that corresponds to the end time t_1 of the unvaried motion), the additional forces must supply the energy:

$$\delta Q = \delta T + \delta_{\text{tot.}} V - \frac{1}{t_1 - t_0}\int_{t_0}^{t_1} \delta_v\Omega\, dt = \frac{1}{t_1 - t_0}\int_{t_0}^{t_1} (\delta T + \delta V)\, dt \tag{246}$$

to the n material points. It is then irrelevant when during the interval $t_1 - t_0$, and in what way, the additional forces act, as long as the only ultimate effect is to generate the varied motion precisely. Then, when the total energy increase of the n points and the total displacement of the v points, and thus, also the energy that is given to them from the outside is determined in that way, so is the energy that is supplied by the additional forces ([1]).

However, $\delta T + \delta V$ is the same quantity that was denoted by δE in formula (223), § 36, and one thus obtains from equation (246):

$$(t_1 - t_0)\ \delta Q = 2\delta\int_{t_1}^{t_0} T\,dt - \sum_{h=1}^{s}(q_h^1\,\delta p_h^1 - q_h^0\,\delta p_h^0)\,. \tag{247}$$

§ 48. Consideration of periodic motions.

In case the unvaried, as well as the varied, motions are periodic, one can easily think of realizing the three types of motion that were treated in the previous paragraphs in temporal succession, namely, the unvaried, the varied, and the gradual transition from the one to the other. If i is the period of the unvaried motion and $i + \delta i$ is that of the varied motion then one can set $t_1 = t_0 + i$, $\delta t_1 = \delta i$. One can then first think of the unvaried motion as taking place several times during the time interval i, then displacing the v points slowly during the time interval i (also $2i$, $3i$), and finally performing the varied motion several times during the time interval $i + \delta i$. One then thinks of all of these processes as taking place one after the other in the course of time, and if one so desires then the difference between temporal changes and variations can drop away completely.

In the case for which the unvaried motion, as well as the varied one, is periodic, as we have seen, the variations at both limits will be equal, and we will have:

$$\delta Q = \frac{2}{t_1 - t_0}\,\delta\left(\int_{t_0}^{t_1} T\,dt\right) = \frac{2}{i}\,\delta(i\bar{T})\,,$$

so

$$\frac{\delta Q}{\bar{T}} = \frac{2}{i}\,\frac{\delta(i\bar{T})}{i\bar{T}} = \delta\ln(i\bar{T})^2\,, \tag{248}$$

in which ln denotes the natural logarithm. A nearly complete analogy with the second law now prevails.

One goes over to increasingly varied paths for which the v points assume increasingly varied positions, but move only infinitely slowly in comparison to the n points, or perhaps one should also make an infinitely small jump by cyclic motion of the n, as

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([1]) $\delta_t\Omega$ will have the same value for all t in the event that the system is a true cyclic one (i.e., in the case where the unvaried motion at the place of each mass that is allowed at that place is immediately replaced with another mass that is equally-arranged, with an equal and equally-directed velocity, such that the v points act upon the latter in precisely the same way that they act upon the former) and in the case where the same thing is also found to occur for the varied motion, and the formula above will also be true when the v points do not move uniformly from the unvaried position to the varied one, so the external work will then be entirely independent of when the displacement of the v points resulted. However, we shall still assume that the total motion of the v points during the time interval $t_1 - t_0$ is very small. Even when it is made in jumps, an infinitely smaller jump can always once more happen after a finite time. Under this condition, one can, as always, consider the quantities that express the influence of the v points to be the slowly-varying parameters.

always after a finite time. In that way, one varies the motion continuously until one obtains finite changes of all quantities, and finally traverses a complete circular process; i.e., one finally returns to the same motion of the *n* points again, coupled with the same positions of the *v* points, without one having to repeat precisely the same motions of the *n* and positions of the *v* points simply in the opposite sequence.

The sum of all of the energy that is thus supplied by the additional forces on the *n* points – which we would like to denote by simply $\int \delta Q$ – will not generally vanish then for all of these variations of the state, even though at the conclusion one has returned to the initial state. By contrast:

$$\int \frac{\delta Q}{\bar{T}}$$

will always be equal to zero. From (248), $\delta Q / \bar{T}$ is then the increase in $\ln(i\bar{T})^2$, so $\int \frac{\delta Q}{\bar{T}}$ is the difference between the values of $\ln(i\bar{T})^2$ at the initial and final states. However, since the initial and final states are identical in the case considered, that difference must be zero.

If one, e.g., first displaces the *v* points (which we would like to call process *A*) then accelerates the motion of the *n* points, then brings the *v* to their old position once more, and finally again removes enough energy from the *n* points and also changes their velocity directions in such a way that their motion again becomes precisely the old one then $\int \delta Q / \bar{T}$ will always vanish. On the contrary, $\int \delta Q$ will not be zero, in general. Naturally, the latter expression will have the same absolute value, but the opposite sign, when one traverses a sequence of states of motion of the *n* points and positions of the *v* points in precisely the opposite way (viz., process *B*).

Process *A* corresponds to the following process: One lets a gas be first expanded, and then one warms it, then compresses it at the higher temperature to the old value, and finally cools it down again, until it has assumed the old state, and everything is done in a reversible way. The direct reverse of such a process will then correspond to process *B*.

The system that we consider differs from warm bodies insofar as its state is by no means determined by its energy and the position of the *v* points (viz., the outer vicinity). For example, the material point *m* in example 2 of § 44 can move in a circular or elliptical path, etc., with the same energy and the same external vicinity.

Since $\int \frac{dQ}{\bar{T}}$ (viz., the entropy) = $2\ln(i\bar{T})$ and any function of it that is multiplied by an integrating factor must again be an integrating factor, *i* will also be an integrating factor of *dQ*. Since *i* is the time inside of which a particle makes a complete circuit, 1 / *i* will be the (whole, rational, or irrational) number of cyclic orbits in a unit interval.

In this paragraph, we understand the phrase "periodic motion" to mean one for which the same values of the rectangular coordinates of all material points repeat after the completion of the period. Whereas, as we saw, periodic motions exhibit a complete analogy with the second law, in the conclusions of § 41 and § 44, we gave an example, which we treated as example 2 of the central motion in an unclosed path, of a non-periodic motion with the following properties: It is otherwise very similar to what was treated in this paragraph and can be converted immediately into a true cyclic (but not generally monocyclic) motion by considering infinitely many mass points that move in the same plane. Thus, $\int \delta Q / \bar{T}$, when it is taken over a complete circular process, will not vanish for them already for fixed positions of the *v* points (e.g., the magnet upon which the values of λ and a depend), and thus, even more so for the varied positions of the *v* points, since, in fact, δQ has no integrating factor whatsoever.

It is only with the assumption of the simultaneous presence of very many systems with all possible surface velocities, amongst which the states are distributed according to the laws of the theory of probability, that the analogy with the second law of the theory of heat can be restored in this case.

The aforementioned flaw in the analogy between warm bodies and the systems of *n* material points that we have considered can also be eliminated with the same assumption, namely, that the state of the former can be determined completely by being given the external situation, along with just one of the values of one variable (e.g., temperature), while for the motion of the system under consideration, along with the positions of the *v* points and the total energy, among other things, the integration constants that determine the initial motion of the *n* points can influence its equations of motion. However, we shall not go further into this here.

§ 49. Theory of cycles.

We now go on to the development of some propositions from the theory of cycles, in particular. As we mentioned already, we understand cyclic coordinates to be roughly ones that, when undifferentiated, enter into either expressions for the *vis viva*, or also into the forces that are in effect, or perhaps the condition equations that are present. As we likewise already mentioned, in the event that no forces exist, in the ordinary sense of mechanics, and no conditions that would restrict the degrees of freedom, the rectangular coordinates will also be be cyclic. By themselves, they do not define any finite motions – i.e., any motions for which the rectangular coordinates and their differential quotients with respect to time are enclosed between finite limits for arbitrary times. On the contrary, the variable that defines the angular position of the centrifugal model that was described in § 44 as example 3, is a cyclic coordinate in the extended *Hertzian* sense, which can still determine a finite motion in all situations, although it can go to infinity with increasing time.

By contrast, a true cyclic coordinate is one that determines a true cyclic motion – i.e., when it varies, while keeping all other coordinates constant, any mass that leaves its position space must be immediately replaced with another moving mass with the same properties and the same speed in the same direction, which is the earmark of a true cyclic motion.

Let a system be given that fulfills the following conditions:

1. Let some of the variables that determine the position of its material points be cyclic.

2. Let the differential quotients with respect to time (i.e., the rates of change) of the non-cyclic variables that are required in order to determine that position be very small in comparison to the rates of change of the cyclic variables (i.e., the cyclic velocities), in

addition. For that reason, the latter variables are called the *slowly-varying* variables or *parameters*.

3. Let the cyclic accelerations be very small, in any case, in comparison to the cyclic velocities – i.e., let the changes in the cyclic velocities that occur in time while those of the absolute values of the cyclic coordinates have already changed very noticeably still be very small. We then call the system a *cyclic system*, or more briefly, a *cycle*.

If its motion is finite, in addition, then we will call it a *finite cycle*. If the cyclic variables are nothing but true cyclic ones then we shall call it a *true cycle*. If only one independent cyclic variable is present then the system is called a *monocycle*. If there are two then it will be called a *bicycle*, and otherwise it will be generally called a *polycycle*. When n independent cyclic variables are present, it will be an *n-cycle*.

As long as the parameters remain constant, no variation of the externally-perceptible state of a true cycle will be noticed, despite the agitated motion that takes place inside of it. This is exhibited by warm bodies, by wires that have constant electrical currents flowing through them, but also by an absolutely symmetric top that rotates around its axis or a completely homogeneous fluid that flows in a tube that returns to itself. By contrast, if the cyclic velocities and the parameters change slowly then it will correspond to a gas that is heated slowly, or reversibly expanded and compressed. Another example is the slow change of intensity or mechanical change of position of a wire that has an electrical current flowing in it, or the slow motion or deformation of a rotating body or a ponderable fluid that flows in a channel.

One can apply the general formulas that were developed in §§ 45-47 to cycles and make many simplifications in them. The cycle should again consist of n material points whose position is then determined, partially by cyclic and partially by slowly-varying coordinates; we would like to denote the former by p_b and the latter by p_a . No further condition equations should exist between them. These n material points correspond to the ones that we also called the n material points in §§ 45-47.

Since the p_b do not enter into the expression for T undifferentiated, the forces that act upon the cyclic coordinates p_b (viz., the cyclic forces):

$$P_b = \frac{dq_b}{dt}, \tag{249}$$

in which:

$$q_b = \frac{\partial T}{\partial p_b'}. \tag{250}$$

The forces P_b correspond to the additional forces of §§ 45-47. The energy that is supplied to the cycle by it shall be called the *cyclically-supplied energy;* it corresponds to the supplied heat.

Moreover, since the terms that contain two derivatives of the p_b or p_a with respect to time, or which are of order 1 or even 2 in the p_a', can be regarded as vanishing in comparison to the ones that merely include p_a and p_b', the forces that act upon the parameters p_a will be:

$$P_a = -\frac{\partial T}{\partial p_b}. \tag{251}$$

Since T is a homogeneous, quadratic function of the p_b', all p_b' and p_a can be constant, so one will have $p_b'' = p_a' = 0$ when all P_b vanish. By contrast, the P_a will generally have non-zero values if this is to occur. We now distinguish different classes according to the forces that act upon the parameters p_a. They can originate, partly from the interaction of the n points and partly from the action of other material points that correspond to the n' points in §§ 45-47, which one fixes in space, once and for all, and which shall be called fixed, material points. Moreover, here, like there, they can also be counted with the n material points. These, partly from the interaction of the n points, partly from the effects of fixed material points that might be present on the forces that originate from them, which we shall refer to a *internal forces*, shall, in any event, have a scleronomic force function, which we shall again denote by F. The total portion of P_a that arises from this source is then $-\partial F / \partial p_a$.

Thus, further forces must be added in order for the p_a to remain constant, in general, which we denote by $\mathfrak{P}_a$ and which we call forces that act upon the parameters externally. The material points from which they original will correspond to the v material points of §§ 45-47 and shall also be once more called the v material points. They will be regarded as external for the system considered.

If one has $p_b'' = p_a' = 0$ exactly, so the cyclic motion is completely stationary, then the position of the v points will also remain completely unchanged and must be such that one has:

$$P_a = \mathfrak{P}_a - \frac{\partial F}{\partial p_a} = -\frac{\partial T}{\partial p_a} \tag{252}$$

exactly. This corresponds to what we called the unvaried motion in §§ 45-47. Here, in the theory of cycles, if we again set:

$$T - F = H, \qquad T + F = E, \tag{253}$$

in which the symbols H and E have the same meaning as in §§ 45-47, but a somewhat different one from what they will have in the remaining sections of this book, then we can also write equations (249) and (252) as:

$$\mathfrak{P}_a = -\frac{\partial H}{\partial p_a}, \qquad\qquad P_a = \frac{d}{dt}\frac{\partial H}{\partial p_b'}. \tag{254}$$

If the slowly-varying parameters and the values of the p_b' are to change very slowly then the $\mathfrak{P}_a$ must be slightly different from values that were given by the equations above, and the P_b must be slightly non-zero. The latter forces then correspond to the ones that we have called the additional forces in §§ 45 to 47, so we shall also preserve that name here, as well. The energy that they supply – viz., the cyclically-supplied energy – corresponds to the supplied heat in the theory of heat.

The slow variation that the cyclic motion experiences due to the additional forces, as well as the fact that the forces that are applied to the v points do not have precisely the values $\mathfrak{P}_a$ that are determined by equation (252), corresponds to what we called the variation of the motion in §§ 45-47; however, it is presently unnecessary to regard the gradual variation of the stationary, cyclic motion that occurs due to the slow motion of the v points and the effectiveness of the additional forces as a problem in the calculus of variations and to emphasize the increases that occur in that way by the use of the symbol δ. Moreover, one can regard this gradual change of state as an ordinary motion that comes about under the influence of the additional forces and the change of positions of the v points in the course of time. This is especially close to the notion of true cycles, for which the unvaried motion does not at all represent a visible change of state, such that perceptible temporal variations will first appear because of its slow variation.

The values of T, V, etc., for an arbitrary moment in time of the unvaried motion coincide with the mean values $\bar{T}$, $\bar{V}$, etc., of the same quantities for the unvaried motion. It is irrelevant at which moment in time of the unvaried motion the variation begins, which has just the character of a mechanical motion that takes place under the influence of given forces and also takes an arbitrarily long time to gradually increase to a finite variation and to end at an arbitrary time. When one suddenly has $p_b'' = p_a' = 0$ in any phase, one will immediately obtain the unvaried motion that one thinks of as corresponding to that phase.

It seems clearly evident here how the variation that was considered in the introduction to this book can quite gradually approximate the character of an ordinary motion that takes place in the course of time for which only individual coordinates are rapidly variable, while the others vary much more slowly.

We would like to still keep the name of variations for the gradual changes in the cyclic motion that occur in time, and also keep the equations of motion for the changes in the p_a that come about due to the fact that the $\mathfrak{P}_a$ do not have precisely the values that were given by the equations (252) and (254) and were first written down in § 53.

§ 50. The integrating factor of the differential of the cyclically-supplied energy.

From equation (249), the work that is performed by the force P_b during the variation of the cyclic motions in the time interval dt is:

$$dQ_b = P_b\, dp_b = P_b\, p_b'\, dt = p_b'\, dq_b\,. \tag{255}$$

The sum dQ of all dQ_b is the total of all works that are performed by all forces P_b (viz., the cyclic or additional forces), which we have called the energy that is supplied to the system cyclically, and which is analogous to the supplied heat.

In case the system is a monocycle, and one writes a single cyclic variable with no index, one will get:

$$dQ = p'\, dq,$$

and since $T = p' q$:

$$\frac{dQ}{T} = \frac{dq}{q}. \tag{256}$$

dQ / T is then a complete differential, and $\ln q$ is the entropy.

This formula can be applied to all true monocycles, inside of which arbitrary masses describe a cyclic motion, such that they all simultaneously return to their starting positions at the same time i inside of them and them begin the same motion again. Such monocycles were referred to as "simple true monocycles" by *Helmholtz*, and one easily sees that they define a special case of the periodic systems that were treated in § 48.

If a cycle possesses different mass systems, each of which describes a repetitive motion in that way, and if the periods i, i_1, i_2, … are different for different systems, then the system will be called an *unfettered*, or only *piecewise-fettered*, polycycle, as long as the durations of these periods depend upon the values of several independent cyclic velocities, in addition to the values of the slowly-varying parameters, while it will be a *completely fettered* polycycle, or a *composed* monocycle when the duration depends upon the values of a single cyclic velocity, in addition to the parameters. If, in the latter case, the number of ratios i_1 / i, i_2 / i, … is finite, and the values of these ratios are completely independent of the those of the parameters (viz., the slowly-varying coordinates) then one can think of these ratios as rational, with no essential change in the mechanical conditions, even if they also have a very large common denominator, and thus find a lengthy span of time inside of which the motion of all mass systems is simultaneously a periodically-repeating one, such that the total mechanical system will represent a simple true monocycle of period J, and everything that comes from such a proof will be applicable.

However, this becomes doubtful when the number of ratios i_1 / i, i_2 / i, … is infinitely large, and is no longer true in any case when the values of these ratios are continuous functions of the parameters, since then the cyclic coordinates, combined with the parameters, will no longer define a system of holonomic coordinates, in general ([1]). However, all of the formulas that were developed are true for only holonomic coordinates. Hence, as we already pointed out in the remark on pp. 16 in § 4, for the entire book, with the exception of §§ 27 and 28, we will always understand generalized coordinates to mean only holonomic, generalized coordinates. We would not like to go into this further here, and we shall carry out the proof of a very general theorem that *Helmholtz* discovered.

([1]) Borchardt's Journal, Bd. 98, 1 Heft, pp. 87, 1885; Wien. Sitz.-Ber. Bd. 111, pp. 1603, 1902. For example, if two bodies that taper down conically on opposite ends have parallel axes of rotation, and a transmission belt or a friction wheel is continuously-displaceable between them, in such a way that sometimes it couples a thick part of the first axis with a thin part of the second one and sometimes the opposite, and one chooses the path s of a point of the first body that does not lie on the rotational axis to be a cyclic coordinate and the displacement a of the belt or the friction wheel to be a slowly-varying one then a point that does not lie on the rotational axis of the second body will generally not arrive at the same place when s first increases by a finite amount σ and the a increases by a likewise finite amount α, or when a first increases by α and then s increases by σ. Thus, in the event that we are dealing with a true cycle, if all velocities also given by a and ds / dt, and thus its entire knowable state, then the positions of all points of the system will still not be determined by being given the values of a and s, so a and s are not holonomic generalized coordinates.

Let an arbitrary, composed, monocyclic (and thus completely fettered, polycyclic) system be given whose slowly-varying coordinates shall be denoted by p_a and whose rapidly-varying coordinate shall be denoted by p. It shall be assumed that for a suitable change in the $\mathfrak{P}_a$ the motion can proceed in precisely the same way, in which all velocities have been multiplied by a constant, but entirely arbitrary number, n that is the same for all velocities, so, to some extent, the time duration of all processes will seem to have been reduced to the n^{th} part. Then, if only a single slowly-varying parameter p_a is present then the total *vis viva* that is contained in the system will always be an integrating denominator of the differential of the externally-supplied energy; by contrast, is several p_a are present then that will also be true whenever that differential possesses an integrating factor at all.

Next, let only a slowly-varying parameter p_a be present, so two genera of state changes will be possible. First, for unchanged path forms, one changes merely the velocity of all moving parts proportionally. The forms of the paths then depend upon only a single independent variable quantity, while the other independent variable quantities determine merely the velocity by which the paths are traversed. Thus, in case one always varies the limits in such a way that the last term in equation (223) vanishes, upon returning to the old path, one must also revert to the original limits, so the two terms in equation (238) on the right-hand side will become identical, and dQ / T will become a complete differential.

In case more than one slowly-varying parameter p_a exists, T will generally no longer be an integrating denominator for dQ, since when one returns to precisely the same state of the system, in general, one will no longer come back to the same integration limits, as long as the limits have always been changed in such a way that the last term in equation (223) vanishes; however, it can also be proved in this case that T must be an integrating denominator in the event that integrating factors of dQ exist, at all. In general, let $dQ = M\,dN$, and choose T and p_a to be the independent variables.

From what was just proved, T must be an integrating denominator of dQ, as long as all p_a are defined up to a constant; if g is any of the indices a and $d\sigma_g$ is the differential that is combined with the factor T then it will follow that:

$$M\left(\frac{\partial N}{\partial T}dT+\frac{\partial N}{\partial p_g}dp_g\right) = T\left(\frac{\partial \sigma_g}{\partial T}dT+\frac{\partial \sigma_g}{\partial p_g}dp_g\right). \tag{257}$$

This equation must be true for all combinations of values of the variables $p_1, p_2, \ldots, p_{g-1}, p_{g+1}, \ldots$, which are assumed to be constant in it. Thus, if M and N are given then σ_g will be determined up to an expression that contains the latter variables. The same thing will be true for any other a index; e.g., h. One then likewise has:

$$M\left(\frac{\partial N}{\partial T}dT+\frac{\partial N}{\partial p_h}dp_h\right) = T\left(\frac{\partial \sigma_h}{\partial T}dT+\frac{\partial \sigma_h}{\partial p_h}dp_h\right), \tag{258}$$

in which, naturally, the identity of σ_g and σ_h has still not been proved. It follows immediately from this and equation (257) that:

(259)
$$\frac{\partial \sigma_g}{\partial T} = \frac{\partial \sigma_h}{\partial T}.$$

σ_g and σ_h can then differ only by quantities that do not contain T, but only the p_a . We would like to set: $\sigma_g = \sigma + \Pi_g$, $\sigma_h = \sigma + \Pi_h$, in which the Π are no longer functions of T; it will then follow from equation (257) that:

(260)
$$M \frac{\partial N}{\partial T} = T \frac{\partial \sigma}{\partial T}$$

and

(261)
$$M \frac{\partial N}{\partial p_g} = T \left(\frac{\partial \sigma}{\partial p_g} + \frac{\partial \Pi_g}{\partial p_g} \right)$$

for every value of g. That will further imply that:

(262)
$$dQ = M\, dN = T \left(d\sigma + \sum \frac{\partial \Pi_g}{\partial p_g} dp_g \right),$$

in which the sum is extended over all possible values of the index g. Since, by convention, dQ has an integrating factor, in any case, it must also possess one when one sets all p_a constant, except for two of them – say, p_g and p_h . If one then divides the differential dQ by T then that will give:

(263)
$$\frac{\partial \sigma}{\partial T} dT + \left(\frac{\partial \sigma}{\partial p_g} + \frac{\partial \Pi_g}{\partial p_g} \right) dp_g + \left(\frac{\partial \sigma}{\partial p_h} + \frac{\partial \Pi_h}{\partial p_h} \right) dp_h = \frac{dQ}{T}.$$

From what was said, this differential expression must also have an integrating factor. If one writes down the known condition for it then one will see that it will follow that either:

$$\frac{\partial \sigma}{\partial T} = 0$$

or

(264)
$$\frac{\partial^2 \Pi_g}{\partial p_g\, \partial p_h} = \frac{\partial^2 \Pi_h}{\partial p_g\, \partial p_h}$$

for all pairs of values of g and h. The first equation can never be fulfilled, since otherwise when one keeps the p_a constant, no energy supply at all would be necessary in order to raise the *vis viva*. Thus, equations (264) must be true, from which it would follow that $\sum \frac{\partial \Pi_g}{\partial p_g} dp_g$ would be the complete differential of a function Π of the p_a, which is why one will then have $dQ = T\, d(\sigma + \Pi)$.

§ 51. Adiabatic and isocyclic motion.

If all $P_b = 0$ for an arbitrary cycle, while the $\mathfrak{P}_a$ are different from the values that were given by equation (252) and (254), such that the p_a change slowly, then one will call the motion an *adiabatic* one. It then follows from equations (249) that the momenta q_b that relate to the cyclic coordinates must all be constant. However, as a result of the slow changing of the parameters, the cyclic velocities p'_b will likewise be slowly-changing. By contrast, when the P_b always have values during the slow changing of the parameters such that the p'_b remain precisely unchanged, one will call the motion an *isocyclic* one.

An example of adiabatic motion is defined by a body that rotates around its axis, or the centrifugal model that was described § 44 as example 3, when a rotational moment never acts around the rotational axis. On the contrary, the centrifugal model exhibits an isocyclic motion when its rotational velocity is kept constant by suitable forces $\mathfrak{P}_b$ that act upon the crank, while the displaceable mass m sometimes approaches the rotational axis and sometimes moves away from it.

Physical analogies for the adiabatic motion are warm bodies, by whose changes of state, heat will either be supplied or removed (hence, the name "adiabatic" for the analogous motions of mechanical cycles, as well), electrical current loops in which unvarying electromotive forces are present, moving statically-charged conductors with a constant quantity of electricity, etc. The corresponding physical processes will be analogous to the isocyclic motions when the temperature of the warm bodies, the current intensity of the electric current, or the potential of the electrostatically-charged conductor are kept constant, respectively. For a rotating body, isocyclic motion occurs when it is connected (i.e., coupled) by a belt or gear with a rotating flywheel of infinite mass or a body that is forced to rotate with constant velocity. Physical analogies are given by a warm body that is well-coupled to an infinite heat reservoir, an electric conductor whose ends are kept at constant potential difference (i.e., they are linked with the binding posts of the power supply), in electrostatics, a body that is conducted to the Earth, which *Helmholtz* also referred to as "coupled" with the Earth, the heat reservoir, etc.

In equation (254), one understands the partial differential quotients to mean that the p'_b are kept constant, so the changes of state have to happen isocyclically. As we already did in § 9, we would like to denote partial differential quotients for which the p'_b are considered to be constant with the index p', while the ones for which q_b are considered to be constant shall be denoted with the index q.

One can then say: H is the force function of the force $\mathfrak{P}_a$ that acts upon the parameters p_a for isocyclic changes of state, so in thermodynamics it will correspond to the isothermal thermodynamic potential. For every isocyclic motion:

$$-\sum_a \frac{\partial H}{\partial p_a} dp_a = - dH = dF - dT$$

is the energy that is supplied by the forces $\mathfrak{P}_a$ that act upon the cycle during the changing of the p_a for the cycle (i.e., the energy that is supplied in the form of external work done).

Since the total energy increment is $dE = dF + dT$, it will follow that $dQ = 2dT$, so the cyclically-supplied energy will then be equal to twice the increment of the kinetic energy.

From equation (63), one has:

$$\frac{\partial_q T}{\partial p_a} = -\frac{\partial_{p'} T}{\partial p_a}, \qquad \text{so} \qquad \frac{\partial_q E}{\partial p_a} = -\frac{\partial_{p'} H}{\partial p_a}.$$

One can then also write the first of equations (254) as:

$$\mathfrak{P}_a = \frac{\partial_q E}{\partial p_a}. \tag{265}$$

The external forces $\mathfrak{P}_a$ can thus also be referred to as the adiabatic partial differential quotients of E with respect to the coordinates p_a, and $-E$, as the adiabatic force function. The total energy that is supplied by the external work that is done is then equal to dE for adiabatic motion, and thus, equal to the total energy increment, which is self-explanatory, since it is the only supply of energy, in this case.

By applying the theorem that we found that the external forces have a force function for adiabatic, as well as isocyclic, changes of state to the theory of heat, we get the following theorem: If a warm, solid body is deformed arbitrarily adiabatically or isocyclically by arbitrary external forces then the deformation work will always be a complete differential, as if the external forces that originated from massive particles at rest were such that equilibrium was preserved when the massive particles of the body were also gripped in agitated thermal motion.

§ 52. Hertz's reciprocal relations.

1. Let the state of a cycle be slowly varied adiabatically, once, in such a way that only the parameters p_a and dp_a increase, and another time, such that only the parameters $p_{a'}$ and $dp_{a'}$ increase. In the first case, the external force $\mathfrak{P}_{a'}$ that acts upon $p_{a'}$ might increase by $d\mathfrak{P}_{a'}$, while in the second case, the external force $\mathfrak{P}_a$ that acts upon p_a might increase by $d\mathfrak{P}_a$, so one will always have:

$$\frac{d\mathfrak{P}_{a'}}{dp_a} = \frac{d\mathfrak{P}_a}{dp_{a'}}.$$

The same thing will also be true when all motions are isocyclic, and we can write these two relations in detail as:

$$\frac{\partial_q \mathfrak{P}_{a'}}{\partial p_a} = \frac{\partial_q \mathfrak{P}_a}{\partial p_{a'}} \qquad \text{and} \qquad \frac{\partial_{p'} \mathfrak{P}_{a'}}{\partial p_a} = \frac{\partial_{p'} \mathfrak{P}_a}{\partial p_{a'}}.$$

The proof follows immediately from equation (254) and (265); i.e., from the fact that the external forces have a force function for the adiabatic changes of state, as well as for the isocyclic ones.

2. Since:

$$\mathfrak{P}_a = -\frac{\partial F}{\partial p_a} - \frac{\partial_q T}{\partial p_a}, \qquad \frac{\partial F}{\partial q_b} = 0, \qquad \text{and} \qquad p_b' = \frac{\partial T}{\partial q_b}$$

[the latter from (60)], one finds that:

(266) $$\frac{\partial \mathfrak{P}_a}{\partial q_b} = -\frac{\partial_q p_b'}{\partial p_a}.$$

Thus, when an increment in a cyclic momentum q_b by dq_b results from an increment in the force $\mathfrak{P}_a$ that acts upon p_a by $d\mathfrak{P}_b$, with constancy of all remaining q and all parameters, an adiabatic increment of p_a by dp_a that results with constancy of the remaining parameters will produce an increment of the cyclic velocity p_b' by dp_b' that is denoted by the opposite of $d\mathfrak{P}_a$ (i.e., a decrement), and indeed the ratios of the increments dq_b and dp_a, which are assumed to be the causes, to the changes ($d\mathfrak{P}_a$ and dp_b'), which are considered to be effects, are equal. (As *Hertz* said, the ratio of the cause and effect is the same in both cases.) Under these special circumstances:

(267) $$\frac{d\mathfrak{P}_a}{dq_b} = -\frac{dp_b'}{dp_a}.$$

Since dp_b' and dq_b must be denoted the same for a monocycle, the following theorem must also be true for it: If an increase in the cyclic velocity p' raises the force on any parameter p_a with constancy of the other parameters then an adiabatic increment of that parameter must reduce the cyclic velocity p' with constancy of the other parameters.

3. If the force P_b produces the increment dq_b that appears in equation (267) in the time interval dt then $dq_b = P_b\, dt$, while, on the other hand, $\mathfrak{P}_a' = d\mathfrak{P}_a / dt$ is the velocity with which the force $\mathfrak{P}_a$ then increases under the circumstances for which equation (267) is true. The left-hand side of equation (267) will then be equal to $\mathfrak{P}_a' / P_b$, and when one reverts to the notation of equation (266) in the right-hand side, for the sake of clarity, it will follow from (267) that:

(268) $$\frac{\mathfrak{P}_a'}{P_b} = -\frac{\partial_q p_b'}{\partial p_a}.$$

In words: If all parameters are constant, and all forces that act upon the cyclic coordinates are zero, except for one (P_b), then the force $\mathfrak{P}_a$ that acts upon the parameter p_a shall increase by $\mathfrak{P}_a'\, dt$ during the time interval dt, and then an adiabatic increment of p_a, with constancy of the remaining parameters, will produce a decrement of p'_b, and the ratios of cause and effect will be equal in both cases, when we consider the force P_b to be the cause and the rate of change $\mathfrak{P}'_a$ of the force $\mathfrak{P}_a$ to be its effect, and on the other hand, the coordinate increment dp_a to be the cause an the decrement $-\ dp'_b$ of the cyclic velocity p'_b to be its effect.

4. Since:

$$\mathfrak{P}_a = -\frac{\partial F}{\partial p_a} + \frac{\partial_{p'} T}{\partial p_a} \qquad \text{and} \qquad q_b = \frac{\partial_{p'} T}{\partial p'_b},$$

one has:

(269)
$$\frac{\partial \mathfrak{P}'_a}{\partial p'} = \frac{\partial_{p'} q_b}{\partial p_a};$$

i.e., if an increment of a cyclic velocity p'_b, with constancy of the remaining cyclic velocities and the parameters, produces an increment in the external force $\mathfrak{P}_a$ that acts upon a parameter p_a then an isocyclic increment of p_a, with constancy of the remaining parameters, will produce an increment of the cyclic momentum q_b that belongs to p'_b, and indeed, as *Hertz* said briefly (once more, for infinitely small increments), the ratio of cause and effect will be the same in both cases. For monocycles, the increment in the cyclic velocity will again have the same sign as that of the cyclic momentum.

5. In precisely the same way that equation (268) was obtained from equation (266), we can also define a new equation from equation (269). When taking the partial differential quotients of the right-hand sides of the latter equation, it is assumed that the $\mathfrak{P}_a$ and P_b have values such that all p'_b and all parameters remain constant, with the exception of a single p_a . The force that then acts upon the cyclic coordinate p_a shall be called P_b, but the increments of p_a and q_b during the time interval dt shall be called dp_a and dq_b , resp. The quotient dq_b / dp_a will then be equal to the quantity that is denoted by $\frac{\partial_{p'} q_b}{\partial p_a}$ in formula (269), but one also has $dp_a = p'_a\ dt$ and $dq_b = P_b\ dt$. Thus:

$$\frac{\partial_{p'} q_b}{\partial p_a} = \frac{P_b}{p'_a},$$

and one can write equation (269) in the form:

(270)
$$\frac{\partial \mathfrak{P}_a}{\partial p'_b} = \frac{P_b}{p'_a}.$$

Expressed in words: If an increase in a cyclic velocity p'_b, with constancy of the remaining cyclic coordinates and the parameters, produces an increment in the force $\mathfrak{P}_b$

that acts upon a parameter p_a then, for the purpose of isocyclic increases in p_a, with constancy of the remaining parameters, a positive force P_b must act in the direction of the cyclic coordinate p_b , and in fact, the ratio of cause and effect will again be the same in both cases, when one refers to the increment in p_b' as the cause and the increment in $\mathfrak{P}_a$ as its effect, and on the other hand, considers the velocity p_a' with which p_a changes isocyclically to be the cause of the necessity of the force P_b and the latter, as its effect.

In the proof and mathematical formulation of this theorem in *Hertz's Prinzipien der Mechanik*, no. (577), pp. 245, the symbol ∂ is all too much.

§ 53. Helmholtz's theorems on mixed cycles.

Helmholtz has explored some considerations of somewhat greater generality. Once more, let a system of n points be given that can interact with n' points that are fixed once and for all, and if one so desires then one can count the latter points with the n points. Let the force function of all of the forces that act between these points be F, and let the kinetic energy of the n be T. This time, as in §§ 45-52, but deviating from our usual notation, we again set:

$$T + F = E,$$
$$T - F = H,$$

such that E is the total energy of the $n + n'$. Among the coordinates that determine the position of the n points, there can be cyclic ones, which we again would like to denote by p_b ; i.e., their values, when differentiated by time, shall enter into T or F. Only the p_b' shall enter into T. Let the total number of these cyclic coordinates be σ. The total force $-\partial F / \partial p_b$ that the $n + n'$ points exert upon any cyclic coordinates must then be equal to zero. The remaining coordinates do not just need to be slowly-varying, so they are then completely arbitrary coordinates. We thus call them *ordinary* coordinates and again denote them by p_h . Let their total number be s. No further condition equations shall exist between the p. We would like to call such a system, which indeed contains cyclic coordinates, but whose remaining coordinates are not considered to be slowly-varying (or at least, not all of them), a *mixed cycle*, and in contrast to that, we would like to call one that contains only cyclic and slowly-varying coordinates a *pure cycle*.

It will be easiest to understand the equations that we shall now develop when we think of the behavior of the system of $n + n'$ points and the force function F, as well as also the motion of the points, and thus that of all coordinates, as functions of time, and ask which forces P_b and $\mathfrak{P}_h$ act upon the cyclic coordinates (must be added to the forces that act upon the ordinary coordinates by means of the force function F) in order to generate the given temporal variation of the coordinates. One must then have:

$$P_b = \frac{d}{dt}\frac{\partial H}{\partial p_b'} \tag{271}$$

for the cyclic coordinates of equations (254), and for the ordinary coordinates, one will have the ordinary *Lagrange* equations:

(272) $$\mathfrak{P}_h = \frac{d}{dt}\frac{\partial H}{\partial p'_h} - \frac{\partial H}{\partial p_h}\,.$$

For every index *b* and *h*, let:

(273) $$\frac{\partial H}{\partial p'} = \frac{\partial T}{\partial p'} = q.$$

The desired forces P_b and $\mathfrak{P}_h$ shall again be applied to arbitrary, external (i.e., from the *v*) material points, with which, we would, moreover, not wish to concern ourselves further. If the values of the *p* are given as functions of time then they can likewise be found as functions of time from equations (271) and (272); i.e., one can answer the question of which external forces P_b and $\mathfrak{P}_h$ must be added at each moment to the one that is required by the force function *F* in order to produce the given motion.

Naturally, the validity of equations (271) and (272) is entirely independent of which quantities one considers to be given in them and which ones are not. These equations are also correct when the P_b and $\mathfrak{P}_h$ are given as functions of time, even when they are given as functions of coordinates, velocities, or otherwise arbitrary quantities, and when one posses the question of determining the motion (i.e., determine the p'' for given initial values of the *p* and p'). It is only in the last case that the left-hand side of equations (271) and (272) will also contain the unknowns that we seek. Indeed, it is actually completely arbitrary which forces one counts as internal for the force function *F* and which one counts as external to the $\mathfrak{P}_h$, whether one counts the bodies from which certain forces originate with the system or regards them as outlying, if that is the only consistent possibility. Thus, that is why, e.g., *Helmholtz*, in the electrodynamics of galvanic resistance of conductors, included merely the $\mathfrak{P}$, since it gave rise to irreversible processes. However, for the sake of greater intuitive appeal, we would always like to pretend that *F* and the motion of the system are given and ask what the P_b and $\mathfrak{P}_h$ would be that would be necessary to produce that.

Equations with the form of equation (272) must also enter in place of equations (252) and (254) when one would like to develop the theory of cycles that was treated in §§ 49-51 without leaving anything out, or when one would like to solve the question of how the slowly-varying parameters change in time under the influence of given $\mathfrak{P}_a$. If one then would like to answer that question then one can obviously not neglect p'_a and p''_a . In equations (271) and (272), however, one no longer finds anything that has been neglected. The condition that the cyclic coordinates enter into either *T* or *F* undifferentiated is not merely realized approximately, but exactly.

Equation (272) has the form that was suggested at the beginning of § 34, which one obtains when one sets $\pi = 0$ in the general *Lagrange* equation (50), and assigns the form (220) to *V*.

Next, we shall have that all $P_b = 0$, such that the cyclic motions proceed adiabatically.

From (271), the quantities:

(274) $$\frac{\partial H}{\partial p'_b} = q_b$$

will then be constant for all time. If the values of these constants are given then the σ quantities p'_b can be eliminated from T by means of the σ equations (274), and thus, also from H. However, since equations (274) contain terms in the first powers of the p' and ones that are free of the p', after that elimination, T will become a function of degree two in the remaining p' (i.e., the p'_h) that is no longer homogeneous, but also contains terms that are linear with respect to the p'_h and contain one of these entirely free terms. The quantities $\mathfrak{H}$ that arise by eliminating the p'_b from H by means of equations (274) will also contain terms that are linear in the p'_h.

An example of this is a system that contains a body that rotates around a principal axis of inertia without friction or opposition, such as the pendulum that we treated in § 22. The angle whose differential quotient with respect to time determines that angular velocity of the rotating body is the p_b in question, and it must be assumed that the forces always act upon only the two vertices of the axes, such that there will never exist a rotational moment that would accelerate or retard the rotation. *Maxwell* imagined the same rotating body that was subject to the same condition in order to explain the magnetism that was present in a volume element of the ether, and in that way explained that the electromagnetic energy of the ether contains terms that are linear in the current strengths, while the purely electromagnetic energy is a homogeneous, quadratic function of the current strengths. Namely, he assumed that the current strengths are the rates of change of the cyclic coordinates.

Since $\mathfrak{H}$ arises from H when one expresses the p'_b in it by means of equations (274) as functions of the p_h and p'_h, one will have:

$$\frac{\partial \mathfrak{H}}{\partial p_h} = \frac{\partial H}{\partial p_h} + \sum_{b=1}^{\sigma} \frac{\partial H}{\partial p'_b} \frac{\partial p'_b}{\partial p_h},$$

$$\frac{\partial \mathfrak{H}}{\partial p'_h} = \frac{\partial H}{\partial p'_h} + \sum_{b=1}^{\sigma} \frac{\partial H}{\partial p'_b} \frac{\partial p'_b}{\partial p'_h},$$

or, due to equations (274):

$$\frac{\partial H}{\partial p_h} = \frac{\partial \mathfrak{H}}{\partial p_h} - \sum_{b=1}^{\sigma} q_b \frac{\partial p'_b}{\partial p_h}, \qquad \frac{\partial H}{\partial p'_h} = \frac{\partial \mathfrak{H}}{\partial p'_h} - \sum_{b=1}^{\sigma} q_b \frac{\partial p'_b}{\partial p'_h}.$$

Therefore, if one sets:

$$H' = \mathfrak{H} - \sum_{b=1}^{\sigma} q_b \, p'_b \tag{275}$$

then:

$$\frac{\partial H'}{\partial p_h} = \frac{\partial H}{\partial p_h}, \qquad \frac{\partial H'}{\partial p'_h} = \frac{\partial H}{\partial p'_h}.$$

Here, the p_b' are thought of as being eliminated from H' by using equations (274), but one considers them to be constant when one forms $\frac{\partial H}{\partial p_h}$ and $\frac{\partial H}{\partial p_h'}$. When one thinks of the p_b' as being expressed in terms of p_h and p_h' by means of equations (274), the general equation of motion (272) will then assume the same form for each p_h:

$$\mathfrak{P}_h = \frac{d}{dt}\frac{\partial H'}{\partial p_h'} - \frac{\partial H'}{\partial p_h}. \tag{276}$$

The quantity H is now replaced with the quantity H' in which the p_b' are thought of as expressed as functions of p_h by means of equations (274), such that H' is not a homogeneous quadratic function of the p_h', in general.

As an example, we once more consider the problem that was treated already in § 22 of the theory of the rotation of a solid body around a fixed point. Let a rigid body be capable of rotating around a fixed point. Let its ellipsoid of inertia relative to that point be an ellipsoid of rotation.

We choose the same notations as before, and the $O\zeta$ axis again coincides with the rotational axis of the ellipsoid of inertia.

The variable B then fulfills the conditions that we imposed upon the coordinates that we now denote by p_b. Thus, if the generalized force $\mathfrak{B}$ that acts upon B is equal to zero at all times then, from (274), one will have:

$$\frac{\partial H}{\partial p_b'} = \frac{\partial H}{\partial B'} = \frac{\partial T}{\partial B'} = \text{const.}$$

If we denote the value of this constant by $-\nu J$ then it will follow from the third of equations (121)

$$c\,A' - B' = \nu,$$

which agrees with (124). If one uses this equation to eliminate B' from the expression (123) for T then it will follow that:

$$H = T = \frac{G}{2}(\gamma^2 A'^2 + C'^2) + \frac{J}{2}\nu^2. \tag{277}$$

In our present calculations, F is the force function for the internal forces of the system, so since it is a single, solid body, it will be equal to zero. The weight, or more generally, the forces $\mathfrak{A}$ and $\mathfrak{C}$, will be regarded as external forces and are included in the other part of the force function $\sum_{h=1}^{s}\mathfrak{P}_h\,p_h$ [equation (220)]. Namely, H is always equal to $T - F$, here, not $T - V$, as was previously true in this book.

The expression (277) is the quantity that was denoted by $\mathfrak{H}$ in equation (275). It follows from this equation that:

$$H' = \mathfrak{H} + v\,J\,B' = \frac{G}{2}(\gamma^2 A'^2 + C'^2) + v\,J\,c\,A', \tag{278}$$

in which the constant $- Jv^2 / 2$ was dropped as superfluous. The generalized forces $\mathfrak{A}$ and $\mathfrak{C}$ that act upon A and C must be derivable from this quantity H' by means of equations that have the *Lagrange* form entirely. One must then have:

$$\left\{ \begin{aligned} \mathfrak{A} &= \frac{d}{dt}\left(\frac{\partial H'}{\partial A'}\right) - \frac{\partial H'}{\partial A}, \\ \mathfrak{C} &= \frac{d}{dt}\left(\frac{\partial H'}{\partial C'}\right) - \frac{\partial H'}{\partial C}. \end{aligned} \right. \tag{279}$$

By substituting the values (278) for H', these equations will, in fact, yield:

$$\left\{ \begin{aligned} \mathfrak{A} &= \frac{d}{dt}(G\gamma^2 A' + vJc), \\ \mathfrak{C} &= G(C'' - c\gamma A'^2) + Jv\gamma A', \end{aligned} \right. \tag{280}$$

in agreement with (125). Equations (279) have entirely the same form as the *Lagrange* equations, but H' now also includes terms that are linear in the velocities. The motions cannot proceed in a precisely invertible way. As we already saw in § 22, for oscillations under which its center of mass moves in a circle, a pendulum that is coupled to a rotating top will have a different period of oscillation for a different orbital direction, as long as the top rotates in the same sense. In complete analogy, the potential will contain an electric current in the presence of a permanent magnet, or will depend upon the electromagnetic rotation of the plane of polarization of light, which are terms that are linear in the current strengths or velocities. This striking analogy is, of course, not a proof that hidden rotational motions will play a role in the latter physical phenomena. However, they would be explained most casually by that hypothesis, and in any event would show that the comparative study of both kinds of phenomena might promise further information. The solid body that is considered in the example that was dealt with is, moreover, a pure monocycle, when the forces $\mathfrak{A}$ and $\mathfrak{C}$ always have just those values that A and C would change very slowly in comparison to B, and otherwise it is a mixed one.

Helmholtz found a case in which H can be a complicated function of the velocities in the following way: Let the expression for the *vis viva* constant of two summands, one of which should contain only a certain velocity p'_a, while the other one contains only the remaining velocities p'_d, such that one then has:

$$\frac{\partial^2 T}{\partial p'_c \, \partial p'_d} = 0.$$

Since F does not include the velocities at all, it will also follow that:

$$\frac{\partial^2 H}{\partial p'_c \, \partial p'_d} = 0.$$

In addition, the external forces that act upon each of the coordinates p_d shall be equal to zero for all time, and thus $\mathfrak{P}_d = 0$. Here, and in all of what follows, we shall no longer speak of cyclic motions.

Now, in any case, motions of the system are possible for which all p'_d vanish, so all p_d will remain constant for all times. Since H contains no term that is linear in any p'_d, one will then also have:

$$\frac{\partial H}{\partial p'_d} = 0 \tag{281}$$

for every index d, and it will follow from equation (272) that:

$$\frac{\partial H}{\partial p_d} = 0 \tag{282}$$

for every index d.

The coordinates p_d can be found as functions of p_c and p'_c from these equations, except for singular cases. If one substitutes the values of the p_d thus-obtained in H then one will obtain a function of the p_c and p'_c that shall be denoted by $\mathfrak{H}$. $\mathfrak{H}$ does not need to be a quadratic function of the p'_c then, but can contain those quantities in an entirely arbitrary way, although it will be an even function of the p'_c. One will then have:

$$\frac{\partial \mathfrak{H}}{\partial p_c} = \frac{\partial H}{\partial p_c} + \sum_d \frac{\partial H}{\partial p_d} \frac{\partial p_d}{\partial p_c},$$

$$\frac{\partial \mathfrak{H}}{\partial p'_c} = \frac{\partial H}{\partial p'_c} + \sum_d \frac{\partial H}{\partial p_d} \frac{\partial p_d}{\partial p'_c},$$

so, due to (281) and (282):

$$\frac{\partial \mathfrak{H}}{\partial p_c} = \frac{\partial H}{\partial p_c}, \qquad \frac{\partial \mathfrak{H}}{\partial p'_c} = \frac{\partial H}{\partial p'_c}, \tag{283}$$

and the *Lagrange* equations for the coordinates p_c experience no change in form when one replaces H with $\mathfrak{H}$ in them. If certain velocities have been eliminated from the expression for H by means of certain equations of motion then *Helmholtz* called the problem in question *incomplete*, since one is restricted to calculating the motions that are linked to those condition equations.

If one seeks mechanical analogies to physical processes then one cannot know from the outset which variables one should place on a parallel to coordinates and which should be analogous to velocities. For example, in electrodynamics, one can draw parallels between the dielectric moments and coordinates and the magnetic ones with velocities, and conversely. Now, in physics, as a rule, the energy is given as a function of the variables experimentally. One can then employ a complete, mechanical problem as an analogy to a physical process only when the experimentally-given expression for the energy is a homogeneous, quadratic function of certain variables, which must then be analogous to velocities. On the contrary, if one chooses an incomplete mechanical problem as an analogy then it can be entirely doubtful which physical variables one should make analogous to velocities and which, to coordinates, since both of them can be included in the expression for energy in an arbitrary form.

§ 54. Helmholtz's reciprocity theorems.

Unlike the *Hertz* reciprocity theorems, these do not refer to cycles, but to completely arbitrary mechanical systems. Whenever an equation of the form (272) is true, it will follow that:

$$\mathfrak{P}_h = -\frac{\partial H}{\partial p_h} + \sum_k \frac{\partial^2 H}{\partial p'_h\,\partial p_k} p'_k + \sum_k \frac{\partial^2 H}{\partial p'_h\,\partial p'_k} p''_k . \tag{284}$$

If certain coordinates are eliminated by the method that was described in the previous paragraph then this equation will remain valid when one understands H to mean the function for which the equations of motion assume precisely the *Lagrange* form, and thus, the quantity H, when equations (276) are true, and the quantity $\mathfrak{H}$, when equations (283) are true. In what follows, we would like to use the same symbol H for all of these quantities, since equations (284) and the ones that we would now like to develop from them are true for all of cases uniformly.

1. From equation (284), the external force $\mathfrak{P}_h$ that must be added to the forces that are determined by the force function F in order to provoke the given temporal change in the coordinate is a linear function of the acceleration p''_h, and one sees immediately that:

$$\frac{\partial \mathfrak{P}_h}{\partial p''_k} = \frac{\partial \mathfrak{P}_h}{\partial p''_h} = \frac{\partial^2 H}{\partial p'_h\,\partial p'_k} . \tag{285}$$

Thus, when an increment in the acceleration p''_k produces an increment in an external force $\mathfrak{P}_h$ that is endowed with another index, with constancy of the coordinates, velocities, and remaining accelerations, then the same increment in the acceleration p''_h

that corresponds to the same coordinate as the force $\mathfrak{P}_h$ will produce an equal increment in the force $\mathfrak{P}_k$ that corresponds to the other coordinate.

Example. From the second of equations (121), it is clear that the acceleration B'' can influence the force $\mathfrak{A}$. Therefore, A'' must also influence $\mathfrak{B}$, and one must have $\frac{\partial \mathfrak{A}}{\partial B''} = \frac{\partial \mathfrak{B}}{\partial A''}$. In fact, from (121), both values will be equal to $- c\, J$.

2. It follows further from equation (284) that:

$$\frac{\partial \mathfrak{P}_h}{\partial p'_k} = -\frac{\partial^2 H}{\partial p_h\, \partial p'_k} + \frac{\partial^2 H}{\partial p'_h\, \partial p_k} + \sum_l \frac{\partial^3 H}{\partial p'_h\, \partial p'_k\, \partial p_l}\, p'_l + \sum_l \frac{\partial^3 H}{\partial p'_h\, \partial p'_k\, \partial p'_l}\, p''_l ,$$

or when one combines the last two sums, as one also must do in order to get from equation (284) back to equation (272):

$$\frac{\partial \mathfrak{P}_h}{\partial p'_k} = -\frac{\partial^2 H}{\partial p_h\, \partial p'_k} + \frac{\partial^2 H}{\partial p'_h\, \partial p_k} + \frac{d}{dt}\left(\frac{\partial^2 H}{\partial p'_h\, \partial p'_k}\right).$$

If one likewise constructs $\partial \mathfrak{P}_k / \partial p'_h$ then one will see that in all cases for which $\partial^2 H / \partial p'_h\, \partial p'_k$ [which, from (285), is equal to $\partial \mathfrak{P}_h / \partial p''_k$ and also equal to $\partial \mathfrak{P}_k / \partial p''_h$] is constant, in particular, whenever it is equal to zero, the following equation will exist:

$$\frac{\partial \mathfrak{P}_h}{\partial p'_k} = -\frac{\partial \mathfrak{P}_k}{\partial p'_h}.$$

In all of these cases then, the following will be true: If a larger velocity p'_k demands a larger $\mathfrak{P}_h$, with equality of all remaining velocities and the values of all coordinates and accelerations, then a larger velocity p'_h will naturally demand a $\mathfrak{P}_k$ that is smaller by the same amount, again with equality of the remaining velocities and the coordinates and accelerations.

Example. If a solid body is capable of rotating around a fixed point in such a way that C, A', and B' are constant, and B' is large in comparison to A' then it will perform a simple precessional motion. The generalized force $\mathfrak{C}$ (viz., the moment of the external force around the OR axis) that acts upon the coordinate C must then be non-zero, and indeed, it must be negative for positive A' and B', and its absolute value must increase with increasing A', such that $\partial \mathfrak{C} / \partial A'$ is negative. Thus, if $\mathfrak{C}$ is generated by the weight, and OZ is the opposite to its direction then the center of mass must lie on the negative $O\zeta$ axis. Furthermore, from (121), one has:

$$\frac{\partial \mathfrak{A}}{\partial C''} = \frac{\partial \mathfrak{C}}{\partial A''} = 0.$$

Thus, $\partial\mathfrak{A} / \partial C'$ must be positive, and its numerical value must be equal to $\partial\mathfrak{C} / \partial A'$. Thus, in order to raise the ratio C' from zero to a small positive value, with no other changes, and then keep it constant for a brief time (i.e., in order to lower the positive $O\zeta$ axis and thus raise the direction of gravity oppositely), a moment $\mathfrak{A}$ would be required that would act in the direction of increasing A, and thus seek to accelerate the precession, which agrees with experience. Naturally, the proof here is applicable only as long as the remaining ratios are still unchanged by the increase in C. Naturally, equations (121) immediately yield:

$$\frac{\partial \mathfrak{A}}{\partial C'} = -\frac{\partial \mathfrak{C}}{\partial A'}.$$

Likewise:

$$\frac{\partial \mathfrak{B}}{\partial C'} = -\frac{\partial \mathfrak{C}}{\partial B'},$$

but not

$$\frac{\partial \mathfrak{A}}{\partial B'} = -\frac{\partial \mathfrak{B}}{\partial A'},$$

since

$$\frac{\partial \mathfrak{A}}{\partial B''} = -\frac{\partial \mathfrak{B}}{\partial A''} = -c\,J.$$

In regard to the proof that this theorem is also true in all of those domains of the theories of heat and electricity and chemistry in which equations are true that are analogous to the mechanical equations that are true for hidden motions, one must refer to the cited original treatise of *Helmholtz*. Here, it shall only be proved, in that regard, that the relationship between the principle of least work and the second law is also true to the extent that most of the relations that one obtains are also proved by means of the second law. (The relation between compression coefficient or elastic moduli and change of temperature under compression or expansion, change of volume under melting and change of melting point under pressure, thermoelectromotor force and the Peltier phenomenon, the production of heat in galvanic elements and the dependency of its electromotor force on temperature, etc. Likewise, I must refer to the original treatise for a series of other propositions and reciprocity theorems, as well as for the further applications in the theory of heat and all applications to electrodynamics, since they would drift too far from the realm of pure mechanics.

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