LECTURES ON THE INTRODUCTION OF SPACE-FILLING MASSES INTO MECHANICS

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FOREWORD

This book emerged from a lecture on "non-rigid systems and the mechanics of H. HERTZ," that was given by the author at the University of Tübingen for a number of years in connection with lectures on the mechanics of material points and rigid systems.

KIRCHHOFF's *Mechanik* and W. THOMSON and TAIT's *Treatise on natural philosophy* had, in contrast to their predecessors, once more taken the position of LAGRANGE's *Mécanique analytique* and included space-filling masses into the scope of mechanics. Other authors followed their example: W. VOIGT, P. APPELL, H. WEBER, *et al.*, have since then developed the increasing volume of research in some splendid works and made it accessible to a broader sphere in the world of mathematics and physics.

However, what seems to be lacking is a brief outline by which the budding mathematician could be introduced to the first principles of that branch of knowledge without going too deeply into the applications, along with acquainting them with the motivating ideas and intuitions that have recently gained their citizenship in it and also extending their effect to pure mechanics, and indeed their foundations have begun to be discussed.

The present small book would like to fill that gap, in that it attempts to treat the mechanics of fluid and elastic masses on the same foundation as the mechanics of material points, and as an aside, undertakes a glimpse into the theory of electrons.

LAGRANGE created the enduring paradigm for the foundation of a uniform treatment of the mechanics of continuous masses. The theory of elasticity can also be constructed upon it when one replaces the preliminary investigations into internal forces of pressure with the assumption of a potential that will then enter into consideration as all that remains of the forces in the applicable principle of mechanics. With that alteration, one will, at the same time, justify the meaning that the concept of energy has taken on since ROBERT MAYER. However, its value will be raised even further by the new conception of the essence of force that physics has developed since then.

The contribution of HEINRICH HERTZ was to basically demand that one had to eliminate the concept of force-at-a-distance from mechanics and take a new approach (*Prinzipien der Mechanik*, Leipzig 1894). The word "force" is absent from the single axiom upon which his theory rests, viz., the "basic law." Pressure and force-at-a-distance were introduced only indirectly and on a later occasion, and the latter, only by its potential, since HERTZ had proved [probably independently of J. J. THOMSON, who had previously ([1]) made a similar remark] that formally the effect of a force could be explained by the assumption of hidden masses whose kinetic energy was the same as the potential energy of the visible masses.

Of course, with that, the concept of force is not actually eliminated from the foundations as long as one does not prove the existence of the hidden masses, or the intermediary that creates the effect of force, or even make it plausible. However, today we are still just as far from a solution to that problem, namely, what it means to the all-important force of gravity, as we would have been

([1]) J. J. THOMSON, *Anwendungen der Dynamik auf Physik und Chemie*, 1888. (German transl., Leipzig, 1890.) Chapter 2.

in the time of FARADAY. For that reason, the mechanics of HERTZ is no longer meaningful as a program, strictly speaking. However, even when one rejects the HERTZian viewpoint, on those and other grounds, one cannot deny that there are certain advantages to its starting point, and many of those advantages are shared by GAUSS's principle of least constraint, moreover.

When the case occurs in which the equations of constraint for a problem are "non-holonomic," i.e., in the form of not-necessarily integrable differential equations [which is an important case that LAGRANGE (*Mécanique analytique*, t. I, Part. 1, Page IV, § **1**) has already considered], the exhibition of the equations between the variations of the coordinates that follows from them, the application of other principles – in particular, HAMILTON's ([1]) – will encounter difficulties that are based upon the fact that those equations cannot be derived from those principles by mere mathematical operations. However, that is possible, with no further assumptions, with GAUSS's principle, because the varied quantities in it are the second derivatives with respect to time. HERTZ seems to have overlooked that fact, perhaps because he (*Mechanik*, Art. 501, 394) derived D'ALEMBERT's principle from LAGRANGE's differential equations, instead of varying the expression for the constraint (*loc. cit.*, Art. 497). However, GIBBS ["Fundamental formulae, etc.," Am. J. Math. **2** (1879)] had already pointed out that fact before HERTZ, along with other advantages of the principles of least constraint. Thus, he also did not meet up with the oft-raised objection that exhibiting and transforming the expression for the constraint that is composed of second differential quotients with respect to time is less computationally convenient than, say, the one for the *vis viva*, to which, e.g., HAMILTON's principle is linked. For that reason ([2]), it would seem that up to now one must also refrain from employing the principle of least constraint at the fundamental level. However, it is worth pointing out that the *variation* of the constraint by no means requires that one must exhibit the expression for the constraint itself, as GIBBS assumed. In what follows, one will see that in no case, not even in the applications to hydrodynamical problems or the theory of elasticity, will the derivation of the differential equations from the principle require more calculation than what it required by HAMILTON's principle.

However, GAUSS's principle has the further advantage over the latter in that since it reduces the natural motion to an ordinary minimum, it makes indispensable the exotic apparatus of the calculus of variations, which rests upon very subtle foundations and is applicable to the non-holonomic equations of constraint in the aforementioned case only when one observes some special precautions, as HÖLDER had shown. However, one is, above all, allowed to ask: What do the equations of motion that are true for a certain time element have to do with a time integral whose limits need to satisfy no other conditions than that they include that time element? However, the formulation of a problem should remain free of exotic gimmicks. That objection also underlies the application of the principle to space-filling masses, where HAMILTON's principle still seems to outshine the others, due to the equal treatment that it lends to time and space coordinates. That is because, here as well, as a result of the partial integration of the time integral, boundary

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([1]) Even when one applies D'ALEMBERT's principle, one cannot always employ the equations between the unvaried quantities for the formation of the equations between the variations. Hence, in the case of incompressible fluid masses, D'ALEMBERT's principle must leave the incompressibility condition (which was mostly derived beforehand) unused. (See KIRCHHOFF, *Mechanik*, 1876, pp. 120; RIEMANN-WEBER, *Partielle Differentialgleichungen der math. Physik*, 1901, vol. II, pp. 428.)

([2]) See, e.g., the report by A. VOSS on the principles of rational mechanics in the *Enzyklopädie der mathematischen Wissenschaften*, Bd., IV, **1**, 1902, pp. 82, 91.

expressions will come into play that are meaningless for the problem, but whose vanishing is by no means always easy to prove ([1]). Finally, one should not overlook the fact that in the case of one-sided constraints, which is are expressed by inequalities, the principle of least constraint is itself superior to that of D'ALEMBERT, as was emphasized by GAUSS (*Werke*, Bd. V, pp. 27), and later once more by GIBBS (*loc. cit.*).

I therefore believed that one should follow HERTZ in one's choice of a fundamental law. It was only later, when I was dealing with the expression for the force function, that I converted the equation of GAUSS's principle into that of D'ALEMBERT's principle by a formal process (which is a path that was already pursued by GIBBS, as I later saw) in order to not stray too far from the usual form.

HERTZ spoke of only discrete mass-points in his mechanics. However, there is no reason to exclude space-filling masses. Conceptual pictures such as methods of proof are also adaptable to it by sensible reinterpretations. Only the definition of the equations of constraint for a "free system" needed to be recast.

Finally, as far as the introduction of the concept of "force" is concerned, just as was true for the forces-at-a-distance for systems of points, it must take the form of the potential (which is indeed what HERTZ ultimately decided). Of course, it would then seem that eliminating force from the fundamental law has been reduced to merely a question of where it appears in the consequences of that law. That is because when one introduces the potential of elastic forces, one will only return once more to the path that MacCULLAGH had already gone down in his treatise "On a dynamical theory of crystalline reflection and refraction" for the elastic force. However, in what follows, it will take the form of a consequence to a certain basic intuition into the essence of force. The isolated potential is only one term in a series, in which one finds arrayed point-like as well as continuous, and fluid as well as solid-elastic, bodies, and even the medium of light itself.

In so doing, just as the basic law will find application to those masses that are distinguished by their specific properties, mechanics itself will merge into a unified whole that might probably satisfy E. MACH's demand of economy of thought.

A further advantage of the path that was mentioned consists of the fact that the various causes of motion and resistances to motion that one combines within the term "force" will be kept separate by the way that they were introduced. Finally, for didactic reasons, one should introduce the cause of motion later on, as W. SCHELL had already remarked when he was drafting up his own textbook.

From what was said, one will get the organization of these lectures by itself. Rather than the concept of "force," the concept of "mass" will be placed at the summit (along with time and space), without giving a basis for its introduction. The first chapter includes a sketch of HERTZ's mechanics of discrete mass-points, from which the equations that originate from the "basic law" will already follow in essentially the same form in which they will be employed in later chapters. In the second and third chapters, which treat the mechanics of space-filling masses, one will restrict oneself almost entirely to the elements and pass over the applications, but without basically

([1]) I recall the case of the boundary expression that appears in the motion of a body in a fluid (KIRCHHOFF, *Mechanik*, Lect. 19, Formula (1), pp. 234, 1876; cf., also RIEMANN-WEBER, *Partielle Differentialgleichungen der math. Physik*, 1901, vol. II, § 162, pp. 426, *et seq*.)

excluding the examples, moreover, which can help to explain the concepts and laws. In that way, the vector calculus proves to be an almost-indispensable instrument for a concise presentation of the many relationships between force fields (which will happen in the fourth chapter). For that reason, in the second and third chapters, not only will use be made of its notations, but to an increasing degree, its operations, as well. In regard to the notations, I shall refer to the *Einführung in die Vektoranalysis* by R. GANS (Leipzig, 1905), which essentially coincides with the ones in the *Enzyklopädie der mathematischen Wissenschaften*, Band V, in its own right.

It might seem surprising that the electromagnetic theory of light will also be taken up in a lecture (in the fourth chapter) that might be preferably directed to a mathematical audience. However, except for the satisfaction that a glimpse into the theory of H. A. LORENTZ (with the many tests that it has already passed) that the non-specialist might also derive, the mathematician must, to a large extent, also address the series of presentations of it that are coupled with the "principle of relativity." That principle might or might not be valid as a consequence of natural law: It represents an extension of the sphere of ideas in theoretical mechanics that has not been valued highly enough. The far-sighted MINKOWSKI was also convinced of that fact, as was manifest in his last scientific utterances.

Since it would not be possible otherwise in a sketch, the appropriate citations are only the proof that readers who might wish to go deeper into the matter will need to do more work of their own. I have not emphasized by own citations over those of others, as some of the experts might have noticed.

In conclusion, it is a pleasant task for me to take this opportunity to express my thanks to my two collaborators in the proof-reading, Herrn Professor Dr. J. SOMMER in Danzig-Langfuhr and Herrn Professor Dr. R. GANS in Tübingen, for their numerous worthwhile remarks and suggestions of a factual and didactic nature, and especially Herrn Professor GANS, for reviewing the last chapter before going to press.

Tübingen in July 1909.

A. Brill.

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Chapter Four

The electromagnetic theory of light.

NOTATIONS

$\equiv$	means "abbreviation for" or "identically equal to"
$\neq$	not equal
d, ∂, δ	total, partial differential, variation, resp.
$[l^\alpha\, m^\beta\, t^\gamma]$	dimensions
d', δ'	not [necessarily] complete differential (variation)
S	sum of three terms that arise from the one written out by permuting the symbols
x_i, y_i, z_i	(i = 1, 2, …, n) rectangular coordinates of a point with mass m_i
p_i, $\mathfrak{p}_i$	(i = 1, 2, …, r, or $\mathfrak{r}$, resp.) r ($\mathfrak{r}$) general (Lagrangian) coordinates.
q_i, $\mathfrak{q}_i$	the corresponding momenta (impulse coordinates)
t, t'	time [positional (proper) time in the moving coordinate system, resp.]
x, y, z	the rectangular coordinates of the spatial element of a continuous (i.e., space-filling) mass or the coordinates of a reference point
a, b, c	the same things at the initial time t = 0 or at another location
T, $\mathfrak{T}$, $\mathfrak{T}_1$	the kinetic energy of a system
v, $\mathfrak{v}$	velocity
f, $\mathfrak{f}$	acceleration. In Chapter IV: force on a unit amount of electricity.
c	(in Chapter I) curvature of the path
T	simply or multiply-connected region of space
Σ	its outer surface
$d\tau$	element of T
$d\sigma$	element of Σ
ds, $d\mathfrak{s}$	line element
n	the normal to a point of Σ that points into T
$\mathfrak{R}$	radius of an infinitely-extended sphere

Fraktur symbols mean vectors (except for $\mathfrak{R}$, $\mathfrak{T}$, cf. *supra*). Their components have indices x, y, z.

$\mathfrak{u} = (u, v, w)$	displacement (with components u, v, w) in an elastic or quasi-elastic medium
$\mathfrak{v} = (u, v, w)$	velocity
$= (\mathfrak{v}_x, \mathfrak{v}_y, \mathfrak{v}_z)$	Chapter II: in fluid medium
div $\mathfrak{v}$	divergence of the vector v
rot $\mathfrak{v}$	rotation of $\mathfrak{v}$, vector with components $\dfrac{\partial w}{\partial y} - \dfrac{\partial v}{\partial z}$, etc.
grad $\mathfrak{v}$	gradient of v, vector with components $\dfrac{\partial u}{\partial x}$, etc.

$\mathfrak{w} = (\xi, \eta, \zeta)$	$= \frac{1}{2}\mathrm{rot}\, \mathfrak{v}$ in a fluid medium
	$= \frac{1}{2}\mathrm{rot}\, \mathfrak{u}$, rotation of the axes of the volume element in an elastic or quasi-elastic medium, where $\mathfrak{u}$ is the displacement
$(\mathfrak{a}, \mathfrak{b})$	scalar (inner) product
$[\mathfrak{a}, \mathfrak{b}]$	vector (outer) product, which is a vector that is perpendicular to $\mathfrak{a}$ and $\mathfrak{b}$ that defines a right-handed system with them and has components $\mathfrak{a}_y\, \mathfrak{b}_z - \mathfrak{b}_y\, \mathfrak{a}_z$, etc.

Dots over variables, such as in $\dot{x}$, $\ddot{x}$, $\dot{\mathfrak{u}}$, $\dot{\xi}$, … mean first (second) differential quotients with respect to time.

$\dot{x}$	$= \dfrac{d}{dt} x(a, b, c, t)$ in the case of fluid masses
$\dot{u}$	$= \dfrac{\partial}{\partial t} u(x, y, z, t)$ in the case of solid-elastic and quasi-elastic masses
X, Y, Z	(except in Chapter IV) components of an external (applied) volume force
$\bar{X}, \bar{Y}, \bar{Z}$	components of the surface traction
$\mathfrak{E} = (X, Y, Z)$	In Chapter IV: the electric field strengths
$\mathfrak{H} = (L, M, N)$	In Chapter IV: the magnetic field strengths
c	(In Chapter IV) speed of light in empty space
ρ	In Chapters II and III: density of a ponderable mass. In Chapter IV: electric mass
$\mathit{\Delta}$, Δ	Lamé differential parameters
$\square$	$= \Delta - \dfrac{\partial^2}{\partial t^2}$

INTRODUCTION

Mechanics is concerned with the change in position of masses and poses the problem of presenting a picture of the processes of motion in Nature (initially, the inanimate ones) that is true enough that it can also encompass the future processes.

What we observe is the relative position of bodies and their motion with respect to each other. Meanwhile, even when only the mutual motions are represented, one cannot do without the assumption of a coordinate system that is, in a certain sense, absolute, namely, one in which the individual bodies already satisfy the law of inertia ([1]). Indeed, the position of such an [arbitrary within certain limits ([2])] "inertial system" ([3]) with respect the celestial masses that surround us is not known. However, experience teaches us that the empirical coordinate systems that astronomers introduce, which are, in a nutshell, rigidly coupled with the fixed stars, can be referred to as inertial systems ([4]). For motions that result within limits that are spatially and temporally sufficiently narrow, any coordinate system that is rigidly fixed in the Earth, which is rotating and orbiting around the Sun, already takes the place of such a thing, and even for the motions of the planets, which are calculated for vast limits in time and space, one refers to a system that has the ecliptic for one coordinate plane, which can itself translate with respect to that inertial system, and an axis that is the slowly-rotating connecting line from the center of the Sun to the Spring equinox.

Just as we can do without an inertial system, we can also do without the assumption of an absolute unit of length and such a unit of time, despite the misgivings we might have about the constancy of either and especially the unit of time that is given to us by the rotation of the Earth.

Above all, no matter what objections that one might raise against the fictions of mechanics that were discussed, and others that have not yet been mentioned, experience teaches us that the picture of processes of motion that is obtained from them and suitably-chosen basis laws corresponds to actual phenomena (and often within further limits of precision in space and time).

That concept of mass also requires one to make an abstraction. When one has to describe the spatial path of a body, it is convenient to restrict oneself, as a first approximation, to determining the path of a point in the body, namely, its center of mass, at which one imagines that the mass of the body has been concentrated. The center of mass, as a *material* (i.e., endowed with mass) *point*, replaces the entire body in that way of looking at things. The motion of a system of bodies will initially be defined by its center of mass, and the picture will then be completed in such a way that one describes the motion of the body and its change of form relative to the path of the center of mass, which is perhaps yet to be justified. For that reason, one prefers to begin with the mechanics of material points, as we shall also do.

([1]) In the broader sense of the planetary systems of NEWTONian mechanics (ANDING, "Über Koordinaten und Zeit," Enzykl. der math. Wiss., VI, 2, 1905), which is a demand whose fulfillment can be tested by astronomical means.

([2]) A motion that results from the principle of inertia in one coordinate system will no longer satisfy that law in another one that rotates or advances non-uniformly with respect to it when one does not add auxiliary forces.

([3]) That terminology goes back to L. LANGE (*Der Bewegungsbegriff*, Leipzig, 1886; cf., also MACH, *Die Mechanik in ihrer Entwicklung*, 4th ed., Leipzig, 1901, pp. 252).

([4]) Cf., H. SEELIGER, "Über die sogenannte absolute Bewegung," Münch. Akad. Ber. **36** (1906), pp. 137.

Similar remarks are also true in regard to all of the concepts and fundamental laws that will be introduced in what follows. When one replaces the complicated, never completely-knowable, connections in nature with simplified ones in order to idealize them in such a way that they will be accessible to computational analysis, one must avoid, once and for all, maintaining more than one reasonably-approximate picture of natural processes. The goal of science must be simply to gradually push back the limits of the approximation by suitably-chosen basic concepts and basic laws and always adding new fictions.

In what follows, we shall assume that *a single concept is known*, which arises from the *elementary* level of mechanics, and which we shall care to discuss in a preliminary lecture that we shall assume has already been resolved. Except for some purely-geometric concepts that are connected with the curvature behavior of the trajectory, they are the following ones:

Center of mass, moment of inertia, ellipsoid of inertia and principal axes of inertia. Some simple laws from the kinematics of rigid systems. The concept of a vector, and finally, the concepts of velocity, *vis viva*, acceleration, force, force function, and work.

The usual units of measurement for the three basic concepts that appear in mechanics, viz., space, mass, and time, are centimeter, gram, and second (C.G.S). Each of the aforementioned quantities, and the ones that are yet-to-be-introduced, and every equation between them will be expressible by a certain dimension in regard to those units that we shall give in the form of a power of the symbols l, m, t (i.e., length, mass, time) enclosed in brackets, as usual. The moment of inertia of a body has the dimension $[l^2 m]$, mass per unit volume, or density, has the dimension $[l^{-3} m]$, for velocity, it is $[l\,t^{-1}]$, acceleration is $[l\,t^{-2}]$, force is $[l\,m\,t^{-2}]$, and work is $[l^2\,m\,t^{-2}]$. A force per unit area (e.g., surface traction or tension) is $[l^{-1}\,m\,t^{-2}]$, etc.

Two points are called *rigidly* connected with each other when their distance is unvarying ([1]). Sometimes, we shall call an equation of constraint that exists between the coordinates of mass-points a *geometric* one when no other of the aforementioned quantities enter into it, and in particular, time. For instance, the rigidity of a *point-system* will be represented by geometric equations of constraint, and likewise the demand that a point should remain on a surface or curve, etc.

One says that a system of n points has *r degrees of freedom* when $3n - r$ equations of constraint exist between its $3n$ coordinates, such that only r of them can be chosen arbitrarily. *A rigid point-system* has six degrees of freedom, because when any three points of the system define a non-equilateral triangle (so their nine coordinates are coupled by three equations), the position of any fourth one will be determined uniquely. One can also say: If one couples that system rigidly to a rectangular coordinate system then all points of the rigid system will be established by the position of that system with respect to another coordinate system that is fixed in space, i.e., its number of degrees of freedom is equal to that of the coordinate system. However, the position of a coordinate system with respect to another one depends upon six constants: the three coordinates of the origin and three (perhaps the EULER) angles that determine the directions of its axes. That number will reduce to three when, e.g., one of the points of the system is fixed, etc.

([1]) We shall omit a discussion of the closely-related topics in geometry. Cf., HÖLDER, *Anschauung und Denken in der Geometrie*, Leipzig, 1900, pp. 5.

In addition to the systems of discrete material points, in what follows, we shall address only those masses that *fill up space continuously*. How one makes the transition from discrete material points to space-filling masses shall remain unmentioned here. Formally, it takes the form of replacing the mass of the material point with a spatial element multiplied by the factor *density* (viz., mass per unit volume), and at the same time, converting the summation over the totality of all mass-points (as it might appear, e.g., in the expressions for the center of mass coordinates or the moment of inertial of a point-system) with an integral that extends over all elements of the space that is filled with mass. That process, in turn, is justified only by its results.

The same things will be true for a rigid continuous mass that were stated above for a rigid point-system, namely, it has six degrees of freedom and its position then depends upon six constants. By contrast, the positions and density distributions of flexible strings and surface, elastic bodies and fluids will be determined by three *functions*, as we will see later on (Art. **17**). Non-rigid continuous masses then possess an *infinitude* of degrees of freedom, in general.

CHAPTER ONE

MATERIAL POINTS AND RIGID MASSES

1. – Velocity and acceleration of material points. Curvature of the path.

If x, y, z are the coordinates of a point relative to a known fixed coordinate system then its motion will be described by three functions of time t that represent those coordinates for each time-point:

$$x = x(t),$$
$$y = y(t),$$
$$z = z(t),$$

where x, y, z might be proper function symbols. The components of its *velocity* at time t are then:

$$\frac{dx}{dt} = \dot{x}, \qquad \frac{dy}{dt} = \dot{y}, \qquad \frac{dz}{dt} = \dot{z},$$

in which, here and in what follows, the first, second, … differential quotients with respect to time will be represented by dots over the function symbol (viz., the dependent variable). The velocity is a vector that we would like to denote by German letters (as with all vectors from now on). Let it be $\mathfrak{v}$. We then represent its length (i.e., magnitude) by:

$$|\mathfrak{v}| = v = \dot{s} = \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}, \tag{1}$$

when ds is the arc-length that is traversed during the time interval dt with a velocity of $\mathfrak{v}$. If v has a value that is unchanging in time, so the *tangential acceleration* vanishes:

$$|\dot{\mathfrak{v}}| = \dot{v} = \ddot{s} = 0,$$

then the motion of the point will be called *uniform.*

The angles α, β, γ that the element ds (i.e., the tangent to the trajectory) define with the coordinate axes are obtained from:

$$\cos\alpha = \frac{dx}{ds} = \frac{\dot{x}}{\dot{s}}, \qquad \cos\beta = \frac{dy}{ds} = \frac{\dot{y}}{\dot{s}}, \qquad \cos\gamma = \frac{dz}{ds} = \frac{\dot{z}}{\dot{s}}.$$

The *acceleration* of the point along its path is a vector $\mathfrak{f}$ with the components $\ddot{x}$, $\ddot{y}$, $\ddot{z}$ whose length is:

$$f = |\,\mathfrak{f}\,| = \sqrt{\ddot{x}^2 + \ddot{y}^2 + \ddot{z}^2}\ . \tag{2}$$

The curvature of the path can also be represented in terms of differential quotients of the coordinates with respect to time. One will get the simplest expression for the radius of curvature ρ when one assumes that the coordinates of the trajectory are represented as functions of the arc-length s, as measured from a fixed location on the path. One will then have:

$$\rho^2 = \frac{1}{\left(\frac{d^2x}{ds^2}\right)^2 + \left(\frac{d^2y}{ds^2}\right)^2 + \left(\frac{d^2z}{ds^2}\right)^2}\,. \tag{3}$$

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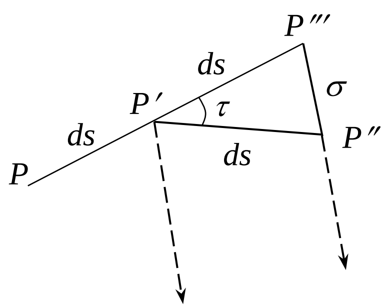

Figure 1.

If the lengths $PP' = P'P'' = ds$ in the accompanying figure are consecutive elements of the space curve, and one lengthens $PP' = P'P''' = ds$ along itself then $\angle P'''P'P'' = \tau$ will be the *contingency angle*. Therefore, when ρ is the principal radius of curvature, for very small values of τ, one will have:

$$\rho\,\tau = ds\,.$$

On the other hand, if $P'''P'' = \sigma$ can be replaced with a circular arc then:

$$ds \cdot \tau = \sigma\,.$$

Now, the coordinates of the points P are the following ones:

P … x, y, z

P' … $x + dx,\ y + dy,\ z + dz = x + \frac{dx}{ds}ds$, etc.

P'' … $x + dx + d\,(x + dx) = x + 2\,dx + d^2x = x + 2\frac{dx}{ds}ds + \frac{d^2x}{ds^2}ds^2$, etc.

P''' … $x + 2\,dx = x + 2\frac{dx}{ds}ds$, etc.

Hence, the following equation will exist for the length σ:

$$\sigma^2 = [(x + 2dx + d^2x) - (x + 2dx)]^2 + [d^2y]^2 + [d^2z]^2\,,$$

or

$$\sigma^2 = \left[\left(\frac{d^2x}{ds^2}\right)^2 + \left(\frac{d^2y}{ds^2}\right)^2 + \left(\frac{d^2z}{ds^2}\right)^2\right]ds^4\,,$$

which then gives:

$$\rho^2 = \frac{ds^2}{\tau^2} = \frac{ds^4}{\sigma^2} = \frac{1}{\left(\frac{d^2x}{ds^2}\right)^2 + \left(\frac{d^2y}{ds^2}\right)^2 + \left(\frac{d^2z}{ds^2}\right)^2} .$$

At the same time, one will get the direction (κ, λ, μ) of the radius of curvature, which agrees with that of $P'''P''$, from:

$$\cos\kappa : \cos\lambda : \cos\mu = \frac{d^2x}{ds^2} : \frac{d^2y}{ds^2} : \frac{d^2z}{ds^2} .$$

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If one now introduces the independent variable t instead of s by means of the relations:

$$\frac{d^2x}{ds^2} = \frac{d}{ds}\left(\frac{\dot{x}}{\dot{s}}\right) = \frac{1}{\dot{s}}\frac{d}{dt}\left(\frac{\dot{x}}{\dot{s}}\right) = \frac{\dot{s}\,\ddot{x} - \dot{x}\,\ddot{s}}{\dot{s}^2}, \tag{4}$$

and one lets S denote (as it always will in what follows) a sum over the three functions x, y, z, only one term of which will be written out, then one will have:

$$\frac{1}{\rho^2} = \mathsf{S}\left(\frac{d^2x}{ds^2}\right)^2 = \mathsf{S}\left(\frac{\dot{s}\,\ddot{x} - \dot{x}\,\ddot{s}}{\dot{s}^2}\right)^2 = \frac{1}{\dot{s}^4}\mathsf{S}\,\ddot{x}^2 + \frac{\ddot{s}^2}{\dot{s}^6}\mathsf{S}\,\dot{x}^2 - \frac{2\dot{s}\,\ddot{s}}{\dot{s}^6}\mathsf{S}\,\dot{x}\,\ddot{x}, \tag{4a}$$

or, since:

$$|\mathfrak{v}|^2 = \mathsf{S}\,\dot{x}^2 = \dot{s}^2, \qquad \mathsf{S}\,\dot{x}\,\ddot{x} = \dot{s}\,\ddot{s} ,$$

one will have:

$$\frac{1}{\rho^2} = \frac{1}{\dot{s}^4}\left(\mathsf{S}\,\ddot{x}^2 - \ddot{s}^2\right) . \tag{5}$$

That equation includes the known fact that the total acceleration $\mathfrak{f}$ can be decomposed into the tangential acceleration $\ddot{s}$ and the normal acceleration v^2 / ρ, which is perpendicular to it.

The curvature c of the path at a location, namely, the inverse value of the radius of curvature:

$$c = \frac{1}{\rho},$$

which is a scalar quantity, is the positive root of the equation:

$$c^2 = \frac{1}{\dot{s}^4}(\ddot{x}^2 + \ddot{y}^2 + \ddot{z}^2 - \ddot{s}^2) = \frac{1}{v^4}(f^2 - \ddot{s}^2) , \tag{6}$$

in which f, v are the (absolute values of the) acceleration and velocity, resp., as above. In the case of uniform motion, the curvature will be proportional to the acceleration, since for $\ddot{s} = 0$:

$$c^2 = \frac{f^2}{v^4} = \frac{1}{v^4}(\ddot{x}^2 + \ddot{y}^2 + \ddot{z}^2) .$$

With HERTZ, we shall now say *the straightest path* of a point [among all possible ones that are allowed by any possible geometric constraint equations that might exist (cf., Intro.)] to mean the one for which the curvature c has the smallest value at every location, so $\ddot{x}$, $\ddot{y}$, $\ddot{z}$ are determined such that:

$$\delta c = 0$$

everywhere for them. The variation symbol δ in that refers to only the acceleration components $\ddot{x}$, …

One might represent a variation like $\delta\ddot{x}$ analytically as a small increase $\varepsilon\,\ddot{\xi}(t)$ in the function $\ddot{x}(t)$, where $\ddot{\xi}(t)$ is an arbitrary (even discontinuous) function and ε is a very small constant. We shall return to that notion in art. **18**, at the end.

We then come to the problem, e.g., of finding the straightest path of a point that moves uniformly on a surface:

$$\varphi(x, y, z) = 0 , \tag{7}$$

so the problem of finding a suitable choice of the functions x (t), y (t), z (t) that fulfill equation (7) of the surface and minimize the quantity c, while at the same time, one has:

$$\dot{v} = \ddot{s} = 0 .$$

Instead of c, one can also minimize $\frac{1}{2}c^2v^4$ *or* $\frac{1}{2}f^2$, which will make that condition unnecessary. Due to the fundamental meaning of the concept of the straightest path, we would like to work through this elementary example. The following relations exist between the quantities $\dot{x}$, $\dot{y}$, $\dot{z}$ and $\ddot{x}$, $\ddot{y}$, $\ddot{z}$, which can be obtained from equation (7) by repeated differentiation:

$$\varphi'(x)\,\dot{x} + \varphi'(y)\,\dot{y} + \varphi'(z)\,\dot{z} = 0 , \tag{8}$$

$$\Phi \equiv \varphi''(x,x)\,\dot{x}^2 + 2\varphi(x,y)\,\dot{x}\,\dot{y} + \cdots + \varphi''(z,z)\,\dot{z}^2 + \varphi'(x)\,\ddot{x} + \varphi'(y)\,\ddot{y} + \varphi'(z)\,\ddot{z} = 0 , \tag{9}$$

in which $\varphi'(x)$, …, $\varphi''(x,x)$, … are the first, second, … partial derivatives of φ with respect to x, …, and Φ will denote an abbreviation for the left-hand side of the last equation. For any given location x, y, z on the path for which, at the same time, $\dot{x}$, $\dot{y}$, $\dot{z}$ are given values that satisfy equation (8), one will have to choose the quantities $\ddot{x}$, $\ddot{y}$, $\ddot{z}$ that correspond to the equation $\Phi = 0$ in such a way that $\frac{1}{2}f^2$ will be a minimum. If one varies $\frac{1}{2}f^2$ and $\Phi = 0$, while x, y, z ; $\dot{x}$, $\dot{y}$, $\dot{z}$ remain constant then one will get:

$$\tfrac{1}{2}\delta(f^2) \equiv \ddot{x}\,\delta\ddot{x} + \ddot{y}\,\delta\ddot{y} + \ddot{z}\,\delta\ddot{z} = 0\,,$$
$$\delta\Phi \equiv \varphi'(x)\,\delta\ddot{x} + \varphi'(y)\,\delta\ddot{y} + \varphi'(z)\,\delta\ddot{z} = 0\,.$$

From the theory of the relative minimum, one now conveniently introduces a yet-to-be-determined function λ of x, y, z, t and arranges that the variations $\delta\ddot{x}$, $\delta\ddot{y}$, $\delta\ddot{z}$ in:

$$\tfrac{1}{2}\delta f^2 + \lambda\,\delta\Phi \equiv (\ddot{x} + \lambda\,\varphi'(x))\,\delta\ddot{x} + (\ddot{y} + \lambda\,\varphi'(y))\,\delta\ddot{y} + (\ddot{z} + \lambda\,\varphi'(z))\,\delta\ddot{z} = 0$$

are mutually independent. That will then give the well-known equations:

$$\ddot{x} + \lambda\,\varphi'(x) = 0\,, \qquad \ddot{y} + \lambda\,\varphi'(y) = 0\,, \qquad \ddot{z} + \lambda\,\varphi'(z) = 0\,, \tag{10}$$

by which (since dt is proportional to ds) one defines a "geodetic" line on the surface, namely, a line for which the direction angle between the surface normal [whose cosines behave like $\varphi'(x)$: $\varphi'(y)$: $\varphi'(z)$] and the principal normal to the trajectory vanishes.

2. – Curvature of the trajectory in the case of discontinuous changes of direction.

For the sake of a later application, we would like to adapt the concept of the curvature of a space curve to the case of a rectilinear polygon with sides of equal length. We imagine that the sides are a sequence of vectors. As in the footnote in art. **1**, the curvature of a continuously-traversed curve will then be defined by:

$$c^2 = \frac{1}{\rho^2} = \frac{\sigma^2}{ds^4}\,,$$

so here as well the curvature c of the polygon at a corner point will be represented by the (vectorial) difference between the sides of the polygon that meet there $\Delta\mathfrak{s}$, $\Delta\mathfrak{s}_1$ (a difference whose absolute value will be taken to be σ). If one lets Δs, Δs_1 denote the sides themselves, and their projections by Δx, …, Δx_1, … then we will have:

$$\sigma^2 = (\Delta x_1 - \Delta x)^2 + (\Delta y_1 - \Delta y)^2 + (\Delta z_1 - \Delta z)^2,$$

and we again set:

$$c^2 = \frac{\sigma^2}{\Delta s^4}\,.$$

If one now assumes that a moving point traverses two sides of the polygon in such a way that its velocity is uniform along each of them, but changes discontinuously under the transition from Δs to Δs_1 in such a way that it covers the two sides in the same time interval Δt then if $\mathfrak{v}$, $\mathfrak{v}_1$ are those velocities, with the components (u, v, w), (u_1, v_1, w_1), and their absolute values are $|\mathfrak{v}|$, $|\mathfrak{v}_1|$, then:

$$c^2 = \frac{\sigma^2}{\Delta s^4} = \frac{\Delta t^2}{\Delta s^4}[(u_1 - u)^2 + (v_1 - v)^2 + (w_1 - w)^2] ,$$

or when one abbreviates the velocity increments (which are all that enters into that) by:

$$u_1 - u = \Delta u , \quad v_1 - v = \Delta v , \quad w_1 - w = \Delta w ,$$

that will be:

$$c^2 = \frac{1}{|\mathfrak{v}|^2 \, \Delta s^2}(\Delta u^2 + \Delta v^2 + \Delta w^2) .$$

The condition for the straightest path is again (art. **1**) fulfilled in such a way that one sets the variation δc, which is obtained by varying the quantities Δu, Δv, Δw (but not $\mathfrak{v}$ or $\Delta\mathfrak{s}$), equal to zero.

Thus, in the case of a discontinuous change in the path direction, the *straightest* path of a point will demand that:

$$\delta\left(|\mathfrak{v}|^2 \, \Delta s^2 \cdot \frac{c^2}{2}\right) = \delta \tfrac{1}{2}(\Delta u^2 + \Delta v^2 + \Delta w^2)$$
$$= \Delta u \; \delta\Delta u + \Delta v \; \delta\Delta v + \Delta w \; \delta\Delta w = 0 ,$$

or when one sets:

$$\mathsf{v}^2 = \Delta u^2 + \Delta v^2 + \Delta w^2 ,$$

that:

$$\delta \tfrac{1}{2}\mathsf{v}^2 = 0 , \tag{1}$$

i.e., that the square of the increment in velocity must be a minimum at the location where the path is discontinuous.

The assumption that was made above that the sides $\Delta\mathfrak{s}$, $\Delta\mathfrak{s}_1$ are both traversed in equal times can be fulfilled by a suitable choice of their lengths. Since those lengths do not appear in the formula (1), that assumption will imply no restriction.

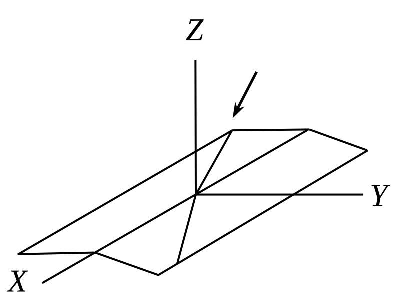

Figure 2.

Example. Find the straightest path of a point that moves rectilinearly with uniform velocity on a half-plane up to its intersection with a second half-plane that is inclined with respect to it and then moves along the latter.

If one lays the line of intersection of the two planes along the X-axis and lets the two planes make equal angles with the XZ-plane then the following equations will exist before and after the transition over the edge:

$$y - \rho z = 0 , \quad y + \rho z = 0 ,$$

in which ρ is a constant. If the velocity components are $u = a$, $w = b$ (a and b constant) before and u_1, v_1, w_1 after then one will have $v - \rho w = 0$, so $v = \rho b$, and $v_1 = - \rho w_1$. The path will then be the shortest one when the quantity:

$$\delta \tfrac{1}{2} v^2 = \delta \tfrac{1}{2}[(u_1 - a)^2 + \rho^2 (b + w_1)^2 + (b - w_1)^2]$$
$$= (u_1 - a)\, \delta u_1 + [\rho^2 (b + w_1) - (b - w_1)]\, \delta w_1$$

vanishes for all values of δu_1 , δw_1 . That will yield:

$$u_1 = a , \qquad w_1 = b \cdot \frac{1-\rho^2}{1+\rho^2} ,$$

so the point will always just cross over the obtuse angle between the two planes.

3. – Motion of a point-system. Velocity, acceleration, curvature of the path.

We shall now adapt some of the concepts that were developed for isolated mass-points in art. **1** to a system of material points. With HERTZ, we define a type of mean, or average, magnitude of the velocity, acceleration, and path curvature of the system by multiplying the square of the velocity, etc., of the point by its mass and adding those products. The positive roots of those sums, divided by the total mass, will then be called the *velocity* of the system (HERTZ, *Mechanik*, arts. 265, 275, 106), its *acceleration* ([1]), etc.

Therefore, the magnitude of the *velocity v of a system of points* with the coordinates x_i, y_i, z_i (i = 1, 2, …, n), masses m_i, and velocities v_i , and a total mass of:

$$m = m_i$$

will be determined by the equation:

$$m v^2 = m \dot{s}^2 = \sum m_i v_i^2 = \sum m_i \dot{s}_i^2 = \sum m_i (\dot{x}_i^2 + \dot{y}_i^2 + \dot{z}_i^2) . \tag{1}$$

The length s of the "path of the system" (viz., the set of all positions that it traverses) is determined as a function of time from the differential equation (1).

The motion will then be called *uniform* when the velocity of the system does not change in time (while the individual point can move non-uniformly), so when:

$$\ddot{s} = 0 ,$$

([1]) We shall not go into the vectorial definitions of those quantities, since they would be irrelevant to what follows.

or when:

$$m\dot{s}^2 = \sum m_i\, v_i^2 = \text{const.} \qquad (i = 1, 2, \ldots, n)$$

is constant. Since the quantity:

$$T = \tfrac{1}{2} m v^2 = \tfrac{1}{2} m \dot{s}^2 = \tfrac{1}{2} \sum m_i\,(\dot{x}_i^2 + \dot{y}_i^2 + \dot{z}_i^2) \tag{1.a}$$

is the *vis viva* or kinetic energy, with the usual terminology, *for uniform motion, the vis viva of the point system will have the same value at each point.*

Addendum. – It is advantageous for one to decompose the kinetic energy of a point-system into two parts that each have a simple interpretation. Namely, if a, b, c are the coordinates of the center of mass of a system of discrete mass-points m_1, m_2, …, m_n , and one displaces the origin of the system of axes to that point, so one sets:

$$x_i = a + \xi_i\,, \qquad y_i = b + \eta_i\,, \qquad z_i = c + \zeta_i\,,$$

then since:

$$a \sum m_i = \sum m_i\, x_i\,, \qquad b \sum m_i = \sum m_i\, y_i\,, \qquad c \sum m_i = \sum m_i\, z_i\,,$$

one will have:

$$\sum m_i\, \xi_i = 0\,, \qquad \sum m_i\, \eta_i = 0\,, \qquad \sum m_i\, \zeta_i = 0\,,$$

as well as:

$$\sum m_i\, \dot{\xi}_i = 0\,, \qquad \text{etc.,}$$

and one will get the following expression for the kinetic energy T :

$$T = \tfrac{1}{2} \sum m_i\, \dot{x}_i^2 = \frac{m}{2}(\dot{a}^2 + \dot{b}^2 + \dot{c}^2) + \tfrac{1}{2} \sum m_i\,(\dot{\xi}_i^2 + \dot{\eta}_i^2 + \dot{\zeta}_i^2)\ . \tag{1.b}$$

Thus, the total *vis viva* of the point-system will decompose into that of the advancing motion of the center of mass and that of the relative motion with respect to the center of mass, which is thought of as fixed.

We further define the magnitude of the *acceleration f of a point-system* by the equation:

$$m\, f^2 = \sum m_i\, f_i^2 = \sum m_i\,(\ddot{x}_i^2 + \ddot{y}_i^2 + \ddot{z}_i^2) \qquad (i = 1, 2, \ldots, n)\,, \tag{2}$$

and the *curvature c of the path of the system* at a given location (s) by:

$$mc^2 = \sum m_i \left[\left(\frac{d^2 x_i}{ds^2}\right) + \left(\frac{d^2 y_i}{ds^2}\right) + \left(\frac{d^2 z_i}{ds^2}\right)\right],$$

in which m is once more the total mass and the variable s is the arc-length of the system, which is used as the independent variable for determining the positions of the individual points x_i, y_i, z_i and is determined from equation (1) and any initial position. If one again introduces time t as the independent variable instead of s then due to (1), one will have:

$$m\,\dot{s}\,\ddot{s} = \sum m_i(\dot{x}_i\,\ddot{x}_i + \dot{y}_i\,\ddot{y}_i + \dot{z}_i\,\ddot{z}_i)\ ,$$

and with the use of formula [(4.a), art. **1**]:

$$mc^2 = \frac{1}{\dot{s}^4}\sum m_i(\ddot{x}_i^2 + \ddot{y}_i^2 + \ddot{z}_i^2) + \frac{\ddot{s}^2}{\dot{s}^6}\sum m_i(\dot{x}_i^2 + \dot{y}_i^2 + \dot{z}_i^2) - 2\frac{\dot{s}\,\ddot{s}}{\dot{s}^6}\sum m_i(\dot{x}_i\,\ddot{x}_i + \dot{y}_i\,\ddot{y}_i + \dot{z}_i\,\ddot{z}_i)$$

or

$$mc^2 = \frac{1}{\dot{s}^4}\left(\sum m_i\ f_i^2 - m\,\ddot{s}^2\right),$$

that is:

$$c^2\,v^4 = f^2 - \ddot{s}^2 .$$

For uniform motion, so when $\ddot{s} = 0$, the curvature will, in turn, be proportional to the acceleration of the system.

4. – The straightest path for a point-system.

The concept of the straightest path that was presented in art. **1** for isolated mass-points can also be adapted to point-systems with no further discussion. *The straightest path of a system* is composed of straightest path-elements. A straightest path-element at a given location (i.e., for a given position and velocity of the system) is one for which *the curvature c* (see previous art.) *has the smallest value* that is allowed by the constraint equations that exist. One might have, say, k constraint equations between the coordinates of the points that have the form:

$$\begin{aligned} &\varphi\,(x_1, y_1, z_1\,;\, x_2, y_2, z_2\,;\, \ldots, z_n) = 0\ ,\\ &\psi\,(x_1, y_1, z_1\,;\, x_2, y_2, z_2\,;\, \ldots, z_n) = 0\ ,\\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned} \tag{1}$$

Upon differentiating them twice with respect to time, one will derive k other ones, the first of which takes the form:

$$\Phi \equiv$$
$$\varphi''(x_1, x_1)\,\dot{x}_1^2 + 2\varphi''(x_1, y_1)\,\dot{x}_1\,\dot{y}_1 + \cdots + \varphi''(z_n, z_n)\,\dot{z}_n^2 + \varphi'(x_1)\,\ddot{x}_1 + \varphi'(y_1)\,\ddot{y}_1 + \cdots + \varphi'(z_n)\,\ddot{z}_n = 0\ . \tag{1.a}$$

The condition for the straightest path (i.e., for an extremum of c, or also, as we would prefer to write in what follows, $\frac{1}{2}mc^2$) then requires that one must, in turn, form the variation $\delta\left(\frac{1}{2}mc^2\right)$ in such a way that *the acceleration components will indeed be varied, but not the coordinates and the velocity components*. Likewise, one varies the *acceleration equations* (1.a) and *chooses them according to the variations of the accelerations*. One then gets:

$$\delta\Phi \equiv \sum[\varphi'(x_i)\,\delta\ddot{x}_i + \varphi'(y_i)\,\delta\ddot{y}_i + \varphi'(z_i)\,\delta\ddot{z}_i] = 0\,. \tag{1.b}$$

Since there are k such equations, only $3n - k$ of the $3n$ quantities $\delta\ddot{x}_1, \ldots, \delta\ddot{z}_n$ are assumed to be arbitrary, while the other ones are determined in terms of them by (1.b). With those values, one has:

$$\delta\left(\tfrac{1}{2}mc^2v^4\right) = \delta\,\tfrac{1}{2}\left[\sum m_i(\ddot{x}_i^2 + \ddot{y}_i^2 + \ddot{z}_i^2) - m\,\ddot{s}^2\right] = 0\,. \tag{2}$$

If one, in turn, combines that equation with equations (1.b), multiplied by the undetermined quantities λ, μ, …, and arranged in terms of $\delta\ddot{x}_1, \ldots, \delta\ddot{z}_n$, and assumes that the k multipliers λ, μ, … are determined in such a way that the coefficients of k of the variations will vanish then since $3n - k$ of them can be taken arbitrarily, the coefficients of the remaining $3n - k$ must also vanish. From a different standpoint: After introducing the λ, μ, …, the $3n$ variations behave as if the were all mutually independent. With that assumption, one will get $3n$ equations from:

$$\delta\left(\tfrac{1}{2}mv^4c^2\right) - \lambda\,\delta\Phi - \mu\,\delta\Psi\,\ldots = 0\,, \tag{3}$$

the first of which reads:

$$m_i\,\ddot{x}_i^2 - \frac{m_i\,\dot{x}_i\,\ddot{s}}{\dot{s}} = \lambda\,\varphi'(x_i) + \mu\,\psi'(x_i) + \cdots, \tag{3.a}$$

or, more briefly [art. **1**, (4)]:

$$\begin{aligned} m_i\,v^2\,\frac{d^2x_i}{ds^2} &= \lambda\,\varphi'(x_i) + \mu\,\psi'(x_i) + \cdots, \\ m_i\,v^2\,\frac{d^2y_i}{ds^2} &= \lambda\,\varphi'(y_i) + \mu\,\psi'(y_i) + \cdots, \\ m_i\,v^2\,\frac{d^2z_i}{ds^2} &= \lambda\,\varphi'(z_i) + \mu\,\psi'(z_i) + \cdots, \end{aligned} \tag{3.b}$$

in which v is defined by the expression (1) in art. **3**. That equation describes the *straightest line*. We shall likewise apply it to the case in which the motion of the system is uniform, so:

$$\ddot{s} = 0\,.$$

That is because the condition for the straightest path, $\delta(\frac{1}{2}mc^2) = 0$, can be replaced with the one for the smallest system acceleration:

$$\delta(\tfrac{1}{2}mv^4c^2) = \delta(\tfrac{1}{2}mf^2) = 0 ,$$

since the last term on the right in (2) indeed vanishes, so the following demand will enter in place of (3):

$$\delta(\tfrac{1}{2}mf^2) - \lambda\,\delta\Phi - \mu\,\delta\Psi - \ldots = 0 , \tag{4}$$

from which one will get:

$$\begin{aligned} m_i\ddot{x}_i &= \lambda\varphi'(x_i) + \mu\psi'(x_i) + \cdots, \\ m_i\ddot{y}_i &= \lambda\varphi'(y_i) + \mu\psi'(y_i) + \cdots, \\ m_i\ddot{z}_i &= \lambda\varphi'(z_i) + \mu\psi'(z_i) + \cdots \end{aligned} \tag{4.a}$$

Equations (4.a) and (1) are necessary and sufficient for determining the 3n coordinates and the k multipliers λ, μ, … as functions of time. The arbitrary constants that appear in the integration can be determined from the 3n coordinates of the initial position of the system and the 3n components of the initial velocities. If one multiplies equations (4.a) by $\dot{x}_i$, $\dot{y}_i$, $\dot{z}_i$, resp., and adds all 3n equations then when one recalls equations (1), or rather their first derivatives with respect to time, one will get:

$$\sum m_i(\ddot{x}_i\,\dot{x}_i + \ddot{y}_i\,\dot{y}_i + \ddot{z}_i\,\dot{z}_i) = \sum m_i\,\dot{s}_i\,\ddot{s}_i = \frac{m}{2}\frac{d}{dt}v^2 = 0 ,$$

or

$$\ddot{s} = 0 .$$

Therefore, *the demand of least acceleration* $\delta f = 0$, or:

$$\delta(\tfrac{1}{2}mf^2) = 0$$

(when the variation is performed in the manner that was given above, namely, for constant coordinates and velocities), can be replaced with the demand of *the straightest path for uniform velocity*, even within the scope of the demand. When one employs a terminology that will be introduced later (art. **15**), one also calls it the *demand of least constraint.*

5. – Example: Rotation of a rigid point-system around its center of mass ([1]).

The number of constraint equations can be reduced or eliminated completely by a convenient choice of coordinates. In the case of a rigid point-system or body, one gets around (cf., Introduction) the Ansatz of constraint equations in such a way that one introduces a coordinate

([1]) From a lecture that was given in the Winter semester of 1898/99.

system that is rigidly coupled with it. We assume that the center of mass of such a thing is fixed, so it possesses only three degrees of freedom, and we then address the problem of determining the straightest path of the system for uniform motion, so the problem of fulfilling the demand of the least constraint. We define the center of mass to be the origin *O* of a coordinate system that is fixed in space and has axes Ξ, H, Z. Let it, as well as a system *X*, *Y*, *Z* with the same origin *O* that is rigidly coupled with the point-system (i.e., body), be a right-handed (English) system (see the Figure).

We shall make that assumption for all of the coordinate systems that are employed in what follows.

The problem is then to minimize the expression $\frac{1}{2}mc^2v^4$ under the assumption that:

$$2T = \sum m_i\,(\dot{\xi}_i^2 + \dot{\eta}_i^2 + \dot{\zeta}_i^2) = h$$

is constant, or since $\ddot{s} = 0$, to minimize the expression:

$$\tfrac{1}{2}m\,f^2 = \sum m_i\,(\dot{\xi}_i^2 + \dot{\eta}_i^2 + \dot{\zeta}_i^2) \;.$$

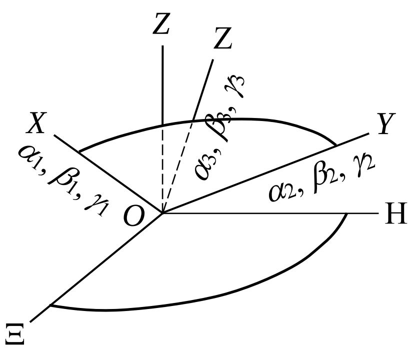

Figure 3.

One next deals with the choice of the three mutually-independent angles ("coordinates," in the broader sense of the word) by which the position of the moving coordinate system is determined with respect to the fixed one. If α_1, β_1, γ_1 are the cosines of the angle that the *X*-axis of the moving coordinate system makes with the axes Ξ, H, Z of the fixed one, and if the quantities α_2, β_2, γ_2 ; α_3, β_3, γ_3 belongs to the *Y* and *Z* axes, resp., then one will have the relations:

$$\begin{aligned}
\xi &= \alpha_1\,x + \alpha_2\,y + \alpha_3\,z\,,\\
\eta &= \beta_1\,x + \beta_2\,y + \beta_3\,z\,,\\
\zeta &= \gamma_1\,x + \gamma_2\,y + \gamma_3\,z\,,
\end{aligned} \tag{1}$$

for which the six known relations exist between the α, β, γ:

$$\begin{aligned}
&\alpha_1^2 + \beta_1^2 + \gamma_1^2 = 1, \qquad && \alpha_2\,\alpha_3 + \beta_2\,\beta_3 + \gamma_2\,\gamma_3 = 0\,,\\
&\alpha_2^2 + \beta_2^2 + \gamma_2^2 = 1, \qquad && \alpha_3\,\alpha_1 + \beta_3\,\beta_1 + \gamma_3\,\gamma_1 = 0\,,\\
&\alpha_3^2 + \beta_3^2 + \gamma_3^2 = 1, \qquad && \alpha_1\,\alpha_2 + \beta_1\,\beta_2 + \gamma_1\,\gamma_2 = 0.
\end{aligned} \tag{2}$$

Since only the angles change in time, when one differentiates (1), one will get:

$$\dot{\xi} = \dot{\alpha}_1 x + \dot{\alpha}_2 y + \dot{\alpha}_3 z, \qquad \ddot{\xi} = \ddot{\alpha}_1 x + \ddot{\alpha}_2 y + \ddot{\alpha}_3 z, \tag{2.a}$$
$$\text{etc.,} \qquad\qquad \text{etc.}$$

One must now express the nine quantities $\dot{\alpha}_1, \ldots$ in terms of three independent ones.

From a known law of the kinematics of rigid bodies, any infinitely-small motion of a body about a fixed point *O* can be represented by an (instantaneous) rotation around an axis *A* that goes through *O*, so by a vector that we would like to replace with its three components in the directions of the *X*, *Y*, *Z* axes of the moving system. However, for a relative motion of the Ξ, H, Z system with respect to the *X*, *Y*, *Z* system, one can also regard *X*, *Y*, *Z* as fixed and Ξ, H, Z as moving. Under an instantaneous rotation of the latter around the *A* axis, a point that is rigidly coupled to Ξ, H, Z and has the coordinates x, y, z will go to the position $x + \delta x, y + \delta y, z + \delta z$. If one decomposes the infinitesimal angle of rotation $\delta'\omega$ into three components $\delta'\varphi$, $\delta'\psi$, $\delta'\chi$, which correspond to rotations about the *X*, *Y*, *Z* axes, resp., then $\delta'\varphi$ will contribute nothing to δx, the quantity $-z\,\delta'\varphi$ to δy, and the quantity $y\,\delta'\varphi$ to δz when $\delta'\varphi$ ($\delta\varphi$ in the Figure) emerges from a right rotation. One gets the contributions that originate in the rotations $\delta'\psi$, $\delta'\chi$ by cyclic permutation. Their composition gives the small coordinate changes that result from the rotation $(\delta'\varphi, \delta'\psi, \delta'\chi)$ of the system Ξ, H, Z around the *A*-axis:

$$\begin{aligned} \delta x &= z\,\delta'\psi - y\,\delta'\chi, \\ \delta y &= x\,\delta'\chi - z\,\delta'\varphi, \\ \delta z &= y\,\delta'\varphi - x\,\delta'\psi. \end{aligned} \tag{3}$$

If one goes from the virtual rotation angles $\delta'\varphi, \ldots$ to the ones $d'\varphi, \ldots$ that actually occur in the time element *dt* then when one correspondingly introduces the components *p*, *q*, *r* of the instantaneous angular velocity, taken with respect to the *X*, *Y*, *Z* axes, by ([1]):

$$\frac{d'\varphi}{dt} = p, \qquad \frac{d'\psi}{dt} = q, \qquad \frac{d'\chi}{dt} = r,$$

one will get the relations:

$$\begin{aligned} \dot{x} &= z\,p - y\,r, \\ \dot{y} &= x\,r - z\,p, \\ \dot{z} &= y\,p - x\,q. \end{aligned} \tag{4}$$

Now let the space-point that is rigidly fixed in the coordinate system Ξ, H, Z be the endpoint of the segment 1 that is carried by the Ξ-axis. For it, one has:

([1]) The prime on the *d*, as it did before with δ, shall express the idea that *p*, *q*, *r* are not differential quotients of a *well-defined* angle with respect to time, since only its *increment* can be defined. One calls such fictitious quantities like the angles φ, ψ, χ *non-holonomic coordinates*. (See, e.g., HAMEL, "Virtuelle Verschiebungen in der Mechanik," Math. Ann. **59**). One always has $\delta d'\varphi \neq d\delta'\varphi$, etc., for them.

$$x = \alpha_1 , \qquad y = \alpha_2 , \qquad z = \alpha_3 .$$

The first column of (4) then implies the following system of equations:

$$\begin{array}{lll} \dot{\alpha}_1 = \alpha_3\, q - \alpha_2\, r, & \dot{\beta}_1 = \beta_3\, q - \beta_2\, r, & \dot{\gamma}_1 = \gamma_3\, q - \gamma_2\, r, \\ \dot{\alpha}_2 = \alpha_1\, r - \alpha_3\, p, & \dot{\beta}_2 = \beta_1\, r - \beta_3\, p, & \dot{\gamma}_2 = \gamma_1\, r - \gamma_3\, p, \\ \dot{\alpha}_3 = \alpha_2\, p - \alpha_1\, q, & \dot{\beta}_3 = \beta_2\, p - \beta_1\, q, & \dot{\gamma}_3 = \gamma_2\, p - \gamma_1\, q, \end{array} \tag{5}$$

by means of which the changes in the nine axis angles are expressed in terms of the components p, q, r of the angular velocity.

Therefore, the latter quantities will serve as the desired three independent quantities for the further analysis.

With the help of equations (5) and the relations (2), as well as by means of the equations that are derived from them by differentiating with respect to time one and two times, we form the following S sums:

$$\begin{gathered} \mathsf{S}\alpha_2\,\dot{\alpha}_3 = \alpha_2\,\dot{\alpha}_3 + \beta_2\,\dot{\beta}_3 + \gamma_2\,\dot{\gamma}_3 = -\,\mathsf{S}\dot{\alpha}_2\,\alpha_3 = p\,, \\ \mathsf{S}\alpha_3\,\dot{\alpha}_1 = -\,\mathsf{S}\dot{\alpha}_3\,\alpha_1 = q\,, \qquad \mathsf{S}\alpha_1\,\dot{\alpha}_2 = -\,\mathsf{S}\dot{\alpha}_1\,\alpha_2 = r\,, \\ \mathsf{S}\dot{\alpha}_1^2 = q^2 + r^2, \qquad \mathsf{S}\dot{\alpha}_2^2 = r^2 + p^2, \qquad \mathsf{S}\dot{\alpha}_3^2 = p^2 + q^2 . \end{gathered} \tag{6}$$

Finally, when one considers just the three quantities $\dot{p}$, $\dot{q}$, $\dot{r}$ in the formation of the variations (art. **4**, at the end):

$$\begin{aligned} \mathsf{S}\ddot{\alpha}_1\,\delta\ddot{\alpha}_1 &= \mathsf{S}(\alpha_3\,\dot{q} + \dot{\alpha}_1\,q - \alpha_2\,\dot{r} - \dot{\alpha}_2\,r)(\dot{\alpha}_3\,\delta q - \alpha_2\,\delta\dot{r}) \\ &= \dot{q}\,\delta\dot{q} + p\,r\,\delta\dot{q} + \dot{r}\,\delta\dot{r} - p\,q\,\delta\dot{r}\,. \end{aligned} \tag{7}$$

If one now addresses the straightest path for uniform motion then one must determine the differential quotients of p, q, r for given values of α_1, β_1, …, γ_3 ; $\dot{\alpha}_1$, $\dot{\beta}_1$, …, $\dot{\gamma}_3$; p, q, r such that:

$$\delta\left(\tfrac{1}{2} m\, f^2\right) = m\, f\, \delta f = \sum m_i (\ddot{\xi}_i\, \delta\ddot{\xi}_i + \ddot{\eta}_i\, \delta\ddot{\eta}_i + \ddot{\zeta}_i\, \delta\ddot{\zeta}_i) = 0\,, \tag{8}$$

in which only the variations of the accelerations do not vanish. Now, due to (2.a), one has:

$$\sum m_i\, \ddot{\xi}_i\, \delta\ddot{\xi}_i = \sum m_i\, (\ddot{\alpha}_1\, x_i + \ddot{\alpha}_2\, y_i + \ddot{\alpha}_3\, z_i)(\delta\ddot{\alpha}_1\, x_i + \delta\ddot{\alpha}_2\, y_i + \delta\ddot{\alpha}_3\, z_i)\,. \tag{8.a}$$

If one chooses the coordinate system that is fixed in the body in such a way that the axes X, Y, Z coincide with the principal axes of inertia and one calls the principal moments of inertia A, B, C then, as is known, one will have:

$$\sum m_i (y_i^2 + z_i^2) = A\,, \qquad \sum m_i\, y_i\, z_i = 0\,,$$
$$\sum m_i (z_i^2 + x_i^2) = B\,, \qquad \sum m_i\, z_i\, x_i = 0\,,$$
$$\sum m_i (x_i^2 + y_i^2) = C\,, \qquad \sum m_i\, x_i\, y_i = 0\,.$$

However, the expression (8.a) goes to:

$$\delta(\tfrac{1}{2} m f^2) = \sum m_i\, x_i^2\, \mathsf{S}\, \ddot{\alpha}_1\, \delta\ddot{\alpha}_1 + \sum m_i\, y_i^2 \cdot \mathsf{S}\, \ddot{\alpha}_2\, \delta\ddot{\alpha}_2 + \sum m_i\, z_i^2 \cdot \mathsf{S}\, \ddot{\alpha}_3\, \delta\ddot{\alpha}_3\,,$$

and with the use of the relation (7), one will have ([1]):

$$\begin{aligned}\delta(\tfrac{1}{2} m f^2) = &\sum m_i\, x_i^2 [(\dot{q} + p\,r)\,\delta\dot{q} + (\dot{r} - p\,q)\,\delta\dot{r}] \\ &+ \sum m_i\, y_i^2 [(\dot{r} + qp)\,\delta\dot{r} + (\dot{p} - q\,r)\,\delta\dot{p}] \\ &+ \sum m_i\, z_i^2 [(\dot{p} + rq)\,\delta\dot{p} + (\dot{q} - rq)\,\delta\dot{q}]\end{aligned}$$

$$= \delta\dot{p}[A\,\dot{p} - (C - B)\,q\,r] + \delta\dot{q}[B\,\dot{q} - (A - C)\,r\,p] + \delta\dot{r}[C\,\dot{r} - (B - A)\,p\,q]\,. \tag{9}$$

If one now sets the coefficients of $\delta\dot{p}$, $\delta\dot{q}$, $\delta\dot{r}$ individually equal to zero then one will get the equations:

$$\begin{aligned} A\,\dot{p} &= (C - B)\,q\,r, \\ B\,\dot{q} &= (A - C)\,r\,p, \\ C\,\dot{r} &= (B - A)\,p\,q \end{aligned} \tag{10}$$

as the condition for the straightest path of uniform motion. Those are the well-known EULER equations, and together with the system (5) they describe the natural motion of a body that rotates about its center of mass.

We will see later (art. **6**, at the end) that one can again arrive at dT / dt backwards from the expression $\delta(\tfrac{1}{2} m f^2)$ by replacing the quantities $\delta\dot{p}$, $\delta\dot{q}$, $\delta\dot{r}$ with p, q, r, resp. If one does that in (9) then one will get:

$$\frac{dT}{dt} = A\,\dot{p}\,p + B\,\dot{q}\,q + C\,\dot{r}\,r\,,$$

and from that:

$$T = \tfrac{1}{2}(A\,p^2 + B\,q^2 + C\,r^2) = h$$

([1]) GIBBS, Am. J. Math. **2** (1879), pp. 64 and APPELL, Journ. f. Math. **121** (1900) exhibit the value of $m\,f^2$, which is laborious to calculate. HERTZ also first constructed f^2 everywhere, instead of representing $2f\,\delta f$ directly.

is equal to a constant quantity, which can also be obtained from (10) upon multiplying those equations by p, q, r and adding them.

We will immediately establish that the natural motion is, above all, represented by the straightest path of uniform motion.

However, before we do that, we shall introduce the concept of a "free system" in the sense that HERTZ gave to that term.

6. – General coordinates. Non-holonomic motion. Straightest path for a point-system.

One can call the nine angles that were introduced in the previous section, which make the equations of constraint for the rigidity of a point-system unnecessary, "coordinates" in the broader sense of the word. Above all, one calls any sort of geometric determining data for the points of a system "general" or "LAGRANGIAN." LAGRANGE employed such things in order to eliminate the equations of constraint that exist between the rectangular coordinate or reduce their number. Theoretically-speaking, one can always achieve that goal in such a way that when:

$$\varphi\,(x_1, y_1, z_1\,;\, x_2, y_2, z_2\,;\, \ldots, z_n) = 0$$

is such an equation of constraint, one can treat the quantity $\varphi = p$ as a coordinate instead of one of the rectangular ones, eliminate it by means of $\varphi = p$, and then solve the problem under the assumption that $p = 0$. However, that process does not generally give coordinates that are exactly suited to the problem. Nonetheless, if one has introduced such things in any way and denotes them by $p_1, p_2, \ldots, p_r$ then one differentiates the equations of transformation:

$$\begin{aligned} x_i &= x_i\,(p_1, p_2, \ldots, p_r)\,, \\ y_i &= y_i\,(p_1, p_2, \ldots, p_r)\,, \\ z_i &= z_i\,(p_1, p_2, \ldots, p_r)\,, \end{aligned} \tag{1}$$

in which x_i, … are again proper function symbols, one and twice with respect to time, and introduces the values thus-obtained into the expressions (art. **3**) for T, f, … That might yield:

$$2T = \sum m_i\,(\dot{x}_i^2 + \dot{y}_i^2 + \dot{z}_i^2) = a_{11}\,\dot{p}_1^2 + 2a_{12}\,\dot{p}_1\,\dot{p}_2 + \cdots + a_{rr}\,\dot{p}_r^2 = \sum\sum a_{ik}\,\dot{p}_i\,\dot{p}_k\,, \tag{2}$$

in which the a_{ik} can include the coordinates $p_1, p_2, \ldots, p_r$ themselves. The expressions for f^2, c^2 will become considerably more complicated ([1]).

It will suffice for us to restrict ourselves to defining the variations of those quantities, as we shall do later on, in which the quantity T is available for simplification.

([1]) Since HERTZ defined the quantity $\delta(c^2)$, instead of $2c\,\delta c$, he was required to perform laborious calculations (art. 108 in *Mechanik*).

The number r of independent coordinates p that remain after all equations of constraint have been eliminated is the *degree of freedom* of the system. However, it is not always possible or advisable to eliminate the equations of constraint completely by introducing new variables. In particular, one will not succeed in doing that when conditions are present that take the form of differential equations that are not integrable and cannot be made integrable in any combination either. For motions of that kind, which HERTZ called *non-holonomic*, the equations of constraint will possibly have the form:

$$\begin{aligned} d'\varphi &\equiv \sum (\varphi_{xi}\, dx_i + \varphi_{yi}\, dy_i + \varphi_{zi}\, dz_i) = 0\,, \\ d'\psi &\equiv \sum (\psi_{xi}\, dx_i + \psi_{yi}\, dy_i + \psi_{zi}\, dz_i) = 0\,, \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned} \tag{3}$$

in which φ_{xi}, φ_{xy}, …, ψ_{xi}, … are functions of the coordinates $x_1, y_1, \ldots, z_n$, and here, as well as in what follows, the prime on d or δ will suggest that not-necessarily-functions $\varphi, \psi, \ldots$ exist whose differentials (variations, resp.) are on the right-hand side. By contrast, the motion will once more be *holonomic* when equations (3) are integrable without restriction, so in particular, when $\varphi, \psi, \ldots$ are partial differential quotients of the same function φ, etc., with respect to the coordinates or can be made into such things by a common multiplier.

One gets the relations between the variations from equations (3) upon dividing by dt, repeatedly differentiating with respect to time, and subsequently varying the second differential quotients. The relations that appear in place of equations (1.b) of art. **4** in that way have the form ([1]):

$$\delta'\ddot{\Phi} \equiv \sum\left\{\varphi_{xi}\,\delta\ddot{x}_i + \varphi_{yi}\,\delta\ddot{y}_i + \varphi_{zi}\,\delta\ddot{z}_i\right\} = 0\,. \tag{3.a}$$

The equations that express our basic law of the straightest path for uniform motion, or as we have also referred to that demand, the condition for least constraint, are then obtained from:

$$\delta\frac{mf^2}{2} - \lambda\,\delta'\ddot{\Phi} - \mu\,\delta'\ddot{\Psi} - \cdots = 0\,, \tag{4}$$

in rectangular coordinates, so from:

$$0 = \sum m_i(\ddot{x}_i\,\delta\ddot{x}_i + \ddot{y}_i\,\delta\ddot{y}_i + \ddot{z}_i\,\delta\ddot{z}_i) - \lambda\sum(\varphi_{xi}\,\delta\ddot{x}_i + \varphi_{yi}\,\delta\ddot{y}_i + \varphi_{zi}\,\delta\ddot{z}_i) - \mu\sum(\varphi_{xi}\,\delta\ddot{x}_i + \cdots) - \cdots,$$

and in the latter case they read:

([1]) The not-entirely-consistent notation $\delta\ddot{\Phi}$ was chosen temporarily in order to express the type of variation.

$$\begin{aligned} m_i \ddot{x}_i &= \lambda \varphi_{xi} + \mu \psi_{xi} + \cdots, \\ m_i \ddot{y}_i &= \lambda \varphi_{yi} + \mu \psi_{yi} + \cdots, \\ m_i \ddot{z}_i &= \lambda \varphi_{zi} + \mu \psi_{zi} + \cdots, \end{aligned} \qquad (i = 1, 2, \ldots, n) , \tag{5}$$

to which one adds the equations of constraint (3), which can be replaced by integral equations in the case of a holonomic motion.

Upon introducing general coordinates $p_1, p_2, \ldots, p_r$, the finite equations of constraint will assume the form:

$$\begin{aligned} \varphi(p_1, p_2, \ldots, p_r) &= 0 , \\ \psi(p_1, p_2, \ldots, p_r) &= 0 , \\ \ldots\ldots\ldots\ldots&\ldots\ldots, \end{aligned} \tag{6}$$

and the differential equations of constraint (3) will go to:

$$\begin{aligned} d'\varphi &\equiv \varphi_1\, dp_1 + \varphi_2\, dp_2 + \ldots + \varphi_r\, dp_r = 0 , \\ d'\psi &\equiv \psi_1\, dp_1 + \psi_2\, dp_2 + \ldots + \psi_r\, dp_r = 0 , \\ \ldots\ldots&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned} \tag{7}$$

so the ones between the variations will take on the form:

$$\delta' \ddot{\Phi} \equiv \varphi_1\, \delta \ddot{p}_1 + \varphi_2\, \delta \ddot{p}_2 + \cdots + \varphi_r\, \delta \ddot{p}_r = 0 , \qquad \text{etc.} \tag{7.a}$$

In order to express the square of the acceleration that appears in the demand of least constraint (art. **4**, at the end):

$$\delta \frac{m f^2}{2} = 0 ,$$

or rather $m f \delta f$, in terms of the coordinates p, one next remarks that:

$$\dot{x}_i = \frac{\partial x_i}{\partial p_1} \dot{p}_1 + \frac{\partial x_i}{\partial p_2} \dot{p}_2 + \cdots + \frac{\partial x_i}{\partial p_r} \dot{p}_r = \sum_k \frac{\partial x_i}{\partial p_k} \dot{p}_k \qquad (k = 1, 2, \ldots, r) \tag{8}$$

will imply that:

$$\frac{\partial \dot{x}_i}{\partial \dot{p}_k} = \frac{\partial x_i}{\partial p_k} , \quad \text{etc.} \tag{9}$$

Furthermore, one has:

$$\frac{\partial \dot{x}_i}{\partial p_k} = \frac{\partial}{\partial p_k} \sum_e \frac{\partial x_i}{\partial p_e} \dot{p}_e = \sum_e \frac{\partial x_i}{\partial p_k\, \partial p_e} \dot{p}_e = \frac{d}{dt}\left(\frac{\partial x_i}{\partial p_k}\right) . \tag{10}$$

Finally, upon differentiating (8), one will get:

$$\ddot{x}_i = \sum_k \sum_e \frac{\partial x_i}{\partial p_k\,\partial p_e}\,\dot{p}_k\,\dot{p}_e + \sum_k \frac{\partial x_i}{\partial p_k}\,\ddot{p}_k \qquad (k, e = 1, 2, \ldots, r)\,,$$

and therefore, when one *varies just the acceleration*, which the aforementioned requirement (art. **4**, at the end) implies:

$$\delta\ddot{x}_i = \sum_k \frac{\partial x_i}{\partial p_k}\,\delta\ddot{p}_k\;. \tag{10.a}$$

Now the expression:

$$\delta\frac{m\,f^2}{2} = \sum m_i\,(\ddot{x}_i\,\delta\ddot{x}_i + \ddot{y}_i\,\delta\ddot{y}_i + \ddot{z}_i\,\delta\ddot{z}_i) \qquad (i = 1, 2, \ldots, n)\,,$$

since one has:

$$\frac{\partial T}{\partial\dot{x}_i} = m_i\,\dot{x}_i\,, \quad \text{etc.,}$$

can also be written:

$$\delta\frac{m\,f^2}{2} = \sum_i\left[\frac{d}{dt}\left(\frac{\partial T}{\partial\dot{x}_i}\right)\delta\ddot{x}_i + \frac{d}{dt}\left(\frac{\partial T}{\partial\dot{y}_i}\right)\delta\ddot{y}_i + \frac{d}{dt}\left(\frac{\partial T}{\partial\dot{z}_i}\right)\delta\ddot{z}_i\right].$$

When one expresses the $\delta\ddot{x}_i$, $\delta\ddot{y}_i$, … in terms of the $\delta\ddot{p}$, due to (10.a), one will then have:

$$\delta\frac{m\,f^2}{2} = \sum_k\sum_i\left[\frac{d}{dt}\left(\frac{\partial T}{\partial\dot{x}_i}\right)\frac{\partial x_i}{\partial p_k}\,\delta\ddot{p}_k + \frac{d}{dt}\left(\frac{\partial T}{\partial\dot{y}_i}\right)\frac{\partial y_i}{\partial p_k}\,\delta\ddot{p}_k + \frac{d}{dt}\left(\frac{\partial T}{\partial\dot{z}_i}\right)\frac{\partial z_i}{\partial p_k}\,\delta\ddot{p}_k\right] = \sum_k P_k\,\delta\ddot{p}_k\,, \tag{10.b}$$

in which one now has:

$$P_k = \sum_i\left[\frac{d}{dt}\left(\frac{\partial T}{\partial\dot{x}_i}\right)\frac{\partial x_i}{\partial p_k} + \cdots\right], \tag{10.c}$$

and the ellipses suggest that one must write out the corresponding terms in y and z. Under the assumption that the x_i, y_i, … are functions of p, one can convert that expression as follows (see ROUTH-SCHEPP, *Dynamik*, Leipzig, 1898, I, § 398):

$$P_k = \frac{d}{dt}\sum_i\left[\frac{\partial T}{\partial\dot{x}_i}\frac{\partial x_i}{\partial p_k} + \cdots\right] - \sum_i\left[\frac{\partial T}{\partial\dot{x}_i}\frac{d}{dt}\left(\frac{\partial x_i}{\partial p_k}\right) + \cdots\right] = \frac{d}{dt}\sum_i\left[\frac{\partial T}{\partial\dot{x}_i}\frac{\partial x_i}{\partial p_k} + \cdots\right] - \sum_i\left[\frac{\partial T}{\partial\dot{x}_i}\frac{\partial\dot{x}_i}{\partial p_k} + \cdots\right],$$

which is justified by equations (9), (10). Since $\partial T / \partial x_i = 0$, one will then get:

$$P_k = \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{p}_k}\right) - \frac{\partial T}{\partial p_k} ,$$

and therefore one will ultimately get the important expression for the variation of the square of the system acceleration f :

$$\delta \frac{m f^2}{2} = \sum_k \left[\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{p}_k}\right) - \frac{\partial T}{\partial p_k}\right] \delta \ddot{p}_k , \tag{11}$$

in which T has the previously-given form:

$$T = \tfrac{1}{2} \sum_i \sum_k a_{ik} \, \dot{p}_i \, \dot{p}_k .$$

It is remarkable that *one can, conversely, again arrive at* $\dot{T}$ *from the function* (11) $\delta \frac{m f^2}{2}$ in such a way that one replaces $\delta \ddot{p}_k$ with $\dot{p}_k$. That is because when one does that in (10.a), $\delta \ddot{p}_k$ will go to $\dot{p}_k$, so $\sum m_i \, \ddot{x}_i \, \delta \ddot{x}_i$, and therefore $\delta \frac{m f^2}{2}$ (in any form) will go to $\dot{T}$. Q.E.D.

If one also introduces the variation into (4) in the transformed form (7.a) then one will get *the equations for the straightest path of uniform motion* (i.e., the condition for least constraint):

$$\sum \left[\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{p}_k}\right) - \frac{\partial T}{\partial p_k} - \lambda \varphi_k - \mu \psi_k - \cdots \right] \delta \ddot{p}_k = 0 , \tag{12}$$

which decomposes into the individual equations:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{p}_k}\right) = \frac{\partial T}{\partial p_k} - \lambda \varphi_k - \mu \psi_k - \cdots \qquad (k = 1, 2, \ldots, r), \tag{13}$$

due to the independence of the variations $\delta \ddot{p}_k$, which is equivalent to the one (5) that was presented before for rectangular coordinates and subsumes it.

Moreover, one will get the uniformity of the motion from equations (5) with no further analysis when one linearly combines them with $\dot{x}_i$, $\dot{y}_i$, $\dot{z}_i$.

7. – Straightest path for a point-system with discontinuous changes in velocity.

In order to also formulate the requirement of the straightest path for a point-system whose direction and velocity *changes suddenly by a finite quantity* at some point along its path, we must go back to the expression for the isolated point that was presented in art. **2**. There, we defined the variation:

$$\delta \tfrac{1}{2} \mathsf{v}^2 = \delta \tfrac{1}{2} (\Delta u^2 + \Delta v^2 + \Delta w^2) .$$

Analogously, in the case of a point-system, the variation:

$$\begin{aligned} \delta \mathsf{T} &= \delta \tfrac{1}{2} m \mathsf{v}^2 = \delta \tfrac{1}{2} \sum m_i \, (\Delta u_i^2 + \Delta v_i^2 + \Delta w_i^2) \\ &= \sum m_i \, (\Delta u_i \, \delta \Delta u_i + \Delta v_i \, \delta \Delta v_i + \Delta w_i \, \delta \Delta w_i) = 0 \end{aligned} \tag{1}$$

is set equal to zero, when (Δu_i , Δv_i , Δw_i) is the change in velocity at the point m_i ($i = 1, 2, \dots, n$).

If constraint equations exist between the coordinates of the point, which will then imply relations for the changes in velocity that take the form:

$$\Phi \equiv \sum (\Phi_{xi} \, \Delta u_i + \Phi_{yi} \, \Delta v_i + \Phi_{xz} \, \Delta w_i) = 0 , \tag{2}$$

then they will again be varied in such a way that the coordinates and the velocities will not involve them, but possibly the changes in velocity Δu_i , … One will then get:

$$\delta \Phi \equiv \sum (\Phi_{xi} \, \delta \, \Delta u_i + \Phi_{yi} \, \delta \, \Delta v_i + \Phi_{xz} \, \delta \, \Delta w_i) = 0 . \tag{3}$$

Those equations can be combined with $\delta \mathsf{T}$, as in art. **6**, at the end, upon multiplying by λ, …, and in that way make the $\delta \, \Delta u_i$ mutually independent.

We can do that, in general, when we again assume that the variables are p_1, p_2, …, p_r , which fulfill all or some of the constraint equations identically, but which we would like to imagine have been chosen in such a way that they themselves do not experience sudden changes, but only their first derivatives. If one introduces them by means of equations (1), (8) of the previous article then the velocities before and after the sudden change will be:

$$u_i = \dot{x}_i = \sum_k \frac{\partial x_i}{\partial p_k} \dot{p}_k$$

and

$$u_i^{(0)} = \dot{x}_i^{(0)} = \sum_k \frac{\partial x_i}{\partial p_k} \dot{p}_k^{(0)} ,$$

resp., then one will have:

$$\Delta \, u_i = \Delta \, \dot{x}_i = \sum_k \frac{\partial x_i}{\partial p_k} \Delta \, \dot{p}_k , \tag{4}$$

and the expression:

$$\mathsf{T} = \tfrac{1}{2} m_i \, (\Delta u_i^2 + \Delta v_i^2 + \Delta w_i^2)$$

will go to:

$$\mathsf{T} = \tfrac{1}{2} \sum \sum a_{ik} \, \Delta \dot{p}_i \, \Delta \dot{p}_k , \tag{5}$$

in which the coefficients a_{ik} are what they were in the expression for the *vis viva T* in the previous article. When no auxiliary conditions appear, the condition for the straightest path will then become

$$0 = \delta \mathsf{T} = \sum\sum a_{ik}\,\Delta\dot{p}_i\,\delta\,\Delta\dot{p}_k = \sum\left(\frac{\partial T}{\partial \dot{p}_k} - \frac{\partial T^{(0)}}{\partial \dot{p}_k^{(0)}}\right)\delta\,\Delta\dot{p}_k = \sum\left(\frac{\partial T}{\partial \dot{p}_k} - \frac{\partial T^{(0)}}{\partial \dot{p}_k^{(0)}}\right)\delta\,(\dot{p}_k - \dot{p}_k^{(0)})\,, \quad (6)$$

in which T ($T^{(0)}$, resp.) is now the expression for the *vis viva* itself when written in terms of the $\dot{p}_k$ ($\dot{p}_k^{(0)}$, resp.), so:

$$T = \tfrac{1}{2}\sum\sum a_{ik}\,\Delta\dot{p}_i\,\Delta\dot{p}_k\,,$$
$$T^{(0)} = \tfrac{1}{2}\sum\sum a_{ik}\,\Delta\dot{p}_i^{(0)}\,\Delta\dot{p}_k^{(0)}\,. \quad (7)$$

If constraint equations such as:

$$\delta\Phi \equiv \Phi_1\,\delta\,\Delta\dot{p}_1 + \Phi_2\,\delta\,\Delta\dot{p}_2 + \cdots + \Phi_r\,\delta\,\Delta\dot{p}_r = 0 \quad (8)$$

exist between the $\delta(\dot{p}_k - \dot{p}_k^{(0)}) = \delta\,\Delta\dot{p}_k$ then the demand that:

$$0 = \delta\mathsf{T} - \lambda\,\delta\Phi - \ldots = 0\,, \quad (9)$$

in which the $\delta\,\Delta\dot{p}_k$ are now mutually-independent, will imply the r equations:

$$\frac{\partial T}{\partial \dot{p}_1} - \frac{\partial T^{(0)}}{\partial \dot{p}_1^{(0)}} = \lambda\,\Phi_1 + \ldots\,,$$
$$\frac{\partial T}{\partial \dot{p}_2} - \frac{\partial T^{(0)}}{\partial \dot{p}_2^{(0)}} = \lambda\,\Phi_2 + \ldots\,, \quad (10)$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

along with those of the constraint equations that emerge from ones in (8) by differentiating, taking the difference, and varying.

8. – The free system. Hertz's fundamental law and LAGRANGE's equations of motion.

With HERTZ, we shall say a *free point-system* to mean one for which only geometric equations of constraint exist between its coordinates, i.e., ones of the form (6), (7) in art. **6**, so finite equations or first-order differential equations of degree one between the coordinates into which time does not enter explicitly.

A "free" system is, for example, the rigid point-system that rotates around a point that was treated in art. **5**. Rigid bodies or systems of them, which might be connected with each other by links, individual points of which can also move on prescribed paths or surfaces, will also define free systems in the HERTZian sense. By contrast, e.g., a system of attracting mass-points is not a free system, since the demand of attraction cannot be put into a geometric form. In arts. **17**, **18**, we will also extend the concept of free systems to non-rigid continuous masses.

The natural motion of such a free system (initially, with a finite number of degrees of freedom) will now be determined by the following single *fundamental law* that HERTZ posed in place of NEWTON's, and more recently posed as an extended axiom (HERTZ, *Mechanik*, art. 309):

Any free system persists in a state of rest or uniform motion along a straightest path.

That axiom combines the law of inertia because it demands uniform motion (i.e., inertia in the broader sense) and the principle of least constraint because it demands the straightest path, and it is initially restricted to the free (force-less) systems. However, the motion of just such a thing was examined in art. **6**, which fulfilled both conditions, and the condition was formulated for it that:

$$\delta\left(\frac{m\,f^2}{2}\right) - \lambda\,\delta'\ddot{\Phi} - \mu\,\delta'\ddot{\Psi} - \cdots = 0\,, \tag{1}$$

whose implementation in general coordinates $p_1, p_2, \ldots, p_r$:

$$\sum\left[\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{p}_k}\right) - \frac{\partial T}{\partial p_k} - \lambda\,\varphi_k - \mu\,\psi_k - \cdots\right]\delta\ddot{p}_k = 0 \qquad (k = 1, 2, \ldots, r) \tag{2}$$

yields the system of equations (art. **6**, at the end):

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{p}_k}\right) = \frac{\partial T}{\partial p_k} + \lambda\,\varphi_k + \mu\,\psi_k + \cdots, \tag{3}$$

in which:

$$T = \tfrac{1}{2}\sum a_{ik}\,\dot{p}_i\,\dot{p}_k \tag{4}$$

is the expression for the *vis viva* of the system, and:

$$\begin{aligned} d'\Phi &\equiv \varphi_1\,dp_1 + \varphi_2\,dp_2 + \ldots + \varphi_r\,dp_r = 0\,, \\ d'\Phi &\equiv \varphi_1\,dp_1 + \varphi_2\,dp_2 + \ldots + \varphi_r\,dp_r = 0\,, \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned} \tag{5}$$

are (non-holonomic or holonomic) *geometric conditions of constraint.* Now, according to the "fundamental law," equations (3) define the natural equations of our point system. They are the well-known *Lagrange equations (of motion) of the second kind* for the case in which no external forces act, while the equivalent system for rectangular coordinates [art. **6**, (5)] are called LAGRANGE's differential equations *of the first kind*.

If one interprets the *completely-arbitrary quantities* $\delta \ddot{p}_k$ *as variations of the coordinates* p_k *themselves and correspondingly replaces them with* p_k [1] then the demand that:

$$\sum \left[\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{p}_k}\right) - \frac{\partial T}{\partial p_k} - \lambda \varphi_k - \mu \psi_k - \cdots \right] \delta p_k = 0 \tag{6}$$

will enter in place of (2), or in rectangular coordinates:

$$\begin{aligned}\sum [(m_i \ddot{x}_i - \lambda \varphi_{xi} - \mu \psi_{xi} - \cdots)\delta x_i + (m_i \ddot{y}_i - \lambda \varphi_{yi} - \mu \psi_{yi} - \cdots)\delta y_i \\ + (m_i \ddot{z}_i - \lambda \varphi_{zi} - \mu \psi_{zi} - \cdots)\delta z_i] = 0 ,\end{aligned} \tag{7}$$

which are equations that, along with the constraints (5) [(3) of art. **6**] above, possess precisely the form in which one is wont to clothe *d'Alembert's principle* for a system of mass-points on which no external forces act. We will come back to that in art. **15** below.

The quantities $\lambda\, \varphi_{xi}$, $\mu\, \psi_{xi}$, … in (7) have the dimension [$l\ m\ t^{-2}$] of a "force" in the usual sense. One can then interpret:

$$\lambda \sqrt{\varphi_{xi}^2 + \varphi_{yi}^2 + \varphi_{zi}^2}$$

as a force (pressure) whose components relate to each other like $\varphi_{xi} : \varphi_{yi} : \varphi_{zi}$, and regard them as the resistance to the motion of the point m_i that is imposed by the condition $d'\varphi = 0$. In the example of art. **1**, λ is the reaction pressure of the surface. We would next like to call upon some further examples for the employment of LAGRANGE's differential equations in the case of holonomic and non-holonomic motions.

---

(1) It should be emphasized that when one follows the path that was chosen here, that will imply, with no further analysis, relations between the variations $\delta p_1, \ldots, \delta p_k$ that also make it possible to go to *Hamilton's principle*, even in the case of non-holonomic motion. That is because when one replaces the $\delta \ddot{p}_k$ with δp_k in (7.a) of art. **6**, one will get the constraint equations:

$$\varphi_1\, \delta p_1 + \varphi_2\, \delta p_2 + \ldots + \varphi_k\, \delta p_k = 0$$

between the variations that one ordinarily infers from the present equations of constraint (5) by simply *converting the d into* δ using an application of D'ALEMBERT's principle (one that is derived from it). One prefers to derive the justification for that process from the prescription of the principal of virtual velocities, in which the displacements δp that are introduced into (6) are thought of as *virtual* ones, i.e., displacements that are compatible with the constraints on the system. However, the latter probably says just: The relations between the virtual displacements must be derivable from those constraint equations by only computable operations. *However, the replacement of d with* δ *is not such an operation:* Nonetheless, the double differentiation with respect to time and repeated variations probably is. That advantage to the principle of least constraint has not, I believe, been emphasized enough up to now, even by GIBBS, who first applied it at a fundamental level.

9. – Examples.

The next simple example will be of service to us later.

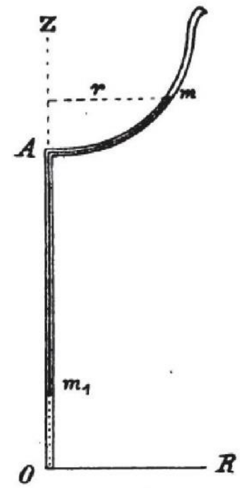

Fig. 5.

1. A material point m moves in the curved part of a tube that is bent into a plane curve and concludes in a straight continuation. In the latter, one finds a second material point m_1 that is coupled with the first one by an inextensible string that runs inside the tube. The straight part of the tube coincides with the Z-axis (see the figure). The system is put into rotation around the latter as a fixed axis by an impact and then left to itself. What motion do the two points exhibit? The tube and string are massless.

If r, z are the coordinates of m in the plane at the time t, ϑ is the angle between the rotating plane and a fixed plane, and s is the length of the curved piece of the string then:

$$ds^2 = dr^2 + dz^2 .$$

The kinetic energy T of the two mass-points is then:

$$T = \tfrac{1}{2}(m\dot{s}^2 + m_1\,\dot{s}^2 + m\,r^2\,\dot{\vartheta}^2) = \tfrac{1}{2}[(m+m_1)(\dot{r}^2 + \dot{z})^2 + m\,r^2\,\dot{\vartheta}^2] , \tag{1}$$

and is indeed equal to a constant when we ignore friction and the action of gravity, which we would like to do. The constraint equations that exist between the coordinates r, z, ϑ of the system is the equation:

$$\varphi\,(r, z) \equiv z - f\,(r) = 0 \tag{2}$$

of the curve into which the tube is bent. The equations of motion that the fundamental law implies:

$$0 = \delta\,\frac{m\,f^2}{2} = \left[\frac{d}{dt}\frac{\partial T}{\partial \dot{r}} - \frac{\partial T}{\partial r}\right]\delta\ddot{r} + \left[\frac{d}{dt}\frac{\partial T}{\partial \dot{z}} - \frac{\partial T}{\partial z}\right]\delta\ddot{z} + \left[\frac{d}{dt}\frac{\partial T}{\partial \dot{\vartheta}} - \frac{\partial T}{\partial \vartheta}\right]\delta\ddot{\vartheta} , \tag{3}$$

when one considers the equations of constraint:

$$-\,f'(r)\,\delta\ddot{r} + \delta\ddot{z} = 0 , \tag{3.a}$$

will read:

$$\begin{aligned}(m+m_1)\,\ddot{r} &= m\,r\,\dot{\vartheta}^2 - \lambda\,f'(r)\\ (m+m_1)\,\ddot{z} &= \lambda ,\\ m\,r^2\,\dot{\vartheta} &= \mathfrak{q}\cdot m ,\end{aligned} \tag{4}$$

in which $\mathfrak{q}$ is an integration constant. The vector λ has the dimension of a force, and since its components in the plane of the tube have the ratio $-f'(r) : 1$, it is perpendicular to the curve itself, so it is the reaction pressure of the tube. Upon eliminating λ from equations (4), one will get:

$$\dot{s}\,\ddot{s}\,(m+m_1) = \frac{m\mathfrak{q}^2\,\dot{r}}{r^3},$$

or when one calculates the coordinate r as a function of the arc-length s, so $r = \psi(s)$, from the equation $z = f(r)$:

$$\frac{m+m_1}{\mathfrak{q}^2 m}\frac{d^2 s}{dt^2} = \frac{1}{r^3}\frac{dr}{ds} = \frac{\psi'(s)}{\psi^3(s)}, \tag{5}$$

in which a known function of s is found on the right-hand side.

On the other hand, upon substituting $\dot{\vartheta}$, r, and $\dot{r}$ in the equation (1) for the *vis viva* $T = h$, one will get:

$$\frac{m+m_1}{\mathfrak{q}^2 m}\dot{r}^2[1+f'(r)^2]+\frac{1}{r^3} = \frac{2h}{\mathfrak{q}^2\,m},$$

and from this, one will get r as a function of t after one assumes some specific form for the function $f(r)$. If one sets the length of the straight part of the tube $OA = l$ and sets the distance from the point m_1 to the endpoint O equal to z_1 then if L is the total length of the string:

$$l - z_1 + s = L,$$

so

$$z_1 = s + \text{const.} \tag{6}$$

Instead of regarding the function $f(r)$ as given, one can also, conversely, assume that:

$$\ddot{z}_1 = \ddot{s}$$

is a certain function of s or $k - s$, in which k is a constant:

$$\ddot{s} = F(k - s) \tag{7}$$

and determine the form of the curved tube from that by substituting (7) in (5), then determining r as a function of s, and then calculating it as a function of z. Now, if the mass-point m, as well as the tube and string, are transparent then the point m_1 would appear to move along the Z-axis as if the arbitrarily-assumed force-at-a-distance $F(k_1 - z_1)$ acted upon it, because due to (6), (7) will imply that:

$$\ddot{z}_1 = F(k_1 - z_1), \tag{7.a}$$

in which k_1 is once more a constant.

2. An example of a non-holonomic motion will be provided by the simple apparatus that was introduced into the study of field measurements not long ago in the form of "PRYTZ's rod planimeter." It consists essentially of a rod whose one end carries a hatchet-shaped support *A* (see the figure) and the other end of which carries a sufficiently-rounded point *B* that is just as big as *A*, and both of them rest upon a horizontal plane *E*. The instrument moves as a result of an impact that is exerted upon the center of mass *S* horizontally. The point *B* can move in all directions, but the blade of the support *A* impedes the sideways displacement of that point and has the effect that *A* can move only forwards in a direction that is determined by the blade. It coincides with direction of the rod *AB*. The impact to the center of mass imparts upon the instrument:

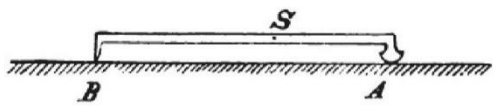

Fig. 6a.

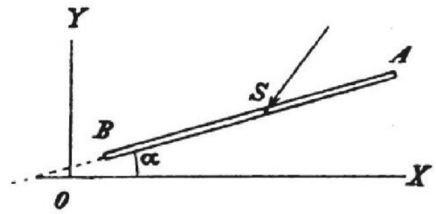

Fig. 6b.

1. An initial rotation around *A* and
2. An advancing motion of *S*.

Let *x*, *y* be the rectangular coordinates of *A*, and let ξ, η be those of the center of mass *S* at any point in time *t* . If $AS = a$ and α is the angle of inclination of the direction of the rod with the *X*-axis then one will have:

$$x - \xi = a \cos \alpha , \qquad y - \eta = a \sin \alpha . \tag{1}$$

The condition that *A* moves in the direction of the blade of the hatchet is expressed by the equation:

$$dy = dx \tan \alpha , \tag{2}$$

which represents a non-holonomic condition. When one sets the mass equal to 1 and *k* is the radius of inertia [art. **3** (1.b)], the *vis viva* of the system will be:

$$T = \tfrac{1}{2}(\dot{\xi}^2 + \dot{\eta}^2 + k^2 \dot{\alpha}^2) . \tag{3}$$

One gets the constraint equation from (1), (2):

$$d'\Phi \equiv d\eta - d\xi \tan \alpha + \frac{a\,d\alpha}{\cos\alpha} = 0 . \tag{4}$$

The fundamental law gives:

$$\delta \frac{m\,f^2}{2} - \lambda\,\delta'\Phi = 0 ,$$

or when expanded [art. **8** (2)]:

$$\left[\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\xi}}\right) + \lambda \tan\alpha\right]\delta\ddot{\xi} + \left[\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\eta}}\right) - \lambda\right]\delta\ddot{\eta} + \left[\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\alpha}}\right) - \frac{\lambda\,\alpha}{\cos\alpha}\right]\delta\ddot{\alpha} = 0 .$$

Therefore, the equations of motion are:

$$\begin{aligned} \ddot{\xi} + \lambda \tan\alpha &= 0 , \\ \ddot{\eta} - \lambda &= 0 , \\ k^2\,\ddot{\alpha} - \frac{a\,\lambda}{\cos\alpha} &= 0 , \end{aligned} \tag{5}$$

to which one adds (4) ([1]):

$$\dot{\eta} - \tan\alpha \cdot \dot{\xi} + \frac{a\,\dot{\alpha}}{\cos\alpha} = 0 . \tag{4.a}$$

The elimination of λ, $\ddot{\eta}$, $\ddot{\xi}$, $\dot{\eta}$, $\dot{\xi}$, in the given sequence, from the equation of *vis viva:*

$$2T = h^2 , \tag{6}$$

equations (5), (4.a), and the differentiated version of equation (4.a) will yield the following differential equation for α:

$$\frac{a^2 + k^2}{a}\ddot{\alpha} = \dot{\alpha}\sqrt{h^2 - (k^2 + a^2)\,\dot{\alpha}^2} ,$$

and by means of (1), (2), (3), one must add to that:

$$\begin{aligned} \frac{\dot{x}^2}{\cos^2\alpha} + (a^2 + k^2)\,\dot{\alpha}^2 &= h^2 , \\ \dot{y} &= \dot{x}\tan\alpha . \end{aligned}$$

If one sets:

$$m^2 = \frac{k^2 + a^2}{a^2}$$

then when one assumes that $a = 0$, $x = \dot{x} = y = 0$ for $t = 0$, one will get:

([1]) If one had one used the non-holonomic relation (4.a) in order to eliminate $\dot{\eta}$ from (3) and applied the fundamental law to that form for T (with just two variables) then, from some basic laws of algebra, that elimination would be equivalent to the addition of $\lambda\,\delta\dfrac{d'\Phi}{dt}$, not of $\lambda\,\delta'\Phi$. However, that would have given a false result, because $\delta\,d'\Phi \neq d\,\delta'\Phi$ since $d'\Phi = 0$ for the *non*-holonomic equation of constraint.

$$h\,t = m^2 a \log\tan\left(\frac{\alpha}{2m}+\frac{\pi}{4}\right),$$

$$x = a\,m\int_0^a \tan\frac{\alpha}{m}\cos\alpha\,d\alpha,$$

$$y = a\,m\int_0^a \tan\frac{\alpha}{m}\sin\alpha\,d\alpha.$$

The accompanying figure, for which I have Herrn Dr. W. REIFF at Kirchheim u. T. to thank, gives the trajectory of the point A for the cases $a = 1$, $m = 4$ (the dashed line), $a = 1$, $m = 9$, and (in the second figure) $a = 1$, $m = 16$, where the latter is drawn in doubled units. It possesses all three asymptotes. The greater m becomes, the greater will be the number of windings about the origin.

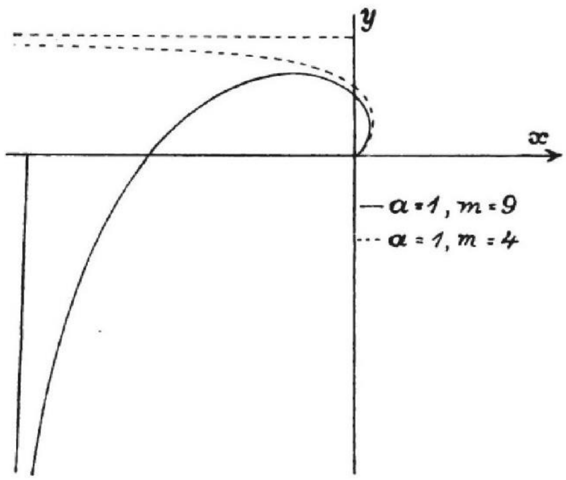

Fig. 7a.

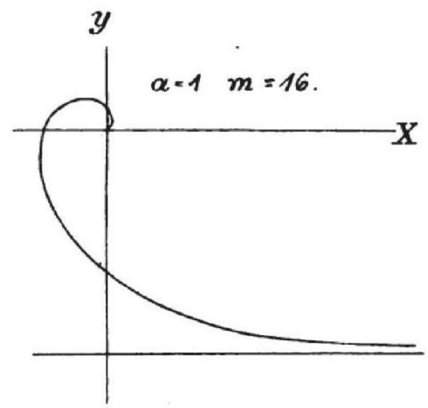

Fig. 7b.

10. – Guided systems. Forces of pressure.

A free system is composed of two parts, A and $\mathfrak{A}$, in a manner that will be given directly, and each of which contains one part of the total mass-points: A contains the points $m_1, m_2, \ldots, m_r$, and $\mathfrak{A}$ contains the points $\mathfrak{m}_1, \mathfrak{m}_2, \ldots, \mathfrak{m}_r$. The expression for the *vis viva*, when written in rectangular coordinates, will then divide into two parts, and the constraint equations will divide into three, namely, one for A alone, one for $\mathfrak{A}$ alone, and one for the two together. If one imagines that the integration of the equations of motion for all points of the free system has been achieved, but only the coordinates of the system $\mathfrak{A}$ are known as a function of time, and they have been substituted in the constraint equations, then the ones for $\mathfrak{A}$ alone will be fulfilled identically, the ones for A will remain unchanged, and the mixed ones will include time explicitly, in addition to the coordinates of the mass-points that appear in A. Of the differential equations of motion, after substituting those coordinates, the ones for the mass-system A can be integrated in their own right, but time will enter into part of the constraint equations. The ones for the system $\mathfrak{A}$ go to relations between the coordinates of the points of A, the multipliers λ, μ, …, and time t, which will be fulfilled identically when one substitutes the values that the first system yields for those values. They can then be eliminated. However, the equations for the mass-system A are implied immediately from $\delta \frac{m f^2}{2} = 0$, when one incorporates only the masses m_1, m_2, …, m_r in m, and admits only those constraint equations into which time enters *explicitly*, but time is *not* varied, as an independent variable. The appearance of time in the condition equations is by itself what makes the motion of the system A what HERTZ called a "guided" motion (so $\mathfrak{A}$ is the "guiding" system), which is distinguished from a free system.

Therefore, if the fundamental law is to also be applicable to a *guided system* by admitting constraint equations of the kind that were just mentioned then the *vis viva* of the guided system by itself will no longer be a constant, as it is for the free (i.e., total) system. Before we infer any further consequences, as an example, we combine the motion that was considered above (art. **9**) with two mass-points m, m_1 that are connected by a string and move inside of a tube that consists of a rectilinear part and a curved one, while it rotates around the straight part. The *vis viva* of the mass-points was [art. **1** (1)]:

$$T = \tfrac{1}{2}[(m+m_1)(\dot{r}^2+\dot{z}^2)+m r^2\,\dot{\vartheta}^2]\ , \tag{1}$$

and the relation:

$$r - F(z) = 0 \tag{2}$$

existed as the constraint equation for the curved part of the tube.

If we now make the assumption that along with the points m, m_1, which will be the system A, one has a system $\mathfrak{A}$ that forces the rotation around the Z-axis to proceed *in a prescribed way* (e.g., uniformly) then a new constraint equation will appear that includes time explicitly:

$$\vartheta - a\,t = 0\ , \tag{3}$$

where a is a constant. The fundamental law now gives:

$$\delta\frac{m f^2}{2} - \lambda(\delta\ddot{r} - F'(z)\,\delta\ddot{z}) - \mu\,\delta\ddot{\vartheta} = 0\ , \tag{4}$$

in which the previous expression [art. **9**, (3)] is substituted for $\delta\,\tfrac{1}{2} m f^2$. The multiplier λ in that means, as it did before, the reaction pressure of the tube on the mass m. μ will also be a vector that has the dimension of a force times a length, so an angular moment. When it acts upon the curved part of the tube, it makes the rotation uniform. Equations (2), (3), together with the three that are implied by (4) and the initial conditions, succeed in determining the quantities r, z, ϑ, λ, μ as functions of t.

However, one can also imagine that this system of equations gives only the quantities λ, μ as function of time and one then substitutes them in (4). Two of the five equations will then become superfluous. In particular, if one drops the constraint equations (2), (3) then the resistance of the tube will be replaced by a *force* or a *pressure* whose direction and magnitude change (as known functions of time) that produces the same effect as that resistance. One can also drop only one of the constraint equations – e.g., (3) – and replace it with the assumption that the quantity μ is a function of time, i.e., the motion of the regulating angular moment. Thus, e.g., $\mu = 0$ will give the previously-treated example (art. **9**).

More generally: For a system of mass-points, one might be led to the following system of LAAGRANGE equations of motion of the second kind by the formulation in art. **6**:

$$\begin{aligned} &-\frac{d}{dt}\frac{\partial T}{\partial \dot{p}_1}+\frac{\partial T}{\partial p_1}+\lambda\,\varphi_1+\mu\,\psi_1+\cdots=0\,,\\ &-\frac{d}{dt}\frac{\partial T}{\partial \dot{p}_1}+\frac{\partial T}{\partial p_1}+\lambda\,\varphi_1+\mu\,\psi_1+\cdots=0\,,\\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned} \tag{5}$$

together with the k constraint equations:

$$\begin{aligned} &d'\Phi \equiv \varphi_1\,dp_1+\varphi_2\,dp_2+\ldots+\varphi_r\,dp_r+\varphi\,dt=0\,,\\ &d'\Psi \equiv \psi_1\,dp_1+\psi_2\,dp_2+\ldots+\psi_r\,dp_r+\psi\,dt=0\,,\\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned} \tag{5.a}$$

which can now also contain time explicitly ([1]) since we do not exclude guiding systems. Let the integration be performed so that it implies certain functions of time for the multipliers λ, μ, …, and the quantities φ_i, ψ_i, φ, ψ in (5). If one imagines that they are known and substitutes them in (5) then the constraint equations (5.a) will become superfluous. Equations (5), by themselves, then represent the motion of a point-system that is influenced by a system of vectors whose components in the direction of the coordinate p_i are represented by $\lambda\,\varphi_i$, ($\mu\,\psi_i$, …, resp.). One can interpret those vectors, whose dimension is that of a force $[l\,m\,t^{-2}]$ when p_i is a length, since $[T] = [l^2\,m\,t^{-2}]$, as *forces of pressure* or resistance that the other systems exert upon the given one, and act in such a way that the constraint equations (5.a) will be fulfilled by themselves. If one now replaces the latter with the total pressure:

$$P_i = \lambda\,\varphi_i + \mu\,\psi_i + \ldots \qquad (i = 1, 2, \ldots, r)$$

in the direction of the coordinate p_i then the motion of the point-system will be defined by the system of equations:

$$-\frac{d}{dt}\frac{\partial T}{\partial \dot{p}_1}+\frac{\partial T}{\partial p_1}+P_1=0\,,$$

([1]) Under a variation, the term in equations (5.a) that is multiplied by dt will drop out after dividing by dt and repeated differentiation with respect to time (art. **8**).

$$-\frac{d}{dt}\frac{\partial T}{\partial \dot{p}_1}+\frac{\partial T}{\partial p_1}+P_1=0\,, \tag{6}$$

…………………………..

just as completely as it was by the simultaneous system (5), (5.a).

Von HELMHOLTZ (Jour. f. Math. **100**, pp. 145; *Wissenschaftliche Abhandlungen*, Bd. 3, pp. 212) called forces (i.e., force components) P_1, P_2, …, P_r that are given as functions of time **Lagrangian forces**. If some of the constraint equations (5.a) express the idea that the coordinates of the system considered coincide with some of the other ones then HERTZ said that the second system was *coupled* with the first one. *The action of one of them on the other* is represented by *forces P* in this case, as well, that are either known as functions of time, as we have assumed up to now, or their magnitudes and directions can be determined in an indirect way, as well will see later (art. **14**).

For each individual system, the forces P are "external" forces; for the total system, they are "internal."

HERTZ had developed the theory of the (pressure) forces thoroughly in arts. 450 to 493 of his *Mechanik*, namely, he also proved the equality of force and reaction force (i.e., action and reaction).

11. – Example of a guided system.

One has von HELMHOLTZ to thank for a remarkable example. He proved that a reciprocity exists for the BOHNENBERGER top, which is a body of rotation with a CARDAN suspension that consists of certain pressures that are exerted on the rings of the frame ("Über das Prinzip der kleinsten Wirkung," Jour. f. Math. **100**, pp. 163; *Wissenschaftliche Abh.*, Bd. 3, pp. 222).

Let L be a fixed circular ring in a vertical plane. The ring M rotates with respect to L about a vertical diameter $Z'O$ of L and defines an angle of φ with it at time t. M carries a horizontal axis OV around which a third ring N rotates. It defines an angle of ϑ with the plane of M. In the plane of N and perpendicular to OV, lies the axis OZ on which a massive circular disc with its center of mass at O is fixed. Let the axes OX, OY in the middle plane of the disc be perpendicular to each other and to the axis OZ of the disc. If one assumes that $OU \perp OV$ in the plane XOY then that plane will be intersected by the plane ZOZ' along OU, since OV is, at the same time, perpendicular to OZ and OZ', and indeed $\sphericalangle\,\vartheta = (OZ, OZ')$. Let the angle between OU and OX be ψ. The angles φ, ψ, ϑ are nothing but the "EULER angles" then. The system is found to be in motion at time t, and let $\dot{\varphi}$, $\dot{\vartheta}$, $\dot{\psi}$ be the angular velocities of the rotations

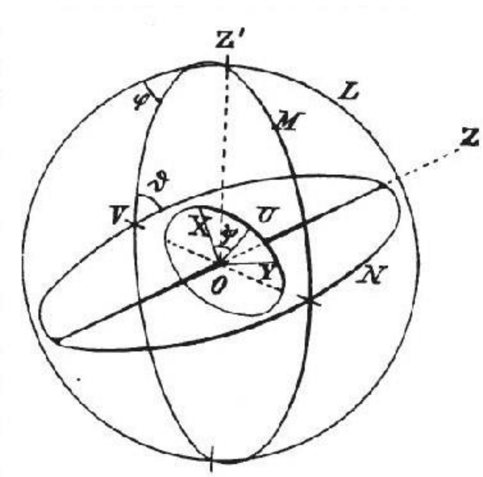

Fig. 8.

around the axes *OZ′*, *OV*, *OZ*, resp. We must next calculate the components *p*, *q*, *r* of the resultant that fall in the directions of the *X*, *Y*, *Z*-axes, resp., from $\dot{\varphi}$, $\dot{\vartheta}$, $\dot{\psi}$.

One projects the component $\dot{\varphi}$ onto the directions *OZ* and *OU* in the plane *ZOZ′* and then obtains $\dot{\varphi}\cos\vartheta$, $\dot{\varphi}\sin\vartheta$, resp. When the component $\dot{\varphi}\sin\vartheta$ is projected into the directions *OX*, *OY*, that will give $\dot{\varphi}\sin\vartheta\cos\psi$, $\dot{\varphi}\sin\vartheta\sin\psi$, resp. Finally, $\dot{\vartheta}$ projects onto the components $\dot{\vartheta}\sin\psi$, $\dot{\vartheta}\cos\psi$ along the directions *OX*, *OY*, resp. One will then have the following relations:

$$\begin{aligned} p &= \dot{\varphi}\sin\vartheta\cos\psi - \dot{\vartheta}\sin\psi, \\ q &= \dot{\varphi}\sin\vartheta\sin\psi + \dot{\vartheta}\cos\psi, \\ r &= \dot{\varphi}\cos\vartheta + \dot{\psi}. \end{aligned} \tag{7}$$

With the use of the expression that we found in art. **5** (at the end) for the *vis viva T* of the rotating motion of a about its center of mass, and if *A*, *B*, *C* are the principal moments of inertia and *A* = *B*, then we will then get the *vis viva* of the rotating disc, as represented in terms of the EULER angles, in the form:

$$T = \tfrac{1}{2}(Ap^2 + Bq^2 + Cr^2) = \tfrac{1}{2}[A\dot{\vartheta}^2 + A(\dot{\varphi}\sin\vartheta)^2 + C(\dot{\varphi}\cos\vartheta + \dot{\psi})^2]. \tag{8}$$

We would now like to assume that the rings *M* and *N* rotate around their axes *OZ′* and *OV* in a prescribed way, such that when φ, ϑ are given functions of time, the following constraint equations will exist:

$$\begin{aligned} \Phi &\equiv \varphi - \varphi(t) = 0, \\ \Theta &\equiv \vartheta - \vartheta(t) = 0. \end{aligned} \tag{9}$$

Our system will then be a guided one, and when we neglect the masses of the rings *M* and *N* and the axis *OZ*, the fundamental law will imply that:

$$\delta\tfrac{1}{2}m f^2 - \lambda\,\delta\ddot{\Phi} - \mu\,\delta\ddot{\Theta} = 0,$$

which will imply the equations of motion [art. **8**, (2), (3)]:

$$\begin{aligned} \lambda &= \frac{d}{dt}[A\dot{\varphi}\sin^2\vartheta + C\cos\vartheta(\dot{\varphi}\cos\vartheta + \dot{\psi})], \\ 0 &= \frac{d}{dt}C(\dot{\varphi}\cos\vartheta + \dot{\psi}), \\ \mu &= \frac{d}{dt}(A\dot{\vartheta}) - A\dot{\varphi}^2\sin\vartheta\cos\vartheta + C\sin\vartheta\,\dot{\varphi}(\dot{\varphi}\cos\vartheta + \dot{\psi}). \end{aligned} \tag{10}$$

In this example, the multipliers λ, μ also have the meaning of moments, i.e., pressure forces at a distance of 1 from the rotational axis. One can succeed in determining either ψ, λ, or μ once one has introduced φ, ϑ into equations (10) as functions of time, as in (9). However, one can also regard the system as a coupled one when one assumes that λ, μ are given functions of time or the angles, after dropping equations (9), and determine φ, ψ, ϑ from (10) and the initial conditions.

The second equation in (10) will then give:

$$\dot{\varphi}\cos\vartheta + \dot{\psi} = \beta,$$

in which β is an integration constant, and will then have the other two equations:

$$\lambda = A\ddot{\varphi}\sin^2\vartheta + 2A\dot{\varphi}\sin\vartheta\cos\vartheta\,\dot{\vartheta} - C\,\beta\sin\vartheta\,\dot{\vartheta},$$
$$\mu = A\ddot{\vartheta} - A\dot{\varphi}^2\sin\vartheta\cos\vartheta + C\,\beta\,\dot{\varphi}\sin\vartheta\ .$$

If one partially differentiates the first one with respect to $\dot{\vartheta}$ and the second one with respect to $\dot{\varphi}$ then that will give:

$$\frac{\partial\lambda}{\partial\dot{\vartheta}} = A\dot{\varphi}\sin 2\vartheta - C\beta\sin\vartheta = -\frac{\partial\mu}{\partial\dot{\varphi}}.$$

Hence, if $\partial\lambda$, $\partial\mu$, $\partial\dot{\vartheta}$ are positive then $\partial\dot{\varphi}$ will be negative, i.e., if the axis of the top moves to a greater distance from the vertical plane (so $\dot{\vartheta}$ will be positive) as a result of an increase in the pressure λ (which will increase the velocity of precession $\dot{\varphi}$) then, conversely, the velocity $\dot{\varphi}$ of precession must be reduced by an increase in the pressure μ (which moves the axis away from the vertical). It is very gratifying to confirm that result by experiment with a model of the BOHNENBERGER top.

12. – Discontinuous motion and impulsive forces.

Just as pressure forces can enter as known functions of time in place of equations of constraint or coupled systems, discontinuously-acting equations of constraint can be replaced with *impacts* (i.e., *impulses*). Namely, if a sudden change of velocity is described by the equations that were posed in art. **7**:

$$\frac{\partial T}{\partial\dot{p}_k} - \frac{\partial T^{(0)}}{\partial\dot{p}_k^{(0)}} = \lambda\,\Phi_k + \mu\,\Psi_k + \ldots = P_k\,, \tag{1}$$

in which T, $T^{(0)}$ are the *vis vivas* for the system before and after the discontinuity in the path velocity occurred, resp., then one can once more regard the vector components $\lambda\,\Phi_k$, $\mu\,\Psi_k$, …, which cause the sudden changes in velocity that are given on the left-hand side, as known quantities that combine into the quantities P_k. P_k then represents an "force of impact" (i.e., impulse components) that acts in the direction of the coordinate p_k, and its dimension is equal to $[l\,m\,t^{-1}]$ when p_k is a linear quantity, so it is a velocity times a mass.

In rectangular coordinates, the equations above will read:

$$\begin{aligned} m_i\,(u_i - u_i^{(0)}) &= X_i\,, \\ m_i\,(v_i - v_i^{(0)}) &= Y_i\,, \qquad (i = 1, 2, \ldots, n) \qquad (2) \\ m_i\,(w_i - w_i^{(0)}) &= Z_i\,, \end{aligned}$$

when (u_i, v_i, w_i), $(u_i^{(0)}, v_i^{(0)}, w_i^{(0)})$ are the velocities of the point with mass m_i before and after the impact, resp., and (X_i, Y_i, Z_i) is the force of impact that acts upon it. When combined into one equation, it will read:

$$\sum[(m_i(u_i - u_i^{(0)} - X_i)\,\delta(u_i - u_i^{(0)}) + (m_i(v_i - v_i^{(0)} - Y_i)\,\delta(v_i - v_i^{(0)}) + (m_i(w_i - w_i^{(0)} - Z_i)\,\delta(w_i - w_i^{(0)})] = 0\,. \qquad (3)$$

For example, if a force of impact whose moment has the components L, M, N acts upon a rigid point-system that can rotate about its center of mass, and if the system was at rest before the impact, so $u_i^{(0)} = v_i^{(0)} = w_i^{(0)} = 0$, then if p, q, r are the components of the angular velocity that is produced, from the fact that:

$$u = z\,q - y\,r\,, \qquad \text{etc.,}$$
$$\delta u = z\;\delta q - y\;\delta r\,, \qquad \text{etc.,}$$

so (3) goes to:

$$\sum\,(m_i\,(z_i\,q - y_i\,r) - X_i)\,(z_i\;\delta q - y_i\;\delta r) + \ldots = 0\,,$$

or, in the notation of art. **5**, when the axes are the principal axes of inertia of a coordinate system that is fixed in the point-system, so $\sum m_i\,y_i\,z_i = 0$, etc., (art. **5**) will imply that:

$$A\,p\;\delta p + B\,q\;\delta q + C\,r\;\delta r - (Y_i\,z_i - Z_i\,y_i)\;\delta p - \ldots = 0\,,$$

$$(A\,p - L)\;\delta p + (Bq - M)\;\delta q + (Cr - N)\;\delta r = 0\,,$$

which will yield the angular velocities that are produced by the force of impact:

$$A\,p = L\,, \qquad B\,q = M\,, \qquad C\,r = N\,.$$

13. – Momenta (impulse coordinates).

The partial differential quotients of the kinetic energy T with respect to the velocity components $\dot{p}_i$ appear in the second form of the LAGRANGE equations (art. **8**). One refers to those quantities:

$$q_i = \frac{\partial T}{\partial \dot{p}_i}$$

as the *momenta* of the system relative to the coordinates p_i (*impulse coordinates*, according to KLEIN and SOMMERFELD, *Theorie des Kreisels*), by analogy with the same constructions for rectangular coordinates, when they have the values $m_i\,\dot{x}_i$, $m_i\,\dot{y}_i$, $m_i\,\dot{z}_i$. It is often advantageous for one to introduce them (all or in part) in place of the $\dot{p}_i$. Just as the coordinates p_i appear in the coefficients a_{ik} of:

$$T = \tfrac{1}{2}\sum\sum a_{ik}\,\dot{p}_i\,\dot{p}_k\,, \tag{1}$$

they can also occur in the b_{ik} of the transformed expression:

$$T = T_1 = \tfrac{1}{2}\sum\sum b_{ik}\,q_i\,q_k\,. \tag{2}$$

The introduction of the q_i results with the help of the equations:

$$q_i = \frac{\partial T}{\partial \dot{p}_i} = a_{1i}\,\dot{p}_1 + a_{2i}\,\dot{p}_2 + \cdots + a_{ri}\,\dot{p}_r = \sum_k a_{ki}\,\dot{p}_k\,. \tag{3}$$

Due to the homogeneity of T relative to the $\dot{p}$, one has:

$$2T = q_1\,\dot{p}_1 + q_2\,\dot{p}_2 + \cdots + a_r\,\dot{p}_r = \sum q_i\,\dot{p}_i\,, \tag{4}$$

from which one gets:

$$2\,dT = \sum q_i\,d\dot{p}_i + \sum \dot{p}_i\,dq_i = \sum \frac{\partial T}{\partial \dot{p}_i}\,d\dot{p}_i + \sum \dot{p}_i\,dq_i\,,$$

by differentiation. On the other hand, differentiating (1) will give:

$$dT = \sum \frac{\partial T}{\partial \dot{p}_i}\,d\dot{p}_i + \sum \frac{\partial T}{\partial p_i}\,dp_i\,.$$

If one subtracts that equation from the previous one then one will get:

$$dT = \sum \dot{p}_i \, dq_i - \sum \frac{\partial T}{\partial p_i} dp_i \,.$$

Finally, upon differentiating (2), one will get:

$$dT_1 = \sum \frac{\partial T_1}{\partial q_i} dq_i + \sum \frac{\partial T_1}{\partial p_i} dp_i \,.$$

From the fact that:

$$dT = dT_1 \,, \tag{5}$$

a comparison with the known equations (JACOBI, *Vorl. über Dynamik*, ed. by CLEBSCH, pp. 353) will yield:

$$\frac{\partial T_1}{\partial q_i} = \dot{p}_i \,, \qquad \frac{\partial T_1}{\partial p_i} = -\frac{\partial T}{\partial p_i} \,. \tag{6}$$

Thus, when the momenta q are introduced into T in place of the velocities $\dot{p}$, along with the r variables p, the partial differential quotients of T with respect to the coordinates themselves – viz., the p – will take the *opposite* sign. That just goes back to the fact that the p enter into the coefficients of the transformation formulas.

The top again gives an example for the introduction of the momenta. If one sets:

$$\frac{\partial T}{\partial \dot{\varphi}} = \Phi \,, \qquad \frac{\partial T}{\partial \dot{\vartheta}} = \Theta \,, \qquad \frac{\partial T}{\partial \dot{\psi}} = \Psi$$

in (8) of art **11** (where Φ, Θ have a different meaning from the one that they had in art. **11**) and solves those equations for $\dot{\varphi}$, $\dot{\psi}$, $\dot{\vartheta}$ then one will get:

$$\dot{\varphi} = \frac{1}{A \sin^2 \vartheta}[\Phi - \Psi \cos \vartheta] \,, \quad \dot{\vartheta} = \frac{\Theta}{A} \,, \qquad \dot{\psi} = \frac{\Psi}{C} \,,$$

and therefore:

$$T_1 = \tfrac{1}{2}\left[\frac{\Theta^2}{A} + \frac{(\Phi - \Psi \cos \vartheta)^2}{A \sin^2 \vartheta} + \frac{\Psi^2}{C}\right] ,$$

from which one easily confirms that:

$$\frac{\partial T}{\partial \vartheta} = -\frac{\partial T_1}{\partial \vartheta} \,.$$

One makes use of the introduction of momenta in the theory of "cyclic systems," to which we will now turn.

14. – Cyclic systems and forces-at-a-distance.

When one of the coordinates p_i itself does not appear in the expression T for the kinetic energy, but only by way of its differential quotients with respect to time $\dot{p}_i$, then, with HERTZ (art. 546), we will call it a *cyclic coordinate*. In the first example of art. **9**, z and ϑ are cyclic coordinates, while φ and ψ are cyclic in the second one (the terminology "cyclic" goes back to von HELMHOLTZ, Jour. f. Math., Bds. 92, 100). If the *cyclic velocities* or "intensities" in a system, i.e., the changes in velocity of the cyclic coordinates are not large compared to the cyclic coordinates, such that the kinetic energy of the system is a homogeneous quadratic function of just the cyclic velocities to a sufficient degree of approximation then it will be called a cyclic system and its motion a cyclic motion. We shall often denote the cyclic coordinates of a system by German symbols $\mathfrak{p}_1, \mathfrak{p}_2, \ldots, \mathfrak{p}_\mathfrak{r}$, and the non-cyclic ones (which are called "slowly-varying parameters" by von HELMHOLTZ), by $p_1, p_2, \ldots, p_r$. A cyclic system is then defined by the assumption:

$$T = \tfrac{1}{2}\sum\sum \mathfrak{a}_{ik}\,\dot{\mathfrak{p}}_i\,\dot{\mathfrak{p}}_k \qquad \text{(approximately)}\,,$$

$$\frac{\partial T}{\partial \mathfrak{p}_k} = 0\,, \qquad \frac{\partial T_1}{\partial \mathfrak{p}_k} = 0\,, \qquad \frac{\partial T_1}{\partial q_k} = \dot{p}_k = 0\,, \qquad \text{(approximately)}\,,$$

when T_1 is the expression for the kinetic energy, as written in terms of the momenta $q_1, q_2, \ldots, q_r$, $\mathfrak{p}_1, \mathfrak{p}_2, \ldots, \mathfrak{p}_\mathfrak{r}$ (instead of the $\dot{p}$, $\dot{\mathfrak{p}}$). In particular, when $\mathfrak{r} = 1, 2$, the system will be called monocyclic (dicyclic, resp.). The first example in art. **9** will be monocyclic when $m\,r\,\dot{\vartheta}^2$ is very large compared to $(m+m_1)\,\dot{s}^2$, and so is the motion of the BOHNENBERGER top shortly after the resulting impetus, so when the angular velocity $\dot{\psi}$ of the disc is very large compared to that $\dot{\varphi}$, $\dot{\vartheta}$ of the rings. We will learn about dicyclic motions later (art. **34**).

If we have held the cyclic velocities constant in some way (e.g., by systems that are coupled with the cyclic ones) then:

$$\dot{\mathfrak{p}}_k = \text{const.} \qquad (i = 1, 2, \ldots, \mathfrak{r})$$

then the system will be called *isocyclic*. By contrast, if no external influences act upon it, so the derivatives of the cyclic moments with respect to time vanish in the LAGRANGE equations for the cyclic subsystem:

$$\frac{d\mathfrak{q}_k}{dt} = \frac{d}{dt}\frac{\partial T}{\partial \dot{\mathfrak{p}}_k} = 0 \qquad (k = 1, 2, \ldots, \mathfrak{r})\,,$$

then, with HERTZ, we call the cyclic system *adiabatic*. The monocyclic motion of the top that was defined above is adiabatic, even when the forces λ, μ are non-zero.

Now, in HERTZ's *Mechanik*, the coupling of an unknown cyclic system with a system of known masses was employed in order to explains the *forces-at-a-distance* that acted upon the latter. A system is composed of a part $\mathfrak{A}$ whose mass-points are in rapid cyclic motion and a part A whose mass-points change position relatively slowly. Let the kinetic energy of the cyclic and non-cyclic subsystems be:

$$\mathfrak{T} = \tfrac{1}{2}\sum\sum \mathfrak{a}_{ik}\, \dot{\mathfrak{p}}_i\, \dot{\mathfrak{p}}_k \qquad (i, k = 1, 2, \ldots, \mathfrak{r}),$$
$$T = \tfrac{1}{2}\sum\sum a_{ik}\, \dot{p}_i\, \dot{p}_k \qquad (i, k = 1, 2, \ldots, r),$$

resp. The kinetic energy of the total system will then be:

$$\mathsf{T} = T + \mathfrak{T},$$

as long as terms with products of the $\dot{p}$ and $\dot{\mathfrak{p}}$ do not appear, which is an assumption that is fulfilled, e.g., in the case of rectangular coordinates. If the total system is free then $\mathsf{T} = h$ will be a constant.

By assumption, the coordinates $\mathfrak{p}_k$ do not appear in the coefficients a_{ik} and $\mathfrak{a}_{ik}$, but possibly the p_k. T and $\mathfrak{T}$ are then functions with the forms:

$$\mathfrak{T} = \mathfrak{T}((\dot{\mathfrak{p}}_k, p_k)), \qquad T = T((\dot{p}_k, p_k)),$$

in which only one representative of the $\mathfrak{r}$ (r, resp.) variables that appear is indicated, which is suggested by the double parentheses. Indeed, some of the coefficients a_{ik} in T might be so large that this function of the slowly-varying parameters p does not vanish in comparison to $\mathfrak{T}$ (which will happen, e.g., in the aforementioned first example in art. **9** when one assumes that $\dot{\vartheta}$ is large compared to $\dot{s}$ and likewise, that m_1 is large compared to m). We would now like to introduce the associated momenta:

$$\mathfrak{q}_k = \frac{\partial \mathfrak{T}}{\partial \dot{\mathfrak{p}}_k} \tag{1}$$

into $\mathfrak{T}$ in place of the $\dot{\mathfrak{p}}_k$, with which, $\mathfrak{T}$ might go to the function $\mathfrak{T}_1$:

$$\mathfrak{T}((\dot{\mathfrak{p}}_k, p_k)) = \mathfrak{T}_1((\mathfrak{q}_k, p_k)) \,. \tag{2}$$

The expression for the total energy will become:

$$\mathsf{T} = T((\dot{p}_k, p_k)) + \mathfrak{T}((\dot{\mathfrak{p}}_k, p_k)) = T((\dot{p}_k, p_k)) + \mathfrak{T}_1((\mathfrak{q}_k, p_k)) \,. \tag{3}$$

For the sake of simplicity, we would like to assume from the outset that the system is free, so no constraint equations will exist for it. The formulation of the fundamental law (arts. **6**, **8**) as:

$$\delta \frac{m f^2}{2} = 0 \tag{4}$$

or

$$\sum_k \left(\frac{d}{dt}\frac{\partial \mathsf{T}}{\partial \dot{p}_k} - \frac{\partial \mathsf{T}}{\partial p_k} \right) \delta \ddot{p}_k + \sum_k \frac{d}{dt}\frac{\partial \mathsf{T}}{\partial \dot{\mathfrak{p}}_k} \cdot \delta \ddot{\mathfrak{p}}_k = 0$$

will then imply the differential equations of motion in the form:

$$\frac{d}{dt}\frac{\partial \mathsf{T}}{\partial \dot{p}_k} = \frac{\partial T}{\partial p_k} + \frac{\partial \mathfrak{T}}{\partial p_k} \qquad (k = 1, 2, \ldots, r) \tag{5}$$

$$\frac{d}{dt}\frac{\partial \mathsf{T}}{\partial \dot{\mathfrak{p}}_k} = \frac{d\mathfrak{q}_k}{dt} = 0 \qquad (k = 1, 2, \ldots, \mathfrak{r})\ . \tag{6}$$

The cyclic part $\mathfrak{A}$ of the system then moves adiabatically. The cyclic momenta $\mathfrak{q}_k$ are constants. *However, the expression $\mathfrak{T}_1$ for the vis viva of that part will then go to a function of just the coordinates p_k* that we would like to denote by $-U$:

$$\mathfrak{T}_1\ ((\mathfrak{q}_k\ ,\ p_k)) = -\ U\ (p_1, p_2, \ldots, p_r)\ ,$$

and the total energy of the system will be:

$$\mathsf{T} = \tfrac{1}{2} \sum \sum a_{ik}\ \dot{p}_i\ \dot{p}_k\ -\ U\ (p_1, p_2, \ldots, p_r) = h\ . \tag{7}$$

In the equations of motion (5), which are all that remain, due to theorem that was proved in art. **13**, eq. (6) about the introduction of the momenta in place of the velocities, one must set:

$$\frac{\partial \mathfrak{T}}{\partial p_k} = -\ \frac{\partial \mathfrak{T}_1}{\partial p_k} = \frac{\partial U}{\partial p_k}\ . \tag{7.a}$$

It will then become:

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{p}_k} = \frac{\partial T}{\partial p_k} + \frac{\partial U}{\partial p_k} \qquad (k = 1, 2, \ldots, r)\ . \tag{8}$$

One will also get that system of equations when one has replaced the condition for the total system:

$$\delta \frac{m f^2}{2} = 0$$

which follows from the fundamental law, with the demand that:

$$\delta \frac{m f^2}{2} - \delta \ddot{U} = 0 \tag{9}$$

for the subsystem A, in which:

$$\delta \ddot{U} = \sum \frac{\partial U}{\partial p_k} \delta \ddot{p}_k \,, \tag{9.a}$$

which is a demand that one can put into the form:

$$\sum m_i \left[\left(\ddot{x}_i - \frac{X_i}{m_i} \right) \delta \ddot{x}_i + \left(\ddot{y}_i - \frac{Y_i}{m_i} \right) \delta \ddot{z}_i + \left(\ddot{z}_i - \frac{Z_i}{m_i} \right) \delta \ddot{z}_i \right] = 0 \tag{10}$$

for rectangular coordinates, in particular, in which X_i, Y_i, Z_i are the partial differential quotients of one and the same function:

$$U = \int \sum \left(X_i \, dx_i + Y_i \, dy_i + Z_i \, dz_i \right) \qquad (i = 1, 2, \ldots, n) \,, \tag{11}$$

which one will now call the *force function* or the *potential for the force* (X_i, Y_i, Z_i).

However, equation (10) is nothing else then than the expression for the *principle of least constraint*, as we will see in the following article for the case in which *forces* act upon the point m_i, so the force (X_i, Y_i, Z_i) acts upon the mass-point m_i .

For rectangular coordinates, the equation of the *vis viva* (7) will assume the form:

$$\mathsf{T} \equiv T - U = h \,, \tag{12}$$

in which:

$$T = \tfrac{1}{2} \sum m_i \, (\dot{x}_i^2 + \dot{y}_i^2 + \dot{z}_i^2) \,,$$

and U is inferred from (11).

15. – The principle of least constraint. D'ALEMBERT's principle. The energy equation.

With that formulation of the effect of the cyclic subsystem $\mathfrak{A}$, we have arrived at the viewpoint of ordinary mechanics. When coupled with the system A, $\mathfrak{A}$ will exhibit its kinetic energy $\mathfrak{T}$ in the form of the force function U. If one then imagines that *the cyclic subsystem is invisible* (i.e., its masses are hidden) then its existence can be recognized only by the influence that it exerts upon the motion of the visible subsystem *A* by the appearance of the force function *U*, as it is manifested by the formal conversion of the *vis viva* of the hidden cyclic system.

The negatively-taken force function, so the work done *against* the active forces, as the *potential energy*, together with the *vis viva* (the *kinetic energy*):

$$T + (-U) = h ,$$

defines the total energy of the point-system *A* (which is constant in the present case). The hidden system, whose kinetic energy provides the potential energy of the visible one, was an adiabatic cyclic one. In that case, one calls the visible system a *conservative* one, which is a terminology that one adopts for any system *on which only forces that possess a force function act.*

A conservative system is defined by, e.g., two mass-points that attract according to the law of gravity and can move in space without friction.

If one now assumes that *equations of constraint* also exist between the coordinates of the visible masses (into which time can enter explicitly) then terms of the form:

$$\lambda\, \varphi_k + \mu\, \psi_k + \ldots$$

will be added to the right-hand sides of equations (8) [art. **6**, at the end], and equation (11) of the previous article will also remain unchanged only in the case in which the constraint equations do not include time.

If one then has a system of mass-points $m_1, m_2, \ldots, m_n$ on which forces act that possess a force function $U(p_1, p_2, \ldots, p_r) = 0$, and LAGRANGIAN forces P_i in the directions of the coordinates p_i (art. **10**) are given as functions of time, and finally if equations of constraint exist:

$$\begin{aligned} d'\Phi &\equiv \varphi_1\, dp_1 + \varphi_2\, dp_2 + \ldots + \varphi_r\, dp_r + \varphi\, dt = 0 , \\ d'\Psi &\equiv \psi_1\, dp_1 + \psi_2\, dp_2 + \ldots + \psi_r\, dp_r + \psi\, dt = 0 , \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned} \tag{1}$$

(in which finite ones can enter their place), in which the $\varphi, \psi, \ldots$ are functions of the coordinates and time, *then the fundamental law will imply the statement* that for all values of the variations $\delta \ddot{p}$, the following equation will exist:

$$\delta \frac{m\, f^2}{2} - \delta\ddot{U} - \sum P_k\, \delta \ddot{p}_k - \lambda\, \delta' \ddot{\Phi} - \mu \delta' \ddot{\Psi} - \cdots = 0 , \tag{2}$$

or

$$\sum \left(\frac{d}{dt} \frac{\partial T}{\partial \dot{p}_k} - \frac{\partial T}{\partial p_k} - \frac{\partial U}{\partial p_k} - P_k - \lambda\, \varphi_k - \mu\, \psi_k - \cdots \right) \delta \ddot{p}_k = 0 \qquad (k = 1, 2, \ldots, r), \tag{3}$$

and in rectangular coordinates, when one combines the potential forces and the LAGRANGIAN forces into X_i, Y_i, Z_i :

$$\sum [(m_i\, \ddot{x}_i - X_i - \lambda\, \varphi_{xi} - \mu \psi_{xi} - \cdots)\, \delta \ddot{x}_i + (m_i\, \ddot{y}_i - Y_i - \lambda\, \varphi_{yi} - \mu \psi_{yi} - \cdots)\, \delta \ddot{y}_i$$
$$+ \; (m_i\, \ddot{z}_i - Z_i - \lambda\, \varphi_{zi} - \mu \psi_{zi} - \cdots)\, \delta \ddot{z}_i] = 0 \qquad (k = 1, 2, \ldots, r). \tag{4}$$

However, that is precisely the equation that GAUSS's *principle of least constraint* will give when constraint equations of the form:

$$\sum_i (\varphi_{xi}\, dx_i + \varphi_{yi}\, dy_i + \cdots) \; + \varphi\, dt = 0 \,,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \tag{5}$$

exist, and the forces (X_i, Y_i, Z_i) act upon the masses m_i. In order to show that, we recall some known concepts. It follows immediately from equation (4) that the coefficients of $\delta \ddot{x}_i$, $\delta \ddot{y}_i$, $\delta \ddot{z}_i$ must vanish individually. Therefore, the force components (X_i, Y_i, Z_i) differ from the components of the so-called *effective forces*, namely, $(m_i\, \ddot{x}_i, m_i\, \ddot{y}_i, m_i\, \ddot{z}_i)$, only by the quantity ($\lambda\, \varphi_{xi} + \mu\, \psi_{xi} + \ldots$, $\lambda\, \varphi_{yi} + \ldots$, $\lambda\, \varphi_{zi} + \ldots$) .

If one combines the latter quantities (divided by m_i) into a vector that one calls the *constraint* that is exerted by the constraints on the mass-point m_i then it can also be represented by components:

$$\frac{1}{m_i}(m_i\, \ddot{x}_i - X_i)\,, \qquad \frac{1}{m_i}(m_i\, \ddot{y}_i - Y_i)\,, \qquad \frac{1}{m_i}(m_i\, \ddot{z}_i - Z_i)$$

of the acceleration that the point m_i will possess after eliminating the equations of constraint. If one once more defines a mean value (cf. art. **3**) when one adds the squares of the constraint forces that act upon the mass-points, multiplied by those masses, then we would like to understand the *constraint Z of the total system* to mean the quantity that is defined by the following quantity:

$$m\, Z^2 \;=\; \sum m_i \left[\left(\ddot{x}_i - \frac{X_i}{m_i} \right)^2 + \left(\ddot{y}_i - \frac{Y_i}{m_i} \right)^2 + \left(\ddot{z}_i - \frac{Z_i}{m_i} \right)^2 \right] . \tag{6}$$

Now, the fundamental law that GAUSS proposed says that the natural motion of a system will result in such a way that the system constraint assumes a smallest value, so:

$$\delta \frac{m\, Z^2}{2} \;= 0 =\; \sum m_i \left[\left(\ddot{x}_i - \frac{X_i}{m_i} \right) \delta \ddot{x}_i + \left(\ddot{y}_i - \frac{Y_i}{m_i} \right) \delta \ddot{y}_i + \left(\ddot{z}_i - \frac{Z_i}{m_i} \right) \delta \ddot{z}_i \right] = 0 \,. \tag{7}$$

However, that demand, together with the constraint equations for the $\delta \ddot{x}_i$, … [as in art. **6** (3.a)] that are derived from (5) give just equation (4).

As a result, we would like to include not merely the LAGRANGIAN forces (art. **10**), but also the forces that are determined by their force function – in particular, the forces-at-a-distance – in formula (7), without looking for an interpretation for the hidden masses. Indeed, we shall employ *formula* (7), *which expresses GAUSS's principle of least constraint*, as the foundation for the dynamics and statics of a point-system, as well as the processes of motion in space-filling masses, by *introducing* into (7), in the one case, the LAGRANGIAN and potential forces, and in the other, the internal (elastic) forces, *by way of their potential* that is inferred by observation.

If we then assume the usual mechanical viewpoint then we would like to connect up with the relations in it in such a way that we replace the completely-arbitrary increases in the accelerations:

$$\delta \ddot{p}_i\,,\ \delta \ddot{x}_i\,,\ \delta \ddot{y}_i\,,\ \delta \ddot{z}_i$$

in formulas (3), (4) with the corresponding likewise-arbitrary coordinate increases:

$$\delta p_i\,,\ \delta x_i\,,\ \delta y_i\,,\ \delta z_i\,.$$

It will then assume the known form of D'ALEMBERT's principle, which one should probably place at the pinnacle of the dynamics of point-systems [1], namely, in the form:

$$\sum \left(\frac{d}{dt}\frac{\partial T}{\partial \dot{p}_k} - \frac{\partial T}{\partial p_k} - \frac{\partial U}{\partial p_k} - \lambda\,\varphi_k - \mu\,\psi_k - \cdots \right) \delta p_k = 0 \qquad (k = 1, 2, \ldots, r), \tag{8}$$

or in rectangular coordinates:

$$\begin{aligned} \sum [& (m_i\,\ddot{x}_i - X_i - \lambda\,\varphi_{xi} - \mu\,\psi_{xi} \cdots)\,\delta x_i \\ & + (m_i\,\ddot{y}_i - Y_i - \lambda\,\varphi_{yi} - \mu\,\psi_{yi} \cdots)\,\delta y_i + \qquad (i = 1, 2, \ldots, n)\,, \\ & + (m_i\,\ddot{z}_i - Z_i - \lambda\,\varphi_{zi} - \mu\,\psi_{zi} \cdots)\,\delta z_i] = 0\,, \end{aligned} \tag{9}$$

with the constraint equations:

$$\begin{aligned} &\varphi_1\,dp_1 + \varphi_2\,dp_2 + \ldots + \varphi_r\,dp_r + \varphi\,dt = 0\,, \\ &\psi_1\,dp_1 + \psi_2\,dp_2 + \ldots + \psi_r\,dp_r + \psi\,dt = 0\,, \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned} \tag{10}$$

or

$$\begin{aligned} &\sum (\varphi_{xi}\,dx_i + \varphi_{yi}\,dy_i + \varphi_{zi}\,dz_i) + \varphi\,dt = 0\,, \\ &\sum (\psi_{xi}\,dx_i + \ldots \qquad\qquad\qquad) + \psi\,dt = 0\,, \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \end{aligned} \tag{11}$$

respectively. As usual, we then call the quantities:

$$P_k\,\delta p_k\,, \qquad\qquad \frac{\partial U}{\partial p_k}\,\delta p_k$$

[1] We have already pointed out the advantage of the path that we have chosen here.

the *elementary work* that the force P_k (the force that is define by U) does *along the virtual* (i.e., imaginary and compatible with the constraints) *path* δp_k . In rectangular coordinates:

$$X_i\,\delta x_k + Y_i\,\delta y_k + Z_i\,\delta z_k$$

is the *elementary work done by the force* (X_i, Y_i, Z_i) *on the mass-point* m_i *along the virtual path* δs_i, and accordingly call any of the integrals:

$$\int (P_1\,dp_1 + P_2\,dp_2 + \cdots + P_r\,dp_r) \ , \tag{12}$$

$$U = \int \left(\frac{\partial U}{\partial p_1} dp_1 + \cdots \frac{\partial U}{\partial p_r} dp_r \right) = \int \sum_i \left(\frac{\partial U}{\partial x_i} dx_i + \frac{\partial U}{\partial y_i} dy_i + \frac{\partial U}{\partial z_i} dz_i \right) ,$$

$$\int \sum (X_i\,dx_i + Y_i\,dy_i + Z_i\,dz_i) \tag{13}$$

the total work done by those forces along a path that the system describes from an initial position (which is generally not defined more precisely) to a location whose coordinates are given by p, x, y, z. We shall return to that later (art. **29**), and here we shall establish only that the work done can have positive or negative values as a scalar quantity.

As a result, we shall be, for the most part, clothing the problem in the form of D'ALEMBERT's principle, and only when we are dealing with fundamentally new formulations, as in the case of fluid masses, will we revert to the principle of least constraint. Even then (arts. **18**, **19**), we will refer to the quantities:

$$P_k\,\delta \ddot{p}_k \ , \qquad \frac{\partial U}{\partial p_k}\delta \ddot{p}_k \ , \qquad X_i\,\delta \ddot{x}_i + Y_i\,\delta \ddot{y}_i + Z_i\,\delta \ddot{z}_i \ ,$$

etc., as "virtual works" done by the forces P_k , etc., along the path $\delta \ddot{p}_k$ ($\delta \ddot{s}_i$, resp.), just as when the two dots over the symbols p, x, y, z are missing.

If one introduces *the actual displacements* dx_i, dy_i, dz_i *that occur* into the expression for the principle of least constraint (4) in place of the virtual accelerations $\delta \ddot{x}_i$, $\delta \ddot{y}_i$, $\delta \ddot{z}_i$, or when one introduces them in place of the virtual displacements δx_i, δy_i, δz_i in the equation for D'ALEMBERT's principle (9), then as a result of the equations of constraint (11), that will give:

$$\sum m_i (\ddot{x}_i\,dx_i + \ddot{y}_i\,dy_i + \ddot{z}_i\,dz_i) - d'U + (\lambda\,\varphi + \mu\,\psi + \ldots)\,dt = 0 \ , \tag{14}$$

or

$$dT - d'U + (\lambda\,\varphi + \mu\,\psi + \ldots)\,dt = 0 \ ,$$

in which:

$$d'U = \sum (X_i\,dx_i + Y_i\,dy_i + Z_i\,dz_i) \ ,$$

and dT is the increment in the kinetic energy. Now when $d\,U$ is a complete differential, namely, the increase in potential energy, and when the equations of constraint do not include time, so $\varphi = \psi = \ldots = 0$, then (14) will once more go *to the equations for the vis viva – i.e., the "energy equation"* [(12) of the previous article]:

$$T - U = h\,,$$

where h is a constant.

The energy equation is true for any system that is "free" in the sense of HERTZ's fundamental law, so more briefly, the ones upon which no force act, because it formulates just the demand of the fundamental law. However, *internal* forces can act between the mass-points of a free system whose work done, taken negatively ($- U$), will then appear as "potential energy" along with the kinetic. According to the type of force, its contribution to the total energy will be referred to as a special *form of energy*. Just as one speaks of kinetic energy or the energy of gravitation, there is an (internal) elastic energy that is due to the forces in a space-filling medium, such as electromagnetic energy, heat, chemical energy, etc. We shall encounter several of those forms of energy later on, and first of all, we shall deal with the heat that is developed by overcoming the frictional resistance, and which will then be included in the calculations as either the work done by the force of friction or as the mechanical equivalent of the heat developed (JOULE first performed experiments on the conversion of the work done by gravitation into heat in 1850).

However, the sum of all forms of energy that appear in a free system, when each is measured in the same unit, will always be a constant quantity. Temporal changes can occur only in such a way that some forms will increase at the expense of others. Thus, a sliding motion will slow down at the expense of heating of the frictional surface. A bolt gets its elasticity at the expense of the tension in an elastic string or a rotating cable. The distribution of electricity in an electric machine results from the work done on it by rotation, and its recombination will produce light and heat, etc.

Our solar system is very nearly a free system. However, even on Earth, one can delimit such a system to some extent and therefore convince oneself of the validity of the law of conservation of energy. That is how ROBERT MAYER arrived at the formulation of that universal law of nature in 1842.

16. – Examples. Frictional resistance.

The transition from the kinetic energy of hidden masses to the potential energy of visible ones might once again be carried out in the example (art. **9**) of the motion of two mass-points in a tube that are connected by an inextensible string.

In art. **9**, we found that the total energy T of the system is:

$$\mathsf{T} = \tfrac{1}{2}\{(m+m_1)\dot{s}^2 + m r^2 \dot{\vartheta}^2\}\,,$$

in which:

$$\dot{s}^2 = \dot{r}^2 + \dot{z}^2\,.$$

The coordinate ϑ is a cyclic one, and the motion of the mass-point that is found in the curved part of the tube will be an intrinsically cyclic one when one takes care to make $\dot{s}$ small compared to $r\,\dot{\vartheta}^2$, which is certainly possible at the onset of the motion. That will be assumed. The motion will also be adiabatic then, due to the relation [(4), art. **9**]:

$$r\,\dot{\vartheta}^2 = \mathfrak{q}\,,$$

from which one infers $\dot{\vartheta}$ and can substitute it in T. If one now splits T into T and $\mathfrak{T}$, where the *vis viva*:

$$T = \tfrac{1}{2} m \dot{s}^2$$

is associated with the subsystem that consists of m_1, and:

$$\mathfrak{T} = \tfrac{1}{2} m \dot{s}^2 + \tfrac{1}{2} m r^2 \dot{\vartheta}^2 = \tfrac{1}{2} m r^2 \dot{\vartheta}^2 \qquad \text{(approximately)}$$

is associated with the cyclic subsystem that consists of m, and replaces $\dot{\vartheta}$ with the constant $\mathfrak{q}$, by which $\mathfrak{T}$ will go to:

$$\mathfrak{T} = \mathfrak{T}_1 = \tfrac{1}{2} \frac{m \mathfrak{q}^2}{r^2} = -\,U\,,$$

and finally assumes that the mass m_1 is considerably larger than the mass m then it will follow from:

$$T + \mathfrak{T} = h$$

that $\dot{s}$ will also remain small compared to $r\,\dot{\vartheta}$ as long as r is not very large. If one now proposes that the mass m that is in a state of rapid cyclic motion, along with the tube and string, are invisible then the visible subsystem m_1 will not only satisfy the equations of constraint (4) that were presented in art. **9**, in which $m + m_1$ can be replaced with m_1, approximately, but also upon eliminating λ from equation (5) that emerges from them:

$$\frac{m_1}{\mathfrak{q}^2 m} \frac{d^2 s}{dt^2} = \frac{\psi'(s)}{\psi^3(s)} = -\,\tfrac{1}{2} \frac{d}{ds} \psi^{-2}(s)$$

[in which:

$$r = \psi(s)$$

is the equation of the curve $z = f(r)$], one will get an equation of the form:

$$\frac{m_1}{\mathfrak{q}^2 m} \ddot{z}_1 = F(z_1)\,,$$

in which $z_1 = s$ + constant is the distance from the mass-point m to a fixed point on the z-axis. However, in the view of ordinary mechanics, that is the equation of motion for a mass-point that moves under the influence of a force-at-a-distance:

$$\mathfrak{q}^2\, m\, F\,(z_1) = -\tfrac{1}{2}\mathfrak{q}^2 m \frac{d}{dz_1}\left[\frac{1}{\psi^2(z_i)}\right]$$

in the z_1-direction. If one imagines, e.g., that the tube has been bent in such a way that $\psi^2(z_1)$ is proportional to $k - z_1$, where k is a constant, then the motion of the mass m_1 will result in such a way that it is as if the mass were attracted to the point $z_1 = k$ in inverse proportion to the square of the distance.

With the assistance of four-dimensional space, one can reproduce the attraction to a fixed point in a plane, by analogy with the foregoing.

Later on, the motion of fluid masses will give us another example.

Even motions that result with *friction* can be included with that context when the work done by friction can be represented as a function of the coordinates or its differential quotients.

An example is the rectilinear motion of a homogeneous ball that rotates around a horizontal axis and has been carefully placed on a horizontal plane in such a way that it will partly slide and partly roll forwards. Let x be the distance from the center of the ball at time t to its initial position, let ϑ be the angle through which the ball has rotated from its initial position, let m be its mass, and let k be its radius of inertia. The *vis viva* of the ball at time t is then:

$$T = \tfrac{1}{2} m(\dot{x}^2 + k^2\,\dot{\vartheta}^2) \ .$$

We assume that friction is a force that is proportional to the weight mg (g = acceleration of gravity) of the ball and a factor μ (the coefficient of friction) and does an amount of elementary work along the path $\delta\xi$, on which sliding of the surface element of the point of contact between the ball of radius a and the base takes place. The measure of that path is given by:

$$\delta\xi = a\ \delta\vartheta - \delta x\ .$$

Therefore, *as long as* $a\vartheta > x$, i.e., as long as the ball simultaneously slides and rolls, the elementary work done by friction will be:

$$\mu\, mg\ \delta x = -\ \delta\,[mg\ \mu\,(a\ \vartheta - x)]\ ,$$

so the force function itself will be:

$$U = -\ mg\ \mu\,(a\ \vartheta - x)\ ,$$

and the equation of the *vis viva* $T - U = h$ will yield:

$$\tfrac{1}{2}m(\dot{x}^2 + k^2\,\dot{\vartheta}^2) + mg\,\mu\,(a\,\vartheta - x) = h\,.$$

The extended conception of the fundamental law:

$$\delta\frac{m\,f^2}{2} - \delta U = 0$$

yields

$$m\,\ddot{x}\,\delta\ddot{x} + k^2\,m\,\ddot{\vartheta}\,\delta\ddot{\vartheta} + mg\,\mu\,(a\,\delta\ddot{\vartheta} - \delta\ddot{x}) = 0\,,$$

or

$$\ddot{x} = g\,\mu\,,$$
$$k^2\,\ddot{\vartheta} = -\,a\,g\,\mu\,,$$

from which it will follow that:

$$x = g\,\mu\frac{t^2}{2}\,,\qquad k^2\vartheta = -\,a\,g\,\mu\frac{t^2}{2} + k^2\dot{\vartheta}_0\,t$$

when $x = 0$, $\dot{x} = 0$, $\vartheta = 0$, $\dot{\vartheta} = \dot{\vartheta}_0$ for $t = 0$, and:

$$a\,\vartheta - x = -\,\frac{a^2 + k^2}{k^2}\cdot\frac{\mu\,g\,t^2}{2} + a^2\,\dot{\vartheta}_0\,t\,.$$

The kinetic energy of the motion T will be progressively diminished by the work done by friction U until the sliding ceases, which will happen after an interval of time:

$$t = \frac{2a\,k^2\,\dot{\vartheta}^2}{\mu g\,(a^2 + k^2)}\,,$$

in which one has $a\,\dot{\vartheta} = \dot{x}$, and the motion will go to a pure rolling with uniform velocity.

We now consider a motion for which the direction of the resistance due to friction is opposite to the velocity of the moving point and is proportional to it in magnitude. An elementary work will then be done by the resisting force under a small displacement whose magnitude, as calculated for all system points, will be:

$$\delta R = -\sum k_i\,(\dot{x}_i\,\delta x_i + \dot{y}_i\,\delta y_i + \dot{z}_i\,\delta z_i)\qquad (i = 1, 2, \ldots, n)\,,$$

when expressed in rectangular coordinates, in which the k_i are constants. One will then get the fundamental equation in the form:

$$\delta\frac{m\,f^2}{2} - \delta U - \delta R = 0\,,$$

and the equation of the *vis viva* will be:

$$T - U - R = h\,,$$

when U is the force function, so the function:

$$R = -\int \tfrac{1}{2} \sum k_i\,(\dot{x}_i^2 + \dot{y}_i^2 + \dot{z}_i^2)\,dt$$

must then be a negative quantity, because R makes a positive, and indeed ever-increasing, contribution to the total energy h that is equivalent to a *dissipation*, or wasting, of energy. Lord RAYLEIGH, who introduced that function, or rather the function $\dot{R}$ [J. W. STRUTT, Proc. Lond. Math. Soc. **4** (1873)], called the latter the "dissipation function."

One transforms R and δR into general coordinates in a manner that is similar to the way that one transformed $\delta m\, f^2 / 2$ above. One first has:

$$\delta R = \sum_i \left(\frac{\partial \dot{R}}{\partial \dot{x}_i}\,\delta x_i + \frac{\partial \dot{R}}{\partial \dot{y}_i}\,\delta y_i + \frac{\partial \dot{R}}{\partial \dot{z}_i}\,\delta z_i \right) \qquad (i = 1, 2, \ldots, n),$$

where:

$$\dot{R} = -\tfrac{1}{2} \sum k_i\,(\dot{x}_i^2 + \dot{y}_i^2 + \dot{z}_i^2)\,.$$

A transformation of that into the coordinates $p_1, p_2, \ldots, p_r$ might yield:

$$\dot{R} = -\tfrac{1}{2} \sum \sum r_{ik}\,\dot{p}_i\,\dot{p}_k \qquad (i, k = 1, 2, \ldots, r)\,.$$

Therefore, due to [(9), art. **6**]:

$$\delta x_i = \sum \frac{\partial x_i}{\partial p_k}\,\delta p_k = \sum \frac{\partial \dot{x}_i}{\partial \dot{p}_k}\,\delta p_k\,,$$

$$\delta R = \sum_i \sum_k \left[\frac{\partial \dot{R}}{\partial \dot{x}_k}\frac{\partial \dot{x}_i}{\partial \dot{p}_k} + \frac{\partial \dot{R}}{\partial \dot{y}_k}\frac{\partial \dot{y}_i}{\partial \dot{p}_k} + \frac{\partial \dot{R}}{\partial \dot{z}_k}\frac{\partial \dot{z}_i}{\partial \dot{p}_k} \right] \delta p_k \qquad \begin{cases} i = 1, 2, \ldots, n, \\ k = 1, 2, \ldots, r \end{cases}$$

or

$$\delta R = \sum_k \frac{\partial \dot{R}}{\partial \dot{p}_k}\,\delta p_k = \sum_k \dot{R}_k\,\delta p_k\,.$$

Hence, the fundamental law once more implies the equation:

$$\sum \left[\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{p}_i} \right) - \frac{\partial T}{\partial p_i} - \frac{\partial U}{\partial p_i} - \dot{R}_i \right] \delta p_i\,,$$

in which one has:

$$\dot{R}_i = \frac{\partial \dot{R}}{\partial \dot{p}_i} .$$

We will encounter an application of the dissipation function in the chapter on elastic media.

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CHAPTER TWO

SPACE-FILLING MASSES, AND FLUIDS IN PARTICULAR

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§ 17. – Form of the constraint equations ([1]).

We shall now turn from discrete mass-points to the continuous masses that fill up an entire region of space continuously.

> "A continuous mass should not be represented as an infinite set of neighboring points, but in the spirit of the infinitesimal calculus, as something composed of infinitely-small elements with the same character as the mass itself."
>
> (LAGRANGE, *Mécanique analytique* I, sect. IV, § 2)

Thus, it consists of volume elements that are filled with mass. Instead of summing over mass-points, integrals over those elements will occur. Regardless of whether the mass is rigid, elastic, or fluid, the mass-element *dm* is an unvarying quantity. However, the volume element $d\tau$, which contains the mass *dm*, can change in magnitude and form over time, whereby the density ρ (mass per unit volume) will also change. Those three quantities are then coupled with each other by the equation:

$$dm = \rho \, d\tau ,$$

where ρ is a function of position and time.

By an abstraction that is similar to the one that led to the concept of a material point (see Introduction), one will arrive at the concept of lines (i.e., distributed along a line) of masses (e.g., strings, wires, chains) and the masses that are distributed on surfaces. The mass-elements for them are:

$$dm = \rho_s \, ds ,$$
$$dm = \rho_\sigma \, d\sigma ,$$

resp., in which *ds* is the length of the line element, $d\sigma$ is the area of the surface element, and the altered meaning of ρ will be carried in the calculation by the indices s, σ.

If a continuous mass is bounded then its behavior on the boundary must be given. If it is unbounded in one or more directions then its behavior at infinity must be given. The nature of a space-filling mass (or as we will also say, a *medium*) whose motion is under study will be defined by "interior" equations that refer to the possible changes in form and magnitude of the volume

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([1]) See the author's notes "Über die Mechanik von HERTZ" (1900) in Mitt. d. math.-nat. Ver. v. Württemberg and "Über ein Beispiel des Herrn BOLTZMANN zu der Mechanik von HERTZ" in Jahresber. der D. Math. Ver. **8** (1900).

(surface, line) elements that contain mass. As we will see, they take the form of partial differential equations with respect to the coordinates.

We relate a mass-element that has the rectangular coordinates x, y, z at time t to the position a, b, c that is possessed in the same system at time $t = 0$, such that the triple of values a, b, c will then take on the role of indices (which were previously applied to the mass-points). If one then knows x, y, z as functions of a, b, c, and time t :

$$x = x\,(a, b, c, t)\,, \qquad y = y\,(a, b, c, t)\,, \qquad z = z\,(a, b, c, t) \tag{1}$$

then the form and distribution, as well as the density and motion, of the mass will be known at every point in time. A point $a + da$, $b + db$, $c + dc$ that is close to a, b, c has the following coordinates at time t :

$$\begin{aligned} x + dx &= x + \frac{\partial x}{\partial a} da + \frac{\partial x}{\partial b} db + \frac{\partial x}{\partial c} dc\,, \\ y + dy &= y + \frac{\partial y}{\partial a} da + \frac{\partial y}{\partial b} db + \frac{\partial y}{\partial c} dc\,, \\ z + dz &= z + \frac{\partial z}{\partial a} da + \frac{\partial z}{\partial b} db + \frac{\partial z}{\partial c} dc\,. \end{aligned} \tag{2}$$

In particular, at the point a, b, c, in the directions of the coordinate axes to the neighboring point:

$$\begin{matrix} a + da, & b, & c\,, \\ a, & b + db, & c\,, \\ a, & b, & c + dc\,, \end{matrix}$$

the following coordinates will arise:

$$\begin{matrix} x + \dfrac{\partial x}{\partial a} da\,, & y + \dfrac{\partial y}{\partial a} da\,, & z + \dfrac{\partial z}{\partial a} da \\ x + \dfrac{\partial x}{\partial b} db\,, & y + \dfrac{\partial y}{\partial b} db\,, & z + \dfrac{\partial z}{\partial c} dc \\ x + \dfrac{\partial x}{\partial c} dc\,, & y + \dfrac{\partial y}{\partial c} dc\,, & z + \dfrac{\partial z}{\partial c} dc\,. \end{matrix}$$

That demand, which refers to the tetrahedron that is defined by those three axes and the point (x, y, z), is then expressed by an equation between x, y, z; a, b, c, and the first partial differential quotients of the x, y, z with respect to the a, b, c.

Examples:

1. The point (x, y, z) on a *string* has a distance a from the starting point at time $t = 0$. The position of the string is then determined by equations of the form:

$$x = x(a, t), \qquad y = y(a, t), \qquad z = z(a, t).$$

One can pose the requirement that the individual line-elements do not change in length. That will give the relation:

$$\left(\frac{\partial x}{\partial a}\right)^2 + \left(\frac{\partial y}{\partial a}\right)^2 + \left(\frac{\partial z}{\partial a}\right)^2 = 1 .$$

If, say, the principal curvature of the curve of the string is to remain unchanged then the further equation must be satisfied:

$$\left(\frac{\partial^2 x}{\partial a^2}\right)^2 + \left(\frac{\partial^2 y}{\partial a^2}\right)^2 + \left(\frac{\partial^2 z}{\partial a^2}\right)^2 = f(a) ,$$

in which f is a given function.

2. An infinitely-thin elastic *plate* that has a planar form at time $t = 0$ moves in space into a deformed state. Let a point that has the coordinates a, b relative to a coordinate system in the plane of the plate at time $t = 0$ be at the location x, y, z at time t. The position of the bent plate will then be known at any time when one knows the functions:

$$x = x(a, b; t), \qquad y = y(a, b; t), \qquad z = z(a, b; t).$$

One can demand, e.g., that the lengths of the sides of the rectangle in which the lines a = const., b = const. of the planar plate lie remain unchanged under the motion. One must then have:

$$\left(\frac{\partial x}{\partial a}\right)^2 + \left(\frac{\partial y}{\partial a}\right)^2 + \left(\frac{\partial z}{\partial a}\right)^2 = 1 .$$

$$\left(\frac{\partial x}{\partial b}\right)^2 + \left(\frac{\partial y}{\partial b}\right)^2 + \left(\frac{\partial z}{\partial b}\right)^2 = 1 .$$

Should the right angles in the rectangle also remain the same, then the further equation must exist:

$$\frac{\partial x}{\partial a}\frac{\partial x}{\partial b} + \frac{\partial y}{\partial a}\frac{\partial y}{\partial b} + \frac{\partial z}{\partial a}\frac{\partial z}{\partial b} = 0 .$$

As is known, the last three equations together express the condition for the planar plate to be bent without folding or stretching, so it will always remain a developable surface.

By contrast, should only the area of the individual elements remain the same under bending, then since twice the area of a triangle is known to be expressed by the determinant of the rectangular coordinates of the corner point and a series of ones, when one initially restricts oneself to a motion in the plane of the plate itself:

$$da\;db = \begin{vmatrix} x & x + \dfrac{\partial x}{\partial a}da & x + \dfrac{\partial x}{\partial b}db \\ y & y + \dfrac{\partial y}{\partial a}da & y + \dfrac{\partial y}{\partial b}db \\ 1 & 1 & 1 \end{vmatrix} = da\;db \begin{vmatrix} \dfrac{\partial x}{\partial a} & \dfrac{\partial x}{\partial b} \\ \dfrac{\partial y}{\partial a} & \dfrac{\partial y}{\partial b} \end{vmatrix} = da\;db\;\frac{\partial(x,y)}{\partial(a,b)},$$

go one must have:

$$\frac{\partial(x,y)}{\partial(a,b)} = 1\,,$$

in which the symbol on the left-hand side denotes the functional determinant of x, y with respect to a, b.

A surface that bends in space with unvarying areas of its surface elements fulfills an analogous requirement:

$$\left[\frac{\partial(y,z)}{\partial(a,b)}\right]^2 + \left[\frac{\partial(z,x)}{\partial(a,b)}\right]^2 + \left[\frac{\partial(x,y)}{\partial(a,b)}\right]^2 = 1\,.$$

3. Similarly, the volume of the space-element $d\tau$ that emerges from that of $d\tau_0 = da\;db\;dc$ is represented by the determinant:

$$dt = \begin{vmatrix} x & x + \dfrac{\partial x}{\partial a}da & x + \dfrac{\partial x}{\partial b}db & x + \dfrac{\partial x}{\partial c}dc \\ y & y + \dfrac{\partial y}{\partial a}da & y + \dfrac{\partial y}{\partial b}db & y + \dfrac{\partial y}{\partial c}dc \\ z & z + \dfrac{\partial z}{\partial a}da & z + \dfrac{\partial z}{\partial b}db & z + \dfrac{\partial z}{\partial c}dc \\ 1 & 1 & 1 & 1 \end{vmatrix} = R \cdot da\;db\;dc\,,$$

where R is the functional determinant:

$$R = \frac{\partial(x,y,z)}{\partial(a,b,c)} = \begin{vmatrix} \dfrac{\partial x}{\partial a} & \dfrac{\partial x}{\partial b} & \dfrac{\partial x}{\partial c} \\ \dfrac{\partial y}{\partial a} & \dfrac{\partial y}{\partial b} & \dfrac{\partial y}{\partial c} \\ \dfrac{\partial z}{\partial a} & \dfrac{\partial z}{\partial b} & \dfrac{\partial z}{\partial c} \end{vmatrix}. \tag{3}$$

If the mass is *elastic*, i.e., if the volume element $d\tau_0$ that encloses the mass $dm = \rho_0\,d\tau_0$ changes in time, so the volume element:

$$d\tau_0 = da\;db\;dc$$

that enclosed the mass of density ρ_0 at time $t = 0$ has a volume $d\tau$ and a density that is equal to ρ then the following relation will be true:

$$\frac{d\tau}{d\tau_0} = \frac{\rho_0}{\rho} \tag{4}$$

or

$$R - \frac{\rho_0}{\rho} = 0\,. \tag{4.a}$$

The increase in a unit volume or spatial extension (i.e., dilatation) of elastic masses (which can be a positive or negative quantity) is then expressed by:

$$\frac{d\tau - d\tau_0}{d\tau_0} = R - 1\,. \tag{4.b}$$

Equation (4.a) [or (4)] is one of the forms of the so-called *continuity equation*, which expresses the fundamental law of the conservation of mass since it gives the connection between the change in unit volume and density at a location.

If the volume of a volume element remains unchanged at all points of a mass under its motion, so the mass is *incompressible, then one will have the equation of constraint:*

$$R = 1\,. \tag{5}$$

Since that equation includes the coordinates of the point that emerges from the point a, b, c and its neighboring points in the directions of the axes, it has exactly the character of a *geometric equation of constraint*, in the sense that was defined before (art. **8**). That same category includes all equations that were exhibited in this article, as well as all equations that exist between a, b, c, x, y, z, and the partial differential quotients of the x, y, z with respect to a, b, c.

By contrast, one cannot refer to the *continuity equation* (4.a):

$$R - \frac{\rho_0}{\rho} = 0$$

as a geometric equation because although the quantity R has a geometric meaning, the quantity ρ_0 / ρ does not.

All geometric constraint equations in this article are referred to as *holonomic*, in the sense of art. **5**, and indeed they have the character of integral equations that include only the coordinates themselves, but not their differential quotients with respect to time.

It will emerge from the coordinates that were exhibited in (2) of the point that is close to the point x, y, z that the increases dx, dy, dz will have the same order as the very small quantities da, db, dc as long as the differential quotients $\frac{\partial x}{\partial a}, \ldots, \frac{\partial z}{\partial c}$ do not become infinite, which will generally occur only along isolated surfaces, so only as long as the mass does not split and no discontinuity (i.e., vortex) surfaces appear, which we would like to exclude. In general, the *neighboring points* to a point of a continuous mas will *always remain neighboring* points then. Correspondingly, we would like to assume, on the grounds of continuity, that the *surface* is always composed of *the same mass-particles*.

§ 18. – The fundamental law applied to continuous masses. D'ALEMBERT's principle.

The concept of a free system that was presented in art. **8** can now be easily adapted to space-filling masses. If the constraint equations for the interior have the form (see the previous article) ([1]):

$$\varphi\left(x, y, z, a, b, c, \frac{\partial x}{\partial a}, \frac{\partial x}{\partial b}, \ldots, \frac{\partial^2 x}{\partial a^2}, \ldots\right) = 0 \tag{1}$$

(the differential quotient of φ with respect to time can also enter in place of φ) and the boundary conditions consist of only equations between the coordinates of the surface or boundary points then one must think of the system of space-filling masses as "free," in the sense of art. **8**.

The fundamental law says that a free system will remain in a state of rest or move uniformly along the straightest path. The demand of "uniform motion" will be satisfied by the assumption that the *vis viva* T is constant (art. **3**):

$$T = \tfrac{1}{2} m \dot{s}^2 = \tfrac{1}{2}\int (\dot{x}^2 + \dot{y}^2 + \dot{z}^2)\, dm = h\,, \tag{2}$$

in which m is the total mass of the system, $\dot{s}$ is its path velocity, x, y, z are the coordinates of the mass-element dm, $\dot{x}$, $\dot{y}$, $\dot{z}$ are its velocity components ([2]), and h is a constant. The demand of the straightest path gives the statement (art. **4**) that the path curvature, or, since that will be proportional to the acceleration for uniform motion, the quantity $\tfrac{1}{2} m f^2$, must be a minimum, such that one will then have:

$$\delta \frac{m f^2}{2} \equiv \delta \int \tfrac{1}{2} (\ddot{x}^2 + \ddot{y}^2 + \ddot{z}^2)\, dm = \int (\ddot{x}\,\delta\ddot{x} + \ddot{y}\,\delta\ddot{y} + \ddot{z}\,\delta\ddot{z})\, dm = 0\,, \tag{3}$$

in which $\ddot{x}$, $\ddot{y}$, $\ddot{z}$ are the components of the acceleration of the mass-element *dm*.

Equations (1) to (3) together describe the natural motion of the "free" (i.e., unforced) system. The three variations $\delta\ddot{x}$, $\delta\ddot{y}$, $\delta\ddot{z}$ must then satisfy the relations that the constraint equations

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([1]) Their number obviously cannot exceed two if the constraint equations by themselves are not supposed to reduce the degrees of freedom in the system to a finite number.

([2]) Here, and in the following article, we shall denote the differential quotients of functions *x*, *y*, *z*, φ, … of *a*, *b*, *c*, *t with respect to t* by putting a dot over the symbol.

imply. If one differentiates equation (1) twice with respect to time, $\dot{\varphi} = 0$, $\ddot{\varphi} = 0$, and then varies it in the manner that the fundamental law demands, i.e., when one leaves the coordinates and velocities constant, then one will get an equation of the form:

$$\delta\ddot{\varphi} \equiv \frac{\partial\ddot{\varphi}}{\partial\ddot{x}}\delta\ddot{x} + \frac{\partial\ddot{\varphi}}{\partial\ddot{y}}\delta\ddot{y} + \cdots + \frac{\partial\ddot{\varphi}}{\partial\left(\frac{\partial\ddot{x}}{\partial a}\right)}\delta\frac{\partial\ddot{x}}{\partial a} + \cdots = 0\,, \tag{4}$$

which one must imagine has been posed for each volume element $d\tau = da\,db\,dc$, in particular. If one then multiplies $\delta\ddot{\varphi}$ by $\lambda\,d\tau$, where λ is an undetermined function of position and time, and adds, so one thus forms the integral:

$$\delta\ddot{\Phi} \equiv \int \lambda\,\delta\ddot{\varphi}\,d\tau\,,$$

then once again, the quantities $\delta\ddot{x}$, $\delta\ddot{y}$, $\delta\ddot{z}$ will no longer be coupled with each other by (4) in the sum:

$$\delta\frac{m\,f^2}{2} - \delta\ddot{\Phi}\,,$$

because λ will take the place of that relation in the sense that was explained in art. **4**.

If the system is guided along its surface (art. **10**) by another system that exerts a pressure $\bar{X}$, $\bar{Y}$, $\bar{Z}$ per unit area at the location x, y, z then the latter functions will enter into formula (4) of art. **15** as LAGRANGIAN forces, which will then give:

$$0 = \delta\frac{m\,f^2}{2} - \delta\ddot{\Phi} - \delta'\ddot{S} \equiv \int[\rho(\ddot{x}\,\delta\ddot{x} + \ddot{y}\,\delta\ddot{y} + \ddot{z}\,\delta\ddot{z}) - \lambda\,\delta\ddot{\varphi}]\,d\tau - \int(\bar{X}\,\delta\ddot{x} + \bar{Y}\,\delta\ddot{y} + \bar{Z}\,\delta\ddot{z})\,d\sigma\,, \tag{5}$$

in which the integral:

$$\delta'\ddot{S} = \int(\bar{X}\,\delta\ddot{x} + \bar{Y}\,\delta\ddot{y} + \bar{Z}\,\delta\ddot{z})\,d\sigma$$

(the prime on δ again suggests that the expression under the integral sign is not necessarily a complete differential) is extended over the surface of the space that is filled with mass, and $\delta\ddot{x}$, … mean the variations of the accelerations of the mass-element $d\tau$ (the surface-element $d\sigma$, resp.).

The transition to the usual form of D'ALEMBERT's principle now happens once more just as it did for discrete mass-points (arts. **8**, **15**). Due to the independence of the $\delta\ddot{x}$, $\delta\ddot{y}$, $\delta\ddot{z}$ in (5), those quantities can be replaced by any three likewise-independent quantities, e.g., the variations δx, δy, δz of the coordinates themselves. The quantity $\delta\ddot{\varphi}$ will then go to:

$$\delta\varphi \equiv \frac{\partial\varphi}{\partial x}\delta x + \frac{\partial\varphi}{\partial y}\delta y + \cdots + \frac{\partial\varphi}{\partial\left(\dfrac{\partial x}{\partial a}\right)}\delta\frac{\partial x}{\partial a}\,, \tag{6}$$

which is an expression *that could also have been obtained directly from* φ *by varying the coordinates* $x, y, z,$ while $\delta\ddot{\Phi}$ goes to $\delta\Phi$, and (5) goes to:

$$\int[\rho(\ddot{x}\,\delta\ddot{x}+\ddot{y}\,\delta\ddot{y}+\ddot{z}\,\delta\ddot{z})-\lambda\,\delta\ddot{\varphi}]\,d\tau - \int(\bar{X}\,\delta\ddot{x}+\bar{Y}\,\delta\ddot{y}+\bar{Z}\,\delta\ddot{z})\,d\sigma = 0\,. \tag{7}$$

That is the statement of the fundamental law when it has been put into the form of *d'Alembert's principle*. The further treatment of the equation will be directed to the special cases. It will likewise be illuminated by some applications. In general, one can say that initially all of the terms of the form $p\,\delta\dfrac{\partial x}{\partial a}$, etc., that appear in $\lambda\,\delta\varphi$ are converted into ones with the factor δx by partial integration as follows:

$$\int p\,\delta\frac{\partial x}{\partial a}d\tau = \int p\frac{\partial\,\delta x}{\partial a}d\tau = \iint db\,dc\int p\frac{\partial\,\delta x}{\partial a}da = \iint db\,dc[p\,dx] - \int\frac{\partial p}{\partial a}d\tau\cdot\delta x\,, \tag{8}$$

in which the expression in square brackets refers to the surface element that is cut out by the parallelepiped that is erected over *db dc*. In that way, the exchange of variation and differentiation is justified by the interpretation of a variation that was given in art. **1**, by which, e.g.:

$$\delta x = \varepsilon\,\xi\,(a, b, c, t)\,,$$

in which ε is a very small constant and ξ is an arbitrary (even discontinuous) function of a, b, c, t. That is because since the variation refers to neither time nor the initial coordinates, but only the form of the function, one will have:

$$\delta\frac{\partial x}{\partial a} = \frac{\partial(x+\delta x)}{\partial a} - \frac{\partial x}{\partial a} = \frac{\partial\,\delta x}{\partial a} = \varepsilon\lim\left(\frac{\xi_1-\xi}{a_1-a}\right)_{a_1=a},$$

in which $\xi_1 = \xi_1\,(a_1, b_1, \ldots)$.

The same thing will be true of $\lambda\,\delta\ddot{\varphi}$ in (5), except that $x, \ldots$ must be replaced with $\ddot{x}, \ldots$

§ 19. – Body forces and surface forces. Example.

A final extension of formula (7) in the previous article consists of assuming that forces act upon the system whose effect is included in the calculations by their potential energy, or more generally, by the work they do under a displacement, according to the prescription in art. **15**. Forces that act upon space-filling masses are either body (volume) forces or surface forces according to whether they act upon the mass-elements themselves or their surfaces.

We have already included the latter in the formula for the fundamental law in the previous article at the end and represented their virtual work (art. **15**) by:

$$\delta' \ddot{S} = \int (\bar{X}\,\delta\ddot{x} + \bar{Y}\,\delta\ddot{y} + \bar{Z}\,\delta\ddot{z})\,d\sigma \;.$$

In the formula of D'ALEMBERT's principle, it is represented by:

$$\delta' S = \int (\bar{X}\,\delta x + \bar{Y}\,\delta y + \bar{Z}\,\delta z)\,d\sigma \;.$$

The body forces are internal or external according to whether they act between the elements of the masses under investigation or between them and external masses (art. **10**, at end). We count the force of gravity among the latter, which one might also call *impressed* forces. When (X, Y, Z) is the force that acts upon a unit mass, the variation of the work done by external forces can be put into the form:

$$\delta' U = \int (X\,\delta x + Y\,\delta y + Z\,\delta z)\,\rho\,d\tau \,,$$

in which the prime on δ means what it did above. Of the internal forces, we would like to focus on just the *elastic forces*, i.e., the ones that already appear under a change of form (i.e., deformation) of the individual element, to which we would like to include in one special case (art. **37**) a simple rotation (without deformation or displacement), as well as certain *forces of friction.* The work done by internal forces will take a different form depending upon whether the mass that fills up the volume element is solid or fluid, incompressible or elastic. Only later (art. **36**, *et seq.*) will we exhibit the virtual work for the different types of media, and we would like to denote it by δW, since the expression will prove to be a complete differential in all cases. We would also like that symbol to subsume the quantity that was denoted by $\delta\Phi$ in the previous article, which has the same form and is likewise assumed to be known.

One then gets by an application of D'ALEMBERT's principle to the otherwise-free system ([1]):

$$\int \mathsf{S}(\ddot{x}\,\delta x)\,\rho\,d\tau - \delta' U - \delta W - \delta' S = 0 \,, \tag{1}$$

([1]) In what follows, we shall overlook the state of motion of particles that one refers to as heat, since we shall restrict ourselves to motions without changes in temperature, unless stated to the contrary.

in which, here and in what follows, the summation sign S means that two more terms of the same kind (in y and z, here) must be added. When written out:

$$\int [(\ddot{x}-X)\,\delta x+(\ddot{y}-Y)\,\delta y+(\ddot{z}-Z)\,\delta z]\,\rho\,d\tau-\delta W-\int(\bar{X}\,\delta x+\bar{Y}\,\delta y+\bar{Z}\,\delta z)\,\delta\sigma\;=0\,. \tag{2}$$

That formula is true for any system of variations δx, δy, δz .

Upon applying GAUSS's principle, one will get precisely the same formula, only written in terms of $\delta\ddot{x}$, $\delta\ddot{y}$, $\delta\ddot{z}$, instead of δx, δy, δz .

The *motion of an inextensible string* (i.e., a "chain") on a surface when the external force (X, Y, Z) (e.g., gravity) acts on each unit length of it will serve as an example. The condition for inextensibility is (art. **17**):

$$\varphi\equiv\left(\frac{dx}{da}\right)^2+\left(\frac{dy}{da}\right)^2+\left(\frac{dz}{da}\right)^2-1=0\,, \tag{3}$$

in which a is the distance from the point (x, y, z) to the end of the chain. Let the equation of the outer surface be:

$$\psi(x, y, z) = 0\,. \tag{4}$$

Formula (2) will then assume the form:

$$\int[\rho\,\mathrm{S}(\ddot{x}-X)\,\delta x-\tfrac{1}{2}\lambda\,\delta\varphi-\mu\,\delta\psi)\,da\;=\int\mathrm{S}[\rho(\ddot{x}-X)\,\delta x-\tfrac{1}{2}\lambda\frac{dx}{da}\delta\frac{dx}{da}-\frac{\partial\psi}{\partial x}\mu\,\delta x]\,da\;=0\,, \tag{5}$$

in which the summation sign S again has the meaning that it was given above. The term with the factor λ is transformed by partial integration:

$$\int\lambda\,\mathrm{S}\,\frac{dx}{da}\delta\frac{dx}{da}da\;=\int\lambda\,\mathrm{S}\,\frac{dx}{da}\frac{d\,\delta x}{da}da\;=\left[\lambda\,\mathrm{S}\,\frac{dx}{da}\,\delta x\right]_{a=0}^{a=l}-\int\mathrm{S}\,\frac{d}{da}\left(\lambda\frac{dx}{da}\right)\delta x\,da\,, \tag{6}$$

in which l is the length of the chain. Now, $\delta\mathfrak{s}$, $d\mathfrak{s}$ are vectors (with lengths δs, ds) that give the virtual displacement (the element of the chain at the location a, resp.), so the expression in square brackets can be represented by:

$$[\lambda\,\delta s\cos(d\mathfrak{s},\delta\mathfrak{s})]_0^l\,.$$

If one substitutes (6) in (5) and compares each of the coefficients δx, δy, δz to zero, and likewise for the variation expression that is true for the boundary points, then one will get the equations:

$$\rho\,\ddot{x}+\frac{d}{da}\left(\lambda\frac{dx}{da}\right)-\mu\,\psi'(x)\;=X\,,$$

$$\rho\,\ddot{y} + \frac{d}{da}\left(\lambda\,\frac{dy}{da}\right) - \mu\,\psi'(y) = Y\,, \tag{7}$$

$$\rho\,\ddot{z} + \frac{d}{da}\left(\lambda\,\frac{dz}{da}\right) - \mu\,\psi'(z) = Z\,,$$

$$\lambda_l\;\delta\mathfrak{s}_l\cos(d\mathfrak{s}_l\,,\,\delta\,\mathfrak{s}_l) - \lambda_0\;\delta\mathfrak{s}_0\cos(d\mathfrak{s}_0\,,\,\delta\,\mathfrak{s}_0) = 0\,, \tag{8}$$

with a notation that is easy to understand. Along with the three equations (7) for determining the five quantities x, y, z, λ, μ as functions of a and t, (3) and (4) can also be employed. Equation (8) will be fulfilled identically when one perhaps has:

$$\delta\mathfrak{s}_l = 0\,, \qquad \delta\mathfrak{s}_0 = 0\,,$$

individually, so when the two endpoints are fixed, or also when a frictionless guide along a curve is attached to each endpoint that allows only a displacement along that curve, because the two cosines will vanish then, or finally, when the quantities λ_0, λ_l are both zero. The meaning of the vector λ is then inferred from the dimension of $\rho\,\ddot{x}\,da = [m\,l^{-1}\,l\,t^{-2}\,l] = [l\,m\,t^{-2}]$, because ρ has the dimension of mass per unit length = $[m\,l^{-1}]$, so λ has the dimension of a force = mass times acceleration, so it is the tension that acts in the direction $dx : dy : dz$ (an internal force, see prev. art.). The quantities λ_0, λ_l are the tensions that act upon the endpoints.

One will find a use for formulas (7), (8) in the problem of the equilibrium of a string in CLEBSCH, Jour. f. Math., Bd. 57, pp. 93, *et seq.*

§ 20. – Change of form of an elementary parallelepiped inside of a moving mass.

In the introduction to the section on space-filling masses in art. **17**, the state of a mass at time t was related to a certain state that went back to the initial time. However, it is also necessary to connect the states that immediately precede and follow it in time, as one does in geometry when one compares the direction of an element of a curve with the neighboring one and thus concludes its curvature properties. We consider the infinitely-small change in the position and form of a volume element that one has cut out from the mass in the form of a rectangular parallelepiped in the directions of the coordinate planes after an element of time has elapsed.

Let the points of a finite region of space (or also an infinite one) be referred to a rectangular coordinate system X, Y, Z that is fixed in space. One imparts a deformation on that spatial region that consists of a stretching (dilatation) or contraction (negative dilatation) in a given direction without changing the coordinate system. That direction, which might be given by the cosines α_1, β_1, γ_1 of the inclination angles with respect to X, Y, Z, resp., might determine the axis Ξ of a second rectangular coordinate system Ξ, H, Z that possesses the same origin O as X, Y, Z and whose cosines are given by α_1, β_1, γ_1, α_2, β_2, γ_2, α_3, β_3, γ_3. The stretching results in such a way that the plane HZ, and therefore the point O, in particular, does not change position, and that any point with the coordinate x, y, z, which has the coordinates ξ, η, ζ in the system Ξ, H, Z, will possess the coordinates $\xi(1 + \rho_1)$, η, ζ in the latter after the dilatation has been completed. We follow that with a second stretching of the spatial region in the direction of the H-axis by which η might go to $\eta(1+\rho_2)$ and a third one by which ζ might go to $\zeta(1 + \rho_3)$, and ask what the coordinates x_1, y_1, z_1 will be that x, y, z go to after performing the three dilatations.

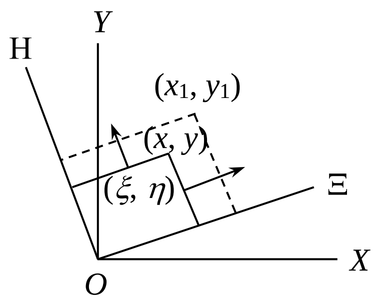

Figure 9.

The accompanying figure illustrates the corresponding process in the plane, so the effect of two mutually-perpendicular dilatations.

In space, one initially has the relations:

$$\begin{aligned} \xi &= \alpha_1\, x + \beta_1\, y + \gamma_1\, z\,, \\ \eta &= \alpha_2\, x + \beta_2\, y + \gamma_2\, z\,, \\ \zeta &= \alpha_3\, x + \beta_3\, y + \gamma_3\, z\,. \end{aligned} \tag{1}$$

After the resulting deformation, this system of equations will enter in place of the latter:

$$\begin{aligned} \xi(1 + \rho_1) &= \alpha_1\, x_1 + \beta_1\, y_1 + \gamma_1\, z_1\,, \\ \eta(1 + \rho_2) &= \alpha_2\, x_1 + \beta_2\, y_1 + \gamma_2\, z_1\,, \\ \zeta(1 + \rho_3) &= \alpha_3\, x_1 + \beta_3\, y_1 + \gamma_3\, z_1\,. \end{aligned} \tag{2}$$

If one multiplies those equations by α_1, α_2, α_3, resp., and adds them then when recalls the known six relations between the nine cosines α, β, γ, one will get the following equations:

$$\begin{aligned} x_1 &= \alpha_1\, \xi(1 + \rho_1) + \alpha_2\, \eta(1 + \rho_2) + \alpha_3\, \zeta(1 + \rho_3)\,, \\ x_2 &= \beta_1\, \xi(1 + \rho_1) + \beta_2\, \eta(1 + \rho_2) + \beta_3\, \zeta(1 + \rho_3)\,, \\ x_3 &= \gamma_1\, \xi(1 + \rho_1) + \gamma_2\, \eta(1 + \rho_2) + \gamma_3\, \zeta(1 + \rho_3)\,, \end{aligned} \tag{3}$$

and from this, once the values of ξ, η, ζ from (1) have been substituted in (3), with the use of the abbreviations:

$$\rho_1 \alpha_1^2 + \rho_2 \alpha_2^2 + \rho_3 \alpha_3^2 = \mathsf{S}\rho\alpha^2\,,$$

$$\rho_1 \alpha_1 \beta_1 + \rho_2 \alpha_2 \beta_2 + \rho_3 \alpha_3 \beta_3 = \mathsf{S}\rho\alpha\beta\,,$$

etc., one will have:

$$\begin{aligned} x_1 &= (1 + \mathsf{S}\rho\alpha^2)\, x + \mathsf{S}\rho\alpha\beta\, y + \mathsf{S}\rho\alpha\gamma\, z\,, \\ y_1 &= \mathsf{S}\rho\beta\alpha\, x + (1 + \mathsf{S}\rho\beta^2)\, y + \mathsf{S}\rho\beta\gamma\, z\,, \\ z_1 &= \mathsf{S}\rho\gamma\alpha\, x + \mathsf{S}\rho\gamma\beta\, y + (1 + \mathsf{S}\rho\gamma^2)\, z\,, \end{aligned} \tag{4}$$

so equations of the form:

$$\begin{aligned} x_1 &= (a_{11} + 1)\, x + a_{12}\, y + a_{13}\, z , \\ y_1 &= a_{21}\, x + (a_{22} + 1)\, y + a_{23}\, z , \\ z_1 &= a_{31}\, x + a_{32}\, y + (a_{33} + 1)\, z , \end{aligned} \tag{4.a}$$

in which $a_{ik} = a_{ki}$. One will then have the theorem:

If a spatial region changes its form as a result of three stretchings or contractions (that follow in any order) *along three mutually-perpendicular directions, while the coordinate system does not change its position, then any spatial point will go from the location* (x, y, z) *to the one* (x_1, y_1, z_1) *in formula* (4.a).

We directly conclude *the converse of that theorem* when we succeed in determining the quantities ρ_1, ρ_2, ρ_3 that were defined by the given transformation (4.a).

A comparison of (4), (4.a) will yield:

$$\begin{aligned} \rho_1\, \alpha_1^2 + \rho_2\, \alpha_2^2 + \rho_3\, \alpha_3^2 &= a_{11} , \\ \rho_1\, \alpha_1\, \beta_1 + \rho_2\, \alpha_2\, \beta_2 + \rho_3\, \alpha_3\, \beta_3 &= a_{12} , \\ \rho_1\, \alpha_1\, \gamma_1 + \rho_2\, \alpha_2\, \gamma_2 + \rho_3\, \alpha_3\, \gamma_3 &= a_{13} , \end{aligned} \tag{5}$$

and two more similar systems. If one multiplies equations (5) by α_1, β_1, γ_1, resp., and adds them then when one recalls the relations between the α, β, γ, one will get the first of the following equations:

$$\begin{aligned} (a_{11} - \rho_1)\, \alpha_1 + a_{12}\, \beta_1 + a_{13}\, \gamma_1 &= 0, \\ a_{21}\, \alpha_1 + (a_{12} - \rho_1)\, \beta_1 + a_{23}\, \gamma_1 &= 0, \\ a_{31}\, \alpha_1 + a_{12}\, \beta_1 + (a_{33} - \rho_1)\, \gamma_1 &= 0. \end{aligned} \tag{6}$$

If one multiplies equations (5) by α_2, β_2, γ_2, resp., then one will get the first of three equations that will emerge from (6) when one replaces α_1, β_1, γ_1, ρ_1 with α_2, β_2, γ_2, ρ_2, resp. Upon eliminating α_1, β_1, γ_1 from (6), one will get the same equation for determining the quantity ρ_1 that one would get by eliminating α_2, β_2, γ_2 from the other three equations: The desired quantities ρ_1, ρ_2, ρ_3 are then roots of the cubic equation:

$$\begin{vmatrix} a_{11} - \rho & a_{12} & a_{13} \\ a_{21} & a_{22} - \rho & a_{23} \\ a_{31} & a_{32} & a_{33} - \rho \end{vmatrix} = 0 , \tag{7}$$

and the values α_1, β_1, γ_1 that are associated with ρ_1 are then inferred from (6), and α_2, β_2, γ_2, etc., are inferred from the analogous equations in ρ_2, etc. We arrange (7) in powers of ρ and get:

$$\rho^3 - \rho^2 (a_{11} + a_{22} + a_{33}) - \rho (a_{22}\, a_{33} + a_{33}\, a_{11} + a_{11}\, a_{22} - a_{23}^2 - a_{31}^2 - a_{12}^2) - \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = 0 .$$

We get the following expressions for the symmetric functions of the roots:

$$\rho_1 + \rho_2 + \rho_3 = a_{11} + a_{22} + a_{33} = \mathsf{S}\, a_{11} ,$$
$$\rho_2\, \rho_3 + \rho_3\, \rho_1 + \rho_1\, \rho_2 = \mathsf{S}\, a_{22}\, a_{33} - \mathsf{S}\, a_{23}^2$$

when S, as always, denotes the sum of three terms that are similar to the one that is written out, from which, we will get:

$$\rho_1^2 + \rho_2^2 + \rho_3^2 = (\mathsf{S}\, \rho_1)^2 - 2\,\mathsf{S}\, \rho_1\, \rho_2 = (\mathsf{S}\, a_{11})^2 - 2\,\mathsf{S}\, a_{22}\, a_{33} + 2\,\mathsf{S}\, a_{23}^2 , \tag{8}$$

which we shall save for later.

When the *lengthenings* of the unit lengths in the directions of the axes Ξ, H, Z are *small enough quantities* that one can neglect their squares and products in comparison to their first powers, as will be assumed in what follows, the expression for the increase in the unit volume – viz., the *spatial dilatation* (expansion), which will would like to denote by Θ – will assume a simple form. Namely, the equation:

$$1 + \Theta = (1 + \rho_1)\,(1 + \rho_2)\,(1 + \rho_3) = 1 + \mathsf{S}\, \rho_1 ,$$

will imply that:

$$\Theta = \rho_1 + \rho_2 + \rho_3 = a_{11} + a_{22} + a_{33} . \tag{8.a}$$

§ 21. – Continuation.

We will now apply the theorem that was proved above to those deformations of a parallelepiped that is a *volume element* (viz., an elementary parallelepiped) that are produced by very small changes in the positions of all points in a space-filling mass. A point *x*, *y*, *z* will go to *x* + *u*, *y* + *v*, *z* + *w* under that change, where the displacement magnitudes:

$$\begin{aligned} u &= \varepsilon \cdot f(x, y, z) , \\ v &= \varepsilon \cdot \varphi(x, y, z) , \\ w &= \varepsilon \cdot \psi(x, y, z) , \end{aligned} \tag{9}$$

are functions f, φ, ψ of the coordinates of the point, multiplied by the very small factor ε ([1]). When regarded in a purely-analytical way, the coordinates of a neighboring point $x + dx$, $y + dy$, $z + dz$ then go to:

$$\begin{aligned} x + u + d\,(x + u) &= x + u + d_1\, x\,, \\ \dots\qquad &= y + v + d_1\, y\,, \\ \dots\qquad &= z + w + d_1\, z\,, \end{aligned}$$

in which:

$$d_1x = dx + du = dx + \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial z}dz\,,$$

or

$$\begin{aligned} d_1x &= \left(1+\frac{\partial u}{\partial x}\right)dx &&+ \frac{\partial u}{\partial x}dy &&+ \frac{\partial u}{\partial x}dz\,, \\ d_1y &= +\frac{\partial v}{\partial x}dx &&+ \left(1+\frac{\partial v}{\partial y}\right)dy &&+ \frac{\partial v}{\partial z}dz\,, \\ d_1z &= +\frac{\partial w}{\partial x}dx &&+ \frac{\partial w}{\partial y}dy &&+ \left(1+\frac{\partial w}{\partial z}\right)dz\,, \end{aligned} \tag{10}$$

and d_1x represents the projection of the diagonal of the deformed elementary parallelepiped onto the X-axis. Those relations can now be regarded as transformation equations for the coordinates dx, dy, dz into the d_1x, d_1y, d_1z, in the sense of the theorem in the foregoing article, since both refer to rectangular coordinate systems whose axes go through the point x, y, z ($x + u$, $y + v$, $z + w$) parallel to the X, Y, Z coordinate axes. We would like to denote the first by K and the second by K'. We can then ask what the change in form would be for the space that is bounded by the points x, y, z when they go to $x + u$, $y + v$, $z + w$. Since, in general, the elements $\frac{\partial u}{\partial y}$, $\frac{\partial v}{\partial x}$, etc. that are pair-wise-symmetric to the diagonal in the transformation determinant are different, the transformation is initially not of the form (4.a) in the previous article, i.e., it is not "affine." We now put equations (10) into the following form:

$$\begin{aligned} d_1x &= \left(1+\frac{\partial u}{\partial x}\right)dx + \tfrac{1}{2}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)dy + \tfrac{1}{2}\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)dz + \left[\tfrac{1}{2}\left(\frac{\partial u}{\partial z}-\frac{\partial w}{\partial x}\right)dz - \tfrac{1}{2}\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)dy\right], \\ d_1y &= \tfrac{1}{2}\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)dx + \left(1+\frac{\partial v}{\partial y}\right)dy + \tfrac{1}{2}\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right)dz + \left[\tfrac{1}{2}\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)dx - \tfrac{1}{2}\left(\frac{\partial w}{\partial y}-\frac{\partial v}{\partial z}\right)dz\right], \\ d_1z &= \tfrac{1}{2}\left(\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}\right)dx + \tfrac{1}{2}\left(\frac{\partial w}{\partial y}+\frac{\partial v}{\partial z}\right)dy + \left(1+\frac{\partial w}{\partial z}\right)dz + \left[\tfrac{1}{2}\left(\frac{\partial w}{\partial y}-\frac{\partial v}{\partial z}\right)dy - \tfrac{1}{2}\left(\frac{\partial u}{\partial z}-\frac{\partial w}{\partial x}\right)dx\right], \end{aligned} \tag{10.a}$$

([1]) The factor ε is assumed to be small enough that the products and second powers of u, v, w, and their derivatives can be neglected in comparison to their first powers.

or with the use of the notations that we shall also use frequently later on:

$$\begin{aligned}
\frac{\partial u}{\partial x} &= x_x ; & \frac{\partial w}{\partial y}+\frac{\partial v}{\partial z} &= y_z = z_y , \\
\frac{\partial v}{\partial y} &= y_y ; & \frac{\partial u}{\partial z}+\frac{\partial w}{\partial x} &= z_x = x_z , \\
\frac{\partial w}{\partial z} &= z_z ; & \frac{\partial v}{\partial x}+\frac{\partial u}{\partial y} &= x_y = y_x ,
\end{aligned} \tag{11}$$

and

$$\begin{aligned}
\tfrac{1}{2}\left(\frac{\partial w}{\partial y}-\frac{\partial v}{\partial z}\right) &= \xi , \\
\tfrac{1}{2}\left(\frac{\partial u}{\partial z}-\frac{\partial w}{\partial x}\right) &= \eta , \\
\tfrac{1}{2}\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right) &= \zeta ,
\end{aligned} \tag{12}$$

into the form:

$$\begin{aligned}
d_1x &= (1+x_x)\,dx + \tfrac{1}{2}x_y\,dy + \tfrac{1}{2}x_z\,dz + [\eta\,dz - \zeta\,dy] , \\
d_1y &= \tfrac{1}{2}y_x\,dx + (1+y_y)\,dy + \tfrac{1}{2}y_z\,dz + [\zeta\,dx - \xi\,dz] , \\
d_1z &= \tfrac{1}{2}z_x\,dx + \tfrac{1}{2}z_y\,dy + (1+z_z)\,dz + [\xi\,dy - \eta\,dx] .
\end{aligned} \tag{13}$$

One can then think of the transformation (13) as being produced by the composition of two transformations of the following form:

$$\begin{aligned}
d_2x &= (1+x_x)\,dx + \tfrac{1}{2}x_y\,dy + \tfrac{1}{2}x_z\,dz , \\
d_2y &= \tfrac{1}{2}y_x\,dx + (1+y_y)\,dy + \tfrac{1}{2}y_z\,dz , \\
d_2z &= \tfrac{1}{2}z_x\,dx + \tfrac{1}{2}z_y\,dy + (1+z_z)\,dz
\end{aligned} \tag{14}$$

and

$$\begin{aligned}
d_1x &= d_2x + \eta\,d_2z - \zeta\,d_2y , \\
d_1y &= d_2y + \zeta\,d_2x - \xi\,d_2z , \\
d_1z &= d_2z + \xi\,d_2y - \eta\,d_2x .
\end{aligned} \tag{15}$$

That is because z_x, …, y_x, … are small quantities of the same order as u, v, … [see formula (9) of this art.] that can be neglected in comparison to η itself, when multiplied by η (ζ, resp.), so upon substituting the values of d_2z, d_2y, the first of equations (15) will go to:

$$d_1x = d_2x + \eta\,dz - \zeta\,dy , \qquad \text{etc.}$$

If one then sets:

$$d_1x = d_2x + d_3x , \qquad \text{etc.,} \tag{16}$$

in which:

$$d_3 x = \eta\, dz - \zeta\, dy ,$$

and correspondingly:

$$\begin{aligned} d_3 y &= \zeta\, dx - \xi\, dz , \\ d_3 z &= \xi\, dy - \eta\, dx , \end{aligned} \tag{17}$$

then the coordinate $d_1 x$, to which dx goes after deformation, will split into:

1. The quantity $d_2 x$. Formulas (14) mean a stretching of the volume element along three mutually-perpendicular directions [cf., (4.a)].

2. The increase $d_3 x$ (whose ratio to $d_2 x$ is very small). The meaning of that component is inferred from a comparison of formulas (17) with the ones (3) in art. **5**. Thus, $d_3 x$ is the increase that the coordinate dx experiences as a result of the components ξ, η, ζ (the axes of the coordinate system K, resp.).

One proves, as above, that one can also perform the infinitesimal changes $d_2 x$, $d_3 x$ in the opposite order. The partial transformation (14) now has the character of an affine one.

One can then state the *theorem:*

The infinitely-small change of state that a spatial region at the location (x, y, z) *experiences as a result of a deformation of the space-filling mass consists of three parts:*

1. A parallel displacement (u, v, w), which we would like to refer to as a vector, and as a result, denote with the German symbol $\mathfrak{u}$ ([1]).

2. A rotation (ξ, η, ζ) around an axis through the point x, y, z (which is likewise a vector).

3. Extensions (or contractions) along three mutually-perpendicular directions, namely, the so-called *principal axes of dilatation* ([2]).

Just like the ξ, η, ζ, the quantities x_x, x_y, … also admit a simple geometric interpretation. The lengths of the edges of the rectangular parallelepiped with the opposite corner points x, y, z and $x + dx$, $y + dy$, $z + dz$ will have changed after the deformation, as well as the angles that they subtend with each other. We would like to determine those quantities. The corner $x + dx$, y, z that is a neighbor to the corner x, y, z will go to a point whose coordinates in the system K' will be obtained when we set $dy = dz = 0$ in formulas (10). We then get the first column of the following table,

([1]) Later, we would also like to denote the components of the velocity vector $\mathfrak{v}$ by u, v, w, so exchanging them with the displacement $\mathfrak{u}$ will be excluded.

([2]) W. VOIGT [Göttinger Nachr. (1900), pp. 120, *et seq.*] proposed to subsume those quantities, and more generally, the x_x, …, y_z, …, under the name of a *tensor-triple*, as a sort of double-sided vector that is combined with others of the same type.

which include the coordinates of the three corner points of the parallelepiped that connect to *x*, *y*, *z after* deformation:

Point:	dx, 0, 0	0, dy, 0	0, 0, dz
x-coordinate	$\left(1+\frac{\partial u}{\partial x}\right)dx$	$\frac{\partial u}{\partial y}dy$	$\frac{\partial u}{\partial z}dz$
y-coordinate	$\frac{\partial v}{\partial x}dx$	$\left(1+\frac{\partial v}{\partial y}\right)dy$	$\frac{\partial v}{\partial z}dz$
z-coordinate	$\frac{\partial w}{\partial x}dx$	$\frac{\partial w}{\partial y}dy$	$\left(1+\frac{\partial w}{\partial z}\right)dz$

One immediately infers the lengths $\overline{dx}$, $\overline{dy}$, $\overline{dz}$ of the edges of the deformed parallelepiped from that table:

$$\overline{dx}^2 = dx^2\left[\left(1+\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial x}\right)^2+\left(\frac{\partial w}{\partial x}\right)^2\right],$$

or when one neglects second-order terms:

$$\overline{dx} = dx\sqrt{1+2\frac{\partial u}{\partial x}} = dx\left(1+\frac{\partial u}{\partial x}\right).$$

Therefore, the (in turn, very small) quantity:

$$x_x = \frac{\partial u}{\partial x} = \frac{\overline{dx}-dx}{dx} \tag{18}$$

means the lengthening that the unit of edge length dx has experienced. The angle $(\overline{dy},\overline{dz})$ that the edges $\overline{dy}$, $\overline{dz}$ of the deformed parallelepiped subtend with each other is calculated from:

$$\cos\,(\overline{dy},\overline{dz}) = \frac{dy\,dz}{\overline{dy}\,\overline{dz}}\left[\frac{\partial u}{\partial y}\frac{\partial u}{\partial z}+\left(1+\frac{\partial v}{\partial y}\right)\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\left(1+\frac{\partial w}{\partial z}\right)\right],$$

or when one neglects second-order terms:

$$\cos\,(\overline{dy},\overline{dz}) = \frac{\partial v}{\partial x}+\frac{\partial w}{\partial y} = y_z = z_y\,. \tag{19}$$

The (very small) *quantities* $z_y = y_z$, etc., *are then the cosines of the angles* that the edges $\overline{dy}$, $\overline{dz}$, etc., subtend with each other after deformation.

We finally define the expression for the spatial dilatation Θ, for which we found above [art. **20**, (8.a)] that:

$$\Theta = a_{11} + a_{22} + a_{33} .$$

With the help of:

$$a_{11} = \frac{\partial u}{\partial x} = x_x , \qquad \text{etc.},$$

$$a_{12} = a_{21} = \tfrac{1}{2} x_y = \tfrac{1}{2} y_x , \qquad \text{etc.},$$

one will get the *spatial dilatation* as:

$$\Theta = \rho_1 + \rho_2 + \rho_3 = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = x_x + y_y + z_z , \tag{20}$$

and formula (8) will yield:

$$\rho_1^2 + \rho_2^2 + \rho_3^2 = \Theta^2 - 2(y_y\, z_z + z_z\, x_x + x_x\, y_y) + \tfrac{1}{2}(y_z^2 + z_x^2 + x_y^2) . \tag{21}$$

From their very meaning, the values of the quantities ρ_1, ρ_2, ρ_3, and their symmetric functions are independent of the position of the coordinate system. One will find other such "invariant" quantities that are composed from the x_x, …, y_z, … given in LOVE, *Lehrbuch der Elastizität*, Ger. by TIMPE, Leipzig, 1907, art. 13.

§ 22. – Application to fluid masses.

The intuitions and formulas that were developed in the previous article are applicable to solid-elastic, as well as fluid masses. However, a distinction is implied by the fact that for bodies that approach a state of rigidity, such as, above all, solid-elastic bodies within the limits of applicability, any two states that it moves through in the course of time will differ from each other only slightly, so as neighboring states, they will admit the immediate application of those formulas ([1]), whereas the displacements of the elements of fluid masses are not constrained by any limits, such that the formulas are applicable to only two states that follow each other immediately in time.

In the case of fluids, it is therefore preferable to refer the very small displacements, rotations, and elongations u, v, w ; ξ, η, ζ ; ρ_1, ρ_2, ρ_3 ; Θ that result inside of the time element dt to that interval of time, if only formally, in such a way that one replaces them with differentials du, dv, …, $d\Theta$ and presents them when divided by dt. In that way, the displacement quantity $\mathfrak{u}$ will then

([1]) When a solid-elastic body is very small with respect to one or two dimensions, such as thin shells or rods, finite displacements, which we excluded in art. **21**, can occur, despite appreciable rigidity to the material. One then cares to employ the formulas of this article only in the immediate neighborhood of a coordinate system (which is assumed to be mobile within the body).

go to the *velocity* $\mathfrak{v}$ of the advancing motion of the fluid. However, when we are dealing with the motion of fluid masses, we would not like to denote the components of the velocity $\mathfrak{v} = \dot{\mathfrak{u}}$ by $\dot{u}$, $\dot{v}$, $\dot{w}$, following general usage, but once more by u, v, w ([1]). Likewise, ξ, η, ζ will go to the components of the *angular velocity* (vortex velocity) of a mass-element at the location x, y, z, which might, in turn, be called ξ, η, ζ ([1]). ρ_1, ρ_2, ρ_3 will become stretchings *per unit time*. Finally, the theorem of the previous article will be adapted to the velocities:

$$\frac{dx_1 - dx}{dt}, \qquad \text{etc.},$$

with which the position and state of the volume change.

Equations (12), (20) of the previous article, which are homogeneous in the quantities that are divided by dt, will now become *relations between the velocity components of the rotating and advancing motion* at a point of the fluid mass with changing its form:

$$\begin{aligned} \xi &= \tfrac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right), \\ \eta &= \tfrac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right), \\ \zeta &= \tfrac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right). \end{aligned} \tag{1}$$

Vector analysis replaces the relation (1) with *one* equation. Namely, if one derives a new vector from $\mathfrak{v}$ that is *the rotation of* $\mathfrak{v}$ (also called the *curl* of $\mathfrak{v}$), whose components are:

$$\text{rot}\,\mathfrak{v} = \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}, \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}, \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right), \tag{1.a}$$

then if ξ, η, ζ are the components of a vector $\mathfrak{w}$:

$$\mathfrak{w} = \tfrac{1}{2}\,\text{rot}\,\mathfrak{v}\,. \tag{2}$$

Whereas the expressions (1) individually depend upon the choice of the coordinate system, by definition, that is not the case for the vectors $\mathfrak{v}$, $\mathfrak{w}$. Therefore, the relations (1) must be true for any arbitrary rectangular system of axes. Vector analysis gives an immediate expression to that *invariance* under a transformation to another system of axes by way of formula (2).

([1]) It will first be in the theory of elastic bodies, so from art. **36** on, that we will once more understand u, v, w to mean displacement components ($\mathfrak{u}$) and ξ, η, ζ to again mean angles.

In that fact, one finds an essential advantage of the vectorial representation, and not just in its conciseness.

Now, if the quantities u, v, w that appear in the expression Θ of the previous article are also velocity components then the spatial dilatation will go to *the rate of increase in the dilatation*. Vector analysis has introduced the terminology *the divergence of the vector* $\mathfrak{v}$ for expressions of that type:

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z} = \text{div}\,\mathfrak{v}\,, \tag{3}$$

which will be justified later on.

The definition of rot $\mathfrak{v}$ (1.a) immediately implies the oft-used identity:

$$\text{div rot}\,\mathfrak{v} = 0\,. \tag{4}$$

§ 23. – Description of fluid motion. The continuity equation.

In the foregoing, the quantities $\mathfrak{v}$, $\mathfrak{w}$, div $\mathfrak{v}$ were regarded as functions of a location x, y, z that is *fixed* in space and time t. On the other hand, in art. **17**, we denoted the spatially-*varying* location that a certain mass-element that was given by its position a, b, c at time $t = 0$ described in time by x, y, z. Both conceptions of the situation can be used as a basis for the description of a fluid motion.

We would next like to relate the two to each other. When one solves the expressions that were assumed in art. **14**:

$$x = x\,(a, b, c, t)\,, \qquad y = y\,(a, b, c, t)\,, \qquad z = z\,(a, b, c, t)\,, \tag{1}$$

which coupled the position x, y, z of a mass-element at time t with the one a, b, c at any initial time t_0 , for a, b, c and substitutes the values obtained:

$$a = a\,(x, y, z, t)\,, \qquad b = b\,(x, y, z, t)\,, \qquad c = c\,(x, y, z, t)\,, \tag{2}$$

in which the a, b, c on the right are again function symbols, in the derivatives for the expresses (1) with respect to time, which we would like to denote by:

$$\dot{x} = U\,(a, b, c, t)\,, \qquad \dot{y} = V\,(a, b, c, t)\,, \qquad \dot{z} = W\,(a, b, c, t)\,, \tag{3}$$

then we will get just the same expressions for the velocity components that are associated with the location x, y, z [they might be denoted by u, v, w in agreement with the notation of the previous article and to distinguish them from the ones in (3)]:

$$\dot{x} = u\,(x, y, z, t)\,, \qquad \dot{y} = v\,(x, y, z, t)\,, \qquad \dot{z} = w\,(x, y, z, t)\,, \tag{4}$$

that were at the basis for the concepts and formulas (1), (2), (3) of the previous article.

$$(\dot{x}, \dot{y}, \dot{z}) = (U, V, W) = (u, v, w) = \mathfrak{v}$$

then means the same vector $\mathfrak{v}$. Only the analytical representations are different. That situation will become noticeable when one deals with the construction of:

$$\dot{\mathfrak{v}} = \frac{d\mathfrak{v}}{dt} .$$

If one regards the *a*, *b*, *c* in equations (1) as given and *t* as variable then they will represent the trajectories that are described by the mass-element that occupied the position *a*, *b*, *c* at time $t = t_0$. Elements along trajectories that are initially neighboring will remain neighboring for all time (art. **17**). Equations (3) give the velocity of each *mass-element*, and the *total* differential quotient ([1]):

$$\dot{\mathfrak{v}} = (\dot{U}, \dot{V}, \dot{W}) = (\ddot{x}, \ddot{y}, \ddot{z})$$

gives the change in *that* velocity, or the acceleration of one and the same mass-element. By contrast, equations (4) represent the velocities of the *different* mass-elements that pass through a given *spatial point* in succession. If one goes from the velocity of one of those elements to that of its following one then since *x*, *y*, *z* will remain unchanged by that, that change will be denoted by the *partial* differential quotients with respect to time:

$$\frac{\partial \mathfrak{v}}{\partial t} = \left(\frac{\partial u}{\partial t}, \frac{\partial v}{\partial t}, \frac{\partial w}{\partial t}\right) .$$

The components of the two vectors $\dot{\mathfrak{v}}$ and $\partial \mathfrak{v} / \partial t$ are connected by three equations, one of which reads:

$$\ddot{x} = \dot{U} = \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} + \frac{\partial u}{\partial z}\frac{dz}{dt} ,$$

or

$$\ddot{x} = \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} u + \frac{\partial u}{\partial y} v + \frac{\partial u}{\partial z} w = \dot{u} , \tag{5}$$

in which *u*, *v*, *w* are defined by (4).

One calls (art. **29**) the vector whose components are $\partial \varphi / \partial x$, etc., the *gradient of the scalar quantity* φ (grad φ). When the bracket $(\mathfrak{a}, \mathfrak{b})$ represents the so-called *scalar* or *inner product*, as usual:

$$(\mathfrak{a}, \mathfrak{b}) = |\,\mathfrak{a}\,|\,|\,\mathfrak{b}\,| \cos (\mathfrak{a}, \mathfrak{b}) , \tag{6}$$

([1]) In what immediately follows, the dots over variables will denote the differential quotients with respect to time that are defined under the assumption that the remaining variables in the function being differentiated are *a*, *b*, *c*.

in which | $\mathfrak{a}$ | is the *magnitude* of $\mathfrak{a}$, then that will explain the operator ($\mathfrak{v}$, grad) = $u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y}+w\frac{\partial}{\partial z}$, with whose help equations (5) can be summarized in one vectorial equation:

$$\dot{\mathfrak{v}} = \frac{\partial \mathfrak{v}}{\partial t} + (\mathfrak{v}, \text{grad})\, \mathfrak{v} \,. \tag{7}$$

Indeed, as was said before, a fluid motion is described completely by equations (1), which give the trajectories of the particles. Equations (4) will then lead the way to them upon integrating the latter under the assumption that $x = a$, $y = b$, $z = c$ at time $t = t_0$. However, since one regards that integration as practicable as a purely-analytic operation, knowing the functions (4) will already suffice. Corresponding to those two conceptions of things, i.e., according to whether one seeks the functions (1) or (4), one can also distinguish the equations of motion that one will obtain by an application of the fundamental law. In this section, we will mostly restrict ourselves to *incompressible fluids* (art. **17**) and consider the *elastic media* only in passing.

If one uses the first conception as a basis, so one employs the components of the acceleration in the form:

$$\dot{\mathfrak{v}} = (\dot{U}, \dot{V}, \dot{W}) \,, \tag{8}$$

then the condition for incompressibility will assume the form (art. **17**):

$$\varphi \equiv R - 1 = 0 \,. \tag{9}$$

However, if one starts from the second conception, so one then represents the acceleration by:

$$\dot{\mathfrak{v}} = (\dot{u}, \dot{v}, \dot{w}) \,,$$

then the incompressibility condition (art. **22**) will become:

$$\varphi \equiv \text{div}\, \mathfrak{v} = \frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z} = 0 \,. \tag{9.a}$$

We shall deal with that later, while exhibiting the hydrodynamical equations of motion in order to construct the expressions $\delta\ddot{\varphi}$ or $\delta\varphi$, according to whether one applies the principle of least constraint or D'ALEMBERT's principle. Indeed, in the second case (9.a) is not a holonomic condition, in the sense of the definition of art. **6**. However, as one knows, it can nonetheless be regarded as such when one relates it the ones in (9). One might likewise do that with the continuity equation, in general.

If one derives the expression that was presented in art. **17**:

$$R = \begin{vmatrix} \frac{\partial x}{\partial a} & \frac{\partial x}{\partial b} & \frac{\partial x}{\partial c} \\ \frac{\partial y}{\partial a} & \frac{\partial y}{\partial b} & \frac{\partial z}{\partial c} \\ \frac{\partial z}{\partial a} & \frac{\partial z}{\partial b} & \frac{\partial z}{\partial c} \end{vmatrix}$$

with respect to t, which will once more be denoted with a dot overhead, as in footnote ([2]) at the beginning of § **18**, and one imagines that the quantities a, b, c in $\dot{R}$ can be expressed in terms of x, y, z with the help of (2), just as in R itself, then since one has, e.g.:

$$\frac{\partial \dot{x}}{\partial a} = \frac{\partial u}{\partial a} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial a} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial a} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial a} ,$$

one will get R as a sum S of three determinants, each of which can be again split into a product of two of them. When one performs the operation for just one determinant that will give:

$$\dot{R} = \mathsf{S}\begin{vmatrix} \frac{\partial \dot{x}}{\partial a} & \frac{\partial \dot{x}}{\partial b} & \frac{\partial \dot{x}}{\partial c} \\ \frac{\partial y}{\partial a} & \frac{\partial y}{\partial b} & \frac{\partial y}{\partial c} \\ \frac{\partial z}{\partial a} & \frac{\partial z}{\partial b} & \frac{\partial z}{\partial c} \end{vmatrix} = \mathsf{S}\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}\begin{vmatrix} \frac{\partial x}{\partial a} & \frac{\partial x}{\partial b} & \frac{\partial x}{\partial c} \\ \frac{\partial y}{\partial a} & \frac{\partial y}{\partial b} & \frac{\partial y}{\partial c} \\ \frac{\partial z}{\partial a} & \frac{\partial z}{\partial b} & \frac{\partial z}{\partial c} \end{vmatrix} = R\,\mathsf{S}\frac{\partial u}{\partial x} ,$$

$$\dot{R} = R\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) \equiv R \operatorname{div} \mathfrak{v} . \tag{10}$$

With that, by logarithmic differentiation with respect to time, the continuity that was presented in art. **17** (4.a):

$$R - \frac{\rho_0}{\rho} = 0$$

will take on the following form:

$$\frac{\dot{R}}{R} + \frac{\dot{\rho}}{\rho} \equiv \operatorname{div} \mathfrak{v} + \frac{\dot{\rho}}{\rho} = 0 ,$$

or when written out ([1]):

([1]) One will get another form of the continuity equation when one introduces:

$$\frac{d\rho}{dt} = \frac{\partial \rho}{\partial t} + \frac{\partial \rho}{\partial x} u + \frac{\partial \rho}{\partial y} v + \frac{\partial \rho}{\partial z} w$$

into (11). That will give:

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}+\frac{1}{\rho}\frac{d\rho}{dt} = 0\,, \tag{11}$$

which is a form for the continuity equation that is employed in the theory of fluids, because it includes the velocities, and once more asserts the conservation of mass.

In exactly the same way that one proved the formula (10), one can prove that the variation of the second derivative of R with respect to time can be expressed in terms of the divergence of the variation of $\mathfrak{v}$ as follows:

$$\delta\ddot{R} = \mathsf{S}\begin{vmatrix} \frac{\partial\,\delta\ddot{x}}{\partial a} & \frac{\partial\,\delta\ddot{x}}{\partial b} & \frac{\partial\,\delta\ddot{x}}{\partial c} \\ \frac{\partial\,\delta y}{\partial a} & \frac{\partial\,\delta y}{\partial b} & \frac{\partial\,\delta y}{\partial c} \\ \frac{\partial\,\delta z}{\partial a} & \frac{\partial\,\delta z}{\partial b} & \frac{\partial\,\delta z}{\partial c} \end{vmatrix} = R\left(\frac{\partial\,\delta\ddot{x}}{\partial x}+\frac{\partial\,\delta\ddot{y}}{\partial y}+\frac{\partial\,\delta\ddot{z}}{\partial z}\right) = R\,\mathrm{div}\,\delta\dot{\mathfrak{v}}\,.$$

Now, if the medium is not compressible, so the condition equation is $R - 1 = 0$, then differentiating twice and perform another variation of that equation will produce:

$$\delta\frac{d^2}{dt^2}\,(R-1) = 0\,, \tag{12}$$

and an application of D'ALEMBERT's principle will give:

$$\delta R = 0\,. \tag{12.a}$$

We would like to pass over a direct derivation of the differential equations that is based upon that notion, which would then establish the x, y, z as functions of a, b, c, t ([1]), and move on to a second one for which ([2]):

$$\dot{\varphi} \equiv \mathrm{div}\,\mathfrak{v} = 0 \tag{13}$$

is the incompressibility condition. Since, one has, in general (10):

$$\frac{d}{dt}(R-1) = R\,\mathrm{div}\,\mathfrak{v}\,,$$

$$\frac{\partial(\rho u)}{\partial x}+\frac{\partial(\rho v)}{\partial y}+\frac{\partial(\rho w)}{\partial z}+\frac{\partial\rho}{\partial t} = \mathrm{div}\,(\rho\,\mathfrak{v}) + \frac{\partial\rho}{\partial t} = 0\,.$$

([1]) One finds it carried out by J. LARMOR in BASSET, *Hydrodynamics* (two vols., 1888), v. I, pp. 32.

([2]) We put a dot over φ in order to suggest that we are dealing with an equations in velocities.

just as one has $R - 1 = 0$ or $\dot{R}\,dt = 0$, one also has div $\mathfrak{v} = 0$ as the geometric, and indeed, *holonomic*, equation of constraint. In order to apply the principle of least constraint, one must define the first differential quotient of (13) with respect to t and its variation, so $\delta\ddot{\varphi} = \delta\,\mathrm{div}\,\dot{\mathfrak{v}}$, multiply it by λ and integrate over the mass (art. **18**), add it to $\delta(m\,f^2/2)$, and set it equal to zero. We will do that in art. **25**. However, if one employs D'ALEMBERT's principle, instead of it, then one must also formally replace the equation of constraint:

$$\mathrm{div}\;\delta\dot{\mathfrak{v}} \equiv \frac{\partial\,\delta\ddot{x}}{\partial x} + \frac{\partial\,\delta\ddot{y}}{\partial y} + \frac{\partial\,\delta\ddot{z}}{\partial z} = 0 \tag{14}$$

[which corresponds to the formal connection between formulas (5), (7) of art. **18**] with:

$$\mathrm{div}\;\delta\,\mathfrak{s} \equiv \frac{\partial\,\delta x}{\partial x} + \frac{\partial\,\delta y}{\partial y} + \frac{\partial\,\delta z}{\partial z} = 0\,, \tag{15}$$

in which:

$$\delta\,\mathfrak{s} = (\delta x,\ \delta y,\ \delta z)$$

is the variation of the displacement of the mass-element that occupied the location x, y, z at time t. Hence, div $\delta\,\mathfrak{s} = \delta\Theta$ is the dilatation of a unit volume (art. **21**) that results from a (virtual) displacement, and $\mathfrak{s}$ is the vector that previously denoted by (u, v, w).

In the case of *elastic media*, when one always varies only the accelerations and notes that $\delta R = 0$ will imply that $\delta\dot{R} = 0$ and also $\delta\rho = 0$ will imply that $\delta\dot{\rho} = 0$, the continuity equation will yield:

$$\frac{\delta\ddot{R}}{R} + \frac{\delta\ddot{\rho}}{\rho} = 0\,, \tag{16}$$

which is a relation that can again be replaced with:

$$\frac{\delta R}{R} + \frac{\delta\rho}{\rho} = 0\,, \tag{16.a}$$

when one goes over to D'ALEMBERT's principle, or with:

$$\mathrm{div}\;\delta\mathfrak{s} + \frac{\delta\rho}{\rho} = \delta\Theta + \frac{\delta\rho}{\rho} = \frac{\partial\,\delta x}{\partial x} + \frac{\partial\,\delta y}{\partial y} + \frac{\partial\,\delta z}{\partial z} + \frac{\delta\rho}{\rho} = 0\,, \tag{17}$$

resp.

24. – An integral relation. Gauss's law.

The proof of the differential equations of hydrodynamics that will be given in art. **25** is based upon an important relation between a volume integral and a surface integral that appears frequently.

In a space T whose outer surface Σ (analogous to the boundary contour of the accompanying planar surface patch) can also be decomposed into pieces, let 𝔳 be a vector with the components u, v, w that are continuous, and just like:

$$\operatorname{div} \mathfrak{v} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z},$$

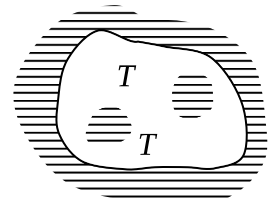

Figure 10.

they are finite, single-valued functions of position x, y, z in T. If λ is such a scalar function then the equation will exist:

$$\begin{aligned}\int \lambda \operatorname{div} \mathfrak{v}\, d\tau &= \int \lambda \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) d\tau \\ &= - \int \left(\frac{\partial \lambda}{\partial x} u + \frac{\partial \lambda}{\partial y} v + \frac{\partial \lambda}{\partial z} w \right) d\tau - \int \lambda [u \cos(n, x) + v \cos(n, y) + w \cos(n, z)]\, d\sigma, \end{aligned} \tag{1}$$

which can be proved directly, in which the spatial integral extends over T and the surface integral extends over the surface Σ. (n, x) is the angle between the inward-pointing normal to the space T and the X-axis.

In order to prove that, we convert the term on the left into:

$$\int \lambda \frac{\partial u}{\partial x} d\tau = \iint dy\, dz \int \lambda \frac{\partial u}{\partial x} dx$$

by partial integration. One has:

$$\int \lambda \frac{\partial u}{\partial x} dx = [\lambda\, u] - \int \frac{\partial \lambda}{\partial x} u\, dx,$$

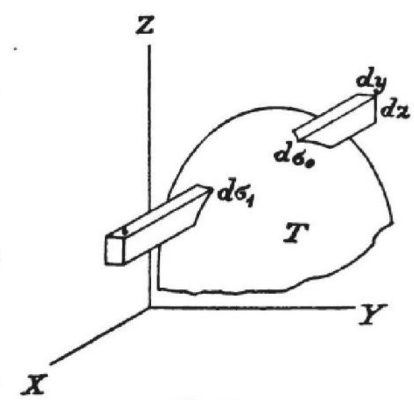

Fig. 11.

in which the square bracket means that they are taken for those (two or a larger even number of) locations, which are chosen to be upper (lower, resp.) limits, at which the parallelepiped (see the figure) that is erected over the rectangle dy, dz parallel to the X-axis goes through the outer surface Σ of T. If one denotes those locations with the indices 0, 1, 2, … then one will have:

$$\int \lambda \frac{\partial u}{\partial x} d\tau = - \int \frac{\partial \lambda}{\partial x} u\, d\tau + \iint [-\lambda_0 u_0 + \lambda_1 u_1 - \lambda_2 u_2 + \cdots]\, dy\, dz\ . \tag{2}$$

The base surface *dy dz* of the parallelepiped can now be regarded as the projection of the surface elements $d\sigma_0$, $d\sigma_1$, $d\sigma_2$, … that intersect it, and one therefore sets:

$$dy\,dz = d\sigma_0 \cos(n_0, x) = -\,d\sigma_1 \cos(n_1, x) = \dots,$$

in which *n* is the normal to the surface element *ds* that point *inward* to T. With that, the last integral (2) will go to:

$$\iint [-\lambda_0\,u_0 + \lambda_1\,u_1 - \cdots]\,dy\,dz = -\int \lambda\,u \cos(n, x)\,d\sigma\,,$$

which extends over all surface elements $d\sigma$, but each of them only once.

One thus obtains the first term on the right-hand side (1).

For $\lambda = 1$, the relation that was proved will go to:

$$\int \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) d\tau = -\int \lambda\,u \cos(n, x)\,d\sigma\,,$$

which can also be presented in the form:

$$\int \operatorname{div} \mathfrak{v}\,d\tau = -\int |\mathfrak{v}| \cos(\mathfrak{v}, n)\,d\sigma\,,$$

or

$$\int \operatorname{div} \mathfrak{v}\,d\tau = -\int \mathfrak{v}_n\,d\sigma\,, \tag{3}$$

where $|\,\mathfrak{v}\,|$ is the absolute value of $\mathfrak{v}$ and $\mathfrak{v}_n$ is the projection of $\mathfrak{v}$ onto the normal that points into T.

That relation, which belongs to the foundations of vector analysis, is called GAUSS's *theorem* (GAUSS's *Werke*, V, pp. 211). One thus converts an oft-appearing (cf., e.g., arts. **27**, **30**) volume integral into an integral that extends over the outer surface of the volume.

The general relation (1), in vectorial form, reads:

$$\int \lambda \operatorname{div} \mathfrak{v}\,d\tau = -\int (\operatorname{grad} \lambda, \mathfrak{v})\,d\tau - \int \lambda\,\mathfrak{v}_n\,d\sigma\,, \tag{4}$$

in which (grad λ, $\mathfrak{v}$) once more means the scalar product [art. **23**, (6)].

§ 25. – The equations of motion for incompressible fluids.

We shall now apply the formulas of art. **19** to incompressible masses upon which external (applied) forces per unit mass (X, Y, Z) might act. However, we would not like to appeal to

D'ALEMBERT's principle, but to the principle of least constraint [arts. **14**, (10), **18**, (5)], in the form:

$$\int [\rho(\ddot{x}-X)\,\delta\ddot{x}+\rho(\ddot{y}-Y)\,\delta\ddot{y}+\rho(\ddot{z}-Z)\,\delta\ddot{z}-\lambda\,\delta\ddot{\varphi}]\,d\tau-\delta\ddot{S} = 0\,, \tag{1}$$

in which the acceleration components $\ddot{x} = du / dt$, etc., are thought of as functions of x, y, z, t, and ρ is the mass density, which is constant everywhere throughout. The condition for incompressibility:

$$\dot{\varphi} = \text{div}\,\mathfrak{v} = 0 \tag{2}$$

will imply (art. **23**) the relation between the variations ([1]):

$$\delta\ddot{\varphi} = \frac{\partial\,\delta\ddot{x}}{\partial x}+\frac{\partial\,\delta\ddot{y}}{\partial y}+\frac{\partial\,\delta\ddot{z}}{\partial z}$$

$$= \frac{\partial}{\partial x}\,\delta\frac{du}{dt}+\frac{\partial}{\partial y}\,\delta\frac{dv}{dt}+\frac{\partial}{\partial z}\,\delta\frac{dw}{dt} = 0\,. \tag{2.a}$$

Let the volume element be:

$$d\tau = dx\,dy\,dz\,,$$

so the integration extends over the space that is filled with fluid at time t. One treats the term $\int\lambda\,\delta\ddot{\varphi}\,d\tau$ with the process that was given at the end of art. **18**.

If one replaces $\mathfrak{v}$ with $\delta\dot{\mathfrak{v}}$ in formula (1) of the previous article then that will give:

$$\int\lambda\,\delta\ddot{\varphi}\,d\tau = \int\lambda\left(\frac{\partial\,\delta\dot{u}}{\partial x}+\frac{\partial\,\delta\dot{v}}{\partial y}+\frac{\partial\,\delta\dot{w}}{\partial z}\right)d\tau$$

$$= -\int\left(\frac{\partial\lambda}{\partial x}\,\delta\dot{u}+\frac{\partial\lambda}{\partial y}\,\delta\dot{v}+\frac{\partial\lambda}{\partial z}\,\delta\dot{w}\right)d\tau-\int\lambda[\delta\dot{u}\cos(n,x)+\delta\dot{v}\cos(n,y)+\delta\dot{w}\cos(n,z)]\,d\sigma\,, \tag{3}$$

in which the last integral extends over the surface of the volume T.

The expression $\delta\ddot{S} = \bar{X}\,\delta\ddot{x}+\bar{Y}\,\delta\ddot{y}+\bar{Z}\,\delta\ddot{z}$ in (1) will drop out when the surface of the fluid is free, or when it is defined by a fixed smooth wall. That is because in the former case, $\bar{X} = \bar{Y} = \bar{Z} = 0$, while in the latter, since the pressure on the fluid is normal to the surface element, it will behave like:

$$\bar{X} : \bar{Y} : \bar{Z} = \cos(n, x) : \cos(n, y) : \cos(n, z)\,.$$

([1]) With the use of D'ALEMBERT's principle, that relation will be replaced by the following one (art. **23**, at the end):

$$\delta\varphi \equiv \text{div}\,\delta\mathfrak{s} = \frac{\partial\,\delta x}{\partial x}+\frac{\partial\,\delta y}{\partial y}+\frac{\partial\,\delta z}{\partial z} = 0\,.$$

One can then think of the term $\delta \ddot{S}$ as being combined with the last term in (3).

If one introduces (3) into (1), combines the terms inside the triple integral that are multiplied by $\delta\dot{u}$, $\delta\dot{v}$, $\delta\dot{w}$, resp., and sets their sums equal to zero, and finally compares the surface integral to zero, as well, then one will get the following system of *differential equations for hydrodynamics:*

$$\begin{aligned} \frac{du}{dt} + \frac{1}{\rho}\frac{\partial\lambda}{\partial x} &= X, \\ \frac{dv}{dt} + \frac{1}{\rho}\frac{\partial\lambda}{\partial y} &= Y, \\ \frac{dw}{dt} + \frac{1}{\rho}\frac{\partial\lambda}{\partial z} &= Z, \end{aligned} \tag{4}$$

and the condition for the surface:

$$\int \lambda[\delta\dot{u}\cos(n,x) + \delta\dot{v}\cos(n,y) + \delta\dot{w}\cos(n,z)]\,d\sigma \equiv \int \lambda\,|\,\delta\dot{\mathfrak{v}}\,|\cos(\delta\dot{\mathfrak{v}},n)\,d\sigma = 0, \tag{5}$$

in which the variation $\delta\dot{\mathfrak{v}}$ is chosen according to the constraints on the problem (art. **4**). In general, the incompressibility condition enters into that:

$$\operatorname{div} \mathfrak{v} = 0. \tag{5.a}$$

The quantity $\partial\lambda / \partial x$ has the dimension of $\rho\dot{u} = \rho\ddot{x}$. λ will then have the dimension:

$$[\lambda] = [l^{-2}\,m\,t^{-2}][l] = \left[\frac{l\,m\,t^{-2}}{l^2}\right]$$

of a force divided by an area, i.e., a *surface pressure* (Intro.). λ is obviously the pressure that is exerted upon a unit area inside of the fluid mass at the location x, y, z, and indeed with any orientation, since λ is a scalar quantity, not a directed one.

Upon introducing the values for $\dot{u}$, $\dot{v}$, $\dot{w}$ in [art. **23**, (5)], equations (4) will take the form:

$$\begin{aligned} \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x}u + \frac{\partial u}{\partial y}v + \frac{\partial u}{\partial z}w + \frac{1}{\rho}\frac{\partial\lambda}{\partial x} &= X, \\ \frac{\partial v}{\partial t} + \frac{\partial v}{\partial x}u + \frac{\partial v}{\partial y}v + \frac{\partial v}{\partial z}w + \frac{1}{\rho}\frac{\partial\lambda}{\partial y} &= Y, \\ \frac{\partial w}{\partial t} + \frac{\partial w}{\partial x}u + \frac{\partial w}{\partial y}v + \frac{\partial w}{\partial z}w + \frac{1}{\rho}\frac{\partial\lambda}{\partial z} &= Z. \end{aligned} \tag{6}$$

They are called the EULER form of the differential equations for hydrodynamics, and in conjunction with the condition (5.a), they serve to determine the velocity components u, v, w, and the pressure λ as continuous and finite functions of the quantities x, y, z, t when the fluid motion is determined by suitable initial and boundary conditions. The outer surface condition (5) cannot contradict the latter.

If the outer surface of the fluid is bounded by a smooth vessel wall that satisfies the equation:

$$\psi(x, y, z) = 0$$

then by differentiating that equation with respect to t twice and subsequent variation, in which however (art. **18**), the coordinates and velocities are *not* varied, one can derive the following condition for the variation $\delta\dot{\mathfrak{v}}$ of the vector $\dot{\mathfrak{v}}$:

$$\frac{\partial\psi}{\partial x}\delta\dot{u} + \frac{\partial\psi}{\partial y}\delta\dot{v} + \frac{\partial\psi}{\partial z}\delta\dot{w} = 0 ,$$

which says that $\delta\dot{\mathfrak{v}}$ must contact the surface. Equation (5) will also be satisfied, since cos ($\delta\dot{\mathfrak{v}}$, n) = 0 . However, $\delta\dot{\mathfrak{v}}$ can have any direction and magnitude at any location where the surface is free. Thus, e.g., equation (5) will also be fulfilled in such a way that $\lambda = 0$, i.e., no surface tension at all is present, so the surface will be free.

One derives the LAGRANGIAN (or first) form of the differential equations of hydrodynamics from the EULERIAN (or second) form when one replaces $\dot{u}$ with $d^2x / dt^2 = \ddot{x}$, etc., in equations (4) and regard x, y, z as functions of a, b, c (as in art. **17**). If one then multiplies the three equations by $\frac{\partial x}{\partial a}$, $\frac{\partial y}{\partial a}$, $\frac{\partial z}{\partial a}$, in succession, and adds them, then one will get:

$$(\ddot{x} - X)\frac{\partial x}{\partial a} + (\ddot{y} - Y)\frac{\partial y}{\partial a} + (\ddot{z} - Z)\frac{\partial z}{\partial a} + \frac{1}{\rho}\frac{\partial\lambda}{\partial a} = 0 .$$

If one assumes that there are external (applied) volume forces for which a force function U exists, such that:

$$X = \frac{\partial U}{\partial x}, \qquad Y = \frac{\partial U}{\partial y}, \qquad Z = \frac{\partial U}{\partial z},$$

and one also thinks of the x, y, z in U as being expressed in terms of a, b, c then that equation will go to the first of the following equations, which are called the LAGRANGIAN *differential equations* of hydrodynamics (although they also go back to EULER):

$$\begin{aligned}
\frac{d^2x}{dt^2}\frac{\partial x}{\partial a} + \frac{d^2y}{dt^2}\frac{\partial y}{\partial a} + \frac{d^2z}{dt^2}\frac{\partial z}{\partial a} + \frac{1}{\rho}\frac{\partial\lambda}{\partial a} &= \frac{\partial U}{\partial a} , \\
\frac{d^2x}{dt^2}\frac{\partial x}{\partial b} + \frac{d^2y}{dt^2}\frac{\partial y}{\partial b} + \frac{d^2z}{dt^2}\frac{\partial z}{\partial b} + \frac{1}{\rho}\frac{\partial\lambda}{\partial b} &= \frac{\partial U}{\partial b} ,
\end{aligned} \tag{7}$$

$$\frac{d^2x}{dt^2}\frac{\partial x}{\partial c}+\frac{d^2y}{dt^2}\frac{\partial y}{\partial c}+\frac{d^2z}{dt^2}\frac{\partial z}{\partial c}+\frac{1}{\rho}\frac{\partial \lambda}{\partial c} = \frac{\partial U}{\partial c}\,.$$

The incompressibility condition will take the form (art. **17**):

$$R - 1 = 0\,.$$

The internal pressure λ can then be calculated from the EULER form (6) of the differential equations, in general, when the fluid motion possesses a "velocity potential" $\varphi\,(x, y, z, t)$ and the external force X, Y, Z possesses a force function. Namely, if u, v, w are partial differential quotients of *one* function, and one introduces:

$$u = \frac{\partial \varphi}{\partial x}, \qquad v = \frac{\partial \varphi}{\partial y}, \qquad w = \frac{\partial \varphi}{\partial z},$$

and

$$X = \frac{\partial U}{\partial x}, \qquad Y = \frac{\partial U}{\partial y}, \qquad Z = \frac{\partial U}{\partial z}$$

into (6) then the first of those equations will go to:

$$\frac{\partial^2 \varphi}{\partial x\,\partial t}+\frac{\partial^2 \varphi}{\partial x^2}\frac{\partial \varphi}{\partial x}+\frac{\partial^2 \varphi}{\partial y\,\partial x}\frac{\partial \varphi}{\partial y}+\frac{\partial^2 \varphi}{\partial z\,\partial x}\frac{\partial \varphi}{\partial z}+\frac{1}{\rho}\frac{\partial \lambda}{\partial x} = \frac{\partial U}{\partial x}\,.$$

If one multiplies that by dx, and the other two, similarly converted, by dy, dz, resp., and adds them then one will get:

$$d\,\frac{\partial \varphi}{\partial t}+\frac{\partial \varphi}{\partial x}\,d\,\frac{\partial \varphi}{\partial x}+\frac{\partial \varphi}{\partial y}\,d\,\frac{\partial \varphi}{\partial y}+\frac{\partial \varphi}{\partial z}\,d\,\frac{\partial \varphi}{\partial z}+\frac{1}{\rho}\,d\lambda = dU$$

as the increase along a line element, and integrating that will give the equation for the pressure λ:

$$\frac{\partial \varphi}{\partial t}+\tfrac{1}{2}\left[\left(\frac{\partial \varphi}{\partial x}\right)^2+\left(\frac{\partial \varphi}{\partial y}\right)^2+\left(\frac{\partial \varphi}{\partial z}\right)^2\right]+\frac{1}{\rho}\lambda = U + \text{const.}, \tag{8}$$

in which the constant can still include time.

§ 26. – Application.

The motion of a fluid that rotates around the axis of a circular-cylindrical vessel that is bounded by two stationary discs might serve as a simple example. We would like to calculate the pressure on the discs and the cylindrical sidewalls, while ignoring the effect of gravity and friction.

We make the cylinder axis the *Z*-axis of a rectangular coordinate system and set:

$$x^2 + y^2 = r^2 .$$

We let the velocity be everywhere parallel to the plane of the discs and equal to $q\,(r) = q$ everywhere at a distance of r from the axis. We will then have:

$$u = -q\frac{y}{r}\,, \qquad v = -q\frac{x}{r}\,, \qquad w = 0\,.$$

Due to the facts that:

$$\frac{\partial u}{\partial x} = -\frac{xy}{r}\frac{\partial}{\partial r}\left(\frac{q}{r}\right), \qquad \frac{\partial u}{\partial y} = -\frac{y^2}{r}\frac{\partial}{\partial r}\left(\frac{q}{r}\right) - \frac{q}{r}\,,$$

$$\frac{\partial v}{\partial x} = \frac{x^2}{r}\frac{\partial}{\partial r}\left(\frac{q}{r}\right) + \frac{q}{r}\,, \qquad \frac{\partial v}{\partial y} = \frac{xy}{r}\frac{\partial}{\partial r}\left(\frac{q}{r}\right),$$

the condition for incompressibility div $\mathfrak{v} = 0$ will be fulfilled. Furthermore, all partial derivatives with respect to time (art. **23**) will vanish for *stationary* motion. Hence, equations (6) of the previous article will give the pressure λ on a unit area:

$$q^2\frac{x}{r^2} - \frac{1}{\rho}\frac{\partial \lambda}{\partial x} = 0\,, \qquad q^2\frac{y}{r^2} - \frac{1}{\rho}\frac{\partial \lambda}{\partial y} = 0\,,$$

which will give:

$$\lambda - \lambda_0 = \int\left(\frac{\partial \lambda}{\partial x}dx + \frac{\partial \lambda}{\partial y}dy\right) = \rho\int_0^r \frac{q^2}{r}dr\,,$$

when λ_0 (≥ 0) is the pressure along the axis ($r = 0$). The pressure:

$$\lambda = \lambda_0 + \rho\int_0^a \frac{q^2}{r}dr$$

acts upon the outer wall of the cylinder, when a is the radius of the upper and lower boundary surface. We shall now make the assumption that the angular velocity ω is the same everywhere, so the fluid rotates like a rigid body. We will then have:

$$q = \omega \cdot r,$$

and the pressure at a distance r will be:

$$\lambda = \lambda_0 + \tfrac{1}{2}\rho\omega^2 r^2 .$$

An infinitely-thin annular piece of the surface of the bottom of radius r and width dr will experience a pressure of $\lambda \cdot 2\pi r\, dr$, and therefore the bottom will experience the total pressure:

$$\Lambda = 2\pi\int_0^a \lambda r\, dr = \pi\left(\rho\frac{\omega^2 a^4}{2} + \lambda_0 a^2\right) .$$

On average, a pressure:

$$\lambda_1 = \frac{\Lambda}{\pi a^2} = \lambda_0 + \frac{\rho\omega^2 a^2}{4}$$

then acts upon a unit area, and on the sidewall:

$$\lambda = \lambda_0 + \frac{\rho\omega^2 a^2}{2} .$$

If the sidewall and top are variable, but in such a way that the cylinder always deforms into a cylinder of the same volume, then an overpressure:

$$\lambda - \lambda_1 = \frac{\rho\omega^2 a^2}{4}$$

would act upon the sidewall such that the top and bottom would approach each other.

If h is the height of the cylinder then the kinetic energy of the fluid mass will be:

$$T = \int \frac{\rho}{2}(u^2 + v^2)\, d\tau = \rho h \int_0^a q^2 r\, \pi\, dr = \rho h \pi \omega^2 \frac{a^4}{4} ,$$

so it is just as large as when the cylinder was rigid, which was to be expected.

If one sets the angular velocity vector that was defined above (art. **22**) for the individual fluid particles equal to:

$$\mathfrak{w} = \tfrac{1}{2} \operatorname{rot} \mathfrak{v}$$

then one will have:

$$\xi = \eta = 0 , \qquad \zeta = \tfrac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = \frac{q}{r} + \frac{r}{2}\frac{\partial}{\partial r}\left(\frac{q}{r}\right) = \omega .$$

Hence, although the fluid mass moves like a rigid body, so the mutual positions and orientations of the individual particles do not change, nonetheless, each of them possess a rotating motion with the same angular velocity ω. In fact, under a single rotation of the cylinder around its axis, each particle will experience a single rotation around its own axis, which is similar to the way that the Moon rotates when it orbits the Earth once.

§ 27. – Kinematics of fluids. Streamlines and vortex lines.

In *Mécanique analytique*, t. II, sect. 10 ([1]), one finds the statement:

> "One has Euler to thank for the first general equations for the motion of fluids, as represented in the clear and simple notation of partial differentials. With that discovery, all of the mechanics of fluids was brought down to a single point in analysis, and when the equations that one obtains are integrable, one can determine completely all states of motion and the effects of arbitrary forces on a moving fluid in any case. However, it is, unfortunately, so rebellious that one can achieve that goal only in very special cases, up to now."

Although that remark still holds true, in essence, nonetheless, since LAGRANGE, hydrodynamics has been enriched by some branches of knowledge that have enlarged the sphere of accessible problems appreciably. Among them, one poses, in particular, the problem of the *geometry of the motion of fluid masses* of separating the temporal changes of state (i.e., velocity) from the *positional* ones, and investigating the mutual positions of the neighboring mass-particles that advance and rotate with the help of the conditions for continuity, conservation of mass, and others.

However, the representation of those relationships will be eased considerably when one appeals to the manner of expression that one finds in *vector analysis*. Hydrodynamics itself has contributed much to the development of that new and powerful branch of mechanics. Arising from concepts and theorems that are drawn from various parts of physics and freed from random details and melded into a unified algorithm, vector analysis has succeeded in making coordinate analysis debatable in broad domains of mechanics.

The new tool takes on ever more meaning in the mechanics of space-filling masses. However, since all of the older literature since LAGRANGE appeals to the language of coordinates, we would like to employ the two representations in tandem without, in that way, underestimating the advantage that vector analysis possesses due to the fact that its statements are mostly independent of the choice of coordinate system, so they have an "invariant" character under changes of coordinates.

The theorems of GAUSS and STOKES, which we will learn about in the following article, belong those tools. In art. **24**, GAUSS's law gave us the means to convert a certain volume integral into a surface integral. It consists of the relation [art. **24**, (3)]:

$$\int \operatorname{div} \mathfrak{v} \, d\tau = - \int \mathfrak{v}_n \, d\sigma \,, \tag{1}$$

([1]) From the German edition by SERVUS, Berlin, 1887.

whose meaning is best understood in the applications that we shall now make.

The trajectory of a fluid particle that was defined above (art. **23**) is described in the course of time. By contrast, a "streamline" has meaning only at a particular moment in time. When one imagines a point x, y, z in the interior of an (elastic or also incompressible) fluid that carries an infinitely-small line element in the direction of the velocity vector and goes from it to a neighboring point, where that construction is repeated, etc., one will then obtain a *streamline*, whose elements satisfy the differential equations:

$$dx : dy : dz = u : v : w .$$

Other starting points produce other streamlines, and an ∞^2-family of them will then run through a space that is filled with a moving fluid. They collectively define what one calls a *vector field*, so they fill up a space in which the vector $\mathfrak{v}$ represents the state of motion. In general, that vector field will change in time. In what follows, we shall initially imagine that a given point in time is fixed.

Two streamlines will run infinitely-close to each other along their entire extent when that is true at one location, since the vector $\mathfrak{v}$ above (art. **17**, at then end) was assumed to be a continuous function of the coordinates. Furthermore, two streamlines can intersect only at a location where $\mathfrak{v} = 0$, and where the direction of advance is consequently undetermined. At any point of a streamline, we construct a surface element dq that is perpendicular to $\mathfrak{v}$ and through whose boundary points we lay streamlines. They define a tube (current tube, current filament) whose interior we think of as filled with streamlines in such a way that at any location, the number of them that meet a cross-section is measured by the quantity $k \mid \mathfrak{v} \mid dq$, where k is a positive constant and $\mid \mathfrak{v} \mid = \sqrt{u^2 + v^2 + w^2}$ is the magnitude of $\mathfrak{v}$ at that location (or rather, the nearest whole number to that quantity). In that way, the fluid mass can be decomposed into current tubes in whose interiors the density of streamlines cannot be variable.

If one bounds any space T in the interior of the fluid mass with the surface Σ then a current tube that enters into T will once more exit it once it has intersected Σ an even number of times. If dq is the cross-section of the tube at a point of intersection and $d\sigma$ is the surface element that is cut out from it, while n is the normal to Σ that points into T, then one will have:

$$dq = \pm\, d\sigma \cos(\mathfrak{v}, n) ,$$

in which the – sign corresponds to negative values of the cosine, so to exiting streamlines.

If one now defines the sum:

$$\int k \mid \mathfrak{v} \mid \cos(\mathfrak{v}, n)\, d\sigma$$

then the positive part of the summand will represent the number of streamlines that enter into T, while the negative part will represent the number that exit it, so the integral is the excess of entering streamlines over the exiting ones. However, from GAUSS's law (cf., *supra*):

$$-k\int |\mathfrak{v}| \cos(\mathfrak{v}, n)\, d\sigma = -k\int \mathfrak{v}_n\, d\sigma = k\int \text{div}\, \mathfrak{v}\, d\tau\,. \tag{2}$$

Therefore, the *volume integral that is extended over* T:

$$k\int \text{div}\, \mathfrak{v}\, d\tau = k\int \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) d\tau \tag{2.a}$$

also represents the excess in the number of exiting streamlines over the entering ones. If T = 1 / *k*, and div $\mathfrak{v}$ is constant inside of the space, which is assumed to be small, then div $\mathfrak{v}$ itself will represent the number of exiting streamlines.

In particular, in an *incompressible fluid*, for which one has:

$$\text{div}\, \mathfrak{v} = 0$$

for any small enough space, so that excess vanishes, a streamline can never begin or end, and one finds just as many streamlines in any cross-section to a current tube. Those tubes must therefore close on themselves into annuli or end on the outer surface.

By contrast, *in elastic fluid masses* with alternating density ρ, at those locations where $\dot{\rho}$ is positive, so the density is thought of as increasing, and where:

$$\frac{\dot{\rho}}{\rho} = -\,\text{div}\, \mathfrak{v}$$

is therefore positive, more streamlines go in than out. One calls those locations *sinks*, while ones where div $\mathfrak{v}$ is positive, so more streamlines *exit* than enter, are *sources*. That explains the word "divergence."

Just as one derives a "mass-point" in the study of ponderable masses as an abstraction from space-filling masses of small dimensions, here one can also arrive at *point-like* sources and sinks. For fluids in reality, sources and sinks appear with only a limited productivity or suction power. If one assumes that they are effective without limit then one must make up one's mind to assume that the influx or outflux has a spatial dimension that is inaccessible. That will be done in the example in art. **32**.

When one bounds a current tube in an *incompressible* fluid (div $\mathfrak{v}$ = 0) with infinitely-small cross-sections with areas q_0 and q_1 that run perpendicular to the stream-filament, and if the absolute values of the stream velocities at those locations are $|\mathfrak{v}_0|$ and $|\mathfrak{v}_1|$, resp., then an application of GAUSS's law to the spatial region T, thus-delimited, will give:

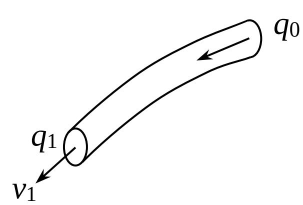

Figure 12.

$$|\mathfrak{v}_0|\, q_0 = |\mathfrak{v}_1|\, q_1\,, \tag{3}$$

because one has cos ($\mathfrak{v}$, n) = 0 along the sidewall of the tube, and the velocity can be regarded as constant in the small elements q_0, q_1 .

The cross-section of a stream-filament is then inversely proportional to the velocity that lives in it.

One can adapt that line of reasoning to any vector *whose divergence is zero* with no further analysis. One calls vectors $\mathfrak{v}$ for which div $\mathfrak{v}$ = 0 *solenoidal vectors* (*ό σωλήν* = tube). Obviously, it is associated with the previously-introduced "rotation" of another vector $\mathfrak{v}$, so:

$$\mathfrak{w} = \tfrac{1}{2} \operatorname{rot} \mathfrak{v} \tag{4}$$

is also a vector with components:

$$\xi = \tfrac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right),$$

$$\eta = \tfrac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right),$$

$$\zeta = \tfrac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$

that defines the angular or vorticial velocity of a fluid particle at the location where $\mathfrak{v} = (u, v, w)$ is the velocity of the advancing motion, as we saw before (art. **22**). As we will see later, we even have that *any* solenoidal vector can be regarded as the rotation of another vector. If one adapts the concepts of streamlines and stream-filaments that were presented for the vector $\mathfrak{v}$ above, along with the theorems that were true for it, to solenoidal vectors then one will arrive at the concepts of *vortex lines*, *vortex filaments*, for which the theorems that von HELMHOLTZ [Jour. f. Math **55** (1858)] proved are true, namely, that vortex lines must either close upon themselves or end at the outer surface of the fluid and that two cross-sections of a vortex filament of very small cross-section are inversely proportional to the angular velocities that live in them. One introduces the terms *vortex strength*, *vortex intensity*, or *vortex moment* for the product of the cross-section with the angular velocity.

Along with those kinematical theorems about vortices, there are also some kinetic ones that follow from the differential equations that we shall address in art. **33**.

§ 28. – Stokes's theorem.

The *potential vector*, whose rotation vanishes, and which will be treated in art. **29**, is, in a certain sense, the opposite of the solenoidal vector, whose divergence is zero. STOKES's theorem has the same significance for the potential vector that GAUSS's law has for the solenoidal vector. We shall present the former theorem. Its proof is based upon a known formula that goes back to CAUCHY (1846) and relates to the conversion of an integral over a planar "simply-connected" surface patch (namely, one whose total boundary consists of a non-self-intersecting curve that closes on itself) into a line integral.

If u, v are two continuous and single-valued functions of x, y at all points of such a planar surface Σ then the integral:

$$\int \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) d\sigma ,$$

which is extended over all surface elements $d\sigma$ on Σ, will become the following line integral:

$$\int (u\,dx + v\,dy) ,$$

which is extended over the entire boundary of Σ. Namely, in order to convert the part of it:

$$\int \frac{\partial v}{\partial x} d\sigma = \iint \frac{\partial v}{\partial x}\, dx\, dy ,$$

we divide the surface into strips that are parallel to the X-axis. The contribution of one of those strips that cuts out the element dy from the Y-axis to the value of the integral is:

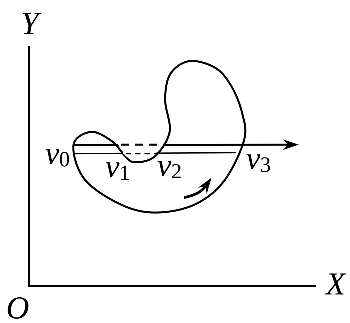

Figure 13.

$$dy \int \frac{\partial v}{\partial x} dx = dy\,(-v_0 + v_1 - v_2 + \dots) ,$$

when v_0, v_1, … mean the values of v at the two or more points whether the strip intersects the boundary curve. Upon integrating over all y that come under consideration, we will obviously exhaust all elements of the surface Σ and all elements of the boundary, and we will then get:

$$\int \frac{\partial v}{\partial x} d\sigma = \int v\,dy ,$$

when taken around that periphery, in which we understand dy to mean the projection onto the Y-axis of a line element ds of that periphery that goes around it in the sense of a right-hand rotation.

One similarly proves the second part of the following equation:

$$\int \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) d\sigma = \int (u\,dx + v\,dy) . \tag{1}$$

STOKES (SMITH Prize Examination, 1854) adapted that theorem to a curved piece of a surface. He first expressed it for an arbitrarily-placed planar region in space. When a planar triangle *ABC* (see Fig. 14) that is bounded by the coordinate planes of a rectangular system is projected onto those planes, it will give the triangles *OBC*, *OCA*, *OAB*, whose areas are known to be derived from that of *ABC* by multiplying the latter by the cosines of the inclination angles the normal *n* to *ABC* that is dropped from *O* defines with the corresponding coordinate axes. Now, if *u*, *v*, *w* are again three continuous functions of position that we would like to regard as components of the vector $\mathfrak{v}$, and we extend the line integral:

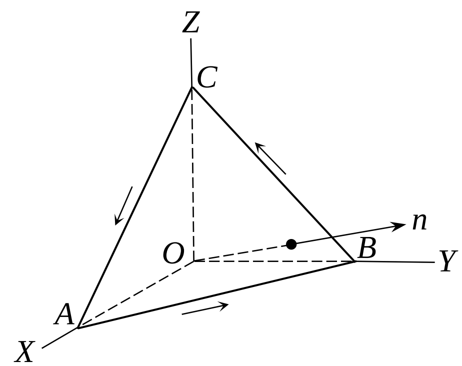

Figure 14.

$$\int (u\,dx + v\,dy + w\,dz)$$

over the peripheries of the three triangles *OBCO*, *OCAO*, *OABO*, in succession, in the sense that is indicated by the symbols then the theorem that was just proved can be applied to any of them: For *OBCO*, we have:

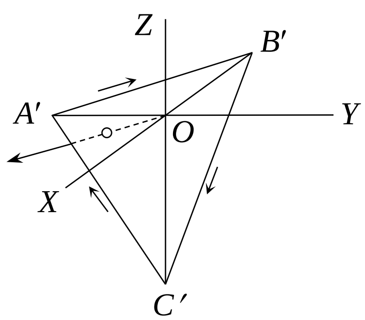

Figure 15.

$$\iint \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} \right) dy\,dz = \int (v\,dy + w\,dz)\,. \tag{2}$$

If one adds the three surface and line integrals thus-obtained and remarks that the latter sum can be replaced by just the line integral over the periphery of the triangle *ABC*, because the positive and negative integrals that are taken along the lines *OA*, *OB*, *OC* will cancel, then when one interprets the left-hand side according to (1) ([1]), that will give the equality of the following integral that is extended over the surface (periphery, resp.) of the triangle *ABC*:

$$\begin{aligned}
&\iint \left[\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} \right) dy\,dz + \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} \right) dz\,dx + \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) dx\,dy \right] \\
&\qquad = \int [2\xi \cos(n, x) + 2\eta \cos(n, y) + 2\zeta \cos(n, z)]\,d\sigma \\
&\qquad = \int (u\,dx + v\,dy + w\,dz)\,,
\end{aligned} \tag{3}$$

([1]) That not-entirely-rigorous conclusion will be flawless when ξ, η, ζ are constants, which is what will happen in the later transition to infinitely-small tetrahedra.

in which, as before, 2ξ, 2η, 2ζ are the components of $2\mathfrak{w}$ = rot $\mathfrak{v}$, $d\sigma$ is an element of the triangle *ABC*, and the direction of the normal *n*, together with the circuit around *ABC*, defines a right-hand system.

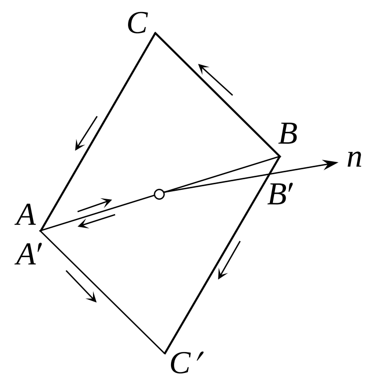

Figure 16.

Formula (3) is clearly also true for the triangle $A'B'C'$ that lies symmetric to *ABC* with respect to the origin, whereby the normal n' will again define a right-handed system with the sense of rotation. If one now reverses the direction of n' (so n' points in the same direction as *n*) and likewise inverts the sense of traversal from $A'B'C'$ to $A'C'B'$ then (3) will remain unchanged. Finally, if one combines the two triangles into a parallelogram by parallel-translating them to an arbitrary location in space, perhaps with a common side *AB* (see Fig. 16), then formula (3) will be true for both triangles, and therefore, since the line integrals along *AB* once more cancel, it will also be true for the parallelogram $AC'BC$.

However, an arbitrary curved surface patch (see Fig. 17) can also be decomposed into infinitesimal parallelograms of the type considered, when one intersects it with two systems of planes parallel to the coordinate planes (the *XZ* and *YZ*-planes, in the case above). If one sums those line integrals along the boundary to the infinitesimal parallelogram then the integrals along the internal lines will, in turn, cancel, and all that will remain is the line integral (3) on the right along the boundary contour of the surface patch, while the left-hand side of formula (3) will refer to the interior of the surface patch.

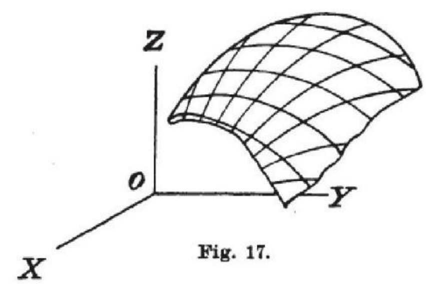

Fig. 17.

With that, the validity of formula (3) is proved for a surface patch of arbitrary form. One can give it the form:

$$\int (\text{rot}\,\mathfrak{v})_n \, d\sigma = \int (\mathfrak{v}, d\mathfrak{s}), \tag{3.a}$$

in which (rot $\mathfrak{v}$)$_n$ means the projection of rot $\mathfrak{v}$ onto the normal *n* that is drawn at the location $d\mathfrak{s}$, and ($\mathfrak{v}$, $d\mathfrak{s}$) is the scalar product (art. **23**) of $\mathfrak{v}$ and the element $d\mathfrak{s}$ of the boundary. That likewise proves the independence of the relation (3) of the coordinate system.

One calls the *relations* (3), (3.a), by which the surface integral on the left is converted into a boundary integral *STOKES's theorem.*

§ 29. – The potential vector.

When the vector $\mathfrak{v}$ possesses a velocity potential φ (*x*, *y*, *z*), so when one has (art. **23**, at the end):

$$u = \frac{\partial \varphi}{\partial x}, \qquad v = \frac{\partial \varphi}{\partial y}, \qquad w = \frac{\partial \varphi}{\partial z}, \tag{1}$$

one calls $\mathfrak{v}$ (art. **23**) the *gradient of* φ :

$$\mathfrak{v} = \text{grad } \varphi . \tag{1.a}$$

We immediately recognize that the vector $\mathfrak{v}$ has a relationship to the scalar function φ that is distributed through space that is independent of the coordinate system. We shall use equation (1.a) as a more concise way of writing (1). It follows from (1) that:

$$\text{rot } \mathfrak{v} = 0 , \tag{2}$$

so we have:

$$\text{rot grad } \varphi = 0 \tag{2.a}$$

for *the scalar function* φ. The property of the vector $\mathfrak{v}$ = grad φ that is expressed by formula (1) explains the terminology "potential vector" for $\mathfrak{v}$. However, in order to define that in a form that is independent of the coordinate system, one starts from the property that that is expressed by equation (2) and calls any vector *whose rotation vanishes a potential vector*. One can see that *conversely,* the property (1.a) that $\mathfrak{v}$ = grad φ *follows from* rot $\mathfrak{v}$ = 0 in the following way:

Let the quantities u, v, w be continuous, finite, single-valued functions of position within a space T that combine into a vector $\mathfrak{v}$ for which rot $\mathfrak{v}$ = 0. Let the spatial region T be *simply connected*, i.e., arranged so that a surface that lies completely within T can be laid through any closed curve that is traversed within T that is bounded by only that curve ([1]). If one extends the integral:

$$\int (u\,dx + v\,dy + w\,dz) = \int (\mathfrak{v}, d\,\mathfrak{s}) \tag{3}$$

along that curve then, from STOKES's theorem (see previous art.), it can be converted into the surface integral:

$$\int (\text{rot } \mathfrak{v})_n \, d\sigma ,$$

so it will vanish when:

$$\text{rot } \mathfrak{v} = 0$$

([1]) The example of a hollow cylinder T that is bounded by two circular rings shows that this requirement cannot always be fulfilled. That is because a circle that is traversed between the cylindrical sidewalls whose center is found on the axis is a curve through which no surface can be laid that is bounded by the curve alone and lies completely inside of T. One can convert T into a *simply-connected* region by a rectangle that lies in a plane through the axis and bounds the four boundary surfaces (as a "cross-section") when one adds it to the boundary (twice). T is called a "doubly-connected spatial region."

everywhere in T. Therefore, the line integral (3) will vanish on *any* closed integration path in T. On the other hand, if one takes the integral (3) from a fixed point M_0 (x_0, y_0, z_0) inside of T to another point M (x, y, z) in T in any way then the value of the integral cannot depend upon the choice of path, because two different paths from M_0 to M can be combined into a closed curve along which the integral will give the value zero, as was shown above. The integral from M_0 to M along the *one* path will then be equal to the integral from M to M_0 along *the other one*, but with a negative sign, or both of the integral from M_0 to M are equal to each other. Therefore, the value of the integral:

$$\int_{M_0}^{M} d\varphi = \int_{M_0}^{M} (u\,dx + v\,dy + w\,dz) = \varphi(x, y, z) - \varphi(x_0, y_0, z_0) \tag{4}$$

depends upon just the chosen boundary points and a single-valued function of them. However, equations (1) follow from (4). Q.E.D.

In particular, if one takes the integral $\int d\varphi$ along a line for which:

$$u = \frac{dx}{ds}, \qquad v = \frac{dy}{ds}, \qquad w = \frac{dz}{ds}$$

(so in the case where φ is a velocity potential, it will be a streamline) then $d\varphi = ds$ will always be positive. Therefore, *a fluid particle will always move from a place of lower velocity potential to one of higher velocity potential.*

One can also interpret u, v, w as the components of a *force*, instead of a velocity. If one then denotes them by X, Y, Z, and the force by $\mathfrak{P}$ then:

$$\varphi = \int_{M_0}^{M} (X\,dx + Y\,dy + Z\,dz) = \int_{M_0}^{M} |\mathfrak{P}|\,ds \cos(\mathfrak{P}, ds) = \int_{M_0}^{M} (\mathfrak{P}, d\mathfrak{s})$$

will be the so-called *force function* or *potential function* (*potential*, in the broader sense). As was mentioned before (art. **15**), it means the work done by the force $\mathfrak{P}$ along the path from M_0 to M and is composed of the *elementary works done* along the infinitesimal path-segments, which are represented by the product of the force, the path-element, and the cosine of the angle between them. One calls the surfaces at whose points the potential φ possesses one and the same numerical value c, so their equations are:

$$\varphi(x, y, z) = c\,,$$

level surfaces. They give one the means to determine the direction of the force $\mathfrak{P}$ at a given location, and therefore to construct the *lines of force*, into which the streamlines go with that interpretation of φ, so the construction of the vector field for $\mathfrak{P} = (X, Y, Z)$. Namely, since the direction of the normal n to the surface φ = const. at a point x, y, z of it is determined from:

$$\cos(n, x) : \cos(n, y) : \cos(n, z) = \frac{\partial\varphi}{\partial x} : \frac{\partial\varphi}{\partial y} : \frac{\partial\varphi}{\partial z} = X : Y : Z ,$$

then the normal will give the direction of the line of force precisely. It coincides with the *direction of the vector* grad φ. On the other hand, since the sum of the squares $\left(\frac{\partial\varphi}{\partial x}\right)^2 + \left(\frac{\partial\varphi}{\partial y}\right)^2 + \left(\frac{\partial\varphi}{\partial z}\right)^2$ is an invariant quantity under a rotation and displacement of the coordinate system, as a simple calculation will show, that will also be the case for grad φ, as was asserted above. If one changes the value of the constant c then one will get a system of level surfaces whose orthogonal trajectories are then the aforementioned lines. Conversely, any function φ that is continuous, finite, and single-valued inside of a region T, along with its first differential quotients, can be employed to determine a potential vector (grad φ). The function φ determines *a scalar field* in T: viz., the totality of all points that can be associated with finite values of φ in a single-valued way. At the same time, a vector field in T is defined by grad φ . If the space T is to not go beyond that field then one must exclude those points from T where the function φ or its differential quotients become infinite by surrounding them with closed surfaces.

We would like to address the construction of such a field T in a special case. Let the potential be:

$$\varphi = \frac{e}{r} + \frac{e_1}{r_1} ,$$

in which e, e_1 are positive or negative constants, and r, r_1 are the distances from an indeterminate point x, y, z (the "reference point") to two fixed centers O, O_1 with the coordinates a, b, c ; a_1, b_1, c_1, so:

$$r^2 = (a - x)^2 + (b - y)^2 + (c - z)^2 ; \qquad r_1^2 = (a_1 - x)^2 + (b_1 - y)^2 + (c_1 - z)^2 .$$

Since φ becomes infinite at O, O_1, we would like to exclude that point from T by placing small balls that surround them that we then count as part of the boundary Σ to the region T, which might extend to infinity everywhere else. The vector $\mathfrak{v} = (u, v, w)$ is now continuous in T, and its direction is undetermined only the "points of indifference," i.e., at the locations where $\mathfrak{v} = 0$ or where, at the same time:

$$\frac{\partial\varphi}{\partial x} = \frac{\partial\varphi}{\partial y} = \frac{\partial\varphi}{\partial z} .$$

In order to ascertain that, we lay the Z-axis along the connecting line OO_1 and the origin of the coordinate system at the point O_1 . One then has:

$$a = b = a_1 = b_1 = c = 0 ,$$

$$r^2 = x^2 + y^2 + z^2 ; \qquad r_1^2 = x^2 + y^2 + (z - c_1)^2 .$$

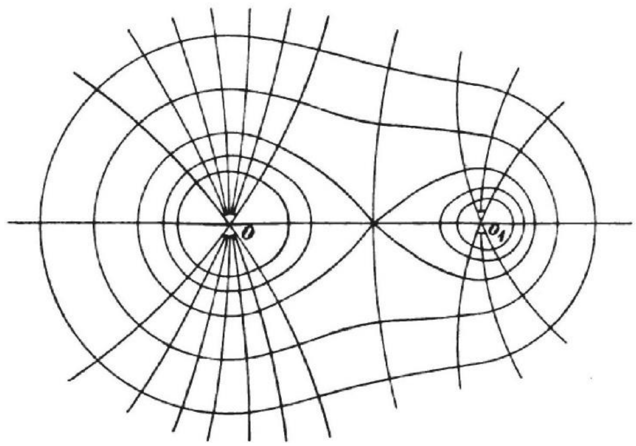

Figure 18.

The level surfaces:

$$\varphi \equiv \frac{e}{r} + \frac{e_1}{r_1} = \text{const.}$$

are surfaces of revolution with the *Z*-axis as the rotational axis, and they can be represented by their meridian curves, some of which are illustrated in the accompanying figure for the cases of $e = 2$, $e_1 = 1$ and $e = 2$, $e_1 = -1$ (from HOLZMÜLLER, *Das Potential*, Leipzig, 1898).

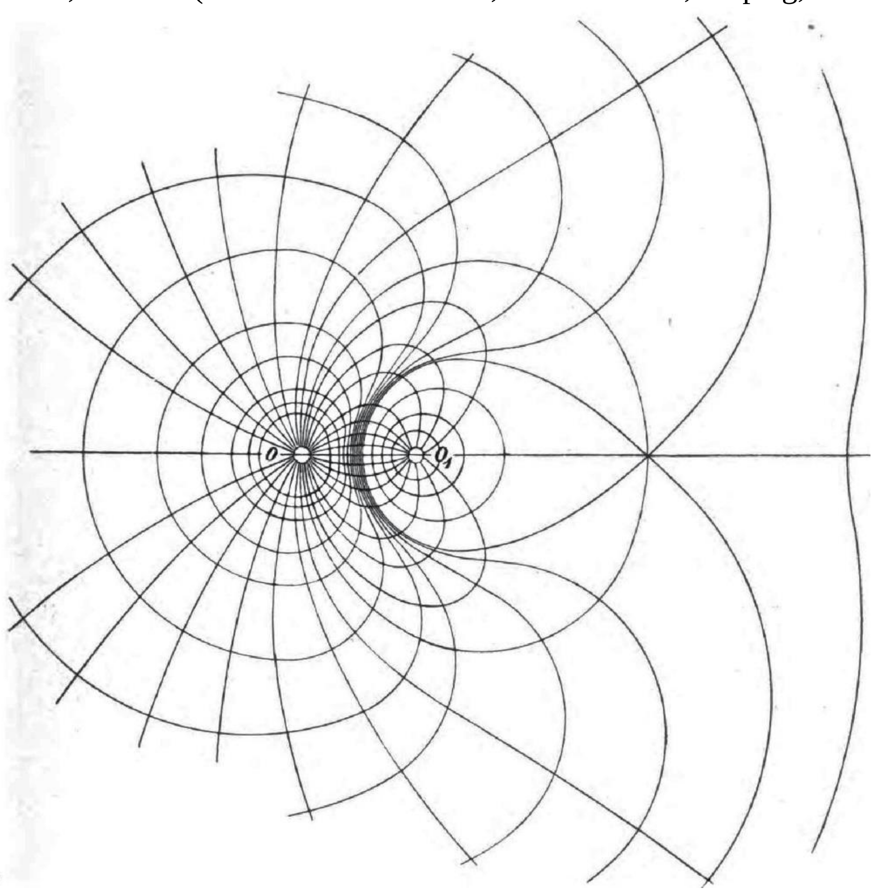

Figure 19.

At the points of indifference, one has:

$$\frac{e\,x}{r^3}+\frac{e_1\,x}{r_1^3}=0\,,\qquad \frac{e\,y}{r^3}+\frac{e_1\,y}{r_1^3}=0\,,\qquad \frac{e\,z}{r^3}+\frac{e_1\,(z-c_1)}{r_1^3}=0\,.$$

The first two expressions vanish only for $x = y = 0$, in addition to $\frac{e}{r^3}=-\frac{e_1}{r_1^3}$ (which will give c_1 = 0 when it is substituted in the third equation). If one then looks at the third equation and observes that one must have $c_1 > z$ when e, e_1 have the same name, so $r_1 = z - c_1$, then one will get:

$$z=\frac{c_1}{1+\lambda}\text{ when }e,\,e_1\text{ have the same name, and }z=\frac{c_1}{1-\lambda}\text{ when }\lambda=+\sqrt{\left|\frac{e_1}{e}\right|}\,.$$

If φ is a velocity potential then the motion when e and e_1 have the same signs (Fig. 18) will consist of either an influx of fluid towards the sinks $r = 0$, $r_1 = 0$ with "suctions" e and e_1 (art. **32**) or an outflux from those sources of "productivities" e and e_1. Fig. 19 corresponds to unequal signs for e and e_1.

§ 30. – Green's theorem. The Newtonian potential.

If φ, ψ are two scalar functions of position that are single-valued, continuous, and finite in any bounded (on the outside and inside) spatial region T, along with their first differential quotients, and the sum S means over the three terms in x, y, z, of which we write out only the first, so:

$$\mathsf{S}\frac{\partial}{\partial x}\left(\varphi\frac{\partial\psi}{\partial x}\right)=\mathsf{S}\frac{\partial\varphi}{\partial x}\frac{\partial\psi}{\partial x}+\varphi\,\mathsf{S}\frac{\partial^2\psi}{\partial x^2}=\mathsf{S}\frac{\partial\varphi}{\partial x}\frac{\partial\psi}{\partial x}+\varphi\Delta\psi\,,$$

in which $\Delta\psi$ is the known LAMÉ differential parameter:

$$\Delta\psi=\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}+\frac{\partial^2\psi}{\partial z^2}=\text{div grad }\psi\,. \tag{1}$$

If one integrates that equation over the entire space T and applies GAUSS's law [art. **24**, (3)]

$$\int\text{div}\,\mathfrak{v}\,d\tau=\int\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)d\tau=-\int\mathfrak{v}_n\,d\sigma$$

to the left-hand side, when referred to the vector φ grad ψ, then one will get:

$$-\int \varphi\left(\frac{\partial\psi}{\partial x}\cos(n,x)+\frac{\partial\psi}{\partial y}\cos(n,y)+\frac{\partial\psi}{\partial z}\cos(n,z)\right)d\sigma = -\int \varphi\frac{\partial\psi}{\partial x}d\sigma$$

$$= \int \mathsf{S}\frac{\partial\varphi}{\partial x}\frac{\partial\psi}{\partial x}d\tau+\int\varphi\,\Delta\psi\,d\tau\,, \tag{2}$$

in which n is the inward-pointing normal to the element $d\sigma$ of the outer surface Σ of the space T (which decomposes into pieces in some situations).

If one subtracts from that equation the one that one obtains from it by switching φ with ψ then that will give the following relation between a surface integral and a spatial integral:

$$\int\left(\psi\frac{\partial\varphi}{\partial x}-\varphi\frac{\partial\psi}{\partial x}\right)d\sigma = \int(\varphi\,\Delta\psi-\psi\,\Delta\varphi)\,d\tau\,. \tag{3}$$

When one sets the functions φ and ψ equal to each other then it will further follow from (2) that:

$$-\int\varphi\frac{\partial\psi}{\partial x}d\sigma = \int\left[\left(\frac{\partial\psi}{\partial x}\right)^2+\left(\frac{\partial\psi}{\partial y}\right)^2+\left(\frac{\partial\psi}{\partial z}\right)^2\right]d\tau+\int\varphi\,\Delta\varphi\,d\tau\,. \tag{4}$$

The relation (3) is called *Green's theorem.* It will serve to represent a function φ in the interior of a space T when one knows the value of $\Delta\varphi$ everywhere on the outer surface of T, along with the values of φ and $\partial\varphi / \partial n$ (which are mutually-independent, moreover).

To that end, one assigns ψ the special value:

$$\psi = \frac{1}{r}\,,$$

where:

$$r^2 = (a-x)^2 + (b-y)^2 + (c-z)^2,$$

and a, b, c are the coordinates of the location of the volume element $d\tau$. If one sets:

$$d\tau = da\;db\;dc\,,$$

in (3), accordingly, then one also writes φ, ψ in terms of a, b, c on the right-hand side, and sets:

$$\Delta\varphi = \frac{\partial^2\varphi}{\partial a^2}+\frac{\partial^2\varphi}{\partial b^2}+\frac{\partial^2\varphi}{\partial c^2} = -\,4\pi\,\mu\,(a,\,b,\,c)\,,$$

$$\Delta\psi = \frac{\partial^2\psi}{\partial a^2}+\frac{\partial^2\psi}{\partial b^2}+\frac{\partial^2\psi}{\partial c^2}\,,$$

in which μ might now be a known function of a, b, c.

Now if P is a point of T in general position whose coordinates x, y, z are to be introduced as the arguments in the function φ to be constructed (i.e., the "reference point") then φ can be ascertained as follows: A minor calculation will next give that:

$$\Delta\psi \equiv \Delta\left(\frac{1}{r}\right) \equiv \frac{\partial^2}{\partial a^2}\left(\frac{1}{r}\right)+\frac{\partial^2}{\partial b^2}\left(\frac{1}{r}\right)+\frac{\partial^2}{\partial c^2}\left(\frac{1}{r}\right) = 0 .$$

Since φ, ψ in (3) were assumed to be finite and continuous functions of a, b, c inside of the region T, but ψ will be infinite for $a = x$, $b = y$, $c = z$, along with its differential quotients, one must exclude the point P and its immediate vicinity from T, perhaps by surrounding P with a ball of very small radius α and not counting it as part of T. The outer surface of that ball must then be counted as part of the boundary Σ of T. In that way, T goes to T', and the surface Σ of T goes to the surface Σ' of T'. An inward-pointing normal to the spherical boundary of T' is then the exterior normal to the spherical surface, and indeed, one has:

$$\frac{\partial\psi}{\partial n} = \frac{\partial\psi}{\partial\alpha} = -\frac{1}{\alpha^2} .$$

The outer surface element $d\sigma$ of the ball can then be replaced with:

$$d\sigma = \alpha^2 d\omega ,$$

in which $d\omega$ is the outer surface element of a ball of radius 1. If one now cuts out the part of the outer surface integral on the left-hand side of (3) that refers to the small ball then (3) will go to:

$$\int\left(\frac{1}{r}\frac{\partial\varphi}{\partial n}-\varphi\frac{\partial}{\partial n}\frac{1}{r}\right)d\sigma+\alpha\int\frac{\partial\varphi}{\partial n}d\omega+4\pi\cdot\varphi(x,y,z) = -\int\frac{\Delta\psi}{r}d\tau ,$$

in which the first integral on the left is now extended over just the original boundary Σ. For vanishingly-small α, that will give:

$$4\pi\,\varphi(x, y, z) = -\int\frac{\Delta\psi}{r}d\tau-\int\left(\frac{1}{r}\frac{\partial\varphi}{\partial n}-\varphi\frac{\partial}{\partial n}\frac{1}{r}\right)d\sigma , \tag{5}$$

which achieves the general representation of φ in the form that was promised. Furthermore, it suffices (KIRCHHOFF, *Mechanik*, Lecture 16, § 5) to know φ on the surface, because $\partial\varphi / \partial n$ can be determined then, or conversely.

If we can now extend the space T *to infinity in all directions* by assuming that the external boundary surface Σ is a ball with an initially large radius $\mathfrak{R}$ whose center lies at a finite point then $d\sigma$ will go to:

$$d\sigma = \mathfrak{R}^2\, d\omega\,, \tag{5.a}$$

in which $d\omega$ is once more the element of the unit sphere. When a, b, c increase without bound and perhaps $\mathfrak{R} = \sqrt{a^2 + b^2 + c^2}$, the quantities:

$$\frac{\mathfrak{R}}{r}, \qquad \mathfrak{R}^2 \frac{\partial}{\partial n}\left(\frac{1}{r}\right)$$

will assume finite values when x, y, z remain finite. Therefore, if we make suitable assumptions about the behavior of $\varphi(a, b, c)$ and $\partial\varphi / \partial n$ at infinity then we can arrange that the outer surface integral on the right-hand side of (5) vanishes. For the moment, we would like to assume that this condition has been fulfilled. We will then get:

$$4\pi\,\varphi(x, y, z) = -\int \frac{\Delta\varphi}{r}\, d\tau \;=\; 4\pi \int \frac{\mu\, d\tau}{r}\,. \tag{6}$$

Now, in order to find the condition to be fulfilled, we multiply both sides of (6) by:

$$\mathfrak{R} = \sqrt{a^2 + b^2 + c^2}\,,$$

and let x, y, z and $\mathfrak{R}$ increase without bound in the integral over a, b, c :

$$\mathfrak{R}\,\varphi(x, y, z) = \int \frac{\mathfrak{R}}{r}\, \mu\, d\tau\,.$$

When μ is non-zero in only some *finite* spatial region, the right-hand side of that, and therefore $\mathfrak{R}\varphi$, will take on a finite value, and likewise the integral of:

$$\mathfrak{R}^2 \frac{\partial\varphi}{\partial l}\,,$$

when l means one of the coordinates x, y, z. Conversely, if one introduces the *condition that:*

$$\mathfrak{R}\,\varphi(x, y, z) \quad \text{and} \quad \mathfrak{R}^2 \frac{\partial\varphi(x, y, z)}{\partial l} \tag{7}$$

(which is written in terms of *a*, *b*, *c*, instead of *x*, *y*, *z*, moreover) *should not become infinite for increasing* $\mathfrak{R}$ into the outer surface integral on the right in (5) then, due to (5.a), the two terms will vanish individually, and (5) will be converted into:

$$\varphi(x, y, z) = \int \frac{\mu\, d\tau}{r}. \tag{8}$$

That expression for φ is derived from just the following *three assumptions:*

1. The function φ and its first differential quotients are continuous and finite functions in infinite space.

2. Its behavior at infinity is prescribed by the assumption (7).

3. The quantity:

$$\Delta\varphi = -4\pi\,\mu$$

is given arbitrarily in the space T (even varying discontinuously), but (due to 2.) vanishes when *x*, *y*, *z* increase without bound.

If one now substitutes the quantity μ, when given as a function of *a*, *b*, *c* ([1]), in the integral over all elements $d\tau = da\, db\, dc$ of infinite space:

$$\varphi(x, y, z) = \int \frac{\mu\, d\tau}{r} \tag{8}$$

(where *x*, *y*, *z* appear only in the quantity *r* on the right then) then that expression will be the only solution of the partial differential equation:

$$\Delta\varphi = \frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} + \frac{\partial^2\varphi}{\partial z^2} = -4\pi\,\mu \tag{8.a}$$

that fulfills those three assumptions simultaneously. In that way, we repeat, that excludes the assumption that $\Delta\varphi$ (along with μ) is distributed discontinuously in infinite space, but not the continuity of φ and $\partial\varphi / \partial n$ ([2]).

The function φ that satisfies the conditions (1)-(3) is called the *NEWTONIAN* or *spatial potential*, and also the "potential" in the narrower sense, and the partial differential equation (8.a) that φ in (8) satisfies is called the LAPLACE-POISSON *equation*.

([1]) For masses that are distributed along surfaces or lines, one should confer the book by RIEMANN-WEBER, *Die partiellen Differentialgleichungen der mathematischen Physik*, Braunschweig, 1900, Bd. 1, pp. 234, *et seq.*, whose presentation essentially follows the one in our book.

([2]) One finds an investigation with all the rigor of proofs in potential theory in HÖLDER's dissertation "Beiträge zur Potentialtheorie," 1882.

The formula (8) finds employment in various domains of mechanics and physics according to the meaning that one ascribes to the function μ. If μ means the density of *ponderable mass* that is distributed in infinite space in any way, so μ will then be a positive quantity or zero for the individual spatial element $d\tau$, then φ will be the (NEWTONIAN) *potential of that mass.* The differential quotients:

$$\frac{\partial \varphi}{\partial x} = -\int \frac{\mu}{r^2}\frac{x-a}{r}d\tau, \qquad \frac{\partial \varphi}{\partial y} = -\int \frac{\mu}{r^2}\frac{y-b}{r}d\tau, \qquad \frac{\partial \varphi}{\partial z} = -\int \frac{\mu}{r^2}\frac{z-c}{r}d\tau,$$

when multiplied by the gravitational constant κ and the mass m at the point x, y, z, will have the meaning of the components of a *force* that one will obtain when one takes the resultant of all elementary forces of attraction $-\kappa\dfrac{m\,\mu\,d\tau}{r^2}$ that are exerted according to the law of gravitation by the masses $\mu\,d\tau$ that are present at each a, b, c at a distance r from the location x, y, z of the reference point with the mass m. By contrast, if μ and m both mean electric or magnetic masses (which are thought of as free), which can also assume negative values, then negative quantities can enter in place of κ; masses of the same kind will repel.

Finally, if φ is *a velocity potential* inside of the space T, so the differential quotients are the components of the velocity $\mathfrak{v}$ of a fluid motion, then it will be irrotational in T, because rot $\mathfrak{v} = 0$ in the case of a potential vector, and the meaning of μ is inferred from (art. **22**, at the end):

$$\operatorname{div} \mathfrak{v} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = \Delta\varphi = -4\pi\,\mu = -\frac{d\log\rho}{dt}. \tag{9}$$

μ is then a quantity that is proportional to a *divergence* (or the increase in the logarithm of the density). If the fluid is *incompressible* then the divergence will be zero at all locations where no sources and sinks are found. However, where such things are found, *μ has the meaning of the productivity of the source or the suction of the sink.* If one knows its distribution in infinite space T then the state of motion of the fluid at any location is determined by the integral of the equation:

$$\Delta\varphi = \operatorname{div} \mathfrak{v},$$

so by the potential:

$$\varphi = \int \frac{\mu\,d\tau}{r} = -\frac{1}{4\pi}\int \frac{\operatorname{div}\mathfrak{v}}{r}d\tau, \tag{10}$$

but only as long as one fulfills the condition that is derived from (7) that the velocity components must vanish for large values of the coordinates $\mathfrak{R}$ in such a way that $\mathfrak{R}^2 u$, $\mathfrak{R}^2 v$, $\mathfrak{R}^2 w$ preserve finite values. We will then say that *the fluid is "at rest at infinity"* (which is a terminology that is also employed when $\mathfrak{R}u$, … merely have a finite value, moreover). When we then imagine that the sources and sinks that are present in an incompressible fluid are surrounded by closed surfaces and applying equation (4) to the space T that is bounded within, but finite or infinite, moreover, it will go to the following one:

$$\int\left[\left(\frac{\partial\varphi}{\partial x}\right)^2+\left(\frac{\partial\varphi}{\partial y}\right)^2+\left(\frac{\partial\varphi}{\partial z}\right)^2\right]d\tau = \int(u^2+v^2+w^2)\,d\tau = -\int\varphi\frac{\partial\varphi}{\partial n}d\sigma, \tag{11}$$

because $\Delta\varphi = 0$ inside of T now. If one multiplies (11) by the factor ρ / 2, where ρ is the (everywhere-constant) density, then the left-hand side will mean the *kinetic energy of the fluid*. Inside of a region that is free of sources and sinks, it is then determined completely by the state on the surface when the fluid motion is stationary, i.e., φ is independent of time.

§ 31. – Composing a vector from a potential vector and a solenoidal one.

In art. **27**, it was remarked that any solenoidal vector $\mathfrak{v}$ = rot $\mathfrak{q}$ satisfies the condition that:

$$\text{div}\ \mathfrak{v} = \text{div rot}\ \mathfrak{q} = 0 .$$

Conversely, *any vector* $\mathfrak{v}$ = (u, v, w) *whose divergence vanishes,* so which satisfies the partial differential equation:

$$\text{div}\ \mathfrak{v} = \frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z} = 0, \tag{1}$$

can be represented as a solenoidal vector, i.e., as the rotation of another vector. The proof of that will be achieved when one can show that the *assumption* that:

$$\mathfrak{v} = \text{rot}\ \mathfrak{q} ,$$

or when L, M, N are the components of $\mathfrak{q}$, the assumption that:

$$\begin{aligned} u &= \frac{\partial N}{\partial y}-\frac{\partial M}{\partial z}, \\ v &= \frac{\partial L}{\partial z}-\frac{\partial N}{\partial x}, \\ w &= \frac{\partial M}{\partial x}-\frac{\partial L}{\partial y} \end{aligned} \tag{2}$$

will allow one to calculate those quantities L, M, N in such a way that the u, v, w that are expressed in terms of them will satisfy equation (1). One next observes that the quantities L, M, N are determined from (2) – if at all – only up to the partial differential quotients of one and the same function. That is because when one knows one system L, M, N that satisfies equations (2), (1):

$$L_1 = L + \frac{\partial \psi}{\partial x}, \quad M_1 = M + \frac{\partial \psi}{\partial y}, \quad N_1 = N + \frac{\partial \psi}{\partial z} \tag{3}$$

will also be such a thing, where ψ is any function of x, y, z. If one now calculates the components 2ξ, 2η, 2ζ of the rotation (art. **22**):

$$2\mathfrak{w} = \text{rot } \mathfrak{v}$$

then one can put the expression for, e.g., 2ξ, into the form:

$$2\xi = \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} = \frac{\partial}{\partial x}\left(\frac{\partial L}{\partial x} + \frac{\partial M}{\partial y} + \frac{\partial N}{\partial z}\right) - \Delta L$$

or

$$2\xi = \frac{\partial}{\partial x} \text{div } \mathfrak{q} - \Delta L, \tag{4}$$

in which one again has:

$$\Delta L = \frac{\partial^2 L}{\partial x^2} + \frac{\partial^2 L}{\partial y^2} + \frac{\partial^2 L}{\partial z^2}.$$

When one combines the formulas for 2ξ, 2η, 2ζ into one:

$$\text{rot } \mathfrak{v} = \text{rot rot } \mathfrak{q} = \text{grad div } \mathfrak{q} - \Delta \mathfrak{q}, \tag{4.a}$$

that will prove an oft-applied *identity of vector analysis*.

One now determines a system of solutions $(L, M, N) = \mathfrak{q}$ from (4), but under the *assumption* that div $\mathfrak{q} = 0$, so from the equations:

$$\Delta L = -2\xi, \quad \Delta M = -2\eta, \quad \Delta N = -2\zeta, \tag{5}$$

i.e., one determines $\mathfrak{q}$ from:

$$\Delta \mathfrak{q} = -\text{rot } \mathfrak{v}, \tag{5.a}$$

under the assumption that for coordinates x, y, z that increase without bound, the quantities:

$$\mathfrak{R}^2 u, \quad \mathfrak{R}^2 v, \quad \mathfrak{R}^2 w, \tag{5.b}$$

in which one has:

$$\mathfrak{R}^2 = x^2 + y^2 + z^2 ,$$

and therefore:

$$\mathfrak{R}^3\,\xi, \quad \mathfrak{R}^3\,\eta, \quad \mathfrak{R}^3\,\zeta$$

will not become infinite, which is why:

$$\mathfrak{R}L, \quad \mathfrak{R}M, \quad \mathfrak{R}N$$

and

$$\mathfrak{R}^2\,\frac{\partial L}{\partial n}, \qquad \mathfrak{R}^2\,\frac{\partial M}{\partial n}, \qquad \mathfrak{R}^2\,\frac{\partial N}{\partial n}$$

will also remain finite. The solutions to the partial differential equations (5) (art. **30**) are then:

$$\begin{aligned} L &= \frac{1}{2\pi}\int\frac{\xi\,d\tau}{r}, \\ M &= \frac{1}{2\pi}\int\frac{\eta\,d\tau}{r}, \\ N &= \frac{1}{2\pi}\int\frac{\zeta\,d\tau}{r}, \end{aligned} \tag{6}$$

or in vector notation:

$$\mathfrak{q} = \frac{1}{4\pi}\int\frac{\operatorname{rot}\mathfrak{v}\,d\tau}{r}, \tag{6.a}$$

where $d\tau = da\;db\;dc$:

$$r^2 = (a-x)^2 + (b-y)^2 + (c-z)^2 ,$$

and in which ξ, η, ζ are written in terms of the coordinates a, b, c of a point in the space T over which the integration extends.

The fact that, conversely, those solutions (6) also satisfy the condition that:

$$\operatorname{div}\mathfrak{q} = \frac{\partial L}{\partial x} + \frac{\partial M}{\partial y} + \frac{\partial N}{\partial z} = 0 \tag{7}$$

is established as follows: Since:

$$\frac{\partial\left(\frac{1}{r}\right)}{\partial x} = -\frac{\partial\left(\frac{1}{r}\right)}{\partial a},$$

and therefore:

$$\frac{\partial\left(\frac{\xi}{r}\right)}{\partial x} = -\,\xi\,\frac{\partial\left(\frac{1}{r}\right)}{\partial a} = -\frac{\partial\left(\frac{\xi}{r}\right)}{\partial a} + \frac{1}{r}\frac{\partial\xi}{\partial a},$$

one will have:

$$\operatorname{div} \mathfrak{q} = -\frac{1}{2\pi}\int d\tau \left[\frac{\partial\left(\frac{\xi}{r}\right)}{\partial a} + \frac{\partial\left(\frac{\xi}{r}\right)}{\partial b} + \frac{\partial\left(\frac{\xi}{r}\right)}{\partial c} \right] + \frac{1}{2\pi}\int \frac{d\tau}{r}\left(\frac{\partial\xi}{\partial a} + \frac{\partial\eta}{\partial b} + \frac{\partial\zeta}{\partial c} \right) .$$

The last integral on the right is zero since div rot $\mathfrak{v} = 0$. We apply GAUSS's law (art. **24**) to the first one and get:

$$\operatorname{div} \mathfrak{q} = \frac{1}{2\pi}\int \frac{d\sigma}{r}[\xi \cos(n, x) + \eta \cos(n, y) + \zeta \cos(n, z)] , \tag{7.a}$$

in which the integration is extended over the infinitely-distant outer surface element:

$$d\sigma = \mathfrak{R}^2\, d\omega$$

of the space T, where $d\omega$ is the element of the unit sphere and $\mathfrak{R}$ increases without bound with r. However, the expression in square brackets vanishes with $\mathfrak{R}\xi$, $\mathfrak{R}\eta$, $\mathfrak{R}\zeta$, and therefore div $\mathfrak{q} = 0$.

However, from the special system of solutions L, M, N of equations (4) thus-obtained, for which div $\mathfrak{q} = 0$, any other one (3) L_1, M_1, N_1 for which div $\mathfrak{q}$ has a given value can be produced by determining the function ψ from the equation:

$$\Delta\psi = \operatorname{div} \mathfrak{q}$$

as in art. **30**. That is because if one knows ψ then one will have, e.g.:

$$\Delta L_1 = \Delta L + \Delta\frac{\partial\psi}{\partial x} = \Delta L + \frac{\partial}{\partial x}\operatorname{div} \mathfrak{q} ,$$

so:

$$-2\xi = \Delta L = \Delta L_1 - \frac{\partial}{\partial x}\operatorname{div} \mathfrak{q} , \text{ etc.,}$$

such that L_1 is the solution of (4). However, the function ψ drops out completely when one constructs u, v, w. Therefore, the formulas (6) already yield a system of solutions. One calls the vector $\mathfrak{q}$ that is defined by (2) the *vector potential* of the vector $\mathfrak{v} = (u, v, w)$, in contrast to the *scalar potential*, as is the case with velocity or force potentials.

One can now prove the following theorem of vector analysis (which is important for the kinematics of fluids):

Any vector can be composed of a potential vector and a solenoidal vector, or when written out in detail:

If u, v, w are three functions of x, y, z that are continuous, finite, and single-valued, along with their differential quotients, and fulfill the condition that (art. **30**) $\mathfrak{R}^2 u$, $\mathfrak{R}^3 \frac{\partial u}{\partial x}$, … keep finite values when $\mathfrak{R}$, etc., increase to infinity then one can always determine four functions ψ, L, M, N in only one way such that the equations are fulfilled:

$$\begin{aligned} u &= \frac{\partial \psi}{\partial x} + \frac{\partial N}{\partial y} - \frac{\partial M}{\partial z}, \\ v &= \frac{\partial \psi}{\partial y} + \frac{\partial L}{\partial z} - \frac{\partial N}{\partial x}, \\ w &= \frac{\partial \psi}{\partial z} + \frac{\partial M}{\partial x} - \frac{\partial L}{\partial y}, \end{aligned} \tag{8}$$

or in vector notation:

$$\mathfrak{v} = \text{grad } \psi + \text{rot } \mathfrak{q} \,. \tag{8.a}$$

This theorem, which was first expressed by CLEBSCH (Jour. f. Math., Bd. 61, pp. 197), can be proved as follows: Upon differentiating equations (8) with respect to x, y, z, resp., and adding them, one will get the equation for ψ:

$$\text{div } \mathfrak{v} = \text{div grad } \psi = \Delta \psi \,. \tag{9}$$

One further gets equations of the form (4) for L, M, N, and from the foregoing, they can be replaced with:

$$\Delta \mathfrak{q} = -\text{ rot } \mathfrak{v} \,, \tag{10}$$

with the condition that:

$$\text{div } \mathfrak{q} = 0 \,.$$

However, the functions (6.a):

$$\mathfrak{q} = -\frac{1}{4\pi} \int \frac{\text{rot } \mathfrak{v}}{r} d\tau$$

and (11)

$$\psi = -\frac{1}{4\pi} \int \frac{\text{div } \mathfrak{v}}{r} d\tau$$

are then the solutions to the differential equations (9), (10) that are determined uniquely on the grounds of the assumption (5.b) ([1]). With their help, from (8), *the quantities u, v, w can be defined* for all locations in space *when one knows the distribution of the quantities* rot $\mathfrak{v}$ *and* div $\mathfrak{v}$ *in space.*

([1]) The fact that a vector $\mathfrak{v}$ whose divergence and rotation are known, and which vanishes at infinity like $1/\mathfrak{R}^2$ is determined uniquely in that way was proved independently of its representation by, e.g., R. GANS, *Einführung in die Vektoranalysis*, Leipzig, 1905, pp. 44. As far as the behavior of $\mathfrak{v}$ at infinity is concerned, according to O.

If $\mathfrak{v}$ means the velocity of the particles of an incompressible fluid that fills up infinite space then that result can be interpreted by saying: If one knows:

1. The distribution of *sources* and *sinks* (div $\mathfrak{v}$),

2. The places where *vortical motion* exists and the components ξ, η, ζ of it, so all of the vortex filaments (rot $\mathfrak{v}$), and

3. *The fact that the fluid is at rest at infinity* (so $\mathfrak{R}^2\,\mathfrak{v}$, $\mathfrak{R}^3\,\frac{\partial \mathfrak{v}}{\partial t}$, remain finite for increasing $\mathfrak{R}$) at a given moment

then the advancing motion $\mathfrak{v} = (u, v, w)$ can be determined *everywhere* by means of formulas (8) and (11).

The state of the fluid at a given point in time is then determined from the distribution of vortices and sinks (sources). Furthermore, the knowledge of $\mathfrak{v}$ will imply that knowledge at the next moment, and in the event that no external forces are acting, one can then ascertain the motion for a finite time interval by continued construction over infinitely-small intervals.

We shall add yet another general remark about the operation $\Delta\varphi$ that was used frequently in this article. Since (9):

$$\Delta\varphi = \text{div grad } \varphi$$

can be represented by a succession of two vector operations, each of which admits an interpretation that is independent of the coordinate system (arts. **27**, **29**), $\Delta\varphi$ is also a quantity that is independent of the position of the rectangular coordinate system, so it is a "differential invariant" under orthogonal transformations. If one then performs an arbitrary rotation and displacement of the coordinate system (say, of X, Y, Z into X', Y', Z'), under which the function $\varphi(x, y, z)$ goes to, say, $\varphi'(x', y', z')$, then:

$$\Delta\varphi = \frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} + \frac{\partial^2\varphi}{\partial z^2}$$

must also necessarily go to:

$$\Delta\varphi' = \frac{\partial^2\varphi'}{\partial x'^2} + \frac{\partial^2\varphi'}{\partial y'^2} + \frac{\partial^2\varphi'}{\partial z'^2} .$$

BLUMENTHAL [Math. Ann. **61** (1905), pp. 235], the vanishing of $\mathfrak{v}$ and its differential quotients already suffices for one to represent it.

§ 32. – Examples of fluid motion with single-valued and multi-valued velocity potentials.

1. Example. – We next apply GREEN's theorem to the current in a fluid with two fixed centers a, b, c ; a_1, b_1, c_1 that was mentioned in art. **29**, which has the potential:

$$\varphi = \frac{e}{r} + \frac{e_1}{r_1}, \tag{1}$$

in which:

$$\begin{aligned} r^2 &= (x-a)^2 + (y-b)^2 + (z-c)^2 , \\ r_1^2 &= (x-a_1)^2 + (y-b_1)^2 + (z-c_1)^2 . \end{aligned} \tag{2}$$

If one surrounds those two centers with very small balls of radii α, α_1 that we would like to call α, α_1 then the equation:

$$\Delta\varphi = 0 \tag{3}$$

will exist in the remaining space T. We now apply formula (11) of art. **30** to the interior of this space T, which is bounded on the inside by the balls α and on the outside by a sphere of infinitely large radius, which represents the total kinetic energy of an incompressible fluid that fills up an infinite space as an integral that is extended over just the outer surface of that space, which consists of the spherical surfaces α, α_1 that bound the space on the inside here, because our function φ fulfills the conditions (7) of art. **30** at infinity. If T is the kinetic energy of the fluid then from that formula (11), we have:

$$T = \int \frac{\rho}{2}(u^2 + v^2 + w^2)\,d\tau = -\frac{\rho}{2}\int \varphi \frac{\partial \varphi}{\partial n}\,d\sigma , \tag{4}$$

in which ρ is the density. Since the radius of the spherical surface α is a very small quantity, the value of φ on it is approximately:

$$\varphi = \frac{e}{\alpha} + \frac{e_1}{R},$$

in which R is the distance from the center of the sphere:

$$R^2 = (a - a_1)^2 + (b - b_1)^2 + (c - c_1)^2. \tag{5}$$

We now form the expression:

$$\varphi \frac{\partial \varphi}{\partial n}$$

for the sphere α. Since the inward-pointing normal to T is the outward-pointing normal to the sphere, one has:

$$dn = d\alpha ,$$

and furthermore (approximately):

$$\frac{\partial\varphi}{\partial n} = -\frac{e}{\alpha^2} - \frac{e_1}{R^2}\frac{\partial r_1}{\partial n},$$

and when $d\omega$ is once more the outer surface element of the unit sphere:

$$-\int \varphi \frac{\partial\varphi}{\partial n} d\sigma = \int\left(\frac{e^2}{\alpha} + \frac{e\,e_1}{R}\right) d\omega + \int\left(\frac{e}{\alpha} + \frac{e_1}{R}\right)\frac{e_1}{R^2}\frac{\partial r_1}{\partial n}\alpha^2 d\omega \ .$$

The last integral on the right is zero, because $\partial r_1 / \partial n$ will have equal, but opposite, values for any two elements of the ball α that lie at opposite ends of a diameter. Ultimately, one will have:

$$-\int \varphi \frac{\partial\varphi}{\partial n} d\sigma = 4\pi\left(\frac{e^2}{\alpha} + \frac{e\,e_1}{R}\right) \tag{6}$$

for the ball α. If one defines the corresponding integral for the ball α_1 and sums then that will give:

$$T = 2\pi\rho\left(\frac{e^2}{\alpha} + \frac{2e\,e_1}{R} + \frac{e_1^2}{\alpha_1}\right) . \tag{7}$$

Since we have contracted the spatially-distributed divergence to a sink (art. **27**), we can once more think of replacing the point-like sink of suction e with a spatial sink that fills up the interior of the ball α continuously and has a suction force per unit volume of ε, so we will then have:

$$e = \tfrac{4}{3}\alpha^3\pi \cdot \varepsilon \, .$$

One proceeds similarly for e_1 . When GAUSS's law is applied to the outer surface of the ball α, when the influx into it is distributed uniformly and regarded as normal to the surface, that will give:

$$\int \Delta\varphi \, d\tau = -4\pi \int \varepsilon \, d\tau = -4\pi e = -\int v \, d\sigma = -v \cdot 4\pi\alpha^2,$$

from which one determines the inflow velocity $v = |\,\mathfrak{v}\,|$ to be:

$$v = \frac{e}{\alpha^2},$$

which was also implied (approximately) by $\varphi = e / \alpha$. The amount of fluid that flows into the sink α per second is then:

$$4\alpha^2\pi\rho v = 4\pi e \rho = \frac{16}{3}\pi^2\varepsilon \cdot \rho \cdot \alpha^3 .$$

Therefore, the kinetic energy per second of the fluid that flows into α will be approximately:

$$\tfrac{1}{2}v^2 \cdot 4\pi e\rho = 2\pi\rho \cdot \frac{e^2}{\alpha^4} = \left(\frac{4\pi\varepsilon}{3}\right)^3 \cdot 2\pi\rho\cdot\alpha^5 ,$$

which will be a quantity that is proportional to the fifth power of the radius of the sphere α and the third power of the force of suction ε per unit volume of the ball.

2. Example. The way that GREEN's theorem was used up to now assumes that the velocity potential φ is single-valued. We would next like to illustrate that fact, and the way that it can be employed when the function φ in formula (11) of art. **30** is a multi-valued or infinite-valued quantity.

In art. **26**, we considered a stationary fluid motion in the interior of a straight cylinder that was closed off by two circular discs whose velocity components was given by:

$$u = -\,q\frac{y}{r}\,, \qquad v = q\frac{x}{r}\,, \qquad w = 0\,,$$

in which q is a function of the distance:

$$r = \sqrt{x^2 + y^2}\,.$$

One can succeed in determining q in such a way that the motion has a potential, such that:

$$d\varphi = u\,dx + v\,dy = \frac{q}{r}(x\,dy - y\,dx)$$

is a complete differential. One solution is:

$$q = \frac{\kappa}{r}\,,$$

in which κ is a constant. As one easily sees, one will then get:

$$\varphi = \kappa \arctan\frac{y}{x}\,. \tag{8}$$

φ will be undetermined along the axis of the cylinder. We shall assume that this is true for the motion by enclosing that axis by a circular cylinder of very small radius α. The velocity $|\,\mathfrak{v}\,|$ will then be determined uniquely inside of a hollow cylinder that is bounded by the outer wall and the cylinder α. However, φ is an infinitely multi-valued function of the argument y / x. When φ is *any* solution of equation (8), a well-defined value will correspond to all of the quantities:

$$\varphi + 2n\,\pi \cdot \kappa\,,$$

where n is a positive or negative whole number. That multi-valuedness is based upon the form of the space for which φ is defined. It is not a "simply-connected space" in the sense of art. **29**, which was already referred to in the footnote there. For that reason, the value of the integral $\int d\varphi$, when it is perhaps taken between the endpoints x_0, y_0, z_0 and x_1, y_1, z_1, depends upon not just those endpoints, but also upon the path that connects them.

Now, in order to make GREEN's theorem useful, one converts the doubly-connected space T into a simply-connected one T′ by introducing a wall – a "cross-section" – that consists (see the footnote in art. **29**) of a rectangle (in a plane through the axis) that must pass through any encircling of the axis inside of T. One must imagine that this cross-section has been doubled as the closing surface in both directions. Now, the function φ will again be single-valued in the cylindrical space T′ that was cut through in that way. However, as we will see, φ will have a different value on both sides of the cut surface. In order to be able to apply GREEN's theorem to the function φ thus-defined, one must add the two sides of the cross-section to the boundary surface of T to get T′. Each of those surfaces will contribute to the expression for the kinetic energy:

$$T = -\frac{\rho}{2}\int \varphi \frac{\partial\varphi}{\partial n}\,d\sigma\,.$$

If one sets:

$$x = r\cos\vartheta\,, \quad y = r\sin\vartheta\,,$$

so (8):

$$\varphi = \kappa \cdot \vartheta\,,$$

then for the outer wall of the cylinder, one will have:

$$\frac{\partial\varphi}{\partial n} = \pm\frac{\partial(\vartheta r)}{\partial r} = 0\,,$$

so the contribution to T is zero. The contribution from the end surface is also zero, for the same reason. When one lays the two sides of the cross-section in the plane $\vartheta = 0$:

$$\frac{\partial\varphi}{\partial n} = \pm\frac{1}{r}\frac{\partial\varphi}{\partial\vartheta} = \pm\frac{\kappa}{r}$$

will have the same value, but opposite signs. By contrast, φ will differ by:

$$\kappa\,(\vartheta + 2\pi) - \kappa\,\vartheta = 2\kappa\,\pi$$

on the two sides. If the cylinder has a height of h, so a surface element of the cross-section will be:

$$d\sigma = h\,dr\,,$$

then the contribution that it makes will be:

$$T = -\frac{\rho}{2} 2\kappa\pi \int \frac{\partial\varphi}{\partial n} d\sigma = \frac{\rho}{2} 2\kappa\pi h \int \frac{\kappa}{r} dr = \rho\pi h \kappa^2 \log\frac{a}{\alpha}\,,$$

when α is the inner radius and a is the outer radius of the cylinder wall, as above.

Furthermore, one can calculate the *vis viva* T more directly in the present case when one introduces the values for u and v into:

$$T = \frac{\rho}{2} \int (u^2 + v^2)\, d\tau$$

and integrate over the volume of the hollow cylinder.

Since one has:

$$\log(x + i\,y) = \log\sqrt{x^2 + y^2} + i \cdot \arctan\frac{y}{x}\,,$$

$$i = \sqrt{-1}\,,$$

from known theorems in the theory of functions, the potential motion:

$$\varphi = \kappa \arctan\frac{y}{x}$$

will be associated with the other one:

$$\varphi = \kappa \log\sqrt{x^2 + y^2}$$

in such a way that the streamlines (in the XY-plane) of the one are the level lines of the other. For the one motion, the Z-axis is the vortex axis, and for the other, it is a line of sinks (or sources). The kinetic energy has the same value in both cases.

The process that was applied here can be generalized.

If a potential function φ is given in a multiply-connected space T for which GREEN's theorem is not immediately applicable, since it is multi-valued, then one will convert it into a single-valued one in such a way that one makes the space T into a simply-connected one T' by a cross-section (cut surface) and then adds that cross-section to the boundary of T' twice. *For two points of T' that lie opposite to each other on the two sides of the same cross-section, the value of the potential φ will differ by a quantity that has the same value along the entire cross-section.*

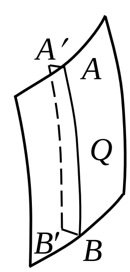

Figure 20.

Namely, if (Fig. 20) A, A' and B, B' are two such point-pairs on one side and the other of the cross-section Q then the value of the integral:

$$\int_{A'}^{B'} d\varphi = \int_{A}^{B} d\varphi$$

along the path $A'B'$ will be equal to the value along the path AB, even when φ has different values at A and A' (B and B', resp.), because the increases $d\varphi$ along infinitely-close line elements have equal values. Therefore, one also has:

$$\varphi(B') - \varphi(B) = \varphi(A') - \varphi(A), \qquad \text{Q.E.D.}$$

When one adds the cross-section, thus-constructed, to the boundary of T, integrals of the form:

$$\frac{\rho}{2} k \int \frac{\partial \varphi}{\partial n} d\sigma$$

will be added to the integral that is extended over the outer surface Σ, where k is that potential difference. In that, the volume integral means the amount that flows through the cross-section per second, since $\partial\varphi / \partial n$ is the velocity in the normal direction.

§ 33. – Continuation. Two vortex rings in a fluid.

A multi-valued velocity potential will appear any time when a fluid mass is penetrated by vortices. That is because, according to art. **27** (at the end), vortex lines either close upon themselves or end on the outer surface of the fluid. The space that left alone by them is therefore no longer simply connected.

Two vortex filaments of very small cross-section might traverse an incompressible fluid that extends to infinite in all directions and is at rest at infinity (art. **30**, at the end) that are closed, but do not intertwine. We would like to call the connecting line between the centers of mass of the cross-sections the *vortex axis*. If we surround each of those filaments with a tubular surface of likewise very small cross-section and excludes it from the rest of space then the infinite space T that is now vortex-free will be triply-connected. The one "cross-section" defines a surface that goes through one of the vortex axes that closes upon itself. We bound it by the wall of the tubular surface (see fig. 21). The other tube likewise bounds the other cross-section. When those cut surfaces, doubly-counted, are added to the outer surface of T, they will bound a simply-connected space T′. From art. **27** (at the end), for each vortex filament, the product of the vortex velocity with the cross-section – viz., the "vortex strength" – is a constant quantity along the vortex that we would like to denote by $2\kappa\pi$ and $2\kappa_1\pi$, resp. However, the velocity vector $\mathfrak{v}$ (always for the given moment) is determined in the entire fluid mass by it and the form of the vertex filament. That is because, from art. **31**, we have:

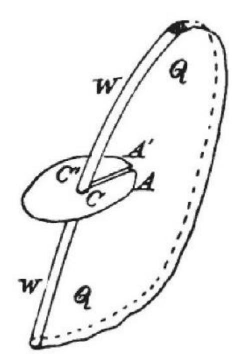

Fig. 21.

$$\mathfrak{v} = \operatorname{rot} \mathfrak{q} \,, \tag{1}$$

where the vector potential $\mathfrak{q}$ with the components L, M, N is calculated from:

$$L = \frac{1}{2\pi}\int \frac{\xi\, d\tau}{r}, \qquad M = \frac{1}{2\pi}\int \frac{\eta\, d\tau}{r}, \qquad N = \frac{1}{2\pi}\int \frac{\zeta\, d\tau}{r}, \tag{2}$$

when the integration extends over the volume of the vortex filament.

On the other hand, $\mathfrak{v}$ also has a potential φ in the vortex-free space, so:

$$\begin{aligned} u &= \frac{\partial\varphi}{\partial x} = \frac{\partial N}{\partial y} - \frac{\partial M}{\partial z}, \\ v &= \frac{\partial\varphi}{\partial y} = \frac{\partial L}{\partial z} - \frac{\partial N}{\partial x}, \\ w &= \frac{\partial\varphi}{\partial z} = \frac{\partial M}{\partial x} - \frac{\partial L}{\partial y}. \end{aligned} \tag{3}$$

Hence, the relation (11) in art. **30** that follows from GREEN's theorem will be applicable, so the kinetic energy can be represented by:

$$T = -\frac{\rho}{2}\int \varphi \frac{\partial\varphi}{\partial n} d\sigma \,, \tag{4}$$

in which the integral on the right extends over the entire boundary of T'. That consists of:

1. The infinitely-distant boundary surface,
2. The wall of the tubular surface, and
3. The cross-sections Q and Q_1, each counted twice.

The contribution from 1. is equal to zero, because the fluid is at rest at infinity. The integral that is extended over the tubular surface:

$$J = -\int \varphi \frac{\partial\varphi}{\partial n} d\sigma \tag{5}$$

also gives zero. That is because if φ_0 (φ_0', resp.) is the greatest absolute value that the quantity φ ($\partial\varphi / \partial n$, resp.) possesses along the wall of the tubular surface then the value of J will remain below:

$$J_0 = -\varphi_0\, \varphi_0' \int d\sigma = -\varphi_0\, \varphi_0'\, \Sigma \,,$$

in which Σ is the area of the tubular surface. However, one can make it small enough that the amount J_0 no longer comes under consideration by a sufficiently-small cross-section of the vortex filament.

As far as the contribution from 3. is concerned, which initially comes from the cut surface Q, any two corresponding points A, A' on the two sides of it will have the same value of $\partial\varphi / \partial n$, but with opposite signs. If the values of φ at A and A' are $\varphi(A)$ and $\varphi(A')$ then (see prev. art., at end) the integral:

$$J = -\int \varphi \frac{\partial\varphi}{\partial n} d\sigma = -[\varphi(A) - \varphi(A')] \int \frac{\partial\varphi}{\partial n} d\sigma \tag{6}$$

is now extended over just one side of Q. The difference in front of the integral can be represented by the line integral:

$$\int_A^{A'} d\varphi = \int_A^{A'} (u\,dx + v\,dy + w\,dz), \tag{7}$$

which is extended along a curve that encircles the vortex that belongs to Q once. One can (Fig. 21) replace that path from A to A' with a narrow path that encircles the vortex W at the location CC' and two mutually-cancelling rectilinear integrals along AC and $C'A'$. However, the integral that encircles the vortex at the location C can be converted into a surface integral that is taken over the cross-section of the vortex filament at C. Namely, (art. **28**), from STOKES's theorem, one has:

$$\begin{aligned}\int_A^{A'} d\varphi = \int_C^{C'} d\varphi &= \int_C^{C'} (u\,dx + v\,dy + w\,dz)\\ &= 2\int [\xi \cos(n, x) + \eta \cos(n, y) + \zeta \cos(n, z)]\,d\sigma \\ &= 2\int \omega\,d\sigma = 2\omega\,q = 4\pi\,\kappa,\end{aligned} \tag{7.a}$$

when ω is the magnitude of the vortex velocity at the location C, so $\omega = |\,\mathfrak{w}\,|$, and $\mathfrak{w} = (\xi, \eta, \zeta)$, q is its vortex cross-section, and $2\pi\,\kappa$ is the "vortex strength", which is constant along the vortex, according to art. **27**. The potential difference across the two sides of the cut surface – or the "circulation" as the English authors say ([1]) – is then equal to twice the vortex strength. On the other hand, one has, in turn:

$$\begin{aligned}\int \frac{\partial\varphi}{\partial n} d\sigma &= \int \left[\frac{\partial\varphi}{\partial x}\cos(n, x) + \frac{\partial\varphi}{\partial y}\cos(n, y) + \frac{\partial\varphi}{\partial z}\cos(n, z)\right] d\sigma \\ &= \int [u \cos(n, x) + v \cos(n, y) + w \cos(n, z)]\,d\sigma\end{aligned}$$

([1]) See, e.g., BASSET, *Hydrodynamics*, Cambridge, 1888, I, pp. 70. – For H. LAMB (*Hydrodynamik*, German trans. by FRIEDEL, Leipzig, 1907), the intensity is twice as big as was assumed in this book.

$$= \int \left[\left(\frac{\partial N}{\partial y} - \frac{\partial M}{\partial z} \right) \cos(n, x) + \cdots \right] d\sigma \tag{8}$$

$$= \int (L\,dx + M\,dy + N\,dz)$$

from STOKES's theorem, in which the line integral is extended along the boundary of the cut surface Q, so (approximately) along the axis of the vortex.

However, the components L, M, N of the vector potential are calculated from:

$$L = \frac{1}{2\pi} \int \frac{\xi\,d\tau}{r}, \qquad \text{etc.}, \tag{9}$$

in which the integral is extended over the vorticial mass that is enclosed by two tubular surfaces. If ds, ds_1 are the line elements of the vortex axes of the vortex filaments Q, Q_1, resp., and q, q_1, resp., are their cross-sections at those locations, while ω, ω_1, resp., are the angular velocities, then, since one has, e.g.:

$$\xi = \omega \frac{dx}{ds},$$

one will have:

$$L = \frac{1}{2\pi} \int \omega \frac{dx}{ds} \frac{q\,ds}{r} + \frac{1}{2\pi} \int \omega_1 \frac{dx_1}{ds_1} \frac{q_1\,ds_1}{r} = \kappa \int \frac{dx}{r} + \kappa_1 \int \frac{dx_1}{r}. \tag{10}$$

With the correspondingly-defined values of M, N, the line integral that is extended along the vortex axis of the first vortex will be:

$$\int (L\,dx + M\,dy + N\,dz)$$

$$= \kappa \iint \frac{dx\,dx' + dy\,dy' + dz\,dz'}{r} + \kappa_1 \iint \frac{dx_1\,dx + dy_1\,dy + dz_1\,dz}{r} \tag{11}$$

$$= \kappa \iint \frac{ds\,ds' \cos(ds, ds')}{r} + \kappa_1 \iint \frac{ds_1\,ds \cos(ds_1, ds)}{r},$$

in which the denominator r is the distance from between the line elements ds, ds' in the numerators, which are both on the same vortex axis, ds_1, ds mean different vortex axes, and the double integral is extended over any pair of elements in the numerator. If one then sets:

$$A = \iint \frac{ds\,ds' \cos(ds, ds')}{r}, \qquad B = \iint \frac{ds_1\,ds \cos(ds_1, ds)}{r}, \qquad C = \iint \frac{ds_1\,ds_1' \cos(ds_1, ds_1')}{r} \tag{12}$$

then one will get the total value for the kinetic energy of the fluid in which one finds two vortex rings:

$$T = 2\pi\rho(\kappa^2 A + 2\kappa\kappa_1 B + \kappa_1^2 C)\,, \tag{13}$$

when expressed in terms of the vortex strengths κ, κ_1 and the "potentials" A, C of the vortices by themselves, while B is potential of the two together. Since the distance r between two elements of the same vortex can be infinitely small, the integrals A, C can take on infinitely-large magnitudes. That fact is based upon the assumption of an infinitely-thin vortex filament, which we made in order to be able to make the boundary of the cross-section coincide with the vortex axis. In the case of a finite (if also smaller) cross-section of the vortex, A and C will be no more infinite than the potential of a mass relative to a point in its interior. Our assumptions represent only an approximation. However, the result suggests that the quantities A, C can be appreciably bigger than B in any event.

In the expression (13) for the kinetic energy, the quantities A, B, C have a purely-geometric nature. They depended upon only the form and position of the tubular surface that enclosed the vortex. As we saw above, no exchange of energy takes place through the tubular surfaces. Now, since the kinetic energy of the otherwise-unperturbed fluid motion does not change in time, and since, as we will see directly, the vortex strengths κ, κ_1 are also unvarying quantities *in time*, one can completely ignore the vortex motion in the tubes, when one is dealing with the motion of the tubes in the fluid, and treat them by themselves on the grounds of the expression for T in (13) and the LAGRANGE equations when one, e.g., assumes that the tubes are rigid rings that move without friction. We shall return to that concept.

Now, since the vortex strengths that were introduced in art. **27** (at the end) also have an unvarying value in time, one sees that the line integral:

$$\int (u\,dx + v\,dy + w\,dz) = \int d\varphi \tag{14}$$

at the time t will be taken along a path $A, \ldots, B$ that connects any sort of elements of the fluid that lie inside of the simply-connected space T' with each other. After a time interval dt has elapsed, those elements will have changed their positions. When the line integral is taken over them, it will have increased by:

$$dt\frac{d}{dt}\int d\varphi = dt\int \frac{d}{dt}(d\varphi) = dt\int d\left(\frac{d\varphi}{dt}\right).$$

Now, from art. **25** (at the end), for any location in an incompressible fluid when no volume forces act upon it:

$$d\left(\frac{d\varphi}{dt}\right) = -d\left|\frac{\mathfrak{v}^2}{2}\right| - \frac{d\lambda}{\rho}\,,$$

in which $|\,\mathfrak{v}\,|$ is the velocity and λ is the pressure. One then has:

$$d\left(\frac{d\varphi}{dt}\right) = d\left[\frac{\partial\varphi}{\partial t}+\frac{\partial\varphi}{\partial x}u+\frac{\partial\varphi}{\partial y}v+\frac{\partial\varphi}{\partial z}w\right] = d\left|\frac{\mathfrak{v}^2}{2}\right|-\frac{d\lambda}{\rho},$$

since $u = \partial\varphi / \partial x$, etc. However, since the quantity:

$$\int_A^B\left[d\left|\frac{\mathfrak{v}^2}{2}\right|-\frac{d\lambda}{\rho}\right] = \left[\frac{u^2+v^2+w^2}{2}-\frac{\lambda}{\rho}\right]_A^B,$$

as a single-valued function of position in T', can depend upon only the limit points A, B, the increase in the line integral (1), when extended over a *closed* curve that is defined by material points, will be:

$$\int d\left(\frac{d\varphi}{dt}\right) = 0$$

for any time element. If the line integral $\int d\varphi$ has a certain value k when taken over one such curve at any point in time then it will keep that value for all time. In particular, *the intensity* (art. **27**) of a vortex filament that (from the above) can be represented by a line integral over a closed path that encircles the vortex filament once will have *a value that is constant for all time,* and not just along the vortex filament.

§ 34. – Two examples of cyclic motion in fluid masses ([1]).

Just as the concept of a "free system" and the geometric equations of constraint can be adapted to space-filling masses, one can also define cyclic fluid motions in the sense of art. **14**.

"Cyclic coordinates" appear in the expression for the (kinetic) energy only in the form of differential quotients with respect to time, so as "cyclic velocities," but not explicitly, and "cyclic" means systems for which the cyclic velocities are so large compared to the non-cyclic ones (the "slowly-varying parameters") that the energy T, in the first approximation, can be regarded as a function of only the former ones.

The following systems then belong to the cyclic, and indeed the dicyclic, adiabatic systems:

1. The fluid motion that was treated in art. **32**, which consists of influx at two spatial points (viz., sinks) of the fluid that filled up infinite space. That is because *the quantities* e, e_1 that represent the forces of suction of the sinks can be regarded as *rates* (perhaps by which the outflowing masses fill a vessel), which can also change in time, and the associated coordinates do not appear in the expression T for the kinetic energy of the moving mass:

([1]) VOLTERRA gave another application of the concept of cyclic motion in fluids, but for ones that move like rigid masses [Annali di mat. (2) **23** (1895) ; cf., also WANGERIN, Univ. Schrift, Halle, 1889], in which he examined the influence of the rotation of the Earth ocean currents.

$$T = 2\pi\rho\left(\frac{e^2}{\alpha} + \frac{2e\,e_1}{R} + \frac{e_1^2}{\alpha_1}\right),$$

in which ρ is the density, and α, α_1 are the radii of the small balls that surround the sinks.

2. The motion that was treated in art. **33**, which consisted of the orbiting of the fluid around two closed rings (or vortex filaments). The kinetic energy was:

$$T = 2\pi\rho(A\kappa^2 + 2B\kappa\kappa_1 + C\kappa_1^2), \tag{1}$$

in which the quantities κ, κ_1 represented the vortex strengths in the case of vortex motion.

The quantities κ, κ_1, as the differential quotients of cyclic coordinates, do not enter in, because they remain constant in time (see prev. art., at end), while the motion is certainly not isocyclic, but adiabatic (see art. **14**). By contrast, the amounts of fluid that flow through the cut surfaces in a unit time are variable, and they can be determined by evaluating the integral:

$$\lambda = \int \frac{\partial\varphi}{\partial n}\,d\sigma \tag{2}$$

(art. **32**, at end) for each of the two cross-sections. We found above [art. **33**, (11)] that those integrals have the values:

$$\begin{aligned}\lambda &= \kappa A + \kappa_1 B,\\ \lambda_1 &= \kappa B + \kappa_1 C\end{aligned} \tag{3}$$

for the cross-sections Q and Q_1, resp. When represented in terms of those quantities, the expression for kinetic energy will read:

$$T = \frac{2\pi\rho}{AC - B^2}(C\lambda^2 - 2B\lambda\lambda_1 + A\lambda_1^2)\ . \tag{4}$$

Now, in the first case, one can imagine that the infinitely-small spherical surfaces α, α_1 are filled with other matter, and in the second case, the infinitely-thin tubes, and then investigate the motion of the system that consists of those masses and the fluid in each case.

We shall do that in both cases.

§ 35. – Hidden cyclic systems ([1]).

We would like to assume that the cyclic system that was described in the previous article, which was a fluid that flowed into two sinks that were surrounded by very small balls α, α_1, is *hidden* from our perception, while the balls might be perceptible. The action of the system will then be noticeable only in the behavior of the balls. Let the space that the balls occupy be filled with the masses m, m_1 . Let the system of those masses, together with the hidden fluid, be a "free system" in the sense of art. **8**, i.e., the motion results in such a way that:

1. The total energy of the system does not change.
2. The system follows the straightest path.

If a, b, c ; a_1, b_1, c_1 are the coordinates of the centers of the sphere at time t, $\dot{a}$, $\dot{b}$, … are the components of their velocities, and R is the distance between then, so:

$$R^2 = (a - a_1)^2 + (b - b_1)^2 + (c - c_1)^2,$$

then the kinetic energy of the balls will be:

$$T = \frac{m}{2}(\dot{a}^2 + \dot{b}^2 + \dot{c}^2) + \frac{m_1}{2}(\dot{a}_1^2 + \dot{b}_1^2 + \dot{c}_1^2) . \tag{5}$$

The kinetic energy $\mathfrak{T}$ of the cyclic fluid motion was found (art. **32**) to be equal to:

$$\mathfrak{T} = 2\pi\,\rho\left(\frac{e^2}{\alpha} + \frac{2e\,e_1}{R} + \frac{e_1^2}{\alpha_1}\right) . \tag{6}$$

Now, the energy of the total system:

$$\mathsf{T} = T + \mathfrak{T} \tag{7}$$

is a constant quantity. The LAGRANGE equations of motion read [art. **14**, (5), (6)]:

$$\frac{d}{dt}\frac{\partial \mathsf{T}}{\partial \dot{a}} = \frac{\partial \mathsf{T}}{\partial a}, \qquad \frac{d}{dt}\frac{\partial \mathsf{T}}{\partial \dot{a}_1} = \frac{\partial \mathsf{T}}{\partial a_1}, \tag{8}$$

$$\text{etc.,} \qquad\qquad \text{etc.,}$$

([1]) I presented the following two examples from HERTZ's *Mechanik* in my lectures from 1901 to 1903 and published the first one in volume **58** of Math. Annalen (1904). Since both of them were already referred to by H. POINCARÉ (although only in passing) in the Appendix "À propos de la théorie de Larmor" in the second edition of his book *Électricité et optique*, 1901, pp. 623, *et seq.*, I have commented on them only briefly.

$$\frac{d}{dt}\frac{\partial \mathfrak{T}}{\partial e} = 0 \, ,$$
$$\frac{d}{dt}\frac{\partial \mathfrak{T}}{\partial e_1} = 0 \, . \tag{8.a}$$

It follows from the last two that:

$$\frac{\partial \mathfrak{T}}{\partial e} = 4\pi\,\rho\left(\frac{e}{\alpha}+\frac{e_1}{R}\right) = \mathfrak{q}\,,$$
$$\frac{\partial \mathfrak{T}}{\partial e_1} = 4\pi\,\rho\left(\frac{e}{R}+\frac{e_1}{\alpha_1}\right) = \mathfrak{q}_1\,, \tag{9}$$

in which $\mathfrak{q}$, $\mathfrak{q}_1$ are constants. The invariability of the impulses (momenta) $\mathfrak{q}$, $\mathfrak{q}_1$, which is due to the adiabatic character of the motion (art. **14**), implies the variability of the flow rates into the sinks while the distance between then R varies. If one solves equations (9) for e and e_1 then that will give:

$$e = \frac{\alpha\,\alpha_1}{4\pi\,\rho}\left(\quad\frac{\mathfrak{q}}{\alpha}-\frac{\mathfrak{q}_1}{R}\right),$$
$$e_1 = \frac{\alpha\,\alpha_1}{4\pi\,\rho}\left(-\frac{\mathfrak{q}}{R}+\frac{\mathfrak{q}_1}{\alpha_1}\right), \tag{9.a}$$

in which the denominator $\left(1-\dfrac{\alpha\,\alpha_1}{R^2}\right)$ can be replaced with 1, due to the smallness of the quantities α, α_1 . However, $\mathfrak{T}$ can be represented as a homogeneous quadratic function of the e's in the form:

$$\mathfrak{T} = \tfrac{1}{2}(e\,\mathfrak{q} + e_1\,\mathfrak{q}_1) \, .$$

When one expresses $\mathfrak{T}$ in terms of only $\mathfrak{q}$, one will then have:

$$\mathfrak{T}\,(e, e_1) = \mathfrak{T}_1\,(\mathfrak{q}, \mathfrak{q}_1) = \frac{\alpha\,\alpha_1}{8\pi\,\rho}\left(\frac{\mathfrak{q}^2}{\alpha_1}-\frac{2\mathfrak{q}\mathfrak{q}_1}{R}+\frac{\mathfrak{q}_1^2}{\alpha}\right) ,$$

or

$$\mathfrak{T}_1 = h - \frac{\alpha\,\mathfrak{q}\cdot\alpha_1\,\mathfrak{q}_1}{4\pi\,\rho}\cdot\frac{1}{R} \, ,$$

in which the index on $\mathfrak{T}_1$ shall mean that $\mathfrak{T}$ is written in terms of the impulses $\mathfrak{q}$, instead of the velocities e, and the constant:

$$h = \frac{1}{4\pi\rho}(\alpha\,\mathfrak{q}^2 + \alpha_1\,\mathfrak{q}_1^2)$$

has an appreciably larger value than the variable terms in $\mathfrak{T}_1$ (for finite values of R).

When one further identifies U with the gravitational potential, following a suggestion of RIEMANN (*Ges. Werke*, ed. by WEBER, pp. 503), and employs an undetermined quantity σ in order to set:

$$m = \alpha\, \mathfrak{q}\, \sigma, \qquad m_1 = \alpha_1\, \mathfrak{q}_1\, \sigma,$$

$$k = \frac{1}{4\pi\rho\,\sigma^2},$$

in which k is the gravitational constant, then [art. **14**, (7.a)] equations (8) will go to the known equations for two masses that attract according to the law of gravitation:

$$m\,\ddot{a} = \frac{\partial U}{\partial a}, \qquad m_1\,\ddot{a}_1 = \frac{\partial U}{\partial a_1},$$
$$\text{etc.,} \qquad\qquad \text{etc.,}$$

where

$$U = \frac{m\,m_1 k}{R} - h$$

is the known gravitational potential. In that way, the phenomena that one observes for two attracting mass-points (e.g., a double star) can be reproduced by the action of two sinks.

One can also employ two sources in their place. The signs of e and e_1, of $\mathfrak{q}$, $\mathfrak{q}_1$, and σ would change simultaneously, but the homogeneous quadratic expressions $\mathfrak{T}$, $\mathfrak{T}_1$, U in those quantities would remain unchanged, just like m, m_1, k. If one now assumes that one has two overlapping fluid systems that move independently of each other and that only the visible masses (the balls α, α_1) are the same form them, and indeed sinks for one of them and sources for the other, then the effects of the two would not cancel, but add together ([1]).

The second example of a cyclic fluid motion, namely, rings immersed in a fluid (art. **33**), differs from the one that was just treated by the fact that for the cyclic system, the expression for the kinetic energy $\mathfrak{T}$ as a function, it is not the vortex strengths $2\pi\kappa$, $2\pi\kappa_1$ that first presented themselves that should be employed, but the amounts of fluid λ, λ_1 that flow through the cross-sections, which are derived from them.

In art. **34**, we obtained the following value for $\mathfrak{T}$:

([1]) That double system is not subject to an objection that was raised in a letter that was written by EHRENFEST against the isolated system, as it was given by the author in Math. Ann. **58**, namely, that the system would shrink as a result of the outflow of mass. In art. **32**, that loss was calculated to be $4\pi\, e\, \rho$ ($4\pi\, e_1\, \rho$, resp.) per second. In the double system, all that would result is a transfer of mass between the mass-points m, m_1 that carries kinetic energy from one space to the other in which the mass-points play the roles of branching points at which the two spaces are connected onto perhaps a four-dimensional space.

Moreover, the mass and energy loss is vanishingly small in comparison to the reserve of infinite space in the case of the isolated system.

$$\mathfrak{T} = \frac{2\pi\rho}{AC - B^2}[\lambda^2 C - 2\lambda\lambda_1 B + \lambda_1^2 A] ,$$

in which the quantities λ can be regarded as velocities (cyclic intensities). One can once more treat the "free system" that is composed of the material rings and the cyclic system. However, we would like to restrict ourselves to the equations that belong to the cyclic subsystem alone:

$$\frac{d}{dt}\left(\frac{\partial\mathfrak{T}}{\partial\lambda}\right) = 0 , \qquad \frac{d}{dt}\left(\frac{\partial\mathfrak{T}}{\partial\lambda_1}\right) = 0 .$$

One then has that:

$$\frac{\partial\mathfrak{T}}{\partial\lambda} = \mathfrak{q} , \qquad \frac{\partial\mathfrak{T}}{\partial\lambda_1} = \mathfrak{q}_1$$

are then constants whose values can again be determined inversely from equations (4) in art. **34**:

$$\mathfrak{q} = 4\pi\,\kappa\,\rho , \qquad \mathfrak{q}_1 = 4\pi\,\kappa_1\,\rho .$$

When expressed in terms of the quantities κ, the kinetic energy has value that is known from art. **34**:

$$\mathfrak{T}(\kappa, \kappa_1) = \mathfrak{T}_1 = 4\pi\,\kappa\,\kappa_1\,\rho \cdot B + h ,$$

where h will be a constant quantity in the case where the rings are rigid. The force function:

$$U = -\,\mathfrak{T}_1 = -\,4\pi\,\kappa\,\kappa_1\,\rho \cdot B + h ,$$

or

$$U = -\,4\kappa\,\kappa_1\,\rho \iint \frac{ds\,ds_1 \cos(ds, ds_1)}{r} + h ,$$

is, as KIRCHHOFF (Jour. f. Math. Bd. 71, pp. 273) emphasized, equal to the potential of the action of two electric currents of intensities κ, κ_1 that flow through the rings upon each other.

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CHAPTER THREE

SOLID-ELASTIC AND QUASI-ELASTIC MASSES

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§ 36. – Solid-elastic masses. Work done by internal forces.

In art. **22**, (elastic) *solid* and *fluid* masses were distinguished in such a way that the former (when they were finitely-extended in any direction) never went very far from a certain equilibrium position, such that the formulas in art. **20** that related to very small changes of state would be generally applicable to them, while for fluid masses, those formulas would only be satisfied by states that follow immediately in time.

Furthermore, the *fluid masses* (art. **23**) were classified into ones with *unvarying densities* (viz., incompressible ones) and *elastic* (viz., compressible) ones according to whether the divergence was everywhere (except for some singular points) zero or non-zero, resp.

A further basis for the classification of space-filling masses is given by the *work* (even when no impressed volume forces act) that is done or consumed by a change of position or state of a volume element that is found inside of the deformed mass. One ascribes it to *internal* (elastic) *forces* that appear under the deformation, and for that reason, one speaks of *internal work*.

If one imagines that a very small disc has been inserted into a space-filling medium at a point then the pressure that acts upon it per unit area (which generally points in a direction that is skew to the surface, so it can be decomposed into a normal and tangential component, even when, say, the medium, and with it, the disc, are found to be in motion) will be equal and opposite on the two sides of that disc, because otherwise a splitting of the mass along the disc and an acceleration of the one part with respect to the other would result, which was excluded in art. **17**, at the end.

One assumes that *elastic forces* will be produced inside of any spatial element in an elastic mass that takes the form of a parallelepiped and is deformed along with it by surface tractions or volume forces that maintain equilibrium in the tensions that the neighboring elements on the boundary surface exert upon each other and can be measured by them. In that way, it will be further assumed of *elastic solid* bodies that there is a state of the mass – viz., the *equilibrium state* – for which no internal elastic forces appear at any location, so all internal pressures will be zero. For *elastic fluid* bodies, the equilibrium state is a limiting state that will occur when the density is zero.

Instead of defining the work done – δw by elastic forces at a distance from the equilibrium state, we shall define the work done δw by the *applied* outer surface forces that neighboring elements exert on the elementary parallelepiped when we first give them *finite* dimensions and appeal to experiments in order to calculate the work that is done by those forces. The elementary work δw (the sum δW of them over the entire mass, resp.) is introduced in the formula for D'ALEMBERT's principle with the same sign as the external volume and surface forces $\delta' U$ and $\delta' S$.

From art. **20**, the most general change of position and form of a spatial element is composed of three elementary motions:

1. A parallel translation,
2. A rotation,
3. A sequence of three extensions (contractions) along three mutually-perpendicular directions, namely, the axes of principal dilatations.

In the case of ponderable masses (which is the only one that we shall deal with for now), only the latter changes will produce work done by elastic forces. We imagine that an elementary cube of edge length 1 has been cut out from the interior of an already-deformed mass (which is at rest or in motion). The outer surface forces P_1, P_2, P_3 that the elongations in the directions of the edges produce might act *perpendicular to the faces of the cube*. That assumption is always fulfilled when the elastic body is "isotropic" (see art. **39**). Accordingly, we will also call the axes of principal dilatation, so the directions of the cube edges, *axes of principal stress*.

Under a further virtual deformation of the mass that follows the aforementioned deformation, the edge lengths might lengthen by $\delta\rho_1$, $\delta\rho_2$, $\delta\rho_3$, resp. The total virtual work that is done by those forces on the cube will then be:

$$\delta w = P_1\,\delta\rho_1 + P_2\,\delta\rho_2 + P_3\,\delta\rho_3\,. \tag{1}$$

We shall neglect the infinitely-small changes of direction that the principal stress axes can experience in that way and assume that the cosines of the small angles that measure them are equal to 1.

As we will infer from experiments later (art. **39**), the stresses P can be represented as linear functions of the dilatations ρ_1, ρ_2, ρ_3 for solid-elastic bodies.

By contrast, for elastic fluids, experiments show that mobility of the particles implies that the three principal stresses are equal to each other at any location. If we then set $P_1 = P_2 = P_3 = P$ then experiments will further show that the *pressure* P inside of an *elastic fluid medium is a function of the density* ρ, $P = f(\rho)$, that vanishes with ρ and is independent of the medium (so it can be assumed to be known). In the case of elastic fluid media, the work done by a virtual deformation is then:

$$\delta w = P\,(\delta\rho_1 + \delta\rho_2 + \delta\rho_3) = P\;\delta\Theta\,, \tag{2}$$

in which $\delta\Theta$ is the virtual increase in the spatial dilatation (art. **21**), or when we use the relation (17) in art. **23** (at the end):

$$\delta w = P\left(\frac{\partial\,\delta x}{\partial x} + \frac{\partial\,\delta y}{\partial y} + \frac{\partial\,\delta z}{\partial z}\right) = P \operatorname{div} \delta\mathfrak{s} = -\,P\,\frac{\delta\rho}{\rho}\,. \tag{2.a}$$

If one lets δW denote the *total amount* of the work done by the applied forces then one must set:

$$\delta W = \int \delta w\, d\tau = \int (P_1\,\delta\rho_1 + P_2\,\delta\rho_2 + P_3\,\delta\rho_3)\, d\tau \tag{3}$$

for *solid-elastic bodies* and:

$$\delta W = \int \delta w\, d\tau = \int P\left(\frac{\partial\,\delta x}{\partial x} + \frac{\partial\,\delta y}{\partial y} + \frac{\partial\,\delta z}{\partial z}\right) d\tau \tag{4}$$

One must introduce those values for δW in the equation for D'ALEMBERT's principle [art. **19**, (2)]:

$$\int [(\ddot{x} - X)\,\delta x + (\ddot{y} - Y)\,\delta y + (\ddot{z} - Z)\,\delta z]\,\rho\, d\tau - \delta W - \delta' S = 0\,, \tag{5}$$

according to the nature of the elastic medium, in which the x, y, z are again thought of functions of a, b, c, and t. Here, as in art. **19**, it is also generally assumed that the motion results with no production of heat. When a loss of energy results from friction (either internal friction, as in viscous fluids, or friction against the environment), we would like to introduce that into the calculations (art. **16**, at the end) as the work done by frictional forces.

As we will see, in the case of solid-elastic bodies, a function W, namely, the potential of the elastic forces, will exist that makes the quantity δW the variation of that function. In the case of elastic fluids, such a thing can be inferred from formula (2.a) with no further analysis when one recalls that P is a function of ρ. In that case, δw clearly has the precisely the same form as in the text and footnote to art. **25** for incompressible fluids $\lambda\ \delta\varphi$, in which $\delta\varphi = 0$ was derived from the condition div $\mathfrak{v} = 0$ for incompressibility (art. **23**). When one sets $P = \lambda$, the formulas that D'ALEMBERT's principle implies will be identical in both cases. Therefore, the further treatment of equation (5) will lead to the same differential equations of motion for the case of elastic fluids as the ones that were presented in art. **25** in (4), (5), except that the continuity equation:

$$\text{div}\ \mathfrak{v} + \frac{\dot{\rho}}{\rho} = 0$$

will enter as a further equation of constraint (in place of div $\mathfrak{v} = 0$), or a relation that exists between pressure and density:

$$P = f(\rho)\,,$$

resp.

§ 37. – The quasi-elastic medium: Work done by internal forces.

With the assumptions that were made in the previous article, the motion or equilibrium inside of ponderable solid-elastic and fluid masses can be described in accordance with experiments. The theory that is founded in that way and developed further in what follows likewise subsumes the study of the rigidity of engineering constructions, such as the study of the propagation of sound in air and water or the vibrations of resonant strings and membranes.

One also seeks to place the propagation of light (and more recently, the state of electromagnetic excitation that relates to the motion of light) on the same foundation when one assumes that an elastic medium is the carrier of the oscillations that emanate from luminous or electrically-excited bodies. However, in a treatise that was directed against CAUCHY ([1]), MAC CULLAGH has already referred to certain contradictions that stand in the way of explaining elliptic polarization when one regards the ether as a solid-elastic body. By contrast, he succeeded in avoiding that complication, along with others that pertain to the behavior of light at the boundary surface between two media, by making the assumption that the carrier of light motion is a medium that possesses a peculiar type of elasticity that is not comparable to the kind that was discussed, namely, the kind that one also takes advantage of in the electromagnetic theory of light as the intermediary of its waves ([2]). Speaking briefly, the *ether*, as we would like to call that intermediary, then has the elastic property that a mere rotation of the volume element (without a change in form or displacement) already requires an expenditure of work.

LORD KELVIN also made such a medium accessible to representation ([3]) when he constructed a framework in which a number of tops were embedded and distributed in a special way that each rotated about its axis in pair-wise opposition to the others. Due to their ambition to maintain the axis of rotation, a resistance opposed any attempt to rotate those tops (when their axes were oriented in different directions) that required an expenditure of work to overcome: The elastic medium that MAC CULLAGH introduced can be regarded as just such a finely-conceived model.

KELVIN called such a medium improperly solid (*quasi-rigid*), or more roughly, "ether." We would like to follow his example, except that we will translate the word "quasi-rigid" into *quasi-elastic*.

We thus assume that the increment of work δw that an applied force does on an elementary parallelepiped under a small increment of rotation is proportional to:

1. The absolute value of the increment of rotation $\delta\mathfrak{w} = (\delta\xi, \delta\eta, \delta\zeta)$, when:

$$\xi = \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right), \qquad \eta = \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right), \qquad \zeta = \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) \tag{1}$$

(in which u, v, w are small displacements) mean the components of the rotation $\mathfrak{w}$ from the equilibrium position.

2. The absolute value of just that rotation $\mathfrak{w}$ from the equilibrium position.

3. The cosine of the angle between the axes of $\mathfrak{w}$ and $\delta\mathfrak{w}$.

([1]) "Notes on some Points in the Theory of Light," Proc. Irish Acad. **2** (1841); *Collected Works of MacCullagh*, pp. 194. The mechanical explanation was given in the treatise "An Essay towards a Dynamical Theory of Crystalline Reflexion and Refraction," Trans. Irish Acad. **21** (1839), pp. 145.

([2]) LORD KELVIN (WILLIAM THOMSON) was probably the first to do that in *Math. and Phys. Papers*, III, pp. 436 (1890), art. 99: "Viscous Liquid, Elastic Solid Ether," and almost simultaneously, A. SOMMERFELD: "Mechanische Darstellung, etc." Ann. Chem. Phys., Series N **46** (1892). R. REIFF: "Elastizität und Elektrizität," Freiburg, 1893 and J. LARMOR: "A Dynamical Theory of Electric and Luminiferous Medium." Proc. Lond. Math. Soc. **54** (1893).

([3]) *Loc. cit.*, art. 100, "On a Gyrostatic Adynamic Constitution for 'Ether.' "

With the vector notation of art. **23**, when B is a positive constant, we then set:

$$-\delta w = 4\,\mathsf{B}\,(\mathfrak{w},\,\delta\,\mathfrak{w}) = 4\,\mathsf{B}\,|\,\mathfrak{w}\,|\,|\,\delta\,\mathfrak{w}\,|\cos(\mathfrak{w},\,\delta\,\mathfrak{w}) = 4\,\mathsf{B}\,(\xi\,\delta\xi + \eta\,\delta\eta + \zeta\,\delta\zeta)$$
$$= 2\mathsf{B}\,\delta(\xi^2+\eta^2+\zeta^2) = 2\,\mathsf{B}\,\delta|\,\mathfrak{w}\,|^2. \tag{2}$$

With that, the variation of the total work W done by internal forces in the quasi-elastic medium is:

$$\delta W = \int \delta w\,d\tau = -\int 4\mathsf{B}(\xi\,\delta\xi + \eta\,\delta\eta + \zeta\,\delta\zeta)\,d\tau = -\int 4\mathsf{B}\,\delta(\xi^2+\eta^2+\zeta^2)\,d\tau\,. \tag{3}$$

That assumption is reproduced by the one that gives one an idea of the nature of the work done on a solid-elastic body (art. **36**). However, all of those assumptions again refer to ideal cases whose properties occur in nature only to a greater or lesser extent.

In the foregoing (where we ignored internal friction), we defined four different media:

1. The incompressible fluid.
2. The elastic fluid.
3. The solid-elastic mass.
4. The quasi-elastic medium.

We shall once more summarize the properties and relationships that have proved (will prove, resp.) to be necessary and sufficient to define the motion of equilibrium state of a mass particle inside of them completely in the table on the following page.

§ 38. – The elastic string.

Just as one deals with the increment $\delta\,w$ in internal work that results from a virtual deformation as a function of only density for fluid masses, and thus represents the deformation that is created by a spatial dilatation, in the case of solid-elastic bodies, one deals with a connection between the linear dilatations ρ_1, ρ_2, ρ_3 and the forces that they produce.

HOOKE's law, in conjunction with an assumption regarding the lateral contraction, yields that relationship. However, before we go into a general discussion of that, we would like to consider a special case, namely, the elastic string, whose treatment can be reduced to that of the non-elastic one (i.e., the chain) in a way that is similar to how the equations of motion for the elastic fluid could be reduced to the ones for the non-elastic fluid in art. **36**.

As in art. **19**, we refer the motion of a homogeneous string to its initial state: The (equilibrium) state of the inextensible string, for which a point that is found at x, y, z at time t might possess a distance of a from its starting point. If the cross-section is equal to 1 throughout then the volume element will be $d\tau = 1 \cdot ds$. One will get the variation δw in the internal work by means of which da goes to ds when P_1 is the tension in the direction of the string element (one of the principal stress axes) and ρ_1 is the elongation of the unit from formula (1) in art. **36**:

Defining properties and relations	*Fluid masses* (without friction) ρ = density, $\mathfrak{v} = (v, v, w)$ = velocity		*Solid-elastic masses* (in the broad sense) ρ = density, $\mathfrak{v} = (v, v, w)$ = displacement from equilibrium	
	incompressible	*compressible* (elastic)	*ponderable isotropic solid-elastic masses*	*quasi-elastic medium* (ether)
1. The *equilibrium position*, for which the work done by internal forces vanishes	There are none		They exist	
2. Expression for the *kinetic energy*	$\frac{\rho}{2}(u^2+v^2+w^2) = \frac{\rho}{2}\lvert\mathfrak{v}\rvert^2$		$\frac{\rho}{2}(\dot{u}^2+\dot{v}^2+\dot{w}^2) = \frac{\rho}{2}\lvert\dot{\mathfrak{u}}\rvert^2 = \frac{\rho}{2}\lvert\mathfrak{v}\rvert^2$	
3. Relation between *spatial extension* (dilatation) Θ = div $\mathfrak{u}$ and the linear dilatation. *The change:* $\dot{\Theta} = \text{div}\,\mathfrak{u} = \frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}$	–	–	$\Theta = \text{div}\,\mathfrak{u} = \rho_1+\rho_2+\rho_3$, where ρ_1, ρ_2, ρ_3 are the extensions in the directions of the principal stress axes, counted from the equilibrium position	div $\mathfrak{u}$ = 0
in Θ and the *change in density* $\dot{\rho}$, resp.	div $\mathfrak{v}$ = 0 (1) ρ = const.	div $\mathfrak{v} + \frac{\dot{\rho}}{\rho} = 0$ (1)	–	–
4. *Changes (deformations)* under which *elastic* (internal) *work* will be done	none	Change in volume of the volume element with the change in density (see. 3)	Linear elongation or contraction in any direction	Rotation
5. The *elastic work* δw done on a volume element by the deformations in 4.	none	$\delta w = P$ div $\delta\mathfrak{s} = -\delta\rho / \rho$, in which P is the pressure. (For $\delta\mathfrak{s}$, cf., footnote)	$\delta w = P_1\,\delta\rho_1 + P_2\,\delta\rho_2 + P_3\,\delta\rho_3$, in which P_1, P_2, P_3 are the stresses in the direction of the principal stress axes	$\delta w = -4B(\xi\,\delta\xi+\eta\,\delta\eta+\zeta\,\delta\zeta) = -2B\,\delta\lvert\mathfrak{w}\rvert^2$, in which $(\xi, \eta, \zeta) = \mathfrak{w} = \frac{1}{2}$ rot $\mathfrak{u}$ means the rotation from the equilibrium position
6. *Relation* between the internal *tensions* and the *deformations* that they produce	none	A relation exists between pressure and density that depends upon the nature of the medium: $f(P, r) = 0$	The relations: $\rho_1 = \frac{1}{E}(-P_1+\mu P_2+\mu P_3)$, $\rho_2 = \frac{1}{E}(-P_2+\mu P_3+\mu P_1)$, $\rho_3 = \frac{1}{E}(-P_3+\mu P_1+\mu P_2)$, exist between the extensions ρ_1, ρ_2, ρ_3, and the stresses in the directions of the principal stress axes (see art. 39), where E is the elastic modulus	One can regard the vector: $(-4B\,\xi, -4B\,\eta, -4B\,\zeta) = -4B\,\mathfrak{w}$ as the angular moment occurs when the volume element is rotated from the equilibrium position through the quantity $\mathfrak{w}$

(1) Upon differentiating these equations with respect to time and varying the accelerations $\dot{u}$, $\dot{v}$, $\dot{w}$, one will get the relations between the variations div $\delta\ddot{\mathfrak{s}} = 0$ (div $\delta\ddot{\mathfrak{s}} + \delta\ddot{\rho} / \rho = 0$, resp.), which can be replaced with: div $\delta\mathfrak{s} = 0$ (div $\delta\mathfrak{s} + \delta\rho / \rho = 0$, resp.).

$$\delta w = P_1\ \delta\rho_1 = P_1\,\delta\left(\frac{ds}{da}-1\right) = \frac{P_1\,\delta(ds^2)}{2\,ds\,da} = P_1\left(\frac{dx}{ds}\,\delta\frac{dx}{da}+\frac{dy}{ds}\,\delta\frac{dy}{da}+\frac{dz}{ds}\,\delta\frac{dz}{da}\right), \tag{1}$$

or when one sets:

$$P_1\frac{da}{ds} = \lambda\,, \tag{2}$$

one will get:

$$\delta w = \lambda\left(\frac{dx}{da}\,\delta\frac{dx}{da}+\frac{dy}{da}\,\delta\frac{dy}{da}+\frac{dz}{da}\,\delta\frac{dz}{da}\right).$$

Thus, the variation of the total work done by internal forces will become:

$$\delta W = \int \delta w\cdot da = \int \lambda\,\mathsf{S}\,\frac{dx}{da}\,\delta\frac{dx}{da}\,da\,, \tag{3}$$

in which S once more means a sum of three similarly-defined terms.

That expression for δW is introduced into equation (2) of art. **19** for D'ALEMBERT's principle, and when one simply replaces the non-elastic string that was treated there with an elastic one, it will enter in place of [formula (5), *loc. cit.*] the term that is derived from the condition for inextensibility:

$$\tfrac{1}{2}\lambda\,\delta\varphi = \lambda\,\mathsf{S}\,\frac{dx}{da}\,\delta\frac{dx}{da}\,da\,. \tag{4}$$

Performing the further calculations will then yield precisely the same form in both cases. At the same time, λ has the meaning of a pressure in both cases, except that the incompressibility condition must be replaced with the continuity equation:

$$\frac{ds}{da} = \frac{\rho_0}{\rho}\,, \tag{5}$$

which says that the mass $\rho_0\,da$ is distributed along the length ds. However, the three differential equations (7) of art. **19**, together with (5), do not suffice to determine the five quantities x, y, z, ρ, λ. One must then know the connection between the elongation of the unit length and the tension P that makes that possible. *It is provided by the aforementioned Hooke law*, which is a law of nature that defines the foundation for the entire theory of elasticity, although for many materials it is valid only within a very narrow range.

In the present case, it says that as long as ds / da differs from 1 by only a very small quantity, the tension and elongation (pressure and shortening) will be proportional quantities ("*ut tension sic vis*"), so that:

$$P = E\left(1-\frac{ds}{da}\right), \tag{6}$$

where E, the "elastic modulus," is a very large positive number that differs with the material ([1]), just like the limits inside of which that formula will be applicable.

We shall not pursue the system of differential equations that are adopted from [art. **19**, (7)] any further, but refer to the theory of vibrating strings in ROUTH's *Dynamik* (German by SCHEPP, Leipzig, 1898, v. II, pp. 465) for examples of its applications.

§ 39. – Deformation and internal forces in isotropic solid-elastic bodies.

For what follows, we shall make the simplifying assumption that in their natural (i.e., equilibrium) states, the solid-elastic media that we shall be concerned with are:

1. *Homogeneous*, i.e., exhibit an equal resistance to any tension or compression at all locations.

2. *Isotropic*, i.e., they exhibit the same behavior in each direction.

For certain materials that are important in engineering, like wire or wood, which can exhibit a different character in the direction of the fiber than they do perpendicular to it, the following analysis can be employed only cautiously.

At a location inside of an isotropic solid-elastic mass, once a deformation has occurred, the elongations of the unit length in the directions of the principal stress axes will amount to ρ_1, ρ_2, ρ_3, resp. (which are very small quantities in comparison to 1). We attribute the elongations that the edges of a cube K that is cut out along the directions will experience to forces of tension that the neighboring elements exert upon each other in the direction of those axes, so perpendicular to the boundary faces of the cube. The extensions ρ_1, ρ_2, ρ_3 depend upon those tensions, which we would like to denote by $-P_1$, $-P_2$, $-P_3$, resp. (as negative compressions). We infer their law of dependency from experiments that are based upon the behavior of a cube of *finite* dimensions whose opposite faces are acted upon by tensions (or compressions) uniformly. That implies the following:

1. For the connection between tension (compression) and extension (shortening, both of which will be combined into "deformation" or "strain," as the English say), one has the aforementioned *Hooke law*, which expresses the proportionality of the two ([2]). More precisely: A parallelepiped of length l and cross-section q on which a tension $-Q$ (negative compression) acts upon two of its opposite base surfaces will experience a lengthening (negative shortening) of λ as a result. When the tension does not exceed a certain limit that depends upon the material ([3]), one will then have the relation:

([1]) The right-hand side of equation (6) will become negative when the string-element is stretched. Therefore, $(-P)$ means a tension, which agrees with the notation of the previous and following articles.

([2]) The limit up to which the proportionality still exists is, e.g., 16 kg per mm^2 for wrought iron.

([3]) The negative sign is based upon the fact that (art. **36**) the applied tension that is introduced here is directed oppositely to the internal elastic one.

$$\lambda = -\frac{Q\,l}{qE}, \tag{1}$$

in which the proportionality factor E, viz., the "modulus of elasticity," is a constant that depends upon the material ([1]). *That relationship will still exist* when the tension becomes a compression, and the elongation then becomes a contraction.

2. The tension ($-Q$) in the direction of an edge produces a contraction of the parallelepiped in the directions that are perpendicular to the direction of the tension, which is a "lateral contraction" (for a compression, it will be a "lateral dilatation") that amounts to a fraction μ of the elongation λ, where μ is once more a quantity that depends upon the material (and most of the time, $\mu = 1/4$). The edge of a cube of unit side is then shortened by a compression $-P$ $(= -Q/q)$, which we now regard, as always, as a surface force per *unit cross-section*,as a result, by:

$$\mu\,\lambda = -\,\mu\frac{P}{E}$$

in the lateral direction.

3. The elongations (contractions) that are produced by three tensions and compressions that act simultaneously in the directions of the edges (and each of the directions perpendicular to them), which are very small in all cases, combine like vectors (viz., the principle of the *superposition of small actions*).

Finally, we emphasize that the volume forces that might act upon the interior of an elementary cube (just like the difference between the surface tractions that act upon each opposing pair of faces) have an order of magnitude that is lower by 1, like the surface tractions themselves.

Based upon those conventions, the elongations ρ_1, ρ_2, ρ_3 that the edges of an elementary cube K of unit edge length whose edges are oriented along the *principal stress axes* (art. **36**) experience under the influence of the surface tractions P_1, P_2, P_3 that act in just those directions, and conversely, the latter will follow when the former is given.

The elongation of the unit edge length in the direction of the force $-P_1$, when one considers that the forces $-P_2, -P_3$ produce simultaneous lateral contractions, amounts to:

$$\rho_1 = -\frac{P_1}{E} + \frac{\mu P_2}{E} + \frac{\mu P_3}{E},$$

in total. One then gets:

$$\rho_1 = \frac{1}{E}(-P_1 + \mu P_2 + \mu P_3),$$

$$\rho_2 = \frac{1}{E}(\mu P_1 - P_2 + \mu P_3), \tag{2}$$

([1]) E.g., for the modulus of elasticity for steel amounts to 21 000 kg per mm^2.

$$\rho_3 = \frac{1}{E}(\mu\, P_1 + \mu\, P_2 - P_3)\,.$$

In order to solve those equations for P, one must multiply them by $(1 - \mu)$, μ, μ, resp., and add them. That will give:

$$\frac{1}{E}[-P_1\,(1-\mu) + 2P_1\,\mu^2] = \rho_1\,(1 - 2\mu) + \mu\,(\rho_1 + \rho_2 + \rho_3)\,,$$

or when one introduces the spatial dilatation (art. **21**):

$$\Theta = \rho_1 + \rho_2 + \rho_3\,, \tag{3}$$

the first of the following equations:

$$\begin{aligned} -P_1 &= \frac{E}{1+\mu}\left(\rho_1 + \frac{\mu}{1-2\mu}\Theta\right), \\ -P_2 &= \frac{E}{1+\mu}\left(\rho_2 + \frac{\mu}{1-2\mu}\Theta\right), \\ -P_3 &= \frac{E}{1+\mu}\left(\rho_3 + \frac{\mu}{1-2\mu}\Theta\right). \end{aligned} \tag{4}$$

Thus, the work that is done by a virtual change $\delta\rho_1$, $\delta\rho_2$, $\delta\rho_3$ in the edge lengths of the already-deformed cube can be represented by:

$$\delta w = P_1\,\delta\rho_1 + P_2\,\delta\rho_2 + P_3\,\delta\rho_3 = -\frac{E}{2(1+\mu)}\left(\delta(\rho_1^2 + \rho_2^2 + \rho_3^2) + \frac{\mu}{1-2\mu}\delta\,\Theta^2\right). \tag{5}$$

That expression does not change when one converts the elongations ρ into contractions $(-\rho)$. Any distance from the equilibrium position is then coupled with an *accumulation* of work when $1 - 2\mu$ is *positive*, which would then be once more implied by the demand that the equilibrium position should be stable. In arts. **20**, **21**, the ρ have the same meaning that they have here. As we did there, we would now like to introduce fixed spatial coordinates along with the principal stress axes (which are generally inclined with respect to them), relative to which the point x, y, z experiences the displacements u, v, w. The expressions $x_x = \partial u / \partial x$, $z_y = \partial w / \partial y + \partial v / \partial z$, etc. (art. **21**) then measure the changes in the edges and angles of an elementary cube K that is cut out from the directions of those *fixed axes*. Once more, let Θ be the spatial dilatation. The two symmetric functions of the ρ that appear in δw can be expressed in terms of those quantities as follows [*loc. cit.*, forms. (20), (21)]:

$$\rho_1^2 + \rho_2^2 + \rho_3^2 = \tfrac{1}{2}\mathsf{S}\, x_y^2 - 2\,\mathsf{S}\, y_y\, z_z + \Theta^2\,, \tag{6}$$

$$\rho_1 + \rho_2 + \rho_3 = \Theta = \mathsf{S}\, x_x\,, \tag{6.a}$$

in which, as before, S denotes the sum over three terms that are defined analogously.

Hence, δw then takes on the form:

$$\delta w = -\tfrac{1}{2} A \delta \Theta^2 - \tfrac{1}{2} B \delta (x_y^2 + y_z^2 + z_x^2 - 4y_y\, z_z - 4z_z\, x_x - 4x_x\, y_y) \;, \tag{7}$$

where

$$A = \frac{E(1-\mu)}{(1+\mu)(1-2\mu)}, \quad B = \frac{E}{2(1+\mu)} \;. \tag{7.a}$$

Not only will the edge lengths of the elementary cube *K* (when oriented along the axes of the fixed coordinate system) be changed by the deformation, but, in general, its edge angles, as well. Clearly, instead of attributing the work that relates to the deformation of a unit volume and its increment δw to an action of the forces on the cube like K, one can also attribute it to an action of forces that are applied to the outer surface of the cube like *K*. However, whereas the forces *P* are perpendicular to the side faces of K, the forces that are applied to the cube *K* are inclined with respect to them, in general.

One finds those surface tractions simply by calculating the coefficients of the expression for δw in (7), when ordered in terms of the quantities δx_x, …, δy_z, … That is because those quantities can be interpreted as the edge elements (angle elements, resp.), along which the normal forces (tangential force-couples, resp.) that are applied to the faces of *K* do work. Now, one has:

$$\delta w = -\,\mathrm{S}\,[A\,\Theta - 2B\,(y_y + z_z)]\;\delta x_x - \mathrm{S}\;B\,y_z\;\delta y_z\;,$$

as one easily confirms by performing the variations in (7), so in the form:

$$\delta w = \mathrm{S}\;X_x\;\delta x_x + \mathrm{S}\;Y_z\;\delta y_z\;. \tag{8}$$

That will imply that the pressure X_x that causes the *elongation* δx_x of the *edge* 1 of the cube *K* that is parallel to the *X*-axis is:

$$X_x = -\,A\,\Theta + 2\,B\,(y_y + z_z) = -\,\Theta\,(A - 2B) - 2B\,\frac{\partial u}{\partial x} = -\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial z}{\partial z}\right)(A - 2B) - 2B\frac{\partial u}{\partial x}\,, \tag{9}$$

and analogously:

$$\begin{aligned} Y_y &= -\Theta(A-2B) - 2B\frac{\partial v}{\partial y}\,, \\ Z_z &= -\Theta(A-2B) - 2B\frac{\partial w}{\partial z}\,. \end{aligned} \tag*{(9)[sic]}$$

Likewise, the force-pairs that produce the *changes in the edge angles* can be represented by:

$$
\begin{aligned}
Y_z = Z_y &= -B\, y_z = -B\left(\frac{\partial w}{\partial y}+\frac{\partial v}{\partial z}\right), \\
Z_x = X_z &= -B\, z_x = -B\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right), \\
X_y = Y_x &= -B\, x_y = -B\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right).
\end{aligned} \tag{9}
$$

One must then represent the fact that three pressure components act upon each of the six boundary faces, which are pair-wise denoted by the attached indices *x*, *y*, *z*, in the directions of the three coordinate axes (e.g., on the faces X_x, Y_x, Z_x, which are perpendicular to the *X*-axis) that are equally large and oppositely-directed for ones that lie in opposition to each other. The components that are perpendicular to the faces, such as X_x, are tensions or compressions in the proper sense. The other ones, such as Y_z, Z_y, … combine with the ones $-Y_z$, $-Z_y$, … that are applied to each opposing face into force-couples, every two of which will rotate in the opposite sense (as is suggested in the accompanying figure), and thus produce a change in the edge angles. One calls forces like Y_z, Z_y whose directions fall along the faces to which they are applied *thrusts* or *shear forces* (tangential forces).

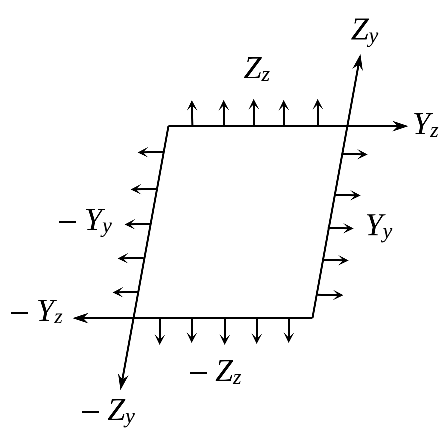

Figure 22.

The variation of the total work δW done by internal forces will be represented by the integral:

$$
\delta W = \int \delta w\, d\tau = \int \mathsf{S}[X_x\,\delta x_x + Y_z\,\delta y_z]\,d\tau = \int \mathsf{S}\left[X_x\,\delta\frac{\partial u}{\partial x} + Y_z\,\delta\left(\frac{\partial w}{\partial y}+\frac{\partial v}{\partial z}\right)\right] d\tau, \tag{11}
$$

in which X_x, …, Y_z, … are the expressions that were given above in (9), (10). We shall make use of that formula in the derivation of the differential equations of motion for a solid-elastic body that now follows.

§ 40. – The differential equations of motion for solid-elastic bodies.

The value of δW that was exhibited in the previous article (at the end) must be substituted in equation (5) of art. **36**, which gives D'ALEMBERT's principle for space-filling masses, except that the coordinates *x*, *y*, *z* in it that were assumed to be variable must be replaced with the displacements *u*, *v*, *w*, because in the previous article, the coordinates *x*, *y*, *z* determined the position of the mass-element in its natural (undeformed) state, so they can be neither functions of time nor varied ones. We then imagine that we have replaced:

$$a, b, c \quad \text{with} \quad x, y, z,$$

$$x, y, z \quad \text{with} \quad x + u, y + v, z + w ,$$

everywhere in formula (5) of art. **36**, in which we represent the displacements u, v, w as functions of x, y, z, t. Previously, we treated the derivatives of x (a, b, c, t) with respect to time as total derivatives. However, since x, y, z now appear along with u, v, w, that would recommend that the differential quotients with respect to time should be denoted by partial derivatives. We then replace:

$$\frac{d^2x}{dt^2} = \ddot{x} \qquad \text{with} \qquad \frac{\partial^2 u}{\partial t^2} = \ddot{u}, \qquad \text{etc.},$$

and in what follows, we would like to denote *partial derivatives with respect to time by an overhead dot*, when that previously denoted total derivatives with respect to time.

One thus obtains from D'ALEMBERT's principle that:

$$\int \rho[(\ddot{u} - X)\,\delta u + (\ddot{v} - Y)\,\delta v + (\ddot{w} - Z)\,\delta w]\,d\tau - \delta W - \delta' S = 0 , \tag{1}$$

in which one now has:

$$\delta W = \int \mathsf{S} \left[X_x\,\delta\frac{\partial u}{\partial x} + Y_z\,\delta\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial x}\right)\right] d\tau . \tag{2}$$

In order to exhibit the differential equations of motion from (1), one must next convert δW into a linear function of δu, δv, δw by partial integration. The moving mass might fill up a space T that can be finite or infinite. From the oft-employed process (cf., e.g., art. **24**), one transforms the integral that is extended over the space T (we restrict ourselves to the first term) as follows:

$$\int X_x\,\delta\left(\frac{\partial u}{\partial x}\right) d\tau = \iint dy\,dz \int X_x \frac{\partial\,\delta u}{\partial x}\,d\tau = \iint dy\,dz\,[X_x\,\delta u] - \int d\tau \frac{\partial X_x}{\partial x}\,\delta u ,$$

in which the term in square brackets is taken at the locations of the outer surface Σ to the space T that is cut out by the parallelepiped over $dy\,dz$. If $d\sigma$ is such a surface element and n is the inward-pointing normal, so:

$$dy\,dz = -\,d\sigma \cos(n, x) ,$$

then the first integral on the right can be converted into an integral over Σ:

$$\iint dy\,dz\,[X_x\,\delta u] = -\int d\sigma\,X_x \cos(n, x)\,\delta u ,$$

so

$$\int X_x\,\delta\left(\frac{\partial u}{\partial x}\right) d\tau = -\int d\sigma\,X_x \cos(n, x)\,\delta u - \int d\tau \frac{\partial X_x}{\partial x}\,\delta u .$$

One similarly finds that:

$$\int Y_z\,\delta\left(\frac{\partial w}{\partial y}+\frac{\partial v}{\partial z}\right)d\tau = -\int d\sigma\,Y_z\,[\cos(n,y)\,\delta w+\cos(n,z)\,\delta v]-\int d\tau\left(\frac{\partial Y_z}{\partial y}\,\delta w+\frac{\partial Y_z}{\partial z}\,\delta v\right).$$

In total, after rearranging the terms on the right, one will then get the work done by internal forces as:

$$\delta W = \int \delta w\,d\tau = \int \mathsf{S}\left[X_x\,\delta\frac{\partial u}{\partial y}+Y_z\,\delta\left(\frac{\partial w}{\partial y}+\frac{\partial v}{\partial z}\right)\right]d\tau$$

$$= -\int \mathsf{S}[X_x\cos(n,y)+X_y\cos(n,y)+X_z\cos(n,z)]\,\delta u\,d\sigma-\int \mathsf{S}\left(\frac{\partial X_x}{\partial x}+\frac{\partial X_y}{\partial y}+\frac{\partial Z_z}{\partial z}\right)\delta u\,d\tau\,, \tag{3}$$

in which S is the oft-used summation sign.

That expression is to be substituted in equation (1) of D'ALEMBERT's principle. One can represent the integral $\delta' S$ that relates to the outer surface in terms of the components $\bar{X}$, $\bar{Y}$, $\bar{Z}$:

$$\delta' S = \int (\bar{X}\,\delta u+\bar{Y}\,\delta v+\bar{Z}\,\delta w)\,d\sigma\,, \tag{4}$$

with which, (1) will take the form:

$$0 = \int \mathsf{S}\left[\rho(\ddot{u}-X)+\frac{\partial X_x}{\partial x}+\frac{\partial X_y}{\partial y}+\frac{\partial X_z}{\partial z}\right]\delta u\,d\tau$$

$$+\int \mathsf{S}[X_x\cos(n,y)+X_y\cos(n,y)+X_z\cos(n,z)-\bar{X}]\,\delta u\,d\sigma\,. \tag{5}$$

If one sets each of the coefficients of the variations δu, … in the space and outer surface integrals equal to zero then that will yield the *differential equations:*

a) For the interior:

$$\begin{aligned}\rho\,\ddot{u} &= \rho\,X-\frac{\partial X_x}{\partial x}-\frac{\partial X_y}{\partial y}-\frac{\partial X_z}{\partial z}\,,\\ \rho\,\ddot{v} &= \rho\,Y-\frac{\partial Y_x}{\partial x}-\frac{\partial Y_y}{\partial y}-\frac{\partial Y_z}{\partial z}\,,\\ \rho\,\ddot{w} &= \rho\,Z-\frac{\partial Z_x}{\partial x}-\frac{\partial Z_y}{\partial y}-\frac{\partial Z_z}{\partial z}\,,\end{aligned} \tag{6}$$

b) For the boundary surface:

$$\begin{aligned}\bar{X} &= X_x\cos(n,x)+X_y\cos(n,y)+X_z\cos(n,z)\,,\\ \bar{Y} &= Y_x\cos(n,x)+Y_y\cos(n,y)+Y_z\cos(n,z)\,,\\ \bar{Z} &= Z_x\cos(n,x)+Z_y\cos(n,y)+Z_z\cos(n,z)\,.\end{aligned} \tag{7}$$

The latter says that a tetrahedron with three faces that run parallel to the coordinate planes and one that belongs to the boundary surface of the body (*ABC*, see the figure) will be in equilibrium under the influence of the stresses that act upon it. Here, as well, the volume forces (X, Y, Z) play no role as opposed to the surface tractions, because when they are calculated for an infinitely-small tetrahedron, they will have a dimension that is lower by 1.

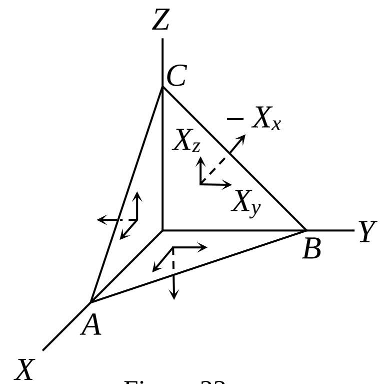

Figure 23.

The equation for the *vis viva* is obtained from (1) when one replaces (art. **15**, at the end) the variations with the displacements that actually occur. When one denotes the increase in kinetic energy by:

$$dT = \tfrac{1}{2}\int \rho\,(\dot{u}^2 + \dot{v}^2 + \dot{w}^2)\,d\tau$$

and the work done by the volume forces by:

$$d'U = \int (X\,dx + Y\,dy + Z\,dz)\,d\tau\,,$$

one will get [1]:

$$dT - d'U - dW - d'S = 0\,.$$

If one substitutes in equations (6) the values for the tension and shear forces X_x, …, Y_z, … that are inferred from [art. **39**, (9), (10)], namely:

$$\begin{aligned} X_x &= -B\frac{\partial u}{\partial x} - (A - 2B)\,\Theta\,, \\ Y_y &= -B\frac{\partial v}{\partial y} - (A - 2B)\,\Theta\,, \\ Z_z &= -B\frac{\partial w}{\partial z} - (A - 2B)\,\Theta\,, \end{aligned} \tag{8}$$

$$\begin{aligned} Z_y = Y_z &= -B\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right), \\ X_z = Z_x &= -B\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right), \\ Y_x = X_y &= -B\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right), \end{aligned} \tag{8.a}$$

where

[1] As before, the prime on d in $d'U$ and $d'S$ means that those expressions are not necessarily differentials.

$$\Theta = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$$

is the spatial dilatation, and the constants A, B are connected with the elastic modulus E and the lateral contraction coefficients μ by the equations:

$$2B = \frac{E}{1+\mu}, \quad A - 2B = \frac{E\mu}{(1+\mu)(1-2\mu)} , \tag{8.b}$$

then when one again employs the differential parameter:

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} ,$$

one will get the following *differential equations for the motion of solid-elastic bodies:*

$$\begin{aligned} \rho \ddot{u} &= B\,\Delta u + (A - B)\frac{\partial \Theta}{\partial x} + \rho X , \\ \rho \ddot{v} &= B\,\Delta v + (A - B)\frac{\partial \Theta}{\partial y} + \rho Y , \\ \rho \ddot{w} &= B\,\Delta w + (A - B)\frac{\partial \Theta}{\partial z} + \rho Z . \end{aligned} \tag{9}$$

One can also give them the form:

$$\begin{aligned} \rho \ddot{u} &= A\frac{\partial \Theta}{\partial x} - 2B\left(\frac{\partial \zeta}{\partial y} - \frac{\partial \eta}{\partial z}\right) + \rho X , \\ \rho \ddot{v} &= A\frac{\partial \Theta}{\partial y} - 2B\left(\frac{\partial \xi}{\partial z} - \frac{\partial \zeta}{\partial x}\right) + \rho Y , \\ \rho \ddot{w} &= A\frac{\partial \Theta}{\partial z} - 2B\left(\frac{\partial \eta}{\partial x} - \frac{\partial \xi}{\partial y}\right) + \rho Z , \end{aligned} \tag{9.a}$$

where

$$\begin{aligned} \xi &= \tfrac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) , \\ \eta &= \tfrac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) , \\ \zeta &= \tfrac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) . \end{aligned}$$

Hence, if one is dealing with the change in form of a known isotropic solid-elastic body on which the volume forces (X, Y, Z) and the surface tractions $(\bar{X}, \bar{Y}, \bar{Z})$ act then equations (9) will give the connection between the former and the displacements (u, v, w) that the point x, y, z experiences, while (8), (8.a) express the connection between the internal forces in the vicinity of the outer surface and the given surface tractions. The arbitrary functions that appear in the partial differential equations (9) upon integration are determined from the former once an assumption about the *initial state* has been made.

Meanwhile, the difficulties that one already encounters in the solution of that problem in simple cases are mostly insurmountable. For that reason, one goes directly to simplifying assumptions when one either restricts oneself to bodies that are very small with respect to one or two dimensions, such as thin plates, shells, and rods, or when one makes other simplifying assumptions on the form of the body and the way that the surface tractions are applied. Thus, if one leaves time out of consideration, so one restricts oneself to static problems, then, as we will see in the next article, in many cases, one will succeed in ascertaining the deformation of a body that is acted upon by forces from the differential equations under special assumptions about the forces and the displacements. Even the propagation of the state of vibration in an unbounded elastic medium (for which, one is then given the displacements at a certain point in time, not the forces) can be predicted with the help of those differential equations, as will be shown in a later article.

Certain problems of hydrodynamics and the theory of elasticity that relate to special bounded bodies or a special assumption on the state of motion make it desirable to know the differential equations in other coordinates besides rectangular ones. Now, instead of transforming the equations of motion that are composed of second differential quotients themselves, it is better to formulate the statement of the fundamental law [art. **25** (1), art. **36**, (5)] in the new coordinates (p_1, p_2, p_3). That is because, on the one hand, from art. **6**, the expression for twice the variation of the constraint $\delta m\, f^2$ transforms as follows:

$$\delta m\, f^2 = \int \delta(\ddot{u}^2 + \ddot{v}^2 + \ddot{w}^2)\, dm = \int \sum \frac{d}{dt}\left(\frac{\partial\, |\mathfrak{v}|^2}{\partial \dot{\pi}_k}\right) \delta \ddot{\pi}_k\, dm,$$

in which:

$$|\mathfrak{v}|^2 = \dot{u}^2 + \dot{v}^2 + \dot{w}^2 = 2\sum\sum \alpha_{ik}\, \dot{\pi}_i\, \dot{\pi}_k$$

is the square of the velocity of the advancing motion of the mass-element dm, π_k is its displacement in the direction of the coordinate p_k , which is a linear function of the u, v, w, and the α_{ik} depend upon just the p (which are time-invariant). On the other hand, in each case, the potential energy of the internal forces is composed of only first-order differential quotients of the displacements u, v, w with respect to the coordinates, and indeed in terms of expressions that one already finds transformed into orthogonal coordinates in LAMÉ, *Leçons sur les coordinates curvilignes*, 1859 and into general coordinates in L. MAURER, "Deform. gekrümmter Platten," Archiv d. Math. u. Phys. (3) **5** (1902). However, we shall not go into the details of that here. (See also a treatise of the author in Math. Ann., Bd. I, pp. 246).

§ 41. – Torsion of a straight cylinder with an elliptical base.

We shall ignore volume forces in what follows, so we will set:

$$X = Y = Z = 0$$

in the equations of the foregoing article. We can next try to make corresponding assumptions about the functions u, v, w of x, y, z, t in equations (9). If we were to substitute any such randomly-found solutions in equations (8) and go to the outer surface by way of (7) then that would yield surface tractions that would probably offer no special physical or engineering interest, since they also most likely could not be realized.

Probably guided by such reasoning, DE SAINT-VENANT embarked upon the following path for determining new and useful solutions of the system of equations ([1]). He assumed *only one part of the surface tractions*, and therefore, *one part of the displacements (de Saint-Venant's problem)* in such a way that both of them together would make the integration of the system of equations (7)-(9) accessible since the remaining ones could be determined directly from them. We would like to show how that can be done in the problem of the twisting (torsion) of a straight cylinder with an elliptical base under the influence of forces that are applied tangentially to the end surfaces.

Let the axis of a straight cylinder with an elliptical base and initially very small height be the Z-axis of a rectangular coordinate system whose origin falls on one of the end surfaces. Let the surface element that includes the origin be rigidly coupled with the deformation of the XY-plane, such that one will have $x + dx = y = 0$, $x = y + dy = dz = 0$, $u = v = w = 0$ for the point $x = y = z = 0$. In other words, one assumes that:

$$u_0 = v_0 = w_0 = \left(\frac{\partial u}{\partial x}\right)_0 = \left(\frac{\partial v}{\partial x}\right)_0 = \left(\frac{\partial w}{\partial x}\right)_0 = \left(\frac{\partial u}{\partial y}\right)_0 = \left(\frac{\partial v}{\partial y}\right)_0 = \left(\frac{\partial w}{\partial y}\right)_0 = 0\,, \tag{1}$$

in which the index zero will mean for:

$$x = y = z = 0\,.$$

We make three further assumptions about the deformed state, and we will then discuss the influence that they would have on formulas (7)-(9) of the previous article.

1. No stresses at all shall act upon the lateral surface of the cylinder. Only forces whose directions fall along the end-surfaces themselves shall act upon those surfaces (so merely shear or tangential forces). Therefore, let:

$$\bar{Z} = 0$$

for all locations on the lateral surface. On the other hand, for them, one has:

([1]) DE SAINT-VENANT, "Mémoire sur la torsion des prismes," Sav. étrang. Mém. **14** (1856). "Flexion des prismes," Liouv. Journ. (2) **1** (1856). See also CLEBSCH, *Elastizität fester Körper*, Leipzig, 1862, pp. 74, *et seq.*

$$\cos(n, z) = 0 ,$$

even for a small deformation. Therefore, due to (7), one will have:

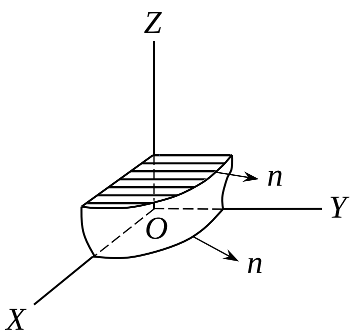

Figure 24.

$$Z_x \cos(n, x) + Z_y \cos(n, y) = 0$$

for any point on the lateral surface, or also, since for any line-element ds of the outer surface (namely, for one such as the boundary curve of the end-surface, see Fig.), one has:

$$dx \cos(n, x) + dy \cos(n, y) + dz \cos(n, z) = 0 ,$$

one will have:

$$Z_x\, dy - Z_y\, dx = 0 . \tag{2}$$

The points of the original end-surface of the cylinder that fell in the *XY*-plane will be only a slight distance from it, because from art. **21**, the displacements u, v, w are very small quantities. Therefore, one will have roughly:

$$\cos(n, x) = 0 , \qquad \cos(n, y) = 0 \tag{3}$$

for them after the deformation, and therefore since $\bar{Z} = 0$, from the assumption (1), that will imply that for all points of the end-cross-section [formula (7) of the previous article], one has:

$$Z_z = 0 . \tag{3.a}$$

2. The longitudinal fibers (i.e., parallel to the axis of the parallelepiped that was cut out) of the cylindrical rod suffer no sort of lateral pressure, and their perhaps rectangular cross-section will remain rectangular.

With that assumption, one will have:

$$X_x = Y_y = 0 , \quad Y_x = X_y = 0 \tag{4}$$

everywhere inside the cylinder.

3. In all of that, the height of the cylinder was assumed to be so small that the magnitudes of the displacements u, v, w were very small quantities (art. **21**), even cross-sections that are at a distance from the *XY*-plane.

Formulas (3), (3.a) will also be true for the other end-surface then, and therefore (3.a), above all.

As one will see, the problem is put into a soluble form by means of equations (1)-(4), in conjunction with (7)-(9) in the previous article.

The system (9) next goes to:

$$\Delta u = 0 , \qquad \Delta v = 0 , \qquad \Delta w = 0 . \tag{5}$$

Due to (3.a), (4), it follows further from the system (8) in the previous article that:

$$\frac{\partial u}{\partial x} = 0 , \qquad \frac{\partial v}{\partial y} = 0 , \qquad \frac{\partial w}{\partial z} = 0 , \tag{6}$$

because the right-hand sides are homogeneous linear functions of those three quantities whose determinant does not vanish. Thus, one also has:

$$\Theta = 0 . \tag{6.a}$$

Equations (8.a) will become:

$$\begin{aligned} B\left(\frac{\partial w}{\partial y}+\frac{\partial v}{\partial z}\right) &= -Z_y = -Y_z , \\ B\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right) &= -X_z = -Z_x , \\ \frac{\partial v}{\partial x}+\frac{\partial u}{\partial y} &= 0 . \end{aligned} \tag{7}$$

Due to (2), for the boundary curve of a previously-planar cross-section:

$$\left(\frac{\partial w}{\partial y}+\frac{\partial v}{\partial z}\right)dy-\left(\frac{\partial w}{\partial y}+\frac{\partial v}{\partial z}\right)dx = 0 . \tag{8}$$

In the end-cross-section $z = 0$ [cf., (7) of the previous article]:

$$X_z = \bar{X} , \qquad Y_z = \bar{Y} , \qquad Z_z = 0 .$$

From the last equation in (7), it will follow for all points in the interior:

$$\frac{\partial^2 v}{\partial x\,\partial y}+\frac{\partial^2 u}{\partial y^2} = 0 , \qquad \frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 u}{\partial x\,\partial y} = 0 ,$$

or due to (6):

$$\frac{\partial^2 u}{\partial y^2} = 0 , \qquad \frac{\partial^2 v}{\partial x^2} = 0 . \tag{9}$$

Due to (6), one then gets from (5) that:

$$\frac{\partial^2 u}{\partial z^2} = 0 , \qquad \frac{\partial^2 v}{\partial z^2} = 0 .$$

Thus, the functions u and v are *linear* in each of x, y, z, and indeed, from (6), u does not contain x, nor does v contain y. One will then have:

$$u = z\,(p\,y + q) + r\,y + s\,,$$
$$v = z\,(p_1\,y + q_1) + r_1\,y + s_1\,,$$

in which the p, q, r, s are constants. Due to the last of equations (7), one then has:

$$p_1 = -p\,, \qquad r_1 = -r\,.$$

Moreover, since from (1), one has:

$$u = v = \frac{\partial u}{\partial y} = \frac{\partial v}{\partial x} = 0$$

for $x = y = z = 0$, one will have:

$$s = s_1 = 0\,, \qquad r = r_1 = 0\,.$$

One will then have:

$$\begin{aligned} u &= z\,(\ \ p\,y + q\,), \\ v &= z\,(-p\,x + q_1). \end{aligned} \tag{10}$$

The function w satisfies [(5), (6)] the partial differential equation:

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = 0\,. \tag{11}$$

Since w is a continuous function of x, y in a neighborhood of $x = y = 0$, in addition, w can be developed in integer powers of x, y. Thus, let:

$$w = \alpha + (\beta\,x + \gamma\,y) + (\delta x^2 + \varepsilon\,x\,y + \zeta\,y^2) + \ldots$$

be an integral of (11), in which the α, β, … no longer depend upon z, due to (6). From (1), one will then have:

$$\alpha = \beta = \gamma = 0\,,$$

so

$$w = \delta x^2 + \varepsilon\,x\,y + \zeta\,y^2 + \ldots \tag{12}$$

However, the function w satisfies yet another boundary equation that one obtains when one enters the value of dy / dx from the equation for the boundary curve for the cross-section for $z = 0$ in (8). That is because when the equation for the boundary ellipse (of the undeformed cylinder) is, say:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\,, \tag{13}$$

from which that of the deformed boundary curve can be obtained by replacing x, y with $x + u$, $y + v$, from (10), the tangent direction for $z = 0$ can be determined from:

$$\frac{x\,dx}{a^2} + \frac{y\,dy}{b^2} = 0 .$$

Hence, with the use of (10), (8) will go to:

$$\left(\frac{\partial w}{\partial x} + p\,y + q\right)\frac{x}{a^2} + \left(\frac{\partial w}{\partial y} - p\,y + q_1\right)\frac{y}{b^2} = 0 , \tag{14}$$

which is an equation that must be fulfilled identically when one introduces (12) with the help of (13). One will then get:

$$[q_1 + (\varepsilon - p)\,x + 2\zeta\,y + \ldots]\,\frac{y}{b^2} + [q + (\varepsilon + p)\,y + 2\delta\,x + \ldots]\,\frac{x}{a^2} = 0 , \tag{15}$$

along with:

$$\frac{y^2}{b^2} + \frac{x^2}{a^2} - 1 = 0 . \tag{13}$$

Equations (15), (13) represent two curves in the XY-plane that must have a common component when (15) does not vanish identically, i.e., for all values of x, y. Obviously, only the latter is possible. One then gets:

$$q = 0 , \quad q_1 = 0 ,$$

$$\frac{\varepsilon - p}{b^2} + \frac{\varepsilon + p}{a^2} = 0 , \qquad \delta = \zeta = \ldots = 0 ,$$

with which:

$$w = \frac{a^2 - b^2}{a^2 + b^2}\,p\,x\,y ,$$

which is an expression that likewise satisfies the differential equation (11) for w. One can also determine w from (11) and the boundary condition (8) along a path that is more connected with the theory of functions (see, e.g., LOVE-TIMPE, *Lehrbuch der Elastizität*, Leipzig, 1907, art. 217).

The system of displacement components is then:

$$\begin{aligned} u &= p\,z\,y , \\ v &= -\,p\,z\,x , \\ w &= \frac{a^2 - b^2}{a^2 + b^2}\,p\,x\,y . \end{aligned} \tag{16}$$

By means of (7), that will yield the tangential forces in any cross-section, and in particular, the external forces that are to be applied to the end cross-section. Namely, one will have [art. **40**, (8)]:

$$X_z = -\frac{E}{1+\mu}\frac{p\,a^2 y}{a^2+b^2}, \qquad Y_z = \frac{E}{1+\mu}\frac{p\,b^2 x}{a^2+b^2}. \tag{17}$$

Since one has:

$$X_z\,dy - Y_z\,dx = 0$$

in the boundary curve of any cross-section, and since the same value of dy / dx that the boundary curve of a cross-section satisfies is also satisfied by each of the curves in the interior that are similar to it, all tangential forces (which are distributed symmetrically with respect to the Z-axis) will act in the direction of the tangents to a system of curves that fill up the interior of the cross-section, similar to the boundary curve. As far as the form of the cylinder after deformation is concerned, the formula (16) for w will give the variation of the z-coordinate of a point x, y in any cross-section. In particular, the original planar cross-section $z = 0$, for which one likewise has $u = v = 0$ (16), will go to the surface that is represented by the last equation in (16):

$$w = \frac{a^2-b^2}{a^2+b^2}\,p\,x\,y,$$

which is a hyperbolic paraboloid. The cross-sections that belong to larger values of z are congruent to it and will be rotated around the Z-axis from it by a small angle. That is because if one sets:

$$x = r\cos\varphi, \qquad y = r\sin\varphi$$

then one will have:

$$u^2+v^2 = p^2 z^2 r^2, \qquad \frac{u}{v} = -\tan\varphi,$$

i.e., for all points of a cross-section that are at an equal distance from the Z-axis, the displacements are proportional to their distance z from the XY-plane and directed like the tangents to a circle. Since:

$$u^2+v^2 = p^2$$

for $r = 1$, $z = 1$, p means the (very small) angle through which any cross-section has been rotated with respect to one at a unit distance from it, so p is inversely proportional to the pitch of the helix to which a fiber that is parallel to the Z-axis will go. Due to the congruence of all deformed cross-sections, one can imagine that the original piece of the cylinder, which is assumed to be short, is connected to them by repeated congruences, so the cylinder will be lengthened arbitrarily, whereby formula (16) will not cease to be valid then.

The "risky" location of the cross-section, namely, the one in which the material's force of resistance to shears is strongest, is calculated from formulas (17). If a is the semi-major axis of the ellipse and b is the semi-minor axis then the expression for X will have its largest value at the location $x = 0$, $y = b$, so at the end of the semi-minor axis.

The total moment that is associated with both end cross-sections and that the torsion in question is able to produce has the value:

$$\int (Y_z x - X_z y)\, d\sigma = \frac{E\, p\, a^2 b^2}{(1+\mu)(a^2+b^2)} \int \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} \right) d\sigma = \frac{E\, p\, a^3 b^3 \pi}{(1+\mu)(a^2+b^2)} \tag{18}$$

when one denotes an element of the end cross-section by $d\sigma$. All that is missing is the proof that the system (16) of displacements u, v, w is *the only one possible* under the assumption that the surface tractions are:

$$\bar{X} = X_z\,, \qquad \bar{Y} = Y_z\,,$$

corresponding to formulas (17) and with the assumed orientation of the cylinder with respect to the coordinate-cross. That will be addressed in the following article.

§ 42. – Uniqueness of the solution.

We will show that, in general ([1]), for a given system of volume forces X, Y, Z and surface tractions $\bar{X}$, $\bar{Y}$, $\bar{Z}$, *only one* system of internal forces $X_x, \ldots, Y_z, \ldots$ (as defined for the equilibrium state, so for $\ddot{u} = \ddot{v} = \ddot{w} = 0$) exists that will satisfy the general equations (6) to (8.a) in art. **40**, i.e., that any two systems of displacements u, v, w and associated $X_x, \ldots, Y_z, \ldots$ that satisfy those equations will differ by only a spatial parallel displacement and rotation of the elastic body (which is then thought of as rigid).

In fact: The system (6) to (8.a) in art. **40** might be satisfied by the quantities:

$$u, v, w, X_x, \ldots, Y_z, \ldots, \tag{A}$$

in addition to:

$$u_1, v_1, w_1, X_{x_1}, \ldots, Y_{z_1}, \ldots \tag{B}$$

If one sets:

$$u_1 - u = u'\,, \quad v_1 - v = v'\,, \quad w_1 - w = w' \tag{1}$$

then the differences:

$$X_{x_1} - X_x = X'_x\,, \qquad \ldots, \qquad Y_{z_1} - Y_z = Y'_z\,, \ \ldots$$

will have the same mechanical interpretation as the individual terms because the partial differential quotients of the u, v, w that they are linear combinations of will themselves transform linearly, i.e., like the u in (1). Now, in the case of equilibrium, so when one sets:

([1]) From CLEBSCH, *Theorie der Elastizität fester Körper*, § **21**, pp. 67.

$$\ddot{u} = \ddot{v} = \ddot{w} = 0 , \tag{2}$$

equations (6) will be satisfied by the system (A), as well as by (B). If one subtracts the corresponding equations from each other then one will get:

$$\begin{aligned} 0 &= \frac{\partial X'_x}{\partial x} + \frac{\partial X'_y}{\partial y} + \frac{\partial X'_z}{\partial z}, \\ 0 &= \frac{\partial Y'_x}{\partial x} + \frac{\partial Y'_y}{\partial y} + \frac{\partial Y'_z}{\partial z}, \\ 0 &= \frac{\partial Z'_x}{\partial x} + \frac{\partial Z'_y}{\partial y} + \frac{\partial Z'_z}{\partial z}. \end{aligned} \tag{3}$$

One likewise gets from (7), art. **40** that:

$$X'_x \cos(n,x) + X'_y \cos(n,y) + X'_z \cos(n,z) = 0 , \tag{4}$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

Now, D'ALEMBERT's principle [(1) of art. **40**] led to equations (6), (7) of that article, and due to (2), it will take the form:

$$\int -\rho\, d\tau\, (X\,\delta u + Y\,\delta v + Z\,\delta w) - \delta W - \delta' S = 0 , \tag{5}$$

in which:

$$\delta' S = \int (\bar{X}\,\delta x + \bar{Y}\,\delta y + \bar{Z}\,\delta z)\, d\sigma .$$

The conditions (5) will also lead to equations of the form (3), (4), but only under the assumption:

$$-\,\delta W = \int -\delta w \cdot d\tau = 0 ,$$

in which:

$$-\,\delta w = \frac{E}{2(1+\mu)} \left[\delta(\rho_1^2 + \rho_2^2 + \rho_3^2) + \frac{\mu}{1-2\mu}\,\delta\,\Theta^2 \right]$$

is an essentially-positive quantity (because, from art. **39**, the coefficient $\mu < 1/2$) that is zero only when one has:

$$\rho_1 = \rho_2 = \rho_3 = 0$$

everywhere in the body, i.e., when no deformation at all exists.

Hence, if the body is coupled with the coordinate system in a certain way (as we assumed for the elliptical cylinder above) then the problem of equilibrium of solid-elastic bodies will be *uniquely-soluble under the assumption that the volume forces and surface tractions* that act upon it are *well-defined*.

§ 43. – Propagation of plane waves in a solid-elastic medium.

We shall make a further application of the differential equations (9.a) in art. **40**, namely, to a solid-elastic medium that extends to infinity and on which no surface tractions and volume forces act. Those equations:

$$\begin{aligned}
\rho\,\ddot{u} &= A\frac{\partial\Theta}{\partial x}-2B\left(\frac{\partial\zeta}{\partial y}-\frac{\partial\eta}{\partial z}\right),\\
\rho\,\ddot{v} &= A\frac{\partial\Theta}{\partial y}-2B\left(\frac{\partial\xi}{\partial z}-\frac{\partial\zeta}{\partial x}\right),\\
\rho\,\ddot{w} &= A\frac{\partial\Theta}{\partial z}-2B\left(\frac{\partial\eta}{\partial x}-\frac{\partial\xi}{\partial y}\right),
\end{aligned}\tag{1}$$

or in vector-analytic form:

$$\rho\,\ddot{\mathfrak{u}} = A \text{ grad } \Theta - 2\,B \text{ rot } \mathfrak{w}\,,$$

in which the solenoidal vector is once more:

$$(\xi,\,\eta,\,\zeta) = \mathfrak{w} = \tfrac{1}{2}\text{ rot } \mathfrak{u}\,,$$

which can be decomposed into a potential vector and a solenoidal vector by way of the one vector, if one recalls what was said in art. **31**. In fact, one will get the same result when one sets:

$$\Theta = \frac{\rho}{A}\frac{d^2\psi}{dt^2},\tag{2}$$

$$-\,\xi = \frac{\rho}{2B}\frac{d^2L}{dt^2}, \qquad -\,\eta = \frac{\rho}{2B}\frac{d^2M}{dt^2}, \qquad -\,\zeta = \frac{\rho}{2B}\frac{d^2N}{dt^2},\tag{2.a}$$

in which L, M, N are the components of a vector potential $\mathfrak{q}$ whose divergence vanishes. That is because equations (1) then go to:

$$\begin{aligned}
\ddot{u} &= \frac{\partial\ddot{\psi}}{\partial x}+\frac{\partial\ddot{N}}{\partial y}-\frac{\partial\ddot{M}}{\partial z},\\
\ddot{v} &= \frac{\partial\ddot{\psi}}{\partial y}+\frac{\partial\ddot{L}}{\partial z}-\frac{\partial\ddot{N}}{\partial x},\\
\ddot{w} &= \frac{\partial\ddot{\psi}}{\partial z}+\frac{\partial\ddot{M}}{\partial x}-\frac{\partial\ddot{L}}{\partial y},
\end{aligned}\tag{3}$$

or more briefly:

$$\ddot{\mathfrak{u}} = \text{grad}\,\ddot{\psi} + \text{rot}\,\ddot{\mathfrak{q}}\,.$$

If one integrates equations (3) over t twice and lets the linear functions of t with arbitrary functions of x, y, z as coefficients that appear under integration enter into u, v, w then that will give:

$$
\begin{aligned}
u &= \frac{\partial\psi}{\partial x} + \frac{\partial N}{\partial y} - \frac{\partial M}{\partial z} \;, \\
v &= \frac{\partial\psi}{\partial y} + \frac{\partial L}{\partial z} - \frac{\partial N}{\partial x} \;, \\
w &= \frac{\partial\psi}{\partial z} + \frac{\partial M}{\partial x} - \frac{\partial L}{\partial y} \;.
\end{aligned} \tag{4}
$$

Now, when the quantities $\mathfrak{R}u$, $\mathfrak{R}v$, $\mathfrak{R}w$ vanish (in which $\mathfrak{R}$ means the distance from the origin, which increases without limit), as was proved before (art. **31**), the most general solution of that system of equations can be reduced to the following one: Since:

$$\Theta = \text{div}\,\mathfrak{u} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = \Delta\psi ,$$

one will have (2):

$$\rho\frac{d^2\psi}{dt^2} = A\,\Delta\psi , \tag{5}$$

and furthermore [art. **31**, form. (10)]:

$$\Delta L = -\,2\xi , \qquad \Delta M = -\,2\eta , \qquad \Delta N = -\,2\zeta , \tag{6}$$

and then, due to (2.a):

$$\rho\frac{d^2 L}{dt^2} = B\,\Delta L \;, \qquad \rho\frac{d^2 M}{dt^2} = B\,\Delta M \;, \qquad \rho\frac{d^2 N}{dt^2} = B\,\Delta N \;, \tag{6.a}$$

in which the condition exists (cf., *supra*) that:

$$\frac{\partial L}{\partial x} + \frac{\partial M}{\partial y} + \frac{\partial N}{\partial z} = \text{div}\,\mathfrak{q} = 0 \;. \tag{6.b}$$

Equations (4), (5), (6) can be replaced with the ones in (1). If one substitutes the solutions of (5), (6) in (4) then (4) will define a system of displacements that satisfy equations (1). We would like to denote the potential vector grad ψ by $\mathfrak{u}'$, and the solenoidal vector rot $\mathfrak{q}$ by $\mathfrak{u}''$. The vector $\mathfrak{u}$ will then split into:

$$\mathfrak{u} = \mathfrak{u}' + \mathfrak{u}'',$$

where:

$$
\begin{aligned}
\mathfrak{u}' &= \text{grad}\,\psi , \\
\mathfrak{u}'' &= \text{rot}\,\mathfrak{q} .
\end{aligned} \tag{7}
$$

In order to arrive at solutions of equations (5), (6) that have some mechanical interpretation, we would like to assume that an energy source is found at some location inside of an infinitely-extended medium that produces vibrations (i.e., oscillating motions) at either one point in time or repeatedly. Any such vibration will produce deformations in the neighboring elements as a result of the deformation that it produces in the surrounding part of the mass, and in that way, they will propagate in space. At a certain time, the vibration will have arrived at all points of a curved surface that surrounds one of the centers of vibration. If one restricts the investigation to the nearest neighbors of a point P on that surface then one can replace it, in the first approximation, with an element of its tangent plane:

$$K \equiv \alpha x + \beta y + \gamma z - C = 0 ,$$

in which we would like to assume that:

$$\alpha^2 + \beta^2 + \gamma^2 = 1 . \tag{8}$$

Now, while the surface spreads out, one can follow those tangent planes for which α, β, γ have the same value, while only the distance C from the origin changes, which will then make C a function of t. We would like to assume:

$$C = V t ,$$

in which V is constant in anisotropic medium, and then test the admissibility of that assumption by substitution. V then means the speed of propagation of the plane $K = 0$. The vector $\mathfrak{u} = (u, v, w)$, which represents the motion of P, will also represent those of the points of the surface that are found in the neighborhood of P. One will then arrive at the introduction of *plane waves*, i.e., particular integrals of the differential equations (1) for which $\mathfrak{u}$ can be assumed to be a function of:

$$K \equiv \alpha x + \beta y + \gamma z - V t ,$$

so the deflection (*Ausschlag*) $\mathfrak{u}$ will be the same *for all* points on the plane $K = 0$ at a time t. We will choose the function of K (or even better, K/V) that defines the state of excitation to be a simple periodic function by setting:

$$\begin{aligned} u = a \cos\, 2\pi n \frac{K}{V} = a \cos 2\pi n \left(t - \frac{\alpha x + \beta y + \gamma z}{V} \right) , \\ v = b \cos 2\pi n \left(t - \frac{\alpha x + \beta y + \gamma z}{V} \right) , \\ w = c \cos 2\pi n \left(t - \frac{\alpha x + \beta y + \gamma z}{V} \right) , \end{aligned} \tag{9}$$

in which a, b, c, n are constants. We shall test the admissibility of the assumption that was made by substituting formulas (9) in the differential equations (1), or also in [art. **40**, (9)]:

$$\rho\,\ddot{u} = B\,\Delta u + (A - B)\,\frac{\partial\Theta}{\partial x}, \quad \text{etc.}$$

When one discards the cosine and sets:

$$a\,\alpha + b\,\beta + c\,\gamma = \delta, \tag{10}$$

after dividing by $-4\pi\,n^2$, one will get:

$$\rho\,a = \frac{1}{V^2}\,B\,a\,(\alpha^2 + \beta^2 + \gamma^2) + \frac{1}{V^2}\,(A - B)\,\alpha\,\delta,$$

or when one recalls (8), the first of the following equations:

$$\begin{aligned} (A - B)\,\alpha\,\delta + (B - \rho V^2)\,a &= 0, \\ (A - B)\,\beta\,\delta + (B - \rho V^2)\,b &= 0, \\ (A - B)\,\gamma\,\delta + (B - \rho V^2)\,c &= 0. \end{aligned} \tag{11}$$

They give the connection between the constants that were introduced that must exist if (9) is to be a system of solutions. Before we go into that, we would like to establish the meaning of those constants.

The ratios:

$$u : v : w = a : b : c$$

give the direction in which the points of the plane $K = 0$ (viz., the "wave plane") and the planes that are parallel to it, on which the motion propagates, displaces. The magnitude of that displacement (viz., the deflection of the oscillation at the location x, y, z):

$$|\,\mathfrak{u}\,| = \sqrt{a^2 + b^2 + c^2}\,\cos 2\pi\,n\left(t - \frac{\alpha x + \beta y + \gamma z}{V}\right) \tag{12}$$

increases from zero to $\sqrt{a^2 + b^2 + c^2}$ (viz., the "amplitude") and once more goes back to zero in a time of $T / 2$ (viz., one-half the "period" or duration of oscillation) that is determined from the condition that the vector $\mathfrak{u}$ takes the same value again, namely, from:

$$\cos 2\pi\,n\left(t - \frac{\alpha x + \beta y + \gamma z}{V}\right) = \cos 2\pi\,n\left(t + T - \frac{\alpha x + \beta y + \gamma z}{V}\right)$$

or from:

$$n\,T = 1. \tag{13}$$

n is then the reciprocal value of the period of oscillation or the number of oscillations that result in a unit time, namely, the "oscillation number." In order to also ascertain the meaning of $V / n = \lambda$, we consider the state of motion that exists on parallel planes *at the same time*.

The distance between the two planes:

$$K = 0 \, , \, K + \lambda \equiv K + \frac{V}{n} = 0$$

has the property that the deflection (u, v, w) is simultaneously the same on both of them, because:

$$2\pi n \frac{K}{V} = 2\pi n \left(K + \frac{1}{V} \right) \cdot \frac{1}{V} - 2\pi \, .$$

For water waves, the distance λ then corresponds to the "wavelength" that exists between the crest and the trough; hence, the terminology.

We now turn to a discussion of formulas (11) between the constants of the solutions (9). If one multiplies (11) by a, b, g, resp., and adds them then one will get:

$$\delta (A - \rho V^2) = 0 \, . \tag{14}$$

That equation will be satisfied in two ways:

a) One sets:

$$\delta = a \, \alpha + b \, \beta + c \, \gamma = 0 \, .$$

The direction u is then perpendicular to that (α, β, γ) of the normals to the wave-planes (viz., the "wave normals"), i.e., the oscillations result *in* the wave-plane, so they are *perpendicular to the direction of propagation;* one has *transverse waves.* If one introduces $\delta = 0$ into (11) then one will have:

$$B = \rho V^2 .$$

Therefore, the speed of propagation V of transverse waves in a medium of density ρ will be:

$$V = \sqrt{\frac{B}{\rho}} \, .$$

The velocity V will get smaller as the medium gets denser. That relation implies an interpretation for the constant B.

Since the expression for the spatial dilatation Θ:

$$\Theta = (a\ \alpha + b\ \beta + c\ \gamma)\frac{2\pi}{\lambda}\sin\frac{2\pi}{\lambda}[Vt - (a\ \alpha + b\ \beta + c\ \gamma)]$$

has the factor δ, δ will vanish with Θ, and conversely. Since Θ = 0, the propagation of transverse waves in an elastic medium will result without spatial expansion or contraction; it behaves like an incompressible mass.

b) Second assumption. One sets (14):

$$A = \rho V^2 .$$

Thus, (11) will imply that:

$$a = \alpha\,\delta, \qquad b = \beta\,\delta, \qquad c = \gamma\,\delta .$$

In this case, the direction of oscillation:

$$u : v : w = \alpha : \beta : \gamma$$

will coincide with that of the wave normal, which results *in the direction of propagation.* One is then dealing with *longitudinal waves* whose speed of propagation:

$$V = \sqrt{\frac{A}{\rho}}$$

once more gives an interpretation of the constant A of the medium.

In case (a), since $\delta = 0$, one has Θ = 0, and (2) makes the (here periodic) function $\psi \equiv 0$, and therefore the vector:

$$\mathfrak{u}' = \text{grad } \psi = 0 .$$

By contrast, in case (b), one has Θ ≡ 0, and indeed:

$$A\,\Theta = \rho\ddot{\psi} = A\,\Delta\psi . \tag{15}$$

One must associate this case of longitudinal oscillations with the vector that was defined in (7) by $\mathfrak{u}'$ = grad ψ, and from the differential equations (1), one will have:

$$\begin{aligned} \rho\,\ddot{u}' &= \rho\frac{\partial\ddot{\psi}}{\partial x} = A\frac{\partial\Theta}{\partial x}, \\ \rho\,\ddot{v}' &= A\frac{\partial\Theta}{\partial y}, \\ \rho\,\ddot{w}' &= A\frac{\partial\Theta}{\partial z}. \end{aligned} \tag{16}$$

By contrast, case (a) associates transverse oscillations with the vector:

$$\mathfrak{u}'' = \text{rot}\,\mathfrak{q}\;,$$

for which, when L, M, N are once more the components of the solenoidal vector q, the equations will exist (1), (6):

$$\begin{aligned}
\rho\,\ddot{u}' &= B\left(\frac{\partial\,\Delta N}{\partial y}-\frac{\partial\,\Delta M}{\partial z}\right) = -\;2B\left(\frac{\partial\zeta}{\partial y}-\frac{\partial\eta}{\partial z}\right)\;,\\
\rho\,\ddot{v}' &= B\left(\frac{\partial\,\Delta L}{\partial z}-\frac{\partial\,\Delta N}{\partial x}\right) = -\;2B\left(\frac{\partial\xi}{\partial z}-\frac{\partial\zeta}{\partial x}\right)\;,\\
\rho\,\ddot{w}' &= B\left(\frac{\partial\,\Delta M}{\partial x}-\frac{\partial\,\Delta L}{\partial y}\right) = -\;2B\left(\frac{\partial\eta}{\partial x}-\frac{\partial\xi}{\partial y}\right)\;.
\end{aligned}\tag{17}$$

In the case of plane waves, the general equations (1) for $\mathfrak{u}$ can be replaced with the ones (16), (17) for the components of $\mathfrak{u}$.

The laws of propagation of plane waves in an infinitely-extended medium have a meaning for spherical waves, as well, or even curvilinear waves in a bounded medium, because one can regard the curved surface, in the first approximation, as a piece of its tangent plane, and in the second approximation, by the latter and its neighboring planes, so the surface is enveloped by its tangent planes. One thinks of a point in the vicinity of a curved surface as the point of intersection of infinitely-many tangent planes whose positions differ from it infinitely little, and each of which is the carrier of motion of the type that was described above. If a tangent plane to such a wave surface is represented at time t by:

$$a\;\alpha + b\;\beta + c\;\gamma - V\,t = 0 \tag{18}$$

then one will have:

$$(\alpha + d\alpha)\;x + (\beta + d\beta)\;y + (\gamma\; + d\gamma)\;z - V\,t = 0$$

for a neighboring location.

If both planes go through the same point x, y, z that is slightly outside the surface then one will also have:

$$a\;d\alpha + b\;d\beta + c\;d\gamma\; = 0 \tag{18.a}$$

for them.

$d\alpha : d\beta : d\gamma$ will assume different values for the infinitely-many tangent planes that all go through x, y, z. Therefore, if (18.a), (18.b) are to both be true for all of them then one must have:

$$x : y : z = \alpha : \beta : \gamma\,.$$

The point of intersection of the tangent planes that are neighbors to the ones in (18) then moves forth in the direction of the *normals* to the latter. One can imagine that only along those lines can the effect of infinitely-many deflections that differ infinitely-little be regarded by the senses and perceived as light rays or sound.

Although equations (15), (16) for the propagation of longitudinal waves were exhibited for *solid-elastic* bodies, they can also be used for the propagation of sound waves in an *elastic fluid* medium. That is because, in comparison to a form of motion that propagates as rapidly as sound waves, the fluid medium can also behave like a solid-elastic body. The fact that, e.g., air can behave like a rigid body, is shown by the explosion of dynamite, whose perturbing effect is felt in all directions simultaneously, and even underneath.

The general formulas (1) were recently applied by E. WIECHERT to the propagation of vibrations like the ones that are produced in earthquakes (WIECHERT and ZÖPRITZ, "Über Erdbebenwellen," Gött. Nachr. 1907). By comparing the results of theoretical analysis with the seismometric recordings of numerous stations that are distributed about the surface of the Earth, the aforementioned researcher proved the appearance of longitudinal, as well as transverse, waves in the Earth's interior.

§ 44. – The elastic theory of light.

FRESNEL's interference experiments proved that light propagates in transverse waves. Once one knows that fact, the efforts of the theoretical physicist must be directed to finding a carrier of that motion that will propagate light from one heavenly body to another in the way that air carries sound, if not theoretically establish the properties of one such medium that one calls the *ether* and test their admissibility. Most experiments can be casually explained when one assumes that the ether is a solid-elastic medium that allows the propagation of only transverse waves. It is only in a few cases, such as the experiments to explain the phenomena for the transfer of light from one medium to another one of different density, as well as the behavior of crystals with elliptic polarization ([1]), that the so-called elastic theory of light breaks down.

We shall go into just the first-mentioned difficulty. Since $\Theta = 0$, the differential equations that are satisfied by the transverse oscillation $\mathfrak{u}$ read [art. **43**, form. (17)]:

$$\begin{aligned}\rho\ddot{u} &= B\,\Delta u = -2B\left[\frac{\partial\zeta}{\partial y}-\frac{\partial\eta}{\partial z}\right],\\ \rho\ddot{v} &= B\,\Delta v = -2B\left[\frac{\partial\xi}{\partial z}-\frac{\partial\zeta}{\partial x}\right],\\ \rho\ddot{w} &= B\,\Delta w = -2B\left[\frac{\partial\eta}{\partial x}-\frac{\partial\xi}{\partial y}\right],\end{aligned} \tag{1}$$

to which the following ones must be added at the boundary:

([1]) The article "Übergang des Lichtes an der Grenze zweier Medien" by DRUDE in WINKELMANN's *Handbuch der Physik* 1st ed., Bd. II, Section 1 was devoted to a detailed study of that question.

$$\begin{aligned} X_x \cos(n, x) + X_y \cos(n, y) + X_z \cos(n, z) &= \bar{X}, \\ Y_x \cos(n, x) + Y_y \cos(n, y) + Y_z \cos(n, z) &= \bar{Y}, \\ Z_x \cos(n, x) + Z_y \cos(n, y) + Z_z \cos(n, z) &= \bar{Z}. \end{aligned} \tag{2}$$

We would like to imagine defining those equations for two media that bound each other (which are infinitely-extended, moreover) and differ by only their elasticity constants B. We assume that the YZ-plane is the separation surface between them, so we set:

$$\cos(n, x) = 1, \quad \cos(n, y) = \cos(n, z) = 0 .$$

Since the tractions $\bar{X}$, $\bar{Y}$, $\bar{Z}$ must be found on both sides of the separation surface in equilibrium (art. **36**), when one distinguishes the second medium by a prime, one will conclude from (2) that:

$$X_x = X'_x, \qquad Y_x = Y'_x, \qquad Z_x = Z'_x \tag{3}$$

for the boundary surface.

We shall now follow the propagation of a plane wave under the transition from one medium to the other. Let the light be homogeneous (corresponding to a certain location on the spectrum), so the oscillation number is given by n, and furthermore, let it be linearly polarized, and indeed the oscillations might result *in* the plane (viz., the plane of incidence) that contain the light ray (viz., the wave normal). We take that plane (the indicated plane in the accompanying figure) to be the plane $z = 0$. We then have $w = 0$ for the incident light wave, and when a is the amplitude and φ is the angle of incidence [(9) of the previous article], we will have:

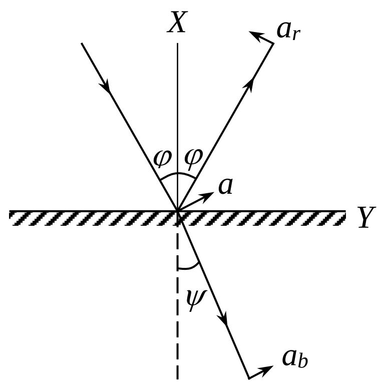

Figure 25.

$$\begin{aligned} u &= a \sin\varphi \cos 2\pi n \left(t - \frac{-x\cos\varphi + y\sin\varphi}{V} \right), \\ v &= a \cos\varphi \cos 2\pi n \left(t - \frac{-x\cos\varphi + y\sin\varphi}{V} \right). \end{aligned} \tag{4}$$

We now infer the following facts from experiments:

1. The incident wave is split into a reflected one (u_r, v_r, w_r) and a refracted one (u_b, v_b, w_b) with speeds of propagation V_r, V_b and oscillation numbers n_r, n_b .

2. The angle of reflection is equal to the angle of incidence.

We will then have:

$$\begin{aligned} u_r &= \quad a_r \sin\varphi \cos 2\pi n_r \left(t - \frac{-x\cos\varphi + y\sin\varphi}{V_r} \right), \\ v_r &= -a_r \cos\varphi \cos 2\pi n_r \left(t - \frac{-x\cos\varphi + y\sin\varphi}{V_r} \right), \end{aligned} \tag{4.a}$$

and

$$\begin{aligned} u_b &= a_b \sin\varphi \cos 2\pi n_b \left(t - \frac{-x\cos\psi + y\sin\psi}{V_b} \right), \\ v_b &= a_b \cos\varphi \cos 2\pi n_b \left(t - \frac{-x\cos\psi + y\sin\psi}{V_b} \right). \end{aligned} \tag{4.b}$$

An oscillation in the boundary surface $x = 0$ can be included in one medium or the other. One concludes that from the following relations ([1]):

$$\bar{u} + \bar{u}_r = \bar{u}_b, \quad \bar{v} + \bar{v}_r = \bar{v}_b, \quad \bar{w} + \bar{w}_r = \bar{w}_b = 0, \tag{5}$$

in which an overbar might suggest the value at the boundary surface $x = 0$.

Since those equations are true for all y and t, when one substitutes the values for u, v, w from (4), (4.a), (4.b) in (5), the cosines must drop out for $x = 0$. That will be possible only when one has both:

$$n = n_r = n_b \tag{6}$$

and

$$\frac{\sin\varphi}{V} = \frac{\sin\varphi}{V_r} = \frac{\sin\psi}{V_b}, \tag{7}$$

so $V_r = V$.

One will then have:

$$\begin{aligned} (a + a_r)\sin\varphi &= a_b \sin\psi, \\ (a - a_r)\cos\varphi &= a_b \cos\psi. \end{aligned} \tag{8}$$

Equation (6) says that the oscillation number – i.e., color – must be preserved under reflection and refraction. (7) is the expression for the law of refraction, according to which:

$$\frac{V}{V_b} = \frac{\sin\varphi}{\sin\psi} = N, \tag{9}$$

i.e., the "refraction quotient" N is independent of the angle of incidence, and it represents the ratio of the speed of propagation of light rays in both medium. Upon solving (8) for a_r, a_b, one will get

([1]) W. VOIGT (Theoret. Physik I, pp. 483) derived those formulas from the demand that kinetic energy cannot be lost under the transition across the boundary surface when no heat is produced.

the known FRESNEL formulas for the amplitudes of the reflected and refracted rays (always for light that oscillates *in* the plane of incidence):

$$\frac{a_b}{a} = \frac{2\sin\varphi\sin\psi}{\sin(\psi+\varphi)}, \qquad \frac{a_r}{a} = \frac{\sin(\psi-\varphi)}{\sin(\psi+\varphi)} \,. \tag{10}$$

Moreover, one gets:

$$\frac{a+a_r}{a_b} = \frac{1}{N} \,. \tag{11}$$

However, the boundary conditions (3) imply relations that partially contradict the formulas (which agrees with experiments). That is because the second equation in (3) will give [art. **39**, (10)]:

$$B\left[\frac{\partial(\bar{u}+\bar{u}_r)}{\partial y}+\frac{\partial(\bar{v}+\bar{v}_r)}{\partial x}\right] = B'\left(\frac{\partial\bar{u}_b}{\partial y}+\frac{\partial\bar{v}_b}{\partial x}\right),$$

so for $y = 0$:

$$\frac{B}{V}(-\sin^2\varphi+\cos^2\varphi)(a+a_r) = \frac{B'}{V_b}(-\sin^2\psi+\cos^2\psi)\,a_b$$

or:

$$\frac{a+a_r}{a_b} = N\,\frac{B'\cos 2\psi}{B\cos 2\varphi} \,. \tag{12}$$

However, that equation contradicts (11), *because:*

$$\frac{\cos 2\psi}{\cos 2\varphi}$$

is not independent of the angle of incidence.

Thus, the assumption that the medium that is the carrier of light oscillations is a solid-elastic body will encounter difficulties. Some have sought to avoid them, and others that come from the theory of dispersion, by assuming that the waves have variable amplitude [K. VONDERMÜHLL, Math. Ann. **5** (1872)]. KIRCHHOFF assumed a type of capillary action between ether and matter at the boundary surface (cf., e.g., his *Vorl. über Optik*, ed. HENSEL, 1891, pp. 143). However, none of those assumptions are satisfied completely (see DRUDE, *loc. cit.*). The new theory of electricity and magnetism ([1]) that CL. MAXWELL created opens the door to a startling way out of that. Based upon FARADAY's picture of electrical forces as polarization states of space (i.e., the ether) that propagate, MAXWELL presented a system of differential equations that say that the propagation is accompanied by certain magnetic phenomena that proceed with the speed of light and make it seem likely that the transverse waves that we attribute to light waves are nothing

([1]) Of the numerous references in H. A. LORENTZ's report on MAXWELL's theory of electromagnetism in the *Enzykl. d. math. Wiss.*, V, pp. 64, we shall single out only: CL. MAXWELL, *A Treatise on Electricity and Magnetism*, 2 vol., Oxford, 3rd ed., 1892.

but electromagnetic waves. That hypothesis was unexpected, but it was soon confirmed brilliantly by HERTZ's experiments on the spreading of the electrical force.

We will learn about MAXWELL's differential equations in the next article and connect them, as far as their derivation is concerned, with the picture of LORD KELVIN in art. **37**, in which the ether is a medium that possesses the above type of resistance to rotation as quasi-elasticity ([1]). The potential of the resisting forces in the ether is just the one that MAC CULLAGH devised before in order to avoid the difficulties that the solid-elastic medium has created in explaining elliptic polarization.

We would now like to draw out the consequences that follow from that assumption about the potential.

§ 45. – The quasi-elastic light medium.

In art. **37**, we defined the work done δw by internal forces under a virtual rotation of a unit volume in a medium that possess the property of *quasi-elasticity*. If u, v, w are very small displacements of a point and:

$$\xi = \tfrac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right),$$

etc., are the components of the rotation then we have found the following expression:

$$\delta W = -4\mathrm{B}\int (\xi\,\delta\xi + \eta\,\delta\eta + \zeta\,\delta\zeta)\,d\tau$$

for the total work done, which was next converted by partial integration into a linear function of δu, δv, δw and then substituted in the expression for D'ALEMBERT's principle (art. **40**):

$$\int \rho(\ddot{u}\,\delta u + \ddot{v}\,\delta v + \ddot{w}\,\delta w)\,d\tau - \delta W - \delta' S = 0\,,$$

in which the outer surface integral:

$$\delta' S = \int (\bar{X}\,\delta u + \bar{Y}\,\delta y + \bar{Z}\,\delta w)\,d\sigma$$

related to the boundary of the space in question. Partial integration (with the use of an abbreviation that has already been used frequently) will give:

$$\delta W = -4\mathrm{B}\int \mathsf{S}\,\xi\,\delta\xi\,d\tau = -2\mathrm{B}\int \mathsf{S}\,\xi\,\delta\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)d\tau = -2\mathrm{B}\int \mathsf{S}\,\xi\left(\frac{\partial\,\delta w}{\partial y} - \frac{\partial\,\delta v}{\partial z}\right)d\tau$$

([1]) One will find a glimpse of the experiments that encountered the aforementioned difficulty in the aforementioned report by H. A. LORENTZ, pp. 136, *et seq.*

$$= -2\mathsf{B}\mathsf{S}\iint dx\,dz[\xi\,\delta w] + 2\mathsf{B}\mathsf{S}\iint dx\,dy[\xi\,\delta v] + 2\mathsf{B}\mathsf{S}\int d\tau\,\frac{\partial\xi}{\partial y}\,\delta w - 2\mathsf{B}\mathsf{S}\int d\tau\,\frac{\partial\xi}{\partial z}\,\delta v\,,$$

in which the square brackets refer to the boundary. If one again forms the sum of the others and introduces the outer surface element:

$$d\sigma = \pm\, dx\; dz \cos\,(n, y) = \dots$$

then one will get:

$$\delta W = -2\mathsf{B}\mathsf{S}\int d\sigma\,\delta u[\zeta\cos(n,y) - \eta\cos(n,z)] - 2\mathsf{B}\mathsf{S}\int d\tau\;\delta u\left(\frac{\partial\zeta}{\partial y} - \frac{\partial\eta}{\partial z}\right). \tag{1}$$

When one compares each of the coefficients of δu, … for the inner as well as the outer surface to zero, D'ALEMBERT's principle will then give the following equations for motion inside of a quasi-elastic medium:

$$\begin{aligned}\rho\,\ddot{u} &= -2\mathsf{B}\left(\frac{\partial\zeta}{\partial y} - \frac{\partial\eta}{\partial z}\right),\\ \rho\,\ddot{v} &= -2\mathsf{B}\left(\frac{\partial\xi}{\partial z} - \frac{\partial\zeta}{\partial x}\right),\\ \rho\,\ddot{w} &= -2\mathsf{B}\left(\frac{\partial\eta}{\partial x} - \frac{\partial\xi}{\partial y}\right),\end{aligned} \tag{2}$$

and for the moment on the outer surface:

$$\begin{aligned}\bar{X} &= -\,2\;\mathsf{B}\;[\eta\cos\,(n, z) - \zeta\cos\,(n, y)]\,,\\ \bar{Y} &= -\,2\;\mathsf{B}\;[\zeta\cos\,(n, x) - \xi\cos\,(n, z)]\,,\\ \bar{Z} &= -\,2\;\mathsf{B}\;[\xi\cos\,(n, y) - \eta\cos\,(n, x)]\,.\end{aligned} \tag{3}$$

Differentiating equations (2) with respect to x, y, z and adding them will give:

$$\frac{\partial^2}{\partial t^2}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) = \frac{\partial^2}{\partial t^2}\operatorname{div}\mathfrak{u} = 0\,.$$

If one assumes that a state of equilibrium is possible in the elastic medium then one can set $\Theta = \dot{\Theta} = 0$ for $t = 0$. The dilatation will then be:

$$\Theta = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = \operatorname{div}\mathfrak{u} = 0 \tag{4}$$

for all time.

If one substitutes the values for ξ, … in (1) then equations (2) will assume the form:

$$\begin{aligned} \rho\,\ddot{u} &= \mathsf{B}\,\Delta u\,, \\ \rho\,\ddot{v} &= \mathsf{B}\,\Delta v\,, \\ \rho\,\ddot{w} &= \mathsf{B}\,\Delta w\,. \end{aligned} \tag{4.a}$$

Equations (2) have the same form for the interior as the ones that we obtained before [art. **44**, (17)] under the assumption of a solid-elastic medium. All of the consequences that were inferred from that alone in the previous article will then remain true, and in particular, equations (4) to (11) and the remarks that were connected with them will also be valid for the present conception of things, as long one only sets the constants B and B equal to each other. Namely:

$$V = \sqrt{\frac{\mathsf{B}}{\rho}} \tag{5}$$

also has the meaning here of the *speed of propagation* of plane waves in a medium of "elasticity" B ("rigidity" for W. THOMSON).

By contrast, equations (3) for the outer surface (viz., the boundary surface between two media) are different from the ones in the case of a solid-elastic body. They no longer contradict experiments now.

Namely, if one multiplies (3) by cos (n, x), cos (n, y), cos (n, z), and adds them then that will give:

$$\bar{X}\ \cos(n, x) + \bar{Y}\ \cos(n, y) + \bar{Z}\ \cos(n, z) = 0\,.$$

A rotational moment (i.e., a force-couple) then acts in the boundary surface whose rotational axis is parallel to the surface element. If one again makes the plane $x = 0$ be the boundary surface then (3) will give the tangential force:

$$\bar{X} = 0\,, \qquad \bar{Y} = -\,2\mathsf{B}\bar{\zeta}\,, \qquad \bar{Z} = 2\mathsf{B}\bar{\eta}\,,$$

when one denotes the values on the surface by an overbar, as before.

The surface tractions that exist on one side of the boundary surface and the other must preserve equilibrium (art. **36**). If one then once more distinguishes one medium from the other by a prime then one will have:

$$\begin{aligned} 2\mathsf{B}\bar{\zeta} &= 2\mathsf{B}'\bar{\zeta}', \\ 2\mathsf{B}\bar{\eta} &= 2\mathsf{B}'\bar{\eta}', \end{aligned} \tag{6}$$

from which it will follow that:

$$\bar{\eta} : \bar{\zeta} = \bar{\eta}' : \bar{\zeta}'\,.$$

The internal forces along the boundary surface then have the same direction on both sides. Since:

$$\mathsf{B}^2(\bar{\eta}^2 + \bar{\zeta}^2) = \mathsf{B}'^2(\bar{\eta}'^2 + \bar{\zeta}'^2)\,,$$

their magnitudes are inversely proportional to the constants B, B' that give a measure of the quasi-elasticities. If one now also forms the boundary equations (6) for the particular solutions (4) that were assumed in the previous article, so for a light wave that oscillated in the plane of incidence, then since:

$$\bar{\eta} = \frac{\partial \bar{w}}{\partial x} - \frac{\partial \bar{u}}{\partial z} = 0 = \bar{\eta}',$$

the second of the equations will be fulfilled. The first one goes to:

$$\frac{\mathsf{B}}{V}(a + a_r)(\cos^2\varphi + \sin^2\varphi) = \frac{\mathsf{B}'}{V_b} a_b (\cos^2\psi + \sin^2\psi) ,$$

or to:

$$\frac{a + a_r}{a_b} = \frac{\mathsf{B}'}{\mathsf{B}} \cdot \frac{V}{V_b} = \frac{1}{N} ,$$

when one sets [art. **44**, (9)]:

$$\frac{\mathsf{B}}{\mathsf{B}'} = N^2 = \frac{V^2}{V_b^2} .$$

However, that agrees with the FRESNEL formulas, which were confirmed by experiments, and with (11) in the previous article.

For later purposes, we add the remark that everywhere inside of the medium the direction of the moment-axis of:

$$\mathsf{B}\,\mathfrak{w} = (\mathsf{B}\,\xi, \mathsf{B}\,\eta, \mathsf{B}\,\zeta)$$

is perpendicular to that of the displacement:

$$\mathfrak{u} = (u, v, w)$$

and, at the same time, the wave normal α, β, γ. That is because if one defines the components ξ, η, ζ of $\mathfrak{w} = \frac{1}{2}\text{rot}\,\mathfrak{u}$ with the help of:

$$u = a \cos 2\pi n \left(t - \frac{\alpha x + \beta y + \gamma z}{V}\right)$$

then that will give:

$$\xi = (c\beta - b\gamma)\frac{2\pi n}{V} \sin 2\pi n \left(t - \frac{\alpha x + \beta y + \gamma z}{V}\right),$$

etc. Therefore, one has both:

$$a\,\xi + b\,\eta + c\,\zeta = 0$$

and

$$\alpha\,\xi + \beta\,\eta + \gamma\,\zeta = 0 .$$

Q.E.D.

Before we turn to a different interpretation of equations (1) to (3), we would like to extend them to the case in which the medium whose state of motion they represent is possessed of not just the property of quasi-elasticity, but also a *resistance* to motion (e.g., stiffness, viscosity) that will absorb part of the energy (viz., convert it into heat) and might be proportional to the velocity $\dot{\mathfrak{u}}$. The work done by a virtual displacement δu, δv, δw of a unit volume will then be:

$$- k(\dot{u}\,\delta u + \dot{v}\,\delta v + \dot{w}\,\delta w) = - k\,\mathsf{S}\,\dot{u}\,\delta u ,$$

in which k is an essentially-positive constant, and therefore the energy loss that must be included in the expression for D'ALEMBERT's principle (art. **16**) will be:

$$\delta R = - \int k\,\mathsf{S}(\dot{u}\,\delta u)\,d\tau , \tag{7}$$

from which one constructs:

$$\int \rho\,\mathsf{S}\,\ddot{u}\,\delta u\,d\tau - \delta W - \delta R - \delta' S = 0 ,$$

or after converting it into the form that was given in (1):

$$\int \mathsf{S}\left[\rho\,\ddot{u} + 2\mathsf{B}\left(\frac{\partial \zeta}{\partial y} - \frac{\partial \eta}{\partial z}\right) + k\,\dot{u}\right]\delta u\,d\tau - \int \mathsf{S}[\bar{\eta}\cos(n, z) - \bar{\zeta}\cos(n, y) + \bar{X}]\,\delta u\,d\sigma = 0 .$$

The equations of motion for an absorbing medium then follow from that:

$$\rho\,\ddot{u} + k\,\dot{u} = 2\mathsf{B}\left(\frac{\partial \eta}{\partial z} - \frac{\partial \zeta}{\partial y}\right) , \tag{8}$$

etc., in which once more sets:

$$\xi = \tfrac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) , \tag{9}$$

etc., and the boundary conditions:

$$\bar{X} = - 2\mathsf{B}[\bar{\eta}\cos(n, z) - \bar{\zeta}\cos(n, y)] , \tag{10}$$

etc. In vector notation, those equations read:

$$\rho\,\ddot{\mathfrak{u}} + k\,\dot{\mathfrak{u}} = 2\,\mathsf{B}\,\mathrm{rot}\,\mathfrak{w} , \tag{8.a}$$

$$\mathfrak{w} = \tfrac{1}{2}\,\mathrm{rot}\,\mathfrak{u} ,$$

$$\bar{\mathfrak{P}} = - 2\,\mathsf{B}\,[\mathfrak{w}, \mathfrak{n}] , \tag{10.a}$$

in which $\bar{\mathfrak{P}} = (\bar{X}, \bar{Y}, \bar{Z})$ is the pressure per unit area on the boundary surface, and $\mathfrak{n}$ is a unit vector in the direction of the normal to the surface element.

When one replaces the variations in the left-hand side of the equation for D'ALEMBERT's principle with the displacements that actually occur, as is known (art. **15**, at the end), it will go to the total increase in energy. In a unit time, that will then amount to:

$$\frac{d}{dt}(T - W - R) - \frac{d'S}{dt} = 0 , \tag{11}$$

where:

$$\begin{aligned} T &= \tfrac{1}{2}\int \rho(\dot{u}^2 + \dot{v}^2 + \dot{w}^2)\,d\tau , \\ W &= -2\mathrm{B}(\xi^2 + \eta^2 + \zeta^2)\,d\tau , \\ \frac{dR}{dt} &= -\tfrac{1}{2}\int k(\dot{u}^2 + \dot{v}^2 + \dot{w}^2)\,d\tau , \\ \frac{d'S}{dt} &= \int (\bar{X}\,\dot{u} + \bar{Y}\,\dot{v} + \bar{Z}\,\dot{w})\,d\sigma \\ &= -2\mathrm{B}\int \begin{vmatrix} \dot{u} & \dot{v} & \dot{w} \\ \bar{\xi} & \bar{\eta} & \bar{\zeta} \\ \cos(n,x) & \cos(n,y) & \cos(n,z) \end{vmatrix} d\sigma . \end{aligned} \tag{12}$$

We shall only mention that this damped form of motion is meaningful in the theory of light-absorbing media. In the following chapter, we will learn about the role that it plays in reinterpreting our system of equations for electromagnetic processes.