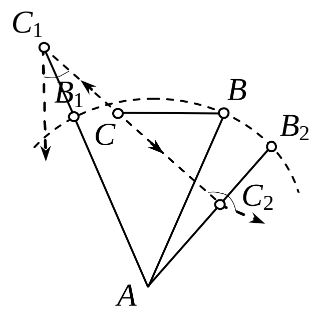
"Über ein Beispiel des Herrn Boltzmann zu der Mechanik von Hertz," Jahresb. Deutsch. Math.-Verein (1900), 200-204.

On Boltzmann's example in regard to Hertz's mechanics

By **A. Brill** in Tübingen

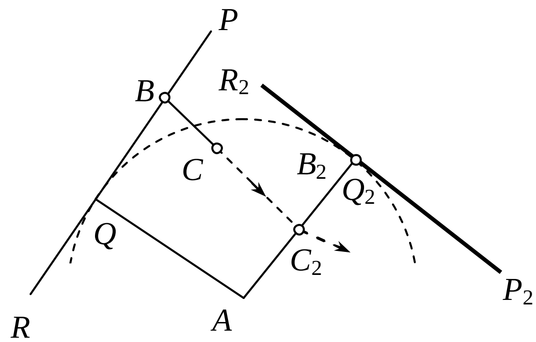
Translated by D. H. Delphenich

In the Jahresbericht der Deutschen Mathematiker-Vereinigung in 1898, **Boltzmann** cited the lack of suitable examples that might make **Hertz**'s book on mechanics more understandable, and then graciously proceeded to give an example of his own. He replaced the motion of a completely-elastic ball in the interior of a hollow sphere in the Hertzian sense that knew of neither forces at a distance nor elastic forces in the usual sense, but only rigid constraints on the masses, with the motion of a system of two massless rods *AB*, *BC* that are coupled to each other by a link *B* and at whose free ends *A*, *C*, one finds mass-points, while the location of the link *B* is a point that has vanishingly-small mass.



In fact, one imagines that the motion of the rod has been simplified in such a way that the one end point *A* is fixed, and writes out the principle of *vis viva* and the law of areas, referred to a planar coordinate that has the point *A* for its origin and its plane is determined by *A* and the initial velocity of *C*. One can then infer the conclusion from that combination of equations that the rectilinear path CC_1 that the end point *C* of the pair of rods sweeps out for a vanishingly-small mass *B* will be converted into another likewise-rectilinear path at the location where the rods define an elongated angle $AB_1C_1 = \pi$ that makes the same angle with the line AB_1C_1 as the one that *C* returns to. At the moment when the path of *C* changes its direction, the infinitely-small mass *B* will gain a *vis viva* whose carrier is otherwise *C* and which helps it to get over the "dead point" in the extended position at that moment by its infinitely-large velocity.

However, that is just the picture of the motion of an elastic ball of radius ρ that moves without the action of forces inside of a hollow sphere of inner radius $AB + BC + \rho$. **Boltzmann** let the rods *AB*, *BC* have equal length so that the center of the ball could go through that of the hollow sphere. However, if one makes $AB \neq BC$ and one chooses the initial direction CC_2 of *C* such that the angle *ABC* between the two rods is equal to zero at that time, then the mass-point C will behave at the location C_2 exactly as it does at the location where $\sphericalangle ABC = \pi$. Namely, it bounces off at the same angle with respect to AC_2 that it approached it with, and the system will behave like a solid elastic ball of radius ρ that appears inside of that hollow sphere on a solid ball of radius $AB - BC - \rho$ ($BC - AB - \rho$, resp.) that is concentric to it.



That modification thus implies the picture of the elastic collision of *two solid balls* that **Boltzmann** required (*loc. cit.*). If one would like to avoid the complications of the hollow sphere then one could let the two rods increase without bound while keeping the same difference in length. One might also require the end point B of the link BC, which carries an infinitely-small mass, instead of remaining on a ball of radius AB, to remain on a rectilinear "guide" (say, a tube with a slit in it) PQR that is kept at a constant distance AQ from the fixed point A by an arm $AQ \perp PQR$ such that it always remains tangent to the ball of radius AQ whose center is A. If the end point C of the rod BC carries a finite mass and $AQ > BC$ then one will again have the picture of the elastic collision of two solid balls. The idea behind the latter arrangement goes back to **Finsterwalder**.

All of those pictures can be reduced to the representation of the collision of smooth balls. They will break down when friction associated with the rotating motion of the balls changes the angle of reflection and the plane of reflection.

Although **Hertz** always spoke only of "rigid constraints," the examples of hidden masses and motions that he might have had in mind when he imagined a replacement for the forces-at-a-distance that occur in nature are however hardly rods or otherwise-discrete mass-systems, but rather matter that fills up space uniformly and can displace in its own right. That is because it was in relation to just that notion that the introduction (*Mechanik*, pp. 31) referred to **Helmholtz**'s theory of hidden motions, which one will find to be developed in his treatises on monocyclic systems and the principle of least action [J. f. Math. **97** (100)], in which it is exemplified by fluids and gaseous bodies. He further cited the representations to which **Maxwell** arrived [in his articles "Über physikalische Kraftlinien" and "A dynamical theory of the electromagnetic field," Trans. Roy. Phil. Soc. London (1864), etc], in which one must think of a "hidden" fluid medium that is switched on between two masses that seem to act upon each other at a distance and which is "coupled" to the visible masses, in **Hertz**'s terminology, and whose vorticial or otherwise cyclic "adiabatic" motions are the carrier of a certain total kinetic energy that ordinary mechanics cares to describe as the potential energy of the visible masses. **Hertz** also referred to **Lord Kelvin**'s vortex theory of atoms, etc.

One does not generally find representations of that sort developed in the text itself. Indeed, **Hertz**'s statements on the form of the permissible equations of motion that define the "rigid" constraints that enter in place of forces for him seem to contradict that assumption, since only either finite equations between the coordinates or homogeneous linear equations between the *differentials* of the coordinates of the system (129), along with a possible position of the system, are permissible, and according to pp. 43 in the introduction, that is the form in which all of the connections in nature must be clothed. If one now assumes that there is a space-filling intermediate medium that is assumed to be incompressible, in any event, then the condition of incompressibility will not, as **Hertz** concluded, already be expressed as a finite equation between the coordinates, but rather the partial differential quotients of the coordinates x, y, z of a system point with respect to its initial values a, b, c will enter into, e.g., **Lagrange**'s incompressibility condition. However, if one, as **Lagrange** himself did, regards that condition as the expression of the purely-geometric fact that the volume of the tetrahedron that is defined by the points (a, b, c), ($a + da$, b, c), (a, $b + db$, c), (a, b, $c + dc$) does not change in time then, as one can easily show (Mitteil. d. math. naturw. Vereins in Württemberg 1900), that equation will also take the form of an equation between the coordinates of four (infinitely-close) system points. It will express only the "rigid constraint between the *smallest parts*" (Intro., pp. 49).

Moreover, one can also make the transition from discrete to continuous mass-points that fill up a line in the system of rods that was considered above without getting close to **Hertz**'s conception of things when one goes from two rods to n of them, and ultimately to an inextensible chain for which one will again have a condition equation in the partial differential quotients (cf., e.g., **Routh-Schepp**, *Dynamik*, II, § **602**).

Hertz might have also imagined that in the vicinity of two colliding elastic bodies a medium that is endowed with properties of the indicated kind would be capable of absorbing the *vis viva* that would be free at the moment of contact for a brief time in order to once more transmit it to the visible masses (no. **733**). He did not explain that any further, but rather referred expressly to "the individual consideration of that special relationship (for collisions) in the realm of general mechanics" (*ibid.*). Nonetheless, one must agree with **Boltzmann** when he placed precisely that special relationship at the center of the discussion of **Hertz**'s mechanics by means of the example that he treated.

Addendum: Later on, a speech was brought to my attention that **Boltzmann** had presented at one of the general sessions of the Munich Congress of Scientists (see these Jahresbericht, pp. 71, *et seq.*) in "Über die Entwicklung der Methoden der theoretischen Physik zu unser Zeit." In it, **Boltzmann** took an entirely different position in regard to the interpretation of hidden masses in **Hertz**'s mechanics when he explained that:

> "The structure of the formerly-useful medium [that is filled with fluid] and also **Maxwell**'s light ether do not need to be endowed with them [the hidden masses], since indeed forces are thought to act in all of those media of the kind that **Hertz** excluded expressly."

Only **Helmholtz** and **Maxwell** have derived the forces that appear in a vorticial incompressible fluid upon which no external forces act from the equations of hydrodynamics and the incompressibility condition. I believe that it needs to be proved that the latter also have the form that **Hertz** allowed. Now, **Lagrange** derived the system of hydrodynamical equations with the help of only **d'Alembert**'s principle on the basis of that one equation. However, **Hertz** also decreed the latter, since the formulation of it in no. **393** likewise implied the equation that **Lagrange** employed when one treated the coordinate increments δp_ρ , not as possible or virtual displacements, but as mutually-independent displacements, from the procedure in *Mécanique analytique*, t. II, sect. IV, no. 11, when one adds the condition equations that exist between them, suitably provided with undetermined multipliers, to the left-hand side. However, if one has put **d'Alembert**'s principle into that form then that would complete the transition from a finite number of mass-points to an infinite number, in the sense of the remark in no. **6** of **Hertz**'s *Mechanik*, and from there to the fluid media, with the help of that condition equation, precisely as **Lagrange** did (*Mécanique analytique*, t. II, sect. IV, no. 17; sect. XI, no. **2**, *et seq.*). It is therefore not clear why Hertz's hidden masses could not also be fluid masses.

Tübingen, 7 November 1899.