"Ueber die Druckkräfte, welche auf Ringe wirksam sind, die in bewegte Flüssigkeit tauchen," J. reine angew. Math. **73** (1871), 111-134.

On the pressure that acts on a ring that is immersed in a moving fluid

(By Herrn **Ludwig Boltzmann** in Graz)

Translated by D. H. Delphenich

In a treatise that appeared in volume 71 of this journal, **Kirchhoff** proved that two infinitely-thin rings in a moving fluid that is at rest at infinity will be subject to a fluid pressure whose moment under any displacement will be equal to the moment of the forces that the rings would exert upon each other if certain electrical currents flowed in them. However, that remark is not true in general (*). When the rings are in motion, the forces that are being compared to each other can still differ by forces that depend upon the motion of the rings, and their moment for any displacement will be equal to zero. In fact, calculation will show that this is the case.

Since **Kirchhoff** produced the proof of the validity of the value that he found for the *vis viva* that is included in the fluid only for the case of a ring with a circular cross-section, I would like to make a start towards the general calculation of the *vis viva* that is included in the fluid for rings with non-circular cross-sections. In so doing, it will be shown how the so-called **Hamilton** principle can be applied to the calculation of the force that acts upon the rings, and it will be shown that the way that **Thomson** and **Tait** first applied that principle to the problems in hydrodynamics is inadmissible, in general, but it will lead to no flawed results in the cases that they considered, due to the vanishing of certain terms. Finally, in conclusion, I would like to carry out the direct determination of the pressure that acts on every surface element of the rings in a somewhat-more-general way that also seems to be the simplest way for arriving at the results that are true for infinitely-thin rings. Let an inviscid fluid be surrounded by a surface O that is closed on all sides. Let arbitrarily-many bodies be immersed in that fluid, so all or some of it will nonetheless fill up a multiply-connected space. A generally-multivalued velocity potential shall exist at each point of the fluid. No forces shall act upon the fluid particles beyond the pressure that prevails in it. We first assume that all of the immersed bodies are found to be at rest. We can provide an expression for the velocity potential that will be very useful in the following investigations in the following way: We shall denote the rectangular coordinates of a point of the fluid by x, y, z, and the components of the velocity c that prevails there by u, v, w. Furthermore, the same quantities at a different point will be denoted by x', y', z', u', v', w'. Finally, let r denote the distance between

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(*) As **Kirchhoff** made me aware of in a conversation on this topic, he himself had already pointed that out.

the two points, and define three quantities *A*, *B*, *C* by the following integrals, which extend over the entire space that is filled with fluid:

$$A = -\frac{1}{4\pi}\iiint \frac{u'}{r}\,dx'\,dy'\,dz'\,, \quad B = -\frac{1}{4\pi}\iiint \frac{v'}{r}\,dx'\,dy'\,dz'\,, \quad C = -\frac{1}{4\pi}\iiint \frac{w'}{r}\,dx'\,dy'\,dz'\,,$$

so we will then have:

$$(1) \qquad u = \frac{d^2A}{dx^2}+\frac{d^2A}{dy^2}+\frac{d^2A}{dz^2}\,, \qquad v = \frac{d^2B}{dx^2}+\frac{d^2B}{dy^2}+\frac{d^2B}{dz^2}\,, \qquad w = \frac{d^2C}{dx^2}+\frac{d^2C}{dy^2}+\frac{d^2C}{dz^2}\,.$$

Moreover:

$$(2) \qquad \frac{dA}{dx}+\frac{dB}{dy}+\frac{dC}{dz} = 0\,.$$

One convinces oneself of the validity of the last equation by transforming each of those expressions in the following way:

$$\frac{dA}{dx} = \frac{1}{4\pi}\iiint \frac{u'(x-x')}{r^3}\,dx'\,dy'\,dz' = \frac{1}{4\pi}\iint \left|\frac{u'}{r}\right| dy'\,dz' - \frac{1}{4\pi}\iiint \frac{1}{r}\cdot\frac{du'}{dx'}\,dx'\,dy'\,dz'\,,$$

and imagining that:

$$\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz} = 0$$

in the entire fluid-filled space, and that $\lambda\,u + \mu\,v + \nu\,w = 0$ on its outer surface when λ, μ, ν are the direction cosines of the surface element in question. The fact that the fluid-filled space is multiply-connected does not alter the validity of those transformations, since *u*, *v*, *w* are everywhere single-valued. If one further sets:

$$L = \frac{dB}{dz}-\frac{dC}{dy}\,, \qquad M = \frac{dC}{dx}-\frac{dA}{dz}\,, \qquad N = \frac{dA}{dy}-\frac{dB}{dx}\,,$$

and transforms the differential quotients of *A*, *B*, *C* in the same way as before, while recalling the equations: $\frac{dv}{dz} = \frac{dw}{dy}$, $\frac{dw}{dx} = \frac{du}{dz}$, $\frac{du}{dy} = \frac{dv}{dx}$, then one will get the following ones:

$$L = \frac{1}{4\pi}\int do\,\frac{\mu\,w_a-\nu\,v_a}{r}\,, \qquad M = \frac{1}{4\pi}\int do\,\frac{\nu\,u_a-\lambda\,w_a}{r}\,, \qquad N = \frac{1}{4\pi}\int do\,\frac{\lambda\,v_a-\mu\,u_a}{r}\,,$$

in which *do* is an element of the fluid surface, u_a, v_a, w_a are the components of the velocity that prevails there, and λ, μ, ν are the cosines of the angles that the normal to *do* that points to the fluid interior defines with the positive coordinate axes. The integration is over the entire fluid surface, so it extends over the surfaces of the immersed bodies, as well as the boundary surface O. However, one has:

$$\frac{dN}{dy}-\frac{dM}{dz} = \frac{d^2A}{dx^2}+\frac{d^2A}{dy^2}+\frac{d^2A}{dz^2}-\frac{d}{dx}\left(\frac{dA}{dx}+\frac{dB}{dy}+\frac{dC}{dz}\right) = u$$

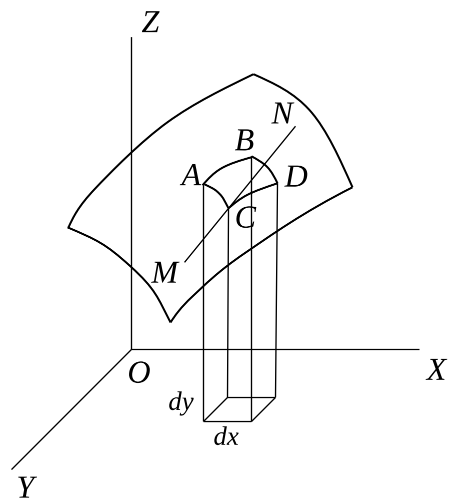
as a consequence of equations (1) and (2). One will then get:

(3) $$u = -\frac{1}{4\pi}\int\frac{do}{r^3}\,[(y-y')(\lambda\,v_a-\mu\,u_a)-(z-z')(\nu\,u_a-\lambda\,w_a)]\ .$$

Corresponding expressions are obtained for v and w. One sees the identity of that calculation with the one that **Helmholtz** performed in his theory of vortex motion. I repeated it here merely to analytically prove the applicability of **Helmholtz**'s formulas for the case of immersed bodies. We can also convince ourselves of that by the following reasoning: We imagine that the surface O is suddenly removed, and our fluid rings are surrounded with fluid of the same type at rest. At the same time, we also remove all of the bodies that are immersed in the fluid and replace them with fluid at rest. Viscosity between the newly-added fluid and the one that was originally present shall exist at one moment, but will again vanish at another. As a result of that, all of the boundary surfaces of the originally-present fluid will be covered by an infinitely-thin layer of rotating fluid particles, and they will become what **Helmholtz** called vortex surfaces. The fluid pressure will generally be different on both sides of any element of the vortex surface, so it will begin to both move and deform. However, we can inhibit that by letting a force act upon each surface element from the outside that brings about equilibrium with the pressures that act upon it. All of the vortex surfaces will then remain at rest, and the motion of the fluid will then be the same as it was in the case where its interior was replaced by the solid body. The pressures that act upon the outer sides of the vortex surfaces are the same as the ones that the surfaces of the solid bodies were exposed to in the former case. However, **Helmholtz**'s formulas are immediately applicable to the latter problem. The expressions that were found for u, v, w are the components of the force that certain electrical currents that flow in the outer surface exert upon a magnetic pole that is found at the point x, y, z per unit volume of magnetic fluid. Namely, if we set:

(4) $g = \mu\,w_a - \nu\,v_a$, $h = \nu\,u_a - \lambda\,w_a$, $k = \lambda\,v_a - \mu\,u_a$, $i = \sqrt{g^2+h^2+k^2}$

then those quantities will fulfill the condition: $g\,\lambda + h\,\mu + k\,\nu = 0$. A line G that is drawn through the point P on the surface whose direction cosines are proportional to the values of g, h, k that prevail there will then be tangent to the surface and likewise perpendicular to the direction of the velocity H of the fluid at the point P. If we regard the surface as a vortex surface then i will be the intensity, and G will be the direction of the vortex filament that goes through P. We would now like to imagine that the total surface of all bodies, as well as the surface O, is covered with electrical currents in such a way that the direction of the current at each point P of the surface is parallel to G, so it will make angles with the coordinate axes whose cosines are g / i, h / i, k / i. However, its intensity is chosen in such a way that when we draw an infinitely-small straight line *dq* parallel to the direction *H*, so it is perpendicular to the direction of the current through *P*, the total intensity of the current that goes through *dq* will be equal to *i dq*. It will simplify the presentation if we imagine that all surfaces are covered with infinitely-many electrical currents that are infinitely dense, and each of which possesses a constant (that is equal for all of them) infinitely-small intensity ε. The number of currents that then go through our line element *dq* is $(i / \varepsilon)\, dq$. It will already follow from the fact that vortex filaments can never terminate that all of the electrical currents that cover the surfaces will be closed, i.e., that none of them can begin or end suddenly. Moreover, one can convince oneself of that in the following way: We consider an element $dx \cdot dy$ in the *xy*-plane and lay a prism through it whose sides are parallel to the *z*-axis. It cuts out the element *ABCD* from any of the surfaces (see the accompanying figure). Let the direction of the current in that element be *MN*. The number of currents that go through *AB* will then be:



$$Z_1 = \frac{i \cdot AB \sin(MN, AB)}{\varepsilon} .$$

The direction cosines of the line are g / i, h / i, k / i. Those of the line *AB*, which is perpendicular to the *y*-axis and the normal to the surface, are: $\frac{\nu}{\sqrt{\lambda^2 + \nu^2}}$, 0, $-\frac{\lambda}{\sqrt{\lambda^2 + \nu^2}}$, from which one finds with no difficulty that:

$$\sin(MN, AB) = \frac{h}{i\sqrt{\lambda^2 + \nu^2}} .$$

Moreover, if *dx* is the projection of *AB* onto the *x*-axis then $AB = dx \sqrt{\lambda^2 + \nu^2} / \nu$. We will then get:

$$Z_1 = \frac{h\, dx}{\nu\, \varepsilon} .$$

We likewise get the value of $Z_2 = \frac{g\, dy}{\nu\, \varepsilon}$ for the number of currents that go through *AC*. The number of electrical currents that exit from *CD* is $Z_1 + \frac{dZ_1}{dy} dy$, and likewise the number of currents that exit through *BD* will be $Z_2 + \frac{dZ_2}{dx} dx$. The excess of the number that exit from the surface element

ABCD over the number that enter it, so the number of currents that begin in that surface element, is then:

$$Z_1 + \frac{dZ_1}{dy}\,dy = \left[\frac{d}{dx}\left(\frac{g}{\nu}\right) + \frac{d}{dy}\left(\frac{h}{\nu}\right)\right]\frac{dx\,dy}{\varepsilon}.$$

However, as a result of equations (4), one has $\frac{g}{\nu} = \frac{\mu}{\nu}w_a - v_a$, $\frac{h}{\nu} = u_a - \frac{\lambda}{\nu}w_a$. If one represents the equation of the surface in question by $z = f(x, y)$ then: $\frac{\lambda}{\nu} = -\frac{df}{dx}$, $\frac{\mu}{\nu} = -\frac{df}{dy}$, and a result:

$$\frac{d}{dx}\left(\frac{g}{\nu}\right) + \frac{d}{dy}\left(\frac{h}{\nu}\right) = \frac{du_a}{dy} - \frac{dv_a}{dx} + \frac{df}{dx}\cdot\frac{dw_a}{dy} - \frac{df}{dy}\cdot\frac{dw_a}{dx}.$$

In forming the partial differential quotients, *z* is regarded as a function of *x* and *y* here, as a result of the equation $z = f(x, y)$, so:

$$\frac{du_a}{dy} = \frac{\partial u_a}{\partial y} + \frac{\partial u_a}{\partial z}\cdot\frac{df}{dy},$$

and similar expressions are obtained for the other partial quotients. The symbol ∂ means a differentiation for which *x*, *y*, and *z* are considered to be independent. If we recall that, in general, the equations:

$$\frac{\partial v_a}{\partial z} = \frac{\partial w_a}{\partial y}, \quad \frac{\partial w_a}{\partial x} = \frac{\partial u_a}{\partial z}, \quad \frac{\partial u_a}{\partial y} = \frac{\partial v_a}{\partial x}$$

will also follow for the surface then we will get:

$$\frac{d}{dx}\left(\frac{g}{\nu}\right) + \frac{d}{dy}\left(\frac{h}{\nu}\right) = 0.$$

No new currents will have their sources in our surface element then, and since that will be true for every element, no current will begin or end suddenly anywhere on the surface, so all current will be closed. If we consider any surface element *do*, define the product of its length with its intensity for every current element that is contained in it, and add all of those products then the sum that we have defined, which we would like to denote by $\sum \varepsilon\, ds$, will be equal to *i do*. Furthermore, when we denote the direction cosines of the current that flows through *do*, so the quantities g / i, h / i, k / i, by *a*, *b*, *c*, respectively, we will have:

$$\sum a\,\varepsilon\,ds = g\,do, \qquad \sum b\,\varepsilon\,ds = h\,do, \qquad \sum c\,\varepsilon\,ds = k\,do,$$

and we will easily convince ourselves that the values for *u*, *v*, *w* that were found in equation (3) are nothing but the components, divided by – 4π, of the electromagnetic effect of all of those currents on a magnetic pole that is found at a point *x*, *y*, *z* per unit volume of magnetic fluid. The velocity potential at the point *x*, *y*, *z* is then the potential of the electrical current at that magnetic pole, divided by – 4π. (The sign of the potential is chosen such that the positive derivatives are equal to the forces.)

The way of distributing electrical currents on surface that we have considered, which we would like to refer to briefly as a minimal configuration, possesses noteworthy property, namely, that the expression:

$$P = \tfrac{1}{2}\sum\sum\iint \frac{\varepsilon^2 \cos\vartheta\, ds\, ds'}{r},$$

so the potential of all currents acting on each other, will be a minimum for it. In it, *ds* and ds' are two current elements, *r* is the distance between then, ϑ is the angle between them, the two integrations are over all elements of any one current, and the summations extend over all currents that are found on the surfaces. We denote the projections of the element *ds* on the coordinate axes by *dx*, *dy*, *dz* and its direction cosines by *a*, *b*, *c*. If the same quantities for ds' are denoted by dx', dy', dz', a', b', c', resp., then we will have

$$P = \frac{\varepsilon^2}{2}\sum\sum\iint \frac{dx\,dx' + dy\,dy' + dz\,dz'}{r}.$$

We now let the positions of all currents on the surface vary by infinitely little. The lengths of the individual currents can change in that way, but they must then remain closed under the introduction of new current elements. The total intensity of all currents that flow through the cross-section of any ring will therefore be invariable. Namely, it will be determined by the increase *k* that the velocity will take on when one orbits around the ring. Namely, if that total intensity is equal to $\mathfrak{I}$, so its potential at a magnetic pole per unit volume of magnetic fluid will increase by – $4\pi\mathfrak{I}$ when one encircles the ring, and since the velocity potential is equal to the potential of the current divided by – 4π, $\mathfrak{I}$ must be equal to *k*. The encircling is counted as positive when it happens from left to right relative to a figure flowing with the current, as seen from the front. Since the variation of *P* that results from the changes in position of all *ds* must be equal to the variation that results from the changes in position of all ds', one will get:

$$\delta P = \varepsilon^2 \sum\sum\iint \left\{\frac{dx'\,d\,\delta x + dy'\,d\,\delta y + dz'\,d\,\delta z}{r} - \frac{(dx\,dx' + dy\,dy' + dz\,dz')[(x-x')\,\delta x + (y-y')\,\delta y + (z-z')\,\delta z]}{r}\right\}.$$

In order to eliminate the differentials of the variations, we partially integrate the first term. When we consider that all of the currents must remain closed, so the variations must be equal at the upper and lower limits, that will give:

$$\delta P = \varepsilon^2 \sum\sum \iint \frac{ds\,ds'}{r^3}\{\delta x[(y-y')\,a'\,b+(z-z')\,a'\,c-(x-x')\,(b\,b'+c\,c')]$$
$$+\ \delta y[(z-z')\,c\,b'+(x-x')\,a\,b'-(y-y')\,(a\,a'+c\,c')]$$
$$+\ \delta z[(x-x')\,a\,c'+(y-y')\,b\,c'-(z-z')\,(a\,a'+b\,b')]\}$$

$$= \sum \int \varepsilon\,ds\{\delta x\,(c\,H-b\,K)+\delta y\,(a\,K-c\,G)+\delta z\,(b\,H-a\,G)\}\,,$$

when we set:

$$G = \sum\int \varepsilon\,ds'\,\frac{(y-y')\,c'-(z-z')\,b'}{r^3} = \int do\,\frac{(y-y')\,k'-(z-z')\,h'}{r^3}\,,$$
$$H = \sum\int \varepsilon\,ds'\,\frac{(z-z')\,a'-(x-x')\,c'}{r^3} = \int do\,\frac{(z-z')\,g'-(x-x')\,k'}{r^3}\,,$$
$$K = \sum\int \varepsilon\,ds'\,\frac{(x-x')\,b'-(y-y')\,a'}{r^3} = \int do\,\frac{(x-x')\,h'-(y-y')\,g'}{r^3}\,.$$

G, *H*, *K* are then nothing but the components of the effect of all currents acting upon a magnetic pole per unit volume of magnetic fluid. They will be indeterminate on the surfaces. Namely, we see that they are equal to $-4\pi u_a$, $-4\pi v_a$, $-4\pi w_a$ on the outer sides. By contrast, they will be zero on the inner sides, since the potential of the current on a magnetic pole is single-valued in the whole interior of the body, and its derivative normal to the surface is equal to zero on the surface. In order to avoid that double-valuedness, we imagine that the current is spread across the surface in a uniform layer of infinitely-small thickness. The values of the quantities *G*, *H*, *K* are then equal to zero on the inner side of the layer and take the values $-4\pi u_a$, $-4\pi v_a$, $-4\pi w_a$ on the outer sides uniformly. The displacements δx, δy, δz shall be taken to be equal for all current elements that lie in a layer. It is easy to prove that we must then set *G*, *H*, *K* equal to their mean values $-2\pi u_a$, $-2\pi v_a$, $-2\pi w_a$, resp. The variations are then subject to the condition that $\lambda\,\delta x + \mu\,\delta y + \nu\,\delta z = 0$ since the currents should not leave the surface. The condition that δP should not vanish will then reduce to:

$$\frac{c\,H-b\,K}{\lambda} = \frac{a\,K-\nu\,G}{\mu} = \frac{b\,G-a\,H}{\nu}\,.$$

That condition is, in fact, fulfilled by the way that we have distributed the current as a minimal configuration, as we see, when we replace *G*, *H*, *K* with the values $-2\pi u_a$, $-2\pi v_a$, $-2\pi w_a$, resp., and imagine that $a = g / i$, $b = h / i$, $c = k / i$. If currents of constant intensity, but varying length, flow on the outer surface (or even the inner one) of an annular body along its centerline, so they are constrained to encircle it, then they will be in (generally labile) equilibrium under the influence of their electrodynamical interaction, so their mutual potential will be a minimum, so they will be found to be in the minimal configuration. They will then be analogous to the distribution of static electric electricity on the outer surface of a conductor. However, in some special cases, the splitting of a current of intensity zero into two equal and opposite ones can perturb that picture, while the analogue is always possible for static electricity. It is immediately clear that when such an equilibrium configuration of currents is possible, their potential must fulfill the requirements that are imposed upon the velocity potential. We can also base the proof of our theorem on that. The effect of the current on a magnetic pole in the interior of the body or outside the surface *O* is therefore zero.

We would now like to calculate the total *vis viva* that is contained in the fluid. If ρ represents the density of the fluid then it will be:

$$T = \rho \iiint dx\, dy\, dz\, \tfrac{1}{2}\left[\left(\frac{d\varphi}{dx}\right)^2 + \left(\frac{d\varphi}{dy}\right)^2 + \left(\frac{d\varphi}{dz}\right)^2\right].$$

φ is the velocity potential, so it is the potential, divided by -4π, of all electrical current at a magnetic pole per unit volume of magnetic fluid. If we now denote the potential, divided by -4π, of a single current at one such magnetic pole by $\delta\varphi$, and denote a summation over all currents on all surfaces by $\sum$ then we will have:

$$\frac{d\varphi}{dx} = \sum \frac{d\,\delta y}{dx}, \qquad \left(\frac{d\varphi}{dx}\right)^2 = \sum \frac{d\,\delta\varphi}{dx} \cdot \sum \frac{d\,\delta'\varphi}{dx} = \sum\sum \frac{d\,\delta\varphi}{dx} \cdot \frac{d\,\delta'\varphi}{dx}.$$

In the last formula, $\delta\varphi$, as well as $\delta'\varphi$, is the potential, divided by -4π, of each individual current (say, $\delta\varphi$ is that of the current *S*, and $\delta'\varphi$ is that of the current S'), and both summations extended over all currents. As a result, we will have:

$$T = \frac{\rho}{2} \sum\sum \iiint dx\, dy\, dz \left[\frac{d\,\delta\varphi}{dx} \cdot \frac{d\,\delta'\varphi}{dx} + \frac{d\,\delta\varphi}{dy} \cdot \frac{d\,\delta'\varphi}{dy} + \frac{d\,\delta\varphi}{dz} \cdot \frac{d\,\delta'\varphi}{dz}\right].$$

However, from **Green**'s theorem:

$$(6) \qquad \iiint dx\, dy\, dz \left(\frac{d\,\delta\varphi}{dx} \cdot \frac{d\,\delta'\varphi}{dx} + \frac{d\,\delta\varphi}{dy} \cdot \frac{d\,\delta'\varphi}{dy} + \frac{d\,\delta\varphi}{dz} \cdot \frac{d\,\delta'\varphi}{dz}\right) = -\int do\, \delta\varphi \frac{d\,\delta'\varphi}{dn} - \int d\omega\, \delta\varphi \frac{d\,\delta'\varphi}{dn},$$

in which *do* is a surface element of the solid body, but $d\omega$ is a surface element of a cross-section by which one can make the space that is filled with fluid simply connected. d / dn means a differentiation in the direction of the normal that points into the fluid. $\delta\varphi$ will have the same value on both sides of a cross-section, with the exception of one of them. That would be the cross-section whose boundary line lies on the same surface of the ring as the current *S*. We can choose the boundary line of that cross-section to be the current itself. The values of $\delta\varphi$ on both sides of that cross-section differ by ε since that is the common intensity of all currents. One then has:

$$\int d\omega\, \delta\varphi \frac{d\,\delta'\varphi}{dn} = \varepsilon \int d\omega \frac{d\,\delta'\varphi}{dn},$$

so it is equal to the mutual potential of the currents S and S', divided by -4π. In so doing, it is assumed that the mutual potential of two currents of intensities J and J' is $J\,J'\iint \frac{ds\,ds'\cos\vartheta}{r}$, in which the intensities are measured in so-called electromagnetic units. When we sum equation (6) over δ and δ' and multiply by $\rho / 2$, we will have:

$$T = -\frac{\rho}{2}\sum\sum\int do\,\delta\varphi\frac{d\,\delta'\varphi}{dn} - \frac{\rho}{2}\sum\sum\int d\omega\,\delta\varphi\frac{d\,\delta'\varphi}{dn}.$$

The second term on the right of this is the mutual potential of all currents, multiplied by $\rho / 4\pi$ since the sum of the potential of each of them on themselves contribute only vanishingly little to it. However, the first term on the right is equal to:

$$-\frac{\rho}{2}\int d\omega\sum\delta\varphi\sum\frac{d\,\delta'\varphi}{dn} = -\frac{\rho}{2}\int do\,\varphi\frac{d\varphi}{dn} = 0\,,$$

since $d\varphi / dn$ vanishes for all surface elements. The total *vis viva* that is contained in the fluid is then equal to the mutual potential of all those currents S, multiplied by $\rho / 4\pi$. The total intensity of any current that flows in a ring must then be equal to the increase k that the velocity potential experiences during a circuit of the ring, and its direction is counted as positive when the circuit takes place from left to right when seen from the front by a figure that flows with the current. The distribution of current is determined by that in such a way that δP will vanish when the surfaces are fixed. If we displace all bodies infinitely-little and imagine that when the current is brought to its new position, that will correspond to a minimal configuration then the increase in P will consist of two parts, one of which originates in the displacement of the currents in the bodies while the distribution on their surface remains unchanged, and the other one stems from the change in that current distribution that is necessary in order for it to produce the minimal configuration that corresponds to the new position of the ring. However, the latter is equal to zero, because we know that δP will be equal to zero for a variation of the distribution of currents with no change in the position of the surface. The variation that the total potential of the currents experiences is then equal to the potential that it would experience if the distribution of currents on the outer surface of the bodies were not changed.

We would now like to go on to the case in which the body is not at rest in the fluid, and indeed we would like to assume that it will change in form, but not in volume, since the calculations will not become any more difficult by doing so. By contrast, the surface O shall be at rest, although its motion would not modify the process of calculation essentially, either. The components of the velocity of any surface element *do* of the body shall be called α, β, γ. For the moment, we would like to merely construct the surface of the body from a substance that is different from the fluid but imagine that its interior is again filled with fluid that is originally at rest, and naturally, as long as the surface of the body moves, that fluid will also be put into motion, and indeed while a single-valued velocity potential exists. We further set:

$$A = -\frac{1}{4\pi}\iiint \frac{u'}{r}dx'\,dy'\,dz', \quad B = -\frac{1}{4\pi}\iiint \frac{v'}{r}dx'\,dy'\,dz', \quad C = -\frac{1}{4\pi}\iiint \frac{w'}{r}dx'\,dy'\,dz',$$

in which the integration extends over all of the fluid that is found inside of the surface *O* outside and inside of the body. One will then have:

$$u = \frac{d^2A}{dx^2}+\frac{d^2A}{dy^2}+\frac{d^2A}{dz^2}, \qquad v = \frac{d^2B}{dx^2}+\frac{d^2B}{dy^2}+\frac{d^2B}{dz^2}, \qquad w = \frac{d^2C}{dx^2}+\frac{d^2C}{dy^2}+\frac{d^2C}{dz^2},$$

$$\frac{dA}{dx}+\frac{dB}{dy}+\frac{dC}{dz} = 0$$

again in the entire fluid. In order to prove the last of those equations, we transform the derivatives of *A*, *B*, *C* as before, for the fluid inside, as well as outside, the body. Now, $\lambda\, u + \mu\, v + \nu\, w$ is indeed no longer equal to zero on its surface. However, that quantity will possess the same value on both sides of the surface, so the terms in the expression for $\frac{dA}{dx}+\frac{dB}{dy}+\frac{dC}{dz}$ that relate to the limits will again drop out. If we keep that fact in mind and transform the three quantities:

$$L = \frac{dB}{dz}-\frac{dC}{dy}, \qquad M = \frac{dC}{dx}-\frac{dA}{dz}, \qquad N = \frac{dA}{dy}-\frac{dB}{dx}$$

then we will get:

$$L = \frac{1}{4\pi}\int do\,\frac{\mu(w_a-w_i)-\nu(v_a-v_i)}{r}, \qquad M = \frac{1}{4\pi}\int do\,\frac{\mu(w_a-w_i)-\nu(v_a-v_i)}{r},$$

$$N = \frac{1}{4\pi}\int do\,\frac{\mu(w_a-w_i)-\nu(v_a-v_i)}{r},$$

in which u_a, v_a, w_a are the velocity components of the fluid on the outer side of *do*, while u_i, v_i, w_i are the components on the inner side. Since we assume that the surface *O* is at rest, we will have $u_i = v_i = w_i = 0$ for it, but u_a, v_a, w_a will be simply the components of the velocities that exist on its elements. As before, that will give:

$$u = \frac{dN}{dy}-\frac{dM}{dz} = -\frac{1}{4\pi}\int do\left[\frac{\lambda(v_a-v_i)-\mu(u_a-u_i)}{r^3}(y-y')-\frac{\nu(u_a-u_i)-\lambda(w_a-w_i)}{r^3}(z-z')\right].$$

One will get corresponding expressions for *v* and *u*. *u*, *v*, *w* are once more the force components of closed electrical currents that flow in the outer surface on a magnetic pole per unit volume of magnetic fluid. The intensity and direction of the current is determined as before, except that the differences $u_a - u_i$, $v_a - v_i$, $w_a - w_i$ will enter in place of u_a, v_a, w_a. If we lay an infinitely-small line *dq* anywhere that is perpendicular to the direction of the current then the intensity of the current that goes through dq will be $i\,dq$. However, the cosines of the angles that the direction of current defines with the coordinate axes are g / i, h / i, k / i, in which we now have:

$$g = \mu\,(w_a - w_i) - \nu\,(v_a - v_i)\,, \quad h = \nu\,(u_a - u_i) - \lambda\,(w_a - w_i)\,, \quad k = \lambda\,(v_a - v_i) - \mu\,(u_a - u_i)\,,$$

$$i = \sqrt{g^2 + h^2 + k^2}\,.$$

If we imagine that the outer surface is covered with infinitely-many currents of constant, but infinitely-small, intensity ε then $(i / \varepsilon)\,dq$ will be the number of currents that go through dq. The potential (divided by -4π) of that current on a magnetic pole per unit volume of the magnetic fluid will then be the velocity potential outside, as well as inside, the body. It is constant outside of O. The potential of all currents on each other, so the expression $P = \frac{1}{2}\varepsilon^2 \sum\sum\iint \frac{ds\,ds'\cos\vartheta}{r}$, will once more exhibit a minimum property. Namely, if we transform its variation as before, that will give:

$$\delta P = \varepsilon \sum \int ds[\delta x(x H - b K) + \delta y(a K - c G) + \delta z(b G - a H)]\,,$$

in which:

$$G = \int do\,\frac{k'(y-y') - h'(z-z')}{r}\,, \quad H = \int do\,\frac{g'(z-z') - k'(x-x')}{r}\,, \quad K = \int do\,\frac{h'(x-x') - g'(y-y')}{r}\,.$$

When all surfaces are imagined to be covered by an infinitely-thin current layer, the quantities G, H, K will again have the same uniform values on the inside of it that they have on the outside. Their mean values, which will be taken here, are: $-2\pi\,(u_a + u_i)$, $-2\pi\,(v_a + v_i)$, $-2\pi\,(w_a + w_i)$. We now choose the three functions α, β, γ of x, y, z such that on the outer surface of any body, they are equal to the velocity components on the corresponding surface element and fulfill the equation: $\frac{d\alpha}{dx} + \frac{d\beta}{dy} + \frac{d\gamma}{dz} = 0$ in the interior of the body and determine three other functions l, m, n of x, y, z by the equations:

$$\frac{dn}{dy} - \frac{dm}{dz} = \alpha\,, \qquad \frac{dl}{dz} - \frac{dn}{dx} = \beta\,, \qquad \frac{dm}{dx} - \frac{dl}{dy} = \gamma\,.$$

If we now denote the expression:

$$\sum \int \varepsilon\,ds\,(a\,l + b\,m + c\,n) = \sum \int \varepsilon\,(l\,dx + m\,dy + n\,dz) \tag{5}$$

by Q, in which the integration extends over all length elements of a current and the summation extends over all currents on all surfaces (l, m, n are generally different for each body), then after eliminating the differentials of the variations:

$$(6)\quad \begin{cases} \delta Q = \sum \int \varepsilon\, ds\Big[\delta x\left(b\dfrac{dm}{dx} + c\dfrac{dn}{dx} - b\dfrac{dl}{dy} - c\dfrac{dl}{dz}\right) \\ \qquad + \delta y\left(c\dfrac{dn}{dy} + a\dfrac{dl}{dy} - c\dfrac{dm}{dz} - a\dfrac{dm}{dx}\right) \\ \qquad + \delta z\left(a\dfrac{dl}{dy} + b\dfrac{dm}{dz} - a\dfrac{dn}{dx} - b\dfrac{dn}{dy}\right)\Big] \\ \quad = \sum \int \varepsilon\, ds\left[\delta x\,(b\,\gamma - c\,\beta) + \delta y\,(c\,\alpha - a\,\gamma) + \delta z\,(a\,\beta - b\,\alpha)\right]. \end{cases}$$

One will then get:

$$\delta(P - 4p\,Q) = 2\pi\,\varepsilon \sum \int ds\,\{\delta x[b\,(w_a + w_i - 2\,\gamma) - c\,(v_a + v_i - 2\,\beta)]$$
$$+\ \delta y[c\,(u_a + u_i - 2\,\alpha) - a\,(w_a + w_i - 2\,\gamma) + \delta z[a\,(v_a + v_i - 2\,\beta) - b\,(u_a + u_i - 2\,\alpha)]\}\,.$$

Now since the velocity components of the fluid inside of the body, as well as outside of it, in the direction of the normal to its outer surface must be equal to the velocity components of the corresponding surface element in that direction, one will have:

$$\lambda\,(u_a + u_i - 2\alpha) + \mu\,(v_a + v_i - 2\alpha) + \nu\,(w_a + w_i - 2\gamma) = 0\,.$$

However, it follows from that equation and the equation $a\,\lambda + b\,\mu + c\,\nu = 0$ that:

$$\frac{b\,(w_a + w_i - 2\gamma) - c\,(v_a + v_i - 2\beta)}{\lambda} = \frac{c\,(u_a + u_i - 2\alpha) - a\,(w_a + w_i - 2\gamma)}{\mu} = \frac{a\,(v_a + v_i - 2\beta) - b\,(u_a + u_i - 2\alpha)}{\nu}.$$

$\delta(P - 4\pi\,Q)$ will then vanish for all δx, δy, δz that satisfy the condition that $\lambda\,\delta x + \mu\,\delta y + \nu\,\delta z = 0$ so for any variation of the position of the current when it does not leave the outer surface and remains closed with unchanged intensity. In the special case where the form of the body does not change, if ξ, η, ζ are the components of the progressive velocity and p, q, r are those of the instantaneous angular velocity then:

$$Q = \varepsilon \sum \int [\eta\, z\, dx + \zeta\, x\, dy + \xi\, y\, dz - \tfrac{1}{2}\, p\,(y^2 + z^2)\, dx - \tfrac{1}{2} q\,(z^2 + x^2)\, dy - \tfrac{1}{2} r\,(x^2 + y^2)\, dz]\,.$$

We would like to refer to the expressions under the integral sign in formula (6) that are multiplied by δx, δy, δz, so the three quantities $(b\,\gamma - c\,\beta)\,\varepsilon\, ds$, $(c\,\alpha - a\,\gamma)\,\varepsilon\, ds$, $(a\,\beta - b\,\alpha)\,\varepsilon\, ds$, as the components of the force that acts on the current element ds whose force potential is Q. The force that such magnetic masses would exert upon the current element is the force with the components α, β, γ would exert upon a magnetic pole. It is perpendicular to the current element ds and its direction of motion, its intensity is $\varepsilon\, ds$ times the components of the velocity of ds, which is perpendicular to *ds*. Since it is always normal to the direction of motion, it will do no work. The distribution of electric current on the body is then the one for which the body would be in equilibrium under the influence of its electrodynamical interaction and the force (times -4π) that acts on every current element as a result of the force potential *Q*.

In the special case where all points on the surface *O* are found at an infinite distance, at least of order *R*, and the space that they enclose is simply connected, the potential of the current that is found on the outer surface of the body at each point of *O* will be infinitely-small of order at least $1/R^2$, so its derivatives will be infinitely-small of order at least $1/R^3$. Therefore, the velocity on the outer surface will also be of that order, so the intensity of the electrical current that flows through it per unit area will be, as well. The electrical current that is found on the unit area will then produce a potential that is infinitely small of order at least $1/R^4$ at finite points. The potential of all currents that are found on *O* will then be infinitely-small of order at least $1/R^2$ at finite points, so its derivatives will vanish with increasing *R*. If we further assume that they are nothing but rings whose cross-section perpendicular to the centerline vanishes then if the constant *k* is to be finite, the velocity on the outer surface of the ring must be infinite and almost perpendicular to the centerline. The fictitious electrical current on the outer surface of the ring must then flow along the centerline, and since the ring is infinitely thin, one can imagine that it flows through the centerline by neglecting infinitely-small quantities. It is now important to point out that in this case, the velocity potential itself is still finite when one is infinitely-close to the outer surface of the ring. In fact, except for a constant, the potential of an electrical current on a magnetic potential per unit volume of magnetic fluid must be equal to at most 4π times the intensity of the current, and since the total intensity of the current that flows through the ring is finite, its potential will be finite, as well. By comparison, its derivatives will become infinite on the outer surface of the ring. We will find a suitable form for the velocity potential φ in this case in the following way: If φ_1 is the velocity potential that would prevail if the ring were at rest in its instantaneous position, while φ_2 is the one that would prevail if the same ring moved in the same way under the existence of a single-valued velocity potential, then $\varphi_1 + \varphi_2$ will fulfill all of the conditions that are imposed on φ. Thus, $\varphi = \varphi_1 + \varphi_2$. Since φ_2 is single-valued, but $d\varphi_2 / dn$ vanishes on the outer surface of the ring, the *vis viva* *T* of the fluid that surrounds the ring will then be:

$$T = -\frac{\rho}{2}\int d\omega\,\varphi_1\frac{d\varphi_1}{dn} - \frac{\rho}{2}\int d\omega\,\varphi_1\frac{d\varphi_2}{dn} - \frac{\rho}{2}\int do\,\varphi\frac{d\varphi_2}{dn}\,.$$

do is once more an element of the outer surface of the ring, and $d\omega$ is an element of a cross-section. If the velocity of the ring is finite then so is the intensity of the current that creates the potential φ_2 per unit area, and since the area that the current flows through is infinitely small, φ_2 itself will be infinitely small. Its derivatives are finite in only an infinitely-small part of space, but otherwise infinitely-small; φ is everywhere-finite. The last two terms in the expression for *T* will vanish then. However, as we saw, the first term $-\frac{\rho}{2}\int d\omega\,\varphi_1\frac{d\varphi_1}{dn}$ is the potential (times $\rho / 4\pi$) of all currents that we must imagine are covering the ring at rest, so the currents that are distributed over it in the minimal configuration. The variation of the *vis viva* will also be equal to the variation of that potential then. However, we found that, first of all, the variation of that potential for the minimal distribution is just as large as it would be if the currents did not change their positions on the ring and the surface *O*, and that secondly, the potential of the current that is found on *O* on the current that is found on the outer surface of the ring will vanish, and thirdly, that we can imagine that all currents flow along the centerline, which will then make its intensity equal to the value of *k* for the ring in question. The increase in the *vis viva* of the fluid is them equal, up to infinitely-small quantities, to $\rho / 4\pi$ times the increase in the mutual potential of the currents that flow along the centerline with intensity *k*. That will then yield the proof of **Kirchhoff**'s theorem for rings of non-circular cross-section, as well. It might still be remarked that currents that do not flow along the centerline, but in such a way that they twine around the centerline solenoidally, would produce a velocity potential that is multivalued in the interior of the ring. It will follow immediately from the equality of the *vis viva* with the work done by that current in its motion that the force components that fall in the direction of motion will be equal. However, in order to also find the force perpendicular to the direction of motion, one must go down a different path. If one would like to start from the *vis viva* then the so-called **Hamilton** principle might serve that purpose. Namely, we form the variation of the quantity:

$$\Omega = \rho \int_0^\tau dt \iiint dx\,dy\,dz\,\frac{u^2 + v^2 + w^2}{2},$$

such that the total state of motion of the body will change arbitrarily-little at every arbitrary time *t*, but the position of the body at the beginning and the end will remain constant, as well as the time interval τ. The triple integral in Ω extends over the entire space that is filled with fluid at every point in time, so the space outside of the body. Ω is then the so-called action of the fluid during the time interval τ. We let δx, δy, δz, $\delta u = \frac{d\,\delta x}{dt}$, $\delta v = \frac{d\,\delta y}{dt}$, $\delta w = \frac{d\,\delta z}{dt}$ denote the variations that the coordinates and velocity components of the fluid-particle that had the coordinates *x*, *y*, *z* at time *t* would experience at that same time by the change in the state of motion of the body. Thus, δu, δv, δw are not variations of the velocity components that prevail at a constant spatial point. We will then have:

$$\begin{aligned}\delta\Omega &= \rho \int_0^\tau dt \iiint dx\,dy\,dz\,(u\,\delta u + v\,\delta v + w\,\delta w) \\ &= \rho \int_0^\tau dt \iiint dx\,dy\,dz \left(u\,\frac{d\,\delta x}{dt} + v\,\frac{d\,\delta y}{dt} + w\,\frac{d\,\delta z}{dt} \right).\end{aligned}$$

If we integrate by parts over *t* here and observe that *u*, *v*, *w* include *t* explicitly, as well as due to the fact that *x*, *y*, *z* are also functions of *t*, then that will give:

$$\delta\Omega = \rho \iiint dx\,dy\,dz \left| u\,\delta u + v\,\delta v + w\,\delta w \right|_0^\tau$$

$$- \rho \int_0^\tau dt \iiint dx\,dy\,dz \left[\delta x \left(\frac{du}{dt} + u\,\frac{du}{dx} + v\,\frac{du}{dy} + w\,\frac{du}{dz} \right) + \delta y \left(\frac{dv}{dt} + u\,\frac{dv}{dx} + v\,\frac{dv}{dy} + w\,\frac{dv}{dz} \right) \right.$$
$$\left. + \ \delta z \left(\frac{dw}{dt} + u\,\frac{dw}{dx} + v\,\frac{dw}{dy} + w\,\frac{dw}{dz} \right) \right].$$

If no forces act upon the fluid from the outside then the terms in the integral on the right-hand side that are multiplied by $\delta\,x$, $\delta\,y$, $\delta\,z$ are equal to $-\frac{dp}{dx}$, $-\frac{dp}{dy}$, $-\frac{dp}{dz}$. If we consider that the displacements $\delta\,x$, $\delta\,y$, $\delta\,z$ must not perturb the continuity of the fluid, so we must have $\frac{d\,\delta x}{dx} + \frac{d\,\delta y}{dy} + \frac{d\,\delta z}{dz} = 0$, and partially integrate the term that is multiplied by δx over x, the one that is multiplied by δy over y, and the one that is multiplied by δz over z, then that will give:

$$\delta\Omega = \rho \iiint dx\,dy\,dz \,\big|\, u\,\delta x + v\,\delta y + w\,\delta z\,\big|_0^\tau - \int_0^\tau dt \int do\; p\,\delta q \cos\eta\;,$$

in which δq is the total displacement of the fluid particle that lies on the outer surface element *do*, and η is the angle between its direction and the normal that is raised over *do* on the side of the fluid. The last term in the expression for $\delta\Omega$ will take the opposite sign when *p* represents the pressure that acts upon outer surface element *do* of the body, not on the fluid. When *X do*, *Y do*, $Z\,do$ are the components of that pressure, it will also be equal to $\int_0^\tau dt \int do\,(X\;\delta x + Y\,\delta y + Z\,\delta z)$ then. If we ignore the first terms in the expression for $\delta\Omega$ then the term that is multiplied by δx will give us the force that acts upon *do* in the direction of the *x*-axis. Similarly, the terms that are multiplied by δy and δz represent the forces that act in the directions of the *y* and *z*-axes, resp., which one likewise recognizes to be **Hamilton**'s principle in the form that was first applied to hydrodynamics by **Thomson** and **Tait**. By contrast, the first term shows that this application is generally incorrect. That is based in the fact that the derivation of **Hamilton**'s principle assumes that the initial and final positions of all material points suffer no variation. However, in our case, that is true for only the points of the solid body, while the final positions of the fluid particles will vary, in general, and therefore the first term. We can again partially-integrate the part of it that is multiplied by δx relative to *x*, etc. It will then go to $-\rho \int do \,\big|\, \varphi\,\delta q \cos\eta\,\big|_0^\tau$, but the integration will not only extend over the outer surface of the body, but also over all cross-sections by which the space that is filled with fluid can be made simply connected. Since we have assumed that the initial and final positions of the body experienced no variation, cos η will vanish for the entire outer surface of the body. The term that originates in the in variation at the limits will vanish then as long as the velocity potential is single-valued. Under that condition, the application of **Hamilton**'s principle, as **Thomson** and **Tait** did, will be, in fact, admissible, except that the vanishing of the term that stems from the variation at the limits cannot be assumed to be obvious, but must first be proved. However, one would be led to an incorrect result in the case where a multivalued velocity potential (or none at all) exists. In the former case, the term that stems from the variation at the limits will be converted into $\sum k \int do\, \rho \,|\, \delta q \cos\eta \,|_0^\tau$, in which k has the previous meaning. The integration extends over all elements of a cross-section, and the $\sum$ sign means a summation over all cross-sections. $\int do\, \rho\, \delta q \cos\eta$ is the mass of the fluid that flows through a cross-section. When the initial position of the fluid particles does not vary, the term that originates in the variation at the limits is then the sum of the products of the fluid masses that go through all cross-sections in the direction of increasing φ with the constant k for the cross-section in question. In order to find the forces that act upon the body by that method, in each special case, one must then convert that term into an integral that is taken for t between zero and τ that includes only δx, δy, δz under the integral sign. I shall again begin with the simpler case in which the outer surfaces of all bodies are at rest. φ will not include time then, and the pressure p that acts upon any outer surface element do will be equal to a constant minus $\frac{1}{2}\rho(u_a^2 + v_a^2 + w_a^2)$. We would now like to calculate the electrodynamical force that the current element that covers the element do exerts upon all of the other ones. Its components in the directions of the three coordinates axes are:

$$X = do\,(k\,H - h\,K)\,, \quad Y = do\,(g\,K - k\,G)\,, \quad Z = do\,(h\,G - g\,H)\,,$$

from the known formulas for the action of closed currents on a current element. With the previously-applied notation, one will then have:

$$G = \int do \frac{k'(y-y') - h'(z-z')}{r^3}, \quad H = \int do \frac{g'(z-z') - k'(y-y')}{r^3}, \quad K = \int do \frac{h'\,(x-x') - g'(y-y')}{r^3}$$

here. As we have seen, the values G, H, K will be indeterminate as long as the currents flow through an absolute surface. By contrast, if we replace the surface with a layer of infinitely-small thickness in which each surface element is filled with currents uniformly then they will increase from zero to $-4\pi\, u_a$, $-4\pi\, v_a$, $-4\pi\, w_a$, resp., from the inner side to the outer side. Thus, the forces that act upon a current element, and which are indeed proportional to the quantities G, H, K will also increase uniformly from the inner side to the outer side, and in order to find the total force that acts upon the current layer that covers the outer surface element, we must replace G, H, K with their mean values $-2\pi\, u_a$, $-2\pi\, v_a$, $-2\pi\, w_a$, resp. When we also replace g, h, k with their values and consider the equation $\lambda\, u_a + \mu\, v_a + \nu\, w_a = 0$, we will then get:

$$X = -\,2\pi\,\lambda\,(u_a^2 + v_a^2 + w_a^2)\,do, \quad Y = -\,2\pi\,\lambda\,(u_a^2 + v_a^2 + w_a^2)\,do, \quad Z = -\,2\pi\,\lambda\,(u_a^2 + v_a^2 + w_a^2)\,do.$$

The electrodynamical effect, like the hydrodynamical pressure, is then perpendicular to do. Its intensity is $2\pi(u_a^2 + v_a^2 + w_a^2)\,do$, so except for a constant, it will be equal to $4\pi / \rho$ times that hydrodynamical pressure. If the bodies are nothing but infinitely-thin rings then the currents that are found on the outer surfaces can be considered to be ones that flow along the centerline with an intensity of k. Furthermore, the effect of the currents that are found on O will vanish when that surface is at infinity. Therefore, in that case, the total effect that any ring experiences from the fluid will be equal, but in the opposite direction, to $\rho / 4\pi$ times the electrodynamical effect that the current that flows along the centerline of that ring with an intensity of k will experience from all other currents that flow in the remaining rings in a similar manner. That will be true when the bodies are at rest. However, things will be more complicated when they are moving, and I would presently like to go into the details of that only in the case of an infinitely-thin ring in an unbounded fluid. As before, we imagine that the interior of the ring is filled a moving fluid for which a single-valued velocity potential exists. If the velocity potential is φ_a for a point on the outer surface of the ring and φ_i on the inner side then φ_a, as well as φ_i, will be finite on each side, so the differential quotients of those quantities with respect to time will be, as well, since the existence of an oscillation with infinitely-small periods would obviously be excluded if the motion of the ring were continuous. The point of the outer surface considered shall advance with the velocity components α, β, γ. The quotient of the total differential of φ_a with the differential of time will then be $\frac{d\varphi_a}{dt} + \alpha \frac{d\varphi_a}{dx} + \beta \frac{d\varphi_a}{dy} + \gamma \frac{d\varphi_a}{dz}$, and likewise, for φ_i, one will have: $\frac{d\varphi_i}{dt} + \alpha \frac{d\varphi_i}{dx} + \beta \frac{d\varphi_i}{dy} + \gamma \frac{d\varphi_i}{dz}$. The difference of the two will then be a finite quantity E in any event. We can then set:

$$\frac{d\varphi_a}{dt} + \alpha \frac{d\varphi_a}{dx} + \beta \frac{d\varphi_a}{dy} + \gamma \frac{d\varphi_a}{dz} - \frac{d\varphi_i}{dt} - \alpha \frac{d\varphi_i}{dx} - \beta \frac{d\varphi_i}{dy} - \gamma \frac{d\varphi_i}{dz} = E .$$

We would now like to denote the pressure that acts upon an element of the outer surface of the ring from the outside by p_a and the pressure that acts upon the same element from inside of the ring of fictitious fluid by p_i . We will then have:

$$p_a - p_i = \rho \frac{d\varphi_i}{dt} + \frac{\rho}{2}\left[\left(\frac{d\varphi_i}{dx}\right)^2 + \left(\frac{d\varphi_i}{dy}\right)^2 + \left(\frac{d\varphi_i}{dz}\right)^2\right] - \rho \frac{d\varphi_i}{dt} - \frac{\rho}{2}\left[\left(\frac{d\varphi_a}{dx}\right)^2 + \left(\frac{d\varphi_a}{dy}\right)^2 + \left(\frac{d\varphi_a}{dz}\right)^2\right] .$$

If we introduce the quantity E here in place of $\frac{d\varphi_a}{dt} - \frac{d\varphi_i}{dt}$ then we will get:

$$p_a - p_i = \frac{\rho}{2}\left[\left(\frac{d\varphi_i}{dx} - \alpha\right)^2 + \left(\frac{d\varphi_i}{dy} - \beta\right)^2 + \left(\frac{d\varphi_i}{dz} - \gamma\right)^2\right] - \frac{\rho}{2}\left[\left(\frac{d\varphi_a}{dx} - \alpha\right)^2 + \left(\frac{d\varphi_a}{dy} - \beta\right)^2 + \left(\frac{d\varphi_a}{dz} - \gamma\right)^2\right] - \rho E \cdot$$

We would once more like to calculate the electrodynamical effect that all of the current elements that cover the outer surface element do experience from the remaining ones, in addition to the forces, times -4π, whose velocity potential is the quantity Q that is determined by the formula (5). We gave the definition of those forces before. The three components of the forces that act upon do can then be calculated precisely as they were in the case of a body at rest. When all of the currents that cover the element do are on its outer side, they will be:

$$X_a = 4\pi\, i\, do\, [b\, (w_a - \gamma) - c\, (v_a - \beta)]\,, \qquad Y_a = 4\pi\, i\, do\, [c\, (u_a - \alpha) - a\, (w_a - \gamma)]\,,$$

$$Z_a = 4\pi\, i\, do\, [a\, (v_a - \beta) - b\, (u_a - \alpha)]\,,$$

in which a, b, c are again the direction cosines of the current in do. We will likewise get:

$$X_i = 4\pi\, i\, do\, [b\, (w_i - \gamma) - c\, (v_i - \beta)]\,, \qquad Y_i = 4\pi\, i\, do\, [c\, (u_i - \alpha) - a\, (w_i - \gamma)]\,,$$

$$Z_i = 4\pi\, i\, do\, [a\, (v_i - \beta) - b\, (u_i - \alpha)]$$

for the inner side. If the currents are in the interior of the layer that covers *do* then part of the current will be counted as internal and the other part as external, and if one is to find the components of all forces that act upon the current elements that cover *do* then one must take the arithmetic means of the values above. They will then possess the values:

$$X = 2\pi\, i\, do\, [b\, (w_a + w_i - 2\gamma) - c\, (v_a + v_i - 2\beta)]\,, \quad Y = 2\pi\, i\, do\, [c\, (u_a + u_i - 1\alpha) - a\, (w_a + w_i - 2\gamma)]\,,$$

$$Z = 2\pi\, i\, do\, [a\, (v_a + v_i - 2\beta) - b\, (u_a + u_i - 2\alpha)]\,.$$

Their resultant is perpendicular to *do*. We would like to denote it by R. To save space, we set:

$$u_i - \alpha = \xi\,, \quad v_i - \beta = \eta\,, \quad w_i - \gamma = \zeta\,, \quad u_a - \alpha = \mathfrak{x}\,, \quad v_a - \beta = \mathfrak{y}\,, \quad w_a - \gamma = \mathfrak{z}\,,$$

$$\mathfrak{y}\,\zeta - \mathfrak{z}\,\eta = \rho\,, \qquad \mathfrak{z}\,\xi - \mathfrak{x}\,\zeta = \sigma\,, \qquad \mathfrak{x}\,\eta - \mathfrak{y}\,\xi = \tau\,,$$

and we will then get:

$$\frac{R}{2\pi\, do} = i\sqrt{(\mathfrak{x}+\xi)^2 + (\mathfrak{y}+\eta)^2 + (\mathfrak{z}+\zeta)^2 - [a\,(\mathfrak{x}+\xi) + b\,(\mathfrak{y}+\eta) + c\,(\mathfrak{z}+\zeta)]^2}\,.$$

We would now like to replace g, h, k with their values in the equation $i = \sqrt{g^2 + h^2 + k^2}$, which will then give:

$$i = \sqrt{(\mathfrak{x}-\xi)^2 + (\mathfrak{y}-\eta)^2 + (\mathfrak{z}-\zeta)^2 - [a\,(\mathfrak{x}-\xi) + b\,(\mathfrak{y}-\eta) + c\,(\mathfrak{z}-\zeta)]^2}\,.$$

However, since the derivative of φ in the direction of the current that flows through *do* cannot have any jumps, we will have:

(7) $$a\,(\mathfrak{x} - \xi) + b\,(\mathfrak{y} - \eta) + c\,(\mathfrak{x} - \zeta) = 0$$

so:

$$i = \sqrt{(\mathfrak{x}-\xi)^2 + (\mathfrak{y}-\eta)^2 + (\mathfrak{z}-\zeta)^2} \, .$$

Since the projection of the velocity in the direction of the normal to *do* inside, as well as outside of the fluid, must be equal to the projection of the velocity of *do* in that same direction, we have:

$$\lambda\,\mathfrak{x} + \mu\,\mathfrak{y} + \nu\,\mathfrak{z} = 0 \, , \quad \lambda\,\xi + \mu\,\eta + \nu\,\zeta = 0 \, .$$

It will then follow from those two equations and the equation $\lambda\,a + \mu\,b + \nu\,c = 0$ that: $a\,\rho + b\,\sigma + c\,\tau = 0$. If we combine that equation with equation (7) then we will get:

$$a = \frac{\sigma(\mathfrak{z}-\zeta) - \tau(\mathfrak{y}-\eta)}{n}, \qquad b = \frac{\tau(\mathfrak{x}-\xi) - \rho(\mathfrak{z}-\zeta)}{n}, \qquad c = \frac{\rho(\mathfrak{y}-\eta) - \sigma(\mathfrak{x}-\xi)}{n},$$

in which:

$$\begin{aligned} n^2 &= [\sigma(\mathfrak{z}-\zeta) - \tau(\mathfrak{y}-\eta)]^2 + [\tau(\mathfrak{x}-\xi) - \rho(\mathfrak{z}-\zeta)]^2 + [\rho(\mathfrak{y}-\eta) - \sigma(\mathfrak{x}-\xi)]^2 \\ &= (\rho^2+\sigma^2+\tau^2)[(\mathfrak{x}-\xi)^2 + (\mathfrak{y}-\eta)^2 + (\mathfrak{z}-\zeta)^2] \, . \end{aligned}$$

Substituting those values will then give:

$$a\,(\mathfrak{x}+\xi) + b\,(\mathfrak{y}+\eta) + c\,(\mathfrak{z}+\zeta) = \frac{2(\rho^2+\sigma^2+\tau^2)}{n} = \frac{2\sqrt{\rho^2+\sigma^2+\tau^2}}{i} \, .$$

When one also replaces *i* with its value, one will then have:

$$\frac{R}{2\pi\,do} = \sqrt{[(\mathfrak{x}+\xi)^2 + (\mathfrak{y}+\eta)^2 + (\mathfrak{z}+\zeta)^2]\,[(\mathfrak{x}-\xi)^2 + (\mathfrak{y}-\eta)^2 + (\mathfrak{z}-\zeta)^2] - 4(\rho^2+\sigma^2+\tau^2)} \, ,$$

and after some reductions:

$$\frac{R}{2\pi\,do} = \mathfrak{x}^2 + \mathfrak{y}^2 + \mathfrak{z}^2 - \xi^2 - \eta^2 - \zeta^2 \, ,$$

so

$$R = 2\pi\,do\left[\left(\frac{d\varphi_a}{dx}-\alpha\right)^2 + \left(\frac{d\varphi_a}{dy}-\beta\right)^2 + \left(\frac{d\varphi_a}{dz}-\gamma\right)^2 - \left(\frac{d\varphi_i}{dx}-\alpha\right)^2 - \left(\frac{d\varphi_i}{dy}-\beta\right)^2 - \left(\frac{d\varphi_i}{dz}-\gamma\right)^2\right],$$

and as a result:

(8) $$p_a\,do = p_i\,do - \frac{\rho R}{4\pi} - \rho\,E\,do \, .$$

If the velocity and acceleration of the ring are finite then the same will be true of p_i ; we likewise know that E is finite. If we would now like to use the expression for $p_a\ do$ to find the forces and force-couples that are affixed to the ring then we must multiply it by certain finite quantities and integrate over the infinitely-small outer surface of the ring. $p_i\ do$, as well as $\rho\ E\ do$, will only produce infinitely-small quantities then. When we neglect infinitely-small quantities, the forces and force-couples that are affixed to the ring will then be just as large as when only the force – $\frac{\rho R}{4\pi}$ acts upon each surface element. That will be – $\frac{\rho}{4\pi}$ times the force that all currents exert upon the current elements that cover the outer surface element do as a result of the potential Q. However, for infinitely-thin rings, not only can all currents be imagined to be concentrated along the centerline, but we can also regard the velocity and acceleration inside of a very short piece of the ring as constant. For the case of a moving infinitely-thin solid ring, the effect that is exerted upon each ring is equal, but oppositely-directed, to the electrodynamical effect, multiplied by $\rho / 4\pi$, that would be exerted upon the current that flows in the ring with an intensity of k by the current that flows in the remaining rings in the same way. But that must be combined with the force with the potential ρQ. If the ring is not rigid, but deformable with no change in length and cross-section, like flexible strings, then when one again seeks the force that is affixed to an element of length $d\sigma$ of the ring that is infinitely-small, but infinitely-long, compared to the dimensions of the cross-section, from the expression (8) for $p_a\ do$, the two terms $p_i\ do$ and – $\rho E\ do$ would once more yield only vanishing contributions. That force (except for the ones that keep any flexible ring in equilibrium) will then be equal to – $\rho / 4\pi$ times the effect of all currents on the current element $d\sigma$, plus the force with potential ρQ. From what we found before, the latter is perpendicular to $d\sigma$ and its direction of motion. Its intensity is equal to $\rho\ k\ d\sigma$ times the component of its velocity that is perpendicular to $d\sigma$. It points towards the side where the direction of motion of the element and that of the fluid are opposite. In conclusion, I shall add that it was assumed in all of this that the pressure that acts upon the fluid at infinity does not become negative on the outer surface of the ring; otherwise, a separation surface would arise. If the ring were infinitely-thin then the velocity of its outer surface would become infinite. The pressure that must exist in the fluid in order for no separation surface to arise would also be infinite then.

Graz, 5 November 1870.
