"Ueber die Transformation der Elasticitätsgleichungen in allgemeinne orthogonale Coordinaten," J. reine u. angew. Math. **76** (1873), 45-58.

On the transformation of the equations of elasticity into general orthogonal coordinates

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Translated by D. H. Delphenich

We can thank **Lamé** for an important result in the theory of the elasticity of solid isotropic (*) bodies that is concerned with the transformation of the differential equations for elastic displacement into a system of general orthogonal coordinates.

That result, which he published in **Liouville**'s Journal [(1841), pp. 52] and also in his celebrated book *Leçons sur les coordinées curvilignes* (pp. 290), and had derived while investing his customary mastery in the calculations, can be expressed in a form that leaves nothing to be desired in the name of simplicity.

Let x, y, z be the rectangular, rectilinear coordinates of a point in a solid elastic body in its original state of elastic equilibrium, and let $x + u$, $y + v$, $z + w$ be its coordinates after an elastic deformation has taken place. As is known, the determination of u, v, w will then depend upon three simultaneous second-order linear partial differential equations that are valid for then entire elastic body and three first-order boundary conditions that are true for the outer surface of the body. If one defines the nine derivatives of the displacements u, v, w with respect to x, y, z, and from them, the six quantities:

$$\frac{\partial u}{\partial x}, \quad \frac{\partial v}{\partial y}, \quad \frac{\partial w}{\partial z},$$

and

$$\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}, \qquad \frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}, \qquad \frac{\partial u}{\partial y}+\frac{\partial v}{\partial x},$$

which one, with **Saint-Venant** (**), can call the three *dilatations* and the three *shears*, then the boundary conditions for elastic bodies of any kind can be represented in terms of those six quantities themselves and the partial differential equations can be represented in terms of the differential quotients with respect to x, y, z.

However, in the case of isotropy, there is a simpler form for the partial differential equations. If one considers, in addition to the six dilatations and shears, the three doubled components of the elementary rotation:

(*) I. e., one whose elasticity is independent of direction.
(**) Liouville's Journal (1863), pp. 260, 262.

$$U = \frac{\partial v}{\partial z} - \frac{\partial w}{\partial y}, \qquad V = \frac{\partial w}{\partial x} - \frac{\partial u}{\partial z}, \qquad W = \frac{\partial u}{\partial y} - \frac{\partial v}{\partial x},$$

and defines the *volume dilatation* from the three dilatations:

$$p = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$$

then the partial differential equations of the elastic displacements of isotropic bodies will have the characteristic property that they can be composed of the differential quotients of the four combinations p, U, V, W.

Having assumed that, **Lamé**'s result expresses the idea that the characteristic property of the partial differential equations that was just explained also remains true for curvilinear orthogonal coordinates. If one forms the expressions for the volume dilatation and the three components of the elementary rotation, which are taken in the direction of increasing coordinates, then one can compose the partial differential equations of the elastic displacements from those four quantities and their differential quotients with respect to the three coordinates.

That makes it clear that a result that can be expressed so simply must be derived without recourse to calculation.

Jacobi has shown (*) that the transformation into general (and in particular, orthogonal) coordinates of the partial differential equations that emerge from the variation of a multiple integral will be simplified immensely when one performs that transformation on the integral, and not on the differential equations.

These ideas, which **Jacobi** set down for the problem of *one* dependent variable, have been extended by **Carl Neumann** to the problem of *three* dependent variables that occurs in the elasticity of isotropic bodies (**). However, in the present case, **Jacobi**'s method by itself will not suffice to yield a satisfactory derivation of the form that **Lamé** found for the transformed equations. Namely, the dilatations and shears, on the one hand, and the elementary rotations, on the other, define two groups of quantities for which there exists a special, and very simple, type of transformation for each of them. By contrast, if one mixes the two groups then one can no longer recognize a simple law in the transformation of such mixed quantities.

In the following pages, I will show that when one properly separates both groups and shifts one's attention from the partial differential quotients to the total differentials under the transformation of coordinates, one will arrive at **Lamé**'s result with almost no calculation.

1. The basic equation of elasticity and the form it takes in the case of isotropy. – **Green** reduced the equations of elasticity to a single equation that expressed the idea that the variation of a triple integral is equivalent to the moment of the given forces. If one

(*) Bd. **36**, pp. 113, of this Journal.
(**) Bd. **57**, pp. 281, of this Journal.

treats the displacements as infinitely small then that basic equation for the case of isotropic bodies will read:

$$(1) \qquad \delta P = K\,\delta\Omega,$$

in **Kirchhoff**'s notation (*). In this:

$$(1') \qquad \delta P = \delta^{(3)}P + \delta^{(2)}P = \int dT\,[X\,\delta u + Y\,\delta v + Z\,\delta w] + \int d\omega[(X)\,\delta u + (Y)\,\delta v + (Z)\,\delta w]$$

means the moment of the given forces, and in fact, it encompasses the integral $\delta^{(3)}P$, which is extended over the volume element dT, of all forces that act upon the interior points of the body, and the integral $\delta^{(2)}P$, which is extended over the outer surface element $d\omega$ of the forces that act upon the points of the boundary of the body. Ω means the integral:

$$(2) \qquad \begin{cases} \Omega = \int dT\,\{\mathfrak{E} + \theta p^2 - \mathfrak{g}\,sp\}, \\ \mathfrak{E} = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial z}\right)^2 + \frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)^2 + \frac{1}{2}\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)^2 + \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)^2, \\ p = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}, \end{cases}$$

that K and $K\theta$ are the two elasticity constants. The term $-\,\mathfrak{g}\,sp$ that occurs in the integral Ω, which is not found in **Kirchhoff**'s investigations, is added in order to include the case that **Duhamel** and **Franz Neumann** examined in which a non-uniformly-distributed heat:

$$s\,(x, y, z)$$

acts upon the individual parts of the elastic body, along with the elastic deformations. The constant $\mathfrak{g}$ that appears as a factor in that term depends upon the linear thermal expansion coefficient $\mathfrak{e}$ of the elastic body by way of the equation (**):

$$\mathfrak{g} = 2\,(1 + 3\theta)\,\mathfrak{e}\,.$$

If one reduces the variation $\delta\Omega$ to its simplest form:

$$\delta\Omega = \delta^{(3)}\Omega + \delta^{(2)}\Omega,$$

(*) Bd. **40**, pp. 55, of this Journal.

(**) Confer **Franz Neumann** for the law of double refraction of light in compressed or non-uniformly heated non-crystalline bodies. Abhandlungen der Berliner Akademie from the year (1841), pp. 100, in which he took $\theta = 1/2$.

according to the prescriptions of the calculus of variations, in which $\delta^{(3)}$ refers to the part of the variation that pertains to the volume integral and $\delta^{(2)}$ refers to the part that pertains to the outer surface integral, then the basic equation (1) will split into the two equations:

$$(1.a) \qquad \delta^{(3)}P = K\,\delta^{(3)}\,\Omega,$$

$$(1.b) \qquad \delta^{(2)}P = K\,\delta^{(2)}\,\Omega\,.$$

That latter, which subsumes the three conditions for the outer surface of the elastic body, cannot be simplified further, in general. By contrast, the former, which subsumes the three partial differential equations, admits an essential simplification, and indeed, by way of a conversion of $\delta^{(3)}\,\Omega$ that is equivalent to the characteristic property of the partial differential equations that relates to the aforementioned case of isotropy.

If one sets:

$$4\,\mathfrak{F} = U^2 + V^2 + W^2, \qquad \mathfrak{G} = \mathfrak{A} + \mathfrak{B} + \mathfrak{C},$$

in which U, V, W mean the doubled components of the elementary rotations

$$U = \frac{\partial v}{\partial z} - \frac{\partial w}{\partial y}, \qquad V = \frac{\partial w}{\partial x} - \frac{\partial u}{\partial z}, \qquad W = \frac{\partial u}{\partial y} - \frac{\partial v}{\partial x},$$

as above, and $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$ mean the second-order functional determinants:

$$\mathfrak{A} = \frac{\partial v}{\partial y}\frac{\partial w}{\partial z} - \frac{\partial v}{\partial z}\frac{\partial w}{\partial y}, \quad \mathfrak{B} = \frac{\partial w}{\partial z}\frac{\partial u}{\partial x} - \frac{\partial w}{\partial x}\frac{\partial u}{\partial z}, \quad \mathfrak{C} = \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x},$$

then $\mathfrak{E}$ can be represented in the form:

$$\mathfrak{E} = p^2 + 2\mathfrak{F} - 2\mathfrak{G}\,.$$

If one introduces the new integrals:

$$\Omega' = \int dT\{2\mathfrak{F} + (1+\theta)\,p^2 - \mathfrak{g}\,sp\}, \qquad \Gamma = \int dT\,\mathfrak{G}$$

then one will have:

$$\Omega = \Omega' - 2\,\Gamma.$$

However, the integral Γ is not a triple (or volume) integral, properly speaking, but an outer surface integral. Namely, just as a simple integral:

$$\int dx \frac{\partial f}{\partial x},$$

in which one finds a differential quotient, is not a proper integral, but depends upon only the values of the function f that it takes on the limits of the integration, so is an n-fold integral:

$$\int dx_1 \cdots dx_n \,\mathfrak{M} = \int dx_1 \cdots dx_n \frac{\partial(f_1, \ldots, f_m)}{\partial(x_1, \ldots, x_m)} \qquad (m \leq n),$$

in which one finds a functional determinant $\mathfrak{M}$ that is formed from m of the n integration variables $x_1, \ldots, x_n$, is not generally a proper n-fold integral, but at most an $(n - 1)$-fold one. Since $\mathfrak{M}$ can be brought into the form (*):

$$\mathfrak{M} = \frac{\partial(f_1\,\mathfrak{M}_1)}{\partial x_1} + \frac{\partial(f_1\,\mathfrak{M}_2)}{\partial x_2} + \cdots + \frac{\partial(f_1\,\mathfrak{M}_m)}{\partial x_m}, \qquad \mathfrak{M}_\mu = \frac{\partial\,\mathfrak{M}}{\partial \frac{\partial f_1}{\partial x_\mu}},$$

the integral in question will not depend upon the values of the functions $f_1, \ldots, f_m$ inside of a domain of integration that is defined in an n-fold continuum, but only upon the values that exist on the boundary of the continuum. For $n = 3$, $m = 2$, the integral (**):

$$\Gamma = \int dx\,dy\,dz\,(\mathfrak{A} + \mathfrak{B} + \mathfrak{C})$$

belongs to that category, so Γ, and therefore $\delta\Gamma$, as well, will merely be an outer surface integral (***). If one recalls the equation:

$$\Omega = \Omega' - 2\,\Gamma$$

and one also performs the splitting of the variation δ into the parts $\delta^{(3)}$ and $\delta^{(2)}$ for the remaining integrals then one will get $\delta\Gamma = 0$ and:

$$\delta^{(3)}\,\Omega = \delta^{(3)}\,\Omega'.$$

(*) **Jacobi**, "Theoria novi multiplicatoris," Bd. **27**, pp. 203, of this Journal.

(**) The value of Γ as an outer surface integral will be given by the formula:

$$2\Gamma = \int d\omega\{[u \cdot p - \varepsilon u]\cos(\nu, x) + [v \cdot p - \varepsilon v]\cos(\nu, y) + [w \cdot p - \varepsilon w]\cos(\nu, z)\}$$

when one forms the expression εf, with:

$$\varepsilon f = \frac{\partial f}{\partial x} u + \frac{\partial f}{\partial y} v + \frac{\partial f}{\partial z} w$$

for any function f of x, y, z and lets (ν, x), (ν, y), (ν, z) denote the angles that the outward-pointing normal to the outer surface element $d\omega$ makes with the positive halves of the coordinate axes, resp.

(***) The special case in which the domain of integration of Γ can be further reduced will not be considered here.

With that conversion, (1.*a*) will go to $\delta^{(3)}P = K\,\delta^{(3)}\,\Omega'$. *In the case of isotropy, there is then the following simplified form that will allow the partial differential equations of elasticity to be summarized in one equation:*

$$\delta^{(3)}P = K\,\delta^{(3)}\,\Omega', \tag{3}$$

$$\left\{\begin{aligned} &\Omega' = \int dT\,\{2\mathfrak{F} + (1+\theta)\,p^2 - \mathfrak{g}\,sp\},\\ &4\mathfrak{F} = U^2 + V^2 + W^2 = \left(\frac{\partial v}{\partial z} - \frac{\partial w}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial x} - \frac{\partial u}{\partial z}\right)^2 + \left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right)^2,\\ &p = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial z}{\partial z}. \end{aligned}\right. \tag{4}$$

2. Transformation of the linear dilatations. – In what follows, as I did before in § **1**, I will need the symbol ε for the operation of elastic variation, such that when f means any function of the coordinates, $f + \varepsilon f$ will mean the value of f after an elastic deformation has taken place.

It is known that the displacements that take place at an infinitely-small distance from the point (x, y, z) can be composed of two types of variations of the element dT, the first of which consists of a linear dilatation without rotations.

Let x', y', z' be the coordinates of a point that belongs to the element dT in its original position, let r be the length of the line that points from (x, y, z) to (x', y', z'), and let α, α_1, α_2 denote the cosine of the angle that this line makes with the directions of increasing x, y, z. r will be converted into $r + \varepsilon r$ under elastic deformation, so as is known, the linear dilatation $\varepsilon r / r$ will be given by the equation:

$$\frac{\varepsilon r}{r} = \sum a_{ik}\,\alpha_i\,\alpha_k \qquad (i, k = 0, 1, 2),$$

in which:

$$\alpha^2 + \alpha_1^2 + \alpha_2^2 = 1$$

and

$$a_{00} = \frac{\partial u}{\partial x}, \qquad a_{11} = \frac{\partial v}{\partial y}, \qquad a_{22} = \frac{\partial w}{\partial z},$$

$$a_{12} = \tfrac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right), \qquad a_{02} = \tfrac{1}{2}\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right), \qquad a_{01} = \tfrac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right).$$

The two expressions $\mathfrak{E}$, p that appear in the integral Ω can be represented in terms of the coefficients a_{ik} in the form (*):

$$\mathfrak{E} = \sum_{ik} a_{ik}^2 ,$$

$$p = \sum_{i} a_{ii} .$$

They are simultaneous invariants for the two quadratic forms:

$$\sum_{ik} a_{ik}\, \alpha_i\, \alpha_k , \qquad \sum_{i} \alpha_i^2$$

and will therefore remain unchanged for all orthogonal transformations; i.e., if one represents the linear dilatation in any rectilinear or curvilinear orthogonal coordinate system in the form:

$$\frac{\varepsilon\, r}{r} = \sum_{ik} b_{ik}\, \beta_i\, \beta_k ,$$

in which β, β_1, β_2 mean the direction cosines that correspond to that coordinate system, then $\mathfrak{E}$ and p will have the expressions in terms of the b_{ik} that they had in terms of the a_{ik} .

Let ρ, ρ_1, ρ_2 be three functions of x, y, z that define an orthogonal coordinate system, so $d\rho$, $d\rho_1$, $d\rho_2$ will be linear functions of dx, dy, dz that satisfy the equation:

$$dx^2 + dy^2 + dz^2 = \frac{d\rho^2}{h^2} + \frac{d\rho_1^2}{h_1^2} + \frac{d\rho_2^2}{h_2^2} .$$

In the rectilinear, rectangular coordinate system of the x, y, z, the u, v, w will likewise be the elastic variations of the coordinates and the displacements in the sense of the latter. In the system of the ρ, ρ_1, ρ_2, that will no longer be the case, since the elastic variations $\varepsilon\rho$, $\varepsilon\rho_1$, $\varepsilon\rho_2$ of the coordinates ρ, ρ_1, ρ_2 and the displacements R, R_1, R_2 in the sense of increasing ρ, ρ_1, ρ_2 are coupled to each other by the equations:

$$R_i = \frac{\varepsilon\, \rho_i}{h_i} .$$

I shall denote the coordinates of the two infinitely-close points (x, y, z) and (x', y', z') by ρ, ρ_1, ρ_2 and ρ', ρ_1', ρ_2', resp., in the new system. The distance between them will then be given by the equation:

(*) The index i or k or both of them will be assigned the values 0, 1, 2 in the simple or double sums that will occur in what follows.

$$r^2 = \left(\frac{\rho' - \rho}{h}\right)^2 + \left(\frac{\rho_1' - \rho_1}{h_1}\right)^2 + \left(\frac{\rho_2' - \rho_2}{h_2}\right)^2 .$$

After the elastic deformation has taken place, r^2 will go to:

$$(r + \varepsilon r)^2 = \left(\frac{\rho' - \rho + \varepsilon\rho' - \varepsilon\rho}{h + \varepsilon h}\right)^2 + \left(\frac{\rho_1' - \rho_1 + \varepsilon\rho_1' - \varepsilon\rho_1}{h_1 + \varepsilon h_1}\right)^2 + \left(\frac{\rho_2' - \rho_2 + \varepsilon\rho_2' - \varepsilon\rho_2}{h_2 + \varepsilon h_2}\right)^2 .$$

Of the three fractions whose squares define the right-hand side of that equation, since elastic variations can be treated as if they were infinitely small, one transforms the first one by means of the equations:

$$\frac{\rho' - \rho + \varepsilon\rho' - \varepsilon\rho}{h + \varepsilon h} = \frac{1}{h}\left\{\left(1 - \frac{\varepsilon h}{h}\right)(\rho' - \rho) + \varepsilon\rho' - \varepsilon\rho\right\},$$

$$\varepsilon\rho' - \varepsilon\rho = \frac{\partial(\varepsilon\rho)}{\partial\rho}(\rho' - \rho) + \frac{\partial(\varepsilon\rho)}{\partial\rho_1}(\rho_1' - \rho_1) + \frac{\partial(\varepsilon\rho)}{\partial\rho_2}(\rho_2' - \rho_2)$$

into

$$\frac{\rho' - \rho + \varepsilon\rho' - \varepsilon\rho}{h + \varepsilon h} = \left(1 + \frac{\partial(\varepsilon\rho)}{\partial\rho} - \frac{\varepsilon h}{h}\right)\frac{\rho' - \rho}{h} + \frac{\partial(\varepsilon\rho)}{\partial\rho_1}\frac{\rho_1' - \rho_1}{h} + \frac{\partial(\varepsilon\rho)}{\partial\rho_2}\frac{\rho_2' - \rho_2}{h}.$$

When one replaces the square of $r + \varepsilon r$ in the equation above with this expression and the two similarly-constructed ones, one will get the following result for the dilatation $\varepsilon r / r$:

$$\frac{\varepsilon r}{r} = \sum_{ik} b_{ik}\,\beta_i\,\beta_k ,$$

where

$$b_{ii} = \frac{\partial(\varepsilon\rho_i)}{\partial\rho_i} - \frac{\varepsilon h_i}{h_i}, \qquad b_{ik} = \tfrac{1}{2}\left(\frac{h_i}{h_k}\frac{\partial(\varepsilon\rho_k)}{\partial\rho_i} + \frac{h_k}{h_i}\frac{\partial(\varepsilon\rho_i)}{\partial\rho_k}\right),$$

$$\beta_i = \frac{\rho_i' - \rho_i}{r\,h_i}, \qquad \beta^2 + \beta_1^2 + \beta_2^2 = 1.$$

Since $\mathfrak{E}$ and p have the same expressions in terms of the b_{ik} that they have in terms of the a_{ik} , one will have:

$$\mathfrak{E} = \sum_{ik} b_{ik}^2 , \qquad p = \sum_i b_{ii} .$$

The quantities that enter into the integral Ω are transformed into the coordinate system of the ρ , ρ_1 , ρ_2 in that way, and when one sets:

$$\mathfrak{r}_i = \varepsilon\,\rho_i\,,$$

to abbreviate, the new expression for that integral will be:

$$(5)\qquad \left\{\begin{aligned} &\Omega = \int dT\,\{\mathfrak{E} + \theta p^2 - \mathfrak{g}\,sp\}, \quad dT = \frac{d\rho\,d\rho_1\,d\rho_2}{\varpi}, \quad \varpi = h\,h_1\,h_2, \\ &\mathfrak{E} = \sum_{ik} b_{ik}^2, \quad b_{ii} = \frac{\partial \mathfrak{r}_i}{\partial \rho_i} - \frac{1}{h_i}\sum_k \frac{\partial h_i}{\partial \rho_k}\,\mathfrak{r}_k\,, \quad b_{ik} = \left(\frac{h_i}{h_k}\frac{\partial \mathfrak{r}_k}{\partial \rho_i} + \frac{h_k}{h_i}\frac{\partial \mathfrak{r}_i}{\partial \rho_k}\right) \\ &p = \sum_i b_{ii} = \varpi \sum_i \frac{\partial}{\partial \rho_i}\left(\frac{\mathfrak{r}_i}{\varpi}\right), \quad \mathfrak{r}_i = h_i R_i\,. \end{aligned}\right.$$

3. Transforming the elementary rotation. – In this paragraph, the second type of change in the volume element *dT* that results from an elastic deformation shall be considered, which consists of a rotation of the element without any linear dilatation. Twice the values of the three components of that rotation around the *x*, *y*, *z* axes are:

$$U = \frac{\partial v}{\partial z} - \frac{\partial w}{\partial y}, \qquad V = \frac{\partial w}{\partial x} - \frac{\partial u}{\partial z}, \qquad W = \frac{\partial u}{\partial y} - \frac{\partial v}{\partial x},$$

so the square of the entire rotation will be identical with the quantity $\mathfrak{F}$ that enters into the integral Ω', which is defined by the equation:

$$4\,\mathfrak{F} = U^2 + V^2 + W^2.$$

The physical meaning of this can already be recognized from the outset in the fact that $\mathfrak{F}$ is an invariant under the transformation into general orthogonal coordinates, but the invariant character of $\mathfrak{F}$ is different from that of the quantities $\mathfrak{E}$, *p* that were transformed above. Further developments will show that the two types of invariability have an adjoint relationship to each other.

If one considers, along with the differentials of the coordinates, any other type of infinitely-small changes that the two systems of quantities *x*, *y*, *z* and ρ, ρ_1, ρ_2 suffer simultaneously and denote those variations by δx, δy, δz and $\delta\rho$, $\delta\rho_1$, $\delta\rho_2$, resp., then those variations will have the same linear relationship to each other that the differentials have, so they will also fulfill the same second-degree condition, and one will have, at the same time:

$$dx^2 + dy^2 + dz^2 = \frac{d\rho^2}{h^2} + \frac{d\rho_1^2}{h_1^2} + \frac{d\rho_2^2}{h_2^2}\,,$$

$$\delta x^2 + \delta y^2 + \delta z^2 = \frac{\delta\rho^2}{h^2} + \frac{\delta\rho_1^2}{h_1^2} + \frac{\delta\rho_2^2}{h_2^2}\,.$$

Furthermore, as one easily convinces oneself, a third relation will follow from the agreement of the linear relations:

$$dx\ \delta x + dy\ \delta y + dz\ \delta z = \frac{d\rho\,\delta\rho}{h^2} + \frac{d\rho_1\,\delta\rho_1}{h_1^2} + \frac{d\rho_2\,\delta\rho_2}{h_2^2}.$$

If one multiplies the first two of those three equations and applies the known formula for representing the product of two sums of three squares as the sum of four squares and subtracts the square of the third equation from the result then that will yield:

$$\mathfrak{X}^2 + \mathfrak{Y}^2 + \mathfrak{Z}^2 = \mathfrak{R}^2 + \mathfrak{R}_1^2 + \mathfrak{R}_2^2,$$

where:

$$\mathfrak{X} = dy\ \delta z - dz\ \delta y, \qquad \mathfrak{Y} = dz\ \delta x - dx\ \delta z, \qquad \mathfrak{Z} = dx\ \delta y - dy\ \delta x,$$

$$\mathfrak{R} = \frac{d\rho_1\,\delta\rho_2 - d\rho_2\,\delta\rho_1}{h_1\,h_2}, \qquad \mathfrak{R}_1 = \frac{d\rho_2\,\delta\rho - d\rho\,\delta\rho_2}{h\,h_2}, \qquad \mathfrak{R}_2 = \frac{d\rho\,\delta\rho_1 - d\rho_1\,\delta\rho}{h\,h_1}.$$

When one lets the variations δ in:

$$dx\ \delta x + dy\ \delta y + dz\ \delta z = \frac{d\rho\,\delta\rho}{h^2} + \frac{d\rho_1\,\delta\rho_1}{h_1^2} + \frac{d\rho_2\,\delta\rho_2}{h_2^2}$$

go to elastic variations ε and sets:

$$\sigma_i = \frac{\varepsilon\,\rho_i}{h_i^2} = \frac{R_i}{h_i},$$

then one will get:

$$u\ dx + v\ dy + w\ dz = \sigma\,d\rho + \sigma_1\,d\rho_1 + \sigma_2\,d\rho_2\,,$$

or, when one sets:

$$f(dx) = u\ dx + v\ dy + w\ dz, \qquad g(d\rho) = \sigma\,d\rho + \sigma_1\,d\rho_1 + \sigma_2\,d\rho_2\,,$$

to abbreviate:

$$f(dx) = g(d\rho).$$

That equation will be satisfied identically by the linear relations between *dx*, *dy*, *dz* and $d\rho$, $d\rho_1$, $d\rho_2$, so they will also be true when they are posed for the differential variations, so one will have:

$$f(\delta x) = g(\delta\rho).$$

When one varies the first of these equations, differentiates the second one, and takes the difference between the two results, that will yield (*):

(*) Confer the treatise of **Lipschitz**, Bd. **70**, pp. 77, of this Journal.

$$\delta f(dx) - df(\delta x) = \delta g(d\rho) - dg(\delta\rho),$$

or, in developed form:

$$\mathfrak{X}\,U + \mathfrak{Y}\,V + \mathfrak{Z}\,W = \mathfrak{R}\,\mathfrak{S} + \mathfrak{R}_1\,\mathfrak{S}_1 + \mathfrak{R}_2\,\mathfrak{S}_2\,,$$

where

$$\mathfrak{S} = h_1\,h_2\left(\frac{\partial\sigma_1}{\partial\rho_2} - \frac{\partial\sigma_2}{\partial\rho_1}\right), \qquad \mathfrak{S}_1 = h\,h_2\left(\frac{\partial\sigma_2}{\partial\rho} - \frac{\partial\sigma}{\partial\rho_2}\right), \qquad \mathfrak{S}_2 = h\,h_1\left(\frac{\partial\sigma}{\partial\rho_1} - \frac{\partial\sigma_1}{\partial\rho}\right).$$

However, one has the following known algebraic theorem:

If $\mathfrak{X}$, $\mathfrak{Y}$, $\mathfrak{Z}$ and $\mathfrak{R}$, $\mathfrak{R}_1$, $\mathfrak{R}_2$ are two systems of variables that depend upon each other linearly and simultaneously satisfy the condition:

$$\mathfrak{X}^2 + \mathfrak{Y}^2 + \mathfrak{Z}^2 = \mathfrak{R}^2 + \mathfrak{R}_1^2 + \mathfrak{R}_2^2\,,$$

and if those two systems are coupled with two other systems U, V, W and $\mathfrak{S}$, $\mathfrak{S}_1$, $\mathfrak{S}_2$ by the identity:

$$\mathfrak{X}U + \mathfrak{Y}V + \mathfrak{Z}W = \mathfrak{R}\mathfrak{S} + \mathfrak{R}_1\,\mathfrak{S}_1 + \mathfrak{R}_2\,\mathfrak{S}_2$$

then the new system will be likewise linearly-dependent and likewise satisfy the condition:

$$U^2 + V^2 + W^2 = \mathfrak{S}^2 + \mathfrak{S}_1^2 + \mathfrak{S}_2^2\,.$$

With that, the transformation of:

$$2\,\mathfrak{F} = \tfrac{1}{2}(U^2 + V^2 + W^2) = \tfrac{1}{2}(\mathfrak{S}^2 + \mathfrak{S}_1^2 + \mathfrak{S}_2^2)$$

into the new coordinates is completed, so the quantities $\tfrac{1}{2}\mathfrak{S}$, $\tfrac{1}{2}\mathfrak{S}_1$, $\tfrac{1}{2}\mathfrak{S}_2$, are, as one easily convinces oneself, the components of the elementary rotation around the directions of increasing ρ, ρ_1, ρ_2. Since p was already transformed in § **2**, all of the quantities that enter into the integral Ω' will be expressed in terms of the new coordinates, and one will get the value:

$$(6)\qquad \left\{\begin{aligned} &\Omega' = \int dT\,\{2\mathfrak{F} + (1+\theta)\,p^2 - \mathfrak{g}\,sp\}, \quad dT = \frac{d\rho\,d\rho_1\,d\rho_2}{\varpi}, \quad \varpi = h\,h_1\,h_2, \\ &4\mathfrak{F} = \sum_i \mathfrak{S}_i^2, \quad \mathfrak{S}_i = h_k\,h_l\left(\frac{\partial\sigma_k}{\partial\rho_l} - \frac{\partial\sigma_l}{\partial\rho_k}\right), \\ &p = \varpi\sum_i \frac{\partial}{\partial\rho_i}\left(\frac{h_i^2\sigma_i}{\varpi}\right), \quad \sigma_i = \frac{R_i}{h_i}, \end{aligned}\right.$$

for Ω', in which $i\ k\ l$ means a positive permutation of the indices 0 1 2.

The transformation of the displacements that is included in the equation that was used above:

$$u\,dx + v\,dy + w\,dz = \sum_i \sigma_i\,dp_i = \sum_i \frac{R_i}{h_i}\,d\rho_i\,,$$

can be posed analogously for the moments of the forces that are given on the boundary, namely:

$$X\,du + Y\,dv + Z\,dw = \sum_i \frac{P_i}{h_i}\,\delta\mathfrak{r}_i = \sum_i P_i\,h_i\,\delta\sigma_i\,,$$

and one transforms the moments of the given forces accordingly by way of the equations:

$$(7)\qquad \begin{cases} \delta^{(2)}P = \int d\omega[(X)\,\delta u + (Y)\,\delta v + (Z)\,\delta w] = \int d\omega \sum_i \frac{(P_i)}{h_i}\,\delta\mathfrak{r}_i\,, \\ \delta^{(3)}P = \int dT\,[X\,\delta u + Y\,\delta v + Z\,\delta w] = \int dT \sum_i P_i\,h_i\,\delta\sigma_i\,, \end{cases}$$

in which P_i and (P_i) mean the components of the given internal and external forces, resp., in the directions of increasing ρ_i .

4. The partial differential equations and boundary conditions in general orthogonal coordinates. – In order to obtain the partial differential equations and boundary conditions in final form, one now comes to the development of the variations $\delta^{(2)}\Omega$ and $\delta^{(3)}\Omega'$. If an integral:

$$\Omega = \int dT\ f\left(\cdots, \rho_k, \ldots, \mathfrak{r}_i, \ldots, \frac{\partial\mathfrak{r}_i}{\partial\rho_k}, \ldots\right), \qquad dT = \frac{d\rho\,d\rho_1\,d\rho_2}{\varpi} \qquad (i, k = 0, 1, 2)$$

is given then the final form of its variation will be known:

$$\delta\Omega = \sum_i \int dT\,\delta\mathfrak{r}_i\left[f^i - \sum_k \varpi\frac{\partial}{\partial\rho_k}\left(\frac{1}{\varpi}\,f_k^i\right)\right] + \sum_i \int d\omega\,\delta\mathfrak{r}_i \sum_k \frac{1}{h_k}\,f_k^i\cos(\nu, \rho_k)\,,$$

where

$$f^i = \frac{\partial f}{\partial\mathfrak{r}_i}, \qquad f_k^i = \frac{\partial f}{\partial\dfrac{\partial\mathfrak{r}_i}{\partial\rho_k}}\,,$$

and (ν, ρ_k) means the angle that the outward-pointing normal to the outer surface element $d\omega$ defines with the direction of increasing ρ_k . One will then have:

$$\delta^{(2)}\Omega = \sum_i \int d\omega\, \delta\mathfrak{r}_i \sum_k \frac{1}{h_k}\, f_k^i \cos(\nu, \rho_k),$$

where:

$$f = \mathfrak{E} + \theta p^2 - \mathfrak{g}\, s\, p.$$

One likewise has:

$$-\delta^{(3)}\Omega' = \sum_i \int dT\, \delta\mathfrak{r}_i \left[f^i - \sum_k \varpi \frac{\partial}{\partial \rho_k}\left(\frac{1}{\varpi} f_k^i\right)\right],$$

$$F = 2\,\mathfrak{F} + (1 + \theta)\, p^2 - \mathfrak{g}\, s\, p,$$

$$F^i = \frac{\partial F}{\partial \sigma_i}, \qquad F_k^i = \frac{\partial F}{\partial \dfrac{\partial \sigma_i}{\partial \rho_k}}.$$

If one replaces $\mathfrak{E}$ and p in f with their expressions:

$$\mathfrak{E} = \sum_{ik} b_{ik}^2, \qquad b_{ii} = \frac{\partial \mathfrak{r}_i}{\partial \rho_i} - \frac{1}{h_i}\sum_k \frac{\partial h_i}{\partial \rho_k}\mathfrak{r}_k, \qquad b_{ik} = \tfrac{1}{2}\left[\frac{h_i}{h_k}\frac{\partial \mathfrak{r}_k}{\partial \rho_i} + \frac{h_k}{h_i}\frac{\partial \mathfrak{r}_i}{\partial \rho_k}\right],$$

$$p = \sum_i b_{ii} = \varpi \sum_i \frac{\partial_i}{\partial \rho_i}\left(\frac{\mathfrak{r}_k}{\varpi}\right),$$

then that will yield the following values for the derivatives f_i^i, f_k^i of f that enter into $\delta^{(2)}\Omega$:

$$f_i^i = 2\,[b_{ii} + \theta p - \tfrac{1}{2}\mathfrak{g}\, s], \qquad f_k^i = 2\, b_{ik}\frac{h_k}{h_i},$$

and therefore:

$$\delta^{(2)}\Omega = 2\sum_i \int d\omega \frac{1}{h_k}\sum_k c_{ik}\cos(\nu, \rho_k),$$

where:

$$c_{ii} = b_{ii} + \theta p - \tfrac{1}{2}\mathfrak{g}\, s, \qquad c_{ik} = b_{ik}.$$

If one substitutes that value of $\delta^{(2)}\Omega$ and the value (7) of $\delta^{(2)}P$ in the basic equation:

$$\delta^{(2)}P = K\,\delta^{(2)}\Omega$$

for the boundary conditions then one will get the three conditions equations for the outer surface, when expressed in the new coordinates, in which all three of them can be represented in the form:

(8) $$[b_{ii} + \theta p - \tfrac{1}{2}\mathfrak{g}\, s] \cos(\nu, \rho_i) + b_{ik} \cos(\nu, \rho_k) + b_{ii} \cos(\nu, \rho_l) = \frac{1}{2K}\,(P_i),$$

in which $i\ k\ l$ means a permutation of the indices 0 1 2, and:

(8′) $$b_{ii} = \frac{\partial \mathfrak{r}_i}{\partial \rho_i} - \frac{1}{h_i}\sum_k \frac{\partial h_i}{\partial \rho_k}\mathfrak{r}_k\,, \qquad b_{ik} = \tfrac{1}{2}\left[\frac{h_i}{h_k}\frac{\partial \mathfrak{r}_k}{\partial \rho_i} + \frac{h_k}{h_i}\frac{\partial \mathfrak{r}_i}{\partial \rho_k}\right],$$

which is a result that will coincide with **Lamé**’s formulas on pp. 281, 282 of his *Leçons sur les coordinées curvilignes* when one sets $s = 0$.

If one sets $\mathfrak{F}$ and p in F equal to their expressions:

$$4\mathfrak{F} = \sum_i \mathfrak{S}_i^2\,, \qquad \mathfrak{S}_i = h_k\, h_l\left(\frac{\partial \sigma_k}{\partial \rho_l} - \frac{\partial \sigma_l}{\partial \rho_k}\right), \qquad p = \varpi \sum_i \left(\frac{h_i^2}{\varpi}\frac{\partial \sigma_i}{\partial \rho_i} + \sigma_i \frac{\partial}{\partial \rho_i}\frac{h_i^2}{\varpi}\right),$$

in which $i\ k\ l$ means a positive permutation of 0 1 2, and introduces the quantity:

$$q = 2\,(1 + \theta)\, p - \mathfrak{g}\, s,$$

then one will get the following values for the derivatives F^i, F_i^i, F_k^i of F that appear in $\delta^{(3)}\Omega'$:

$$F^i = \varpi \frac{\partial}{\partial \rho_i}\frac{h_i^2}{\varpi}\cdot q\,, \qquad F_i^i = h_i^2 q\,, \qquad F_k^i = (i\ k\ l)\, h_i\, h_k\, \mathfrak{S}_l\,,$$

in which $(i\ k\ l)$ means positive or negative unity according to whether ikl is a positive or negative permutation of 0 1 2. One will then get the final expression for $\delta^{(3)}\Omega'$:

$$-\,\delta^{(3)}\Omega' = \sum_i \int dT\, \delta\sigma_i \left\{ h_i^2 \frac{\partial q}{\partial \rho_i} + (ikl)\varpi\left[\frac{\partial}{\partial \rho_k}\frac{\mathfrak{S}_l}{h_l} - \frac{\partial}{\partial \rho_l}\frac{\mathfrak{S}_k}{h_k}\right]\right\}.$$

If one substitutes that value of $\delta^{(3)}\Omega'$ and the value (7) of $\delta^{(3)}P$ in the basic equation:

$$\delta^{(3)}P = K\, \delta^{(3)}\Omega'$$

for the partial differential equations and establishes that $i\ k\ l$ should be a positive permutation of 0 1 2 then one will get three partial differential equations that are expressed in the new coordinates, and all three of them will be represented in the form:

(9) $$\frac{\partial}{\partial \rho_l}\frac{\mathfrak{S}_k}{h_k} - \frac{\partial}{\partial \rho_k}\frac{\mathfrak{S}_l}{h_l} = \frac{h_i}{h_k\, h_l}\frac{\partial q}{\partial \rho_i} + \frac{1}{K}\cdot\frac{P_i}{h_k\, h_l}\,,$$

in which *i k l* means a positive permutation of 0 1 2, and:

$$(9') \qquad \mathfrak{S}_i = h_k \, h_l \left(\frac{\partial \sigma_k}{\partial \rho_l} - \frac{\partial \sigma_l}{\partial \rho_k} \right), \quad q = 2\,(1+\theta)\,p - \mathfrak{g}\,s, \quad p = \varpi \sum_i \frac{\partial}{\partial \rho_i} \left(\frac{h_i^2\,\sigma_i}{\varpi} \right), \quad \sigma_i = \frac{R_i}{h_i}.$$

When one sets $s = 0$, these equations will agree with **Lamé**’s systems (25), (26), (27), pp. 290, 291 in his *Leçons sur les coordinées curvilignes*.

Berlin, January 1873.
