CHAPTER FOUR

THE ELECTROMAGNETIC THEORY OF LIGHT

§ 46. – Transition to the equations of the electromagnetic theory of light.

We now turn to the task of laying the foundations for a new interpretation of the equations of the previous article for the motion of a quasi-elastic medium, but we preface that with the statement that they do not exclude any other explanation for the propagation of light, but only that they include them as a special case. In the foregoing article, we considered a medium that was endowed with the property that any *rotation* of a unit volume $\mathfrak{w} = (\xi, \eta, \zeta)$ – when solved for its connection with the neighboring elements – is opposed by a resistance $\mathsf{B}\,\mathfrak{w}$ that is proportional to $\mathfrak{w}$. Upon applying D'ALEMBERT's principle:

$$\int \rho(\ddot{u}\,\delta u + \ddot{v}\,\delta v + \ddot{w}\,\delta w)\,d\tau + 4\mathsf{B}\int(\xi\,\delta\xi + \eta\,\delta\eta + \zeta\,\delta\zeta)\,d\tau - \int(\bar{X}\,\delta u + \bar{Y}\,\delta v + \bar{Z}\,\delta w)\,d\sigma = 0\,, \qquad (1)$$

in which $\mathfrak{u} = (u, v, w)$ is the displacement of the volume element $d\tau$, and $\bar{X}$, $\bar{Y}$, $\bar{Z}$ are the surface tractions that act upon the boundary (which is initially considered to be finite) of the space, we got equations [(1) to (4.a)] of the previous article:

$$\rho\,\ddot{u} = -\,2\mathsf{B}\left(\frac{\partial\zeta}{\partial y} - \frac{\partial\eta}{\partial z}\right), \qquad \text{etc.,} \qquad (2)$$

and

$$\bar{X} = -\,2\,\mathsf{B}\,[\eta\cos(n, z) - \zeta\cos(n, y)]\,, \qquad \text{etc.} \qquad (3)$$

for the outer surface.

Furthermore, we had:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = \operatorname{div}\mathfrak{u} = 0\,, \qquad (4)$$

while the relation existed between $\mathfrak{u}$ and $\mathfrak{w}$:

$$\mathfrak{w} = \tfrac{1}{2}\operatorname{rot}\mathfrak{u}\,,$$

or when written out in detail:

$$\xi = \tfrac{1}{2}\left(\frac{\partial\zeta}{\partial y} - \frac{\partial\eta}{\partial z}\right), \qquad \text{etc.} \qquad (5)$$

Finally, we directed our attention to the case in which the medium had friction (i.e., viscosity) that opposed the motion $\mathfrak{u}$ with a resistance that was proportional to the velocity $\dot{\mathfrak{u}}$ and in the opposite direction to it. With that assumption, the terms $+ k\dot{u}$, $+ k\dot{v}$, etc., were added to the left-hand sides of equations (2) [previous article, (8)].

As a result of (2), the acceleration will be inseparably coupled with the appearance of the angular moment $\mathrm{B}\,\mathfrak{w}$, and conversely, due to (5), that moment will give rise to a solenoidal distribution of the vector $\mathfrak{u}$ in the field. However, relations of a similar kind exist between electric current and magnetic forces (their increases, resp.) in the electromagnetic field, as it is defined in the neighborhood of a current conductor through which a varying electrical current flows or in a magnetic field that displaces relative to a closed current conductor. If we would like to pursue that analogy further then we must make the acceleration $\ddot{\mathfrak{u}}$ of a force *increase*, e.g., the increase in the electrical force (of the current) be itself proportional to the quantity $\dot{\mathfrak{u}}$, while that moment is assumed to be proportional to the other force (so the magnetic one, here). However, the statements of formulas (2) to (5) can then be compared with the experimentally-established relationships between forces in electromagnetic fields, not just in terms of the external phenomena, but even in terms of the magnitudes and directions of the vectors that come under consideration.

In order to achieve that, we would like to:

1. Interpret the *impulse* (the force of impact) $\rho\,\dot{\mathfrak{u}}$ *as an electric field strength* (i.e., force).

2. Regard the *moment* $\mathrm{B}\,\mathfrak{w}$ as the source of an acceleration in the direction of the axis of rotation (like, say, for a screw) and interpret it as a *magnetic field strength* (force).

3. Regard a resisting (frictional, viscous) medium that perhaps makes up part of a field to be a *conductor* of electricity.

We then identify our quasi-elastic medium with an electromagnetic field, and if $\mathfrak{E} = (X, Y, Z)$ is the electric force (i.e., the force that acts upon a unit amount of electricity) at a location, $\mathfrak{H} = (L, M, N)$ is the magnetic force (which acts upon a positive magnetic pole of unit strength), and a and b are proportionality factors that should prove to be constant inside of a homogeneous non-conductor, for the time being, then we set:

1. $$\rho\,\dot{\mathfrak{u}} = a\,\mathfrak{E}$$

or

$$\rho\,\dot{u} = a\,X\,, \qquad \rho\,\dot{v} = a\,Y\,, \qquad \rho\,\dot{w} = a\,Z\,, \tag{6}$$

2. $$\mathrm{B}\,\omega = -\,b\,\mathfrak{H}$$

or

$$\mathsf{B}\,\xi = -\,b\,L, \qquad \mathsf{B}\,\eta = -\,b\,M, \qquad \mathsf{B}\,\zeta = -\,b\,N\,. \tag{7}$$

When we let the direction of light oscillation $\mathfrak{u}$ coincide with that of the *electric field*, we will be making a selection that the circumstances do *not* necessitate. We could just as well employ the vectors $\mathfrak{E}$ and $\mathfrak{H}$ in the opposite way, i.e., identify the magnetic field strength with $\rho\,\dot{\mathfrak{u}}$ and the electric one with $\mathsf{B}\,\mathfrak{w}$, which is what A. SOMMERFELD, R. REIFF, and J. LARMOR (see the footnote in art. **37**) actually did do. In our way of looking at things, which was advocated by W. THOMSON and L. BOLTZMANN, one can assert that a magnetic force is what produces the rotation of the polarization plane.

If one introduces the expressions (6), (7) into equations (2)-(5), or even better, into the ones that subsume the resisting medium and include them as a special case [prev. art., (8), (9)], then when one has previously differentiated (5) with respect to time, one will get the following system ([1]):

$$\frac{a}{2b}\left(\dot{X} + \frac{k}{\rho}X\right) = \frac{\partial N}{\partial y} - \frac{\partial M}{\partial z}\,, \qquad \text{etc.,} \tag{8}$$

$$\frac{2b\rho}{a\mathsf{B}}\frac{\partial L}{\partial t} = -\left(\frac{\partial Z}{\partial y} - \frac{\partial Y}{\partial z}\right), \qquad \text{etc.} \tag{9}$$

Furthermore, due to (4), one has the relation:

$$\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} + \frac{\partial Z}{\partial z} = 0 \tag{10}$$

at all of those locations where $k = 0$. Finally, one has:

$$\frac{\partial L}{\partial x} + \frac{\partial M}{\partial y} + \frac{\partial N}{\partial z} = 0 \tag{11}$$

everywhere.

The increase in the energy $\mathfrak{E}$ of the total mass per unit time amounts to:

$$\frac{d\mathsf{E}}{dt} = \frac{d}{dt}(T - W - R)$$

$$= \tfrac{1}{2}\frac{d}{dt}\int\left\{\frac{a^2}{\rho}(X^2+Y^2+Z^2) + \frac{4b^2}{\mathsf{B}}(L^2+M^2+N^2)\right\}d\tau + \int\frac{k\,a^2}{\rho^2}(X^2+Y^2+Z^2)\,d\tau \tag{12}$$

([1]) We repeat that, as in art. **40**, the dots over the symbols u, v, … mean partial differential quotients. That will be carried over to all quantities X, …, L, $\mathfrak{E}$, $\mathfrak{H}$, $\mathfrak{A}$, φ that are derived from them in what follows.

and can be represented (*loc. cit.*) by the increase in the surface energy:

$$\frac{d\mathsf{E}}{dt} = \frac{d'S}{dt}. \tag{13}$$

§ 47. – Maxwell's equations.

The system of formulas in the previous article, which we obtained by adapting the ones for the quasi-elastic medium, has the form of known formulas that MAXWELL exhibited for electromagnetic processes, or more precisely, the symmetrically-constructed system of formulas that HERTZ ([1]) and HEAVISIDE ([2]) derived from them. In order to also exhibit that agreement in regard to the constants, we introduce four other (essentially positive) quantities c, ε, μ, λ in place of the quantities a, b, B, k by setting:

$$\begin{aligned} a &= \tfrac{1}{2}\sqrt{\frac{\varepsilon\rho}{\pi}}, & b &= \frac{c}{4}\sqrt{\frac{\rho}{\varepsilon\,\pi}}, \\ \mathsf{B} &= \frac{c^2\rho}{\varepsilon\,\mu}, & k &= \frac{4\pi\,\rho}{\varepsilon}\lambda. \end{aligned} \tag{1}$$

One gets from this that:

$$\begin{aligned} \frac{a}{2b} &= \frac{\varepsilon}{c}, & \frac{2b\rho}{a\mathsf{B}} &= \frac{\mu}{c}, \\ \frac{a^2}{\rho} &= \frac{\varepsilon}{4\pi}, & \frac{4b^2}{\mathsf{B}} &= \frac{\mu}{4\pi}, \\ \frac{k\,a^2}{\rho^2} &= \lambda, & c &= \sqrt{\frac{\mathsf{B}}{\rho}}\sqrt{\varepsilon\mu}\,. \end{aligned} \tag{1.a}$$

If one substitutes those values in equations (8), (9) of the previous article then they will take on the following form:

$$\begin{aligned} \varepsilon\,\dot{X} + 4\pi\,\lambda\,X &= c\left(\frac{\partial N}{\partial y} - \frac{\partial M}{\partial z}\right), \\ \varepsilon\,\dot{Y} + 4\pi\,\lambda\,Y &= c\left(\frac{\partial L}{\partial z} - \frac{\partial N}{\partial x}\right), \\ \varepsilon\,\dot{Z} + 4\pi\,\lambda\,Z &= c\left(\frac{\partial M}{\partial x} - \frac{\partial L}{\partial y}\right), \end{aligned} \tag{2}$$

([1]) *Untersuchungen über die Ausbreitung der electrischen Kraft*, Leipzig, 1892, pp. 215, also Göttinger Nachr. (1890).

([2]) *Electrical Papers*, vol. 2, 1892, also Phil. Mag., Feb. 1888.

$$
\begin{aligned}
\mu \dot{L} &= -c\left(\frac{\partial Z}{\partial y} - \frac{\partial Y}{\partial z}\right), \\
\mu \dot{M} &= -c\left(\frac{\partial X}{\partial z} - \frac{\partial Z}{\partial x}\right), \\
\mu \dot{N} &= -c\left(\frac{\partial Y}{\partial x} - \frac{\partial X}{\partial y}\right),
\end{aligned}
\tag{3}
$$

or, in vector notation:

$$\varepsilon \dot{\mathfrak{E}} + 4\pi \lambda\, \mathfrak{E} = c \operatorname{rot} \mathfrak{H}, \tag{2.a}$$

$$\mu \dot{\mathfrak{H}} = -c \operatorname{rot} \mathfrak{E}, \tag{3.a}$$

in which one has set:

$$
\begin{aligned}
\dot{\mathfrak{u}} &= \tfrac{1}{2}\sqrt{\frac{\varepsilon}{\rho\pi}}\,\mathfrak{E}, \\
\mathfrak{w} &= \tfrac{1}{2}\operatorname{rot}\mathfrak{u} = -\frac{\mu}{4c}\sqrt{\frac{\varepsilon}{\rho\pi}}\,\mathfrak{H},
\end{aligned}
\tag{4}
$$

and $\mathfrak{E} = (X, Y, Z)$ is the electric field strength, while $\mathfrak{H} = (L, M, N)$ is the magnetic field strength. On the boundary, the components of the surface traction $\bar{X}$, $\bar{Y}$, $\bar{Z}$ (which is a notation that naturally has nothing to do with the one for the electric force) can be represented by [art. **46**, (3)]:

$$
\begin{aligned}
\bar{X} &= \frac{c}{2}\sqrt{\frac{\rho}{\varepsilon\pi}}\,[\bar{M}\cos(n,z) - \bar{N}\cos(n,y)], \\
\bar{Y} &= \frac{c}{2}\sqrt{\frac{\rho}{\varepsilon\pi}}\,[\bar{N}\cos(n,x) - \bar{L}\cos(n,z)], \\
\bar{Z} &= \frac{c}{2}\sqrt{\frac{\rho}{\varepsilon\pi}}\,[\bar{L}\cos(n,y) - \bar{M}\cos(n,x)].
\end{aligned}
\tag{5}
$$

The energy equation reads:

$$\frac{d\mathsf{E}}{dt} \equiv \frac{d}{dt}\int \frac{1}{8\pi}\{\varepsilon(X^2+Y^2+Z^2) + \mu(L^2+M^2+N^2)\}d\tau + \int \lambda(X^2+Y^2+Z^2)\,d\tau = \frac{d'S}{dt}. \tag{6}$$

The volume *energy* E is then composed of two essentially different components, namely, of the *electromagnetic energy:*

$$\int \frac{1}{8\pi}\{\varepsilon(X^2+Y^2+Z^2) + \mu(L^2+M^2+N^2)\}d\tau,$$

and the likewise-always-positive quantity:

$$\int_{-\infty}^{t} dt \int \lambda (X^2 + Y^2 + Z^2)\, d\tau ,$$

which is proportional to:

$$\lambda = \frac{\varepsilon k}{4\pi \rho} ,$$

so it is proportional to the resistance k of the absorbing medium, and which can be nothing but so-called JOULE *heat* (art. **15**), since it is generated by the constrained motion of the mass-element through friction.

The increase in the surface energy $d'S / dt$ can be put into the form (art. **45**, at the end):

$$\frac{d'S}{dt} = -2\mathsf{B}\int \begin{vmatrix} \dot{u} & \bar{\xi} & \cos(n,x) \\ \dot{v} & \bar{\eta} & \cos(n,y) \\ \dot{w} & \bar{\zeta} & \cos(n,z) \end{vmatrix} d\sigma = -\frac{c}{4\pi}\int \begin{vmatrix} X & L & \cos(n,x) \\ Y & M & \cos(n,y) \\ Z & N & \cos(n,z) \end{vmatrix} d\sigma$$

$$= \frac{c}{4\pi}\int \mathsf{S}\,\{(Y N - Z M)\cos(n,x)\}\, d\sigma ,$$

in which S once more means the sum of three similarly-constructed terms. Equations (2)-(6) are the ones that MAXWELL had exhibited for the relations between the electric and magnetic force in a field that was defined by non-conductors (i.e., dielectric) and conductors (as well as current-carrying conductors), and into which an external electromagnetic perturbation penetrates.

The quantities ε, μ, c, λ that were introduced have the following meaning for the medium (which is always assumed to be isotropic) ([1]):

ε is the "dielectric constant," which is a positive pure number that has the value 1 in empty space.

μ is the permeability, which is a positive pure number that equals 1 in empty space, and for most bodies it differs very slightly from 1; for paramagnetic bodies, like iron, it is greater than 1, and for diamagnetic ones, like bismuth, it is less than 1.

The meaning of c is given by equation (1.a):

$$c = \sqrt{\varepsilon\mu}\sqrt{\frac{\mathsf{B}}{\rho}} , \tag{7}$$

([1]) See DRUDE, *Optik*, Leipzig, 1900, pp. 249, or ZENNECK, *Electromagnetische Schwingungen*, etc., Stuttgart, 1905, pp. 13.

which goes to:

$$c = \sqrt{\frac{\mathsf{B}}{\rho}} \tag{8}$$

for empty space. Therefore, the (very large) number c (= $3 \cdot 10^{10}$ cm/s) means the *speed of propagation* [art. **45**, (5)] of the (transverse) waves of an electromagnetic disturbance, like *light, in empty space* (i.e., the ether). In a non-conductor with the constants B', ε, μ, the speed of propagation will be:

$$V' = \sqrt{\frac{\mathsf{B}'}{\rho}} = \frac{c}{\sqrt{\varepsilon\,\mu}}\,. \tag{9}$$

We shall come back to that relation in the next article.

Finally, the factor λ is a measure of the amount of heat that is developed in a body that absorbs electromagnetic energy. It is likewise its *electric conductivity*. Namely, whereas an electric wave will go through a non-conductor without producing heat, from the current generally-accepted MAXWELLIAN picture, it is precisely in conductors that the energy of electric motion will be converted into heat. Metals, which are the best conductors, absorb electrical oscillations completely, since they convert it into JOULE heat. For empty space and all non-conductors (in particular), for completely-transparent media, one has $\lambda = 0$.

In that picture, light is only a special form of an electromagnetic disturbance. When the oscillation number of a wave exceeds a certain limit, we perceive it as light. The fastest-oscillating wave that H. HERTZ observed by means of its electromagnetic effects exhibited a frequency of 10^8 oscillations in one second. However, extreme red already has an oscillation number of $395 \cdot 10^{12}$ and violet has one of $758 \cdot 10^{12}$. When one regards light as electromagnetic waves of large oscillation frequency, one also appeals to equations (2), (5) for the propagation of light rays. If we restrict ourselves to non-conductors and set $\lambda = 0$ then we will indeed have derived the aforementioned from formulas (2) to (4.a) of art. **45**, since we only need to replace the constants with other ones by means of formulas (1) of art. **47**, and the vector $\dot{\mathfrak{u}}$ – viz., the change in deflection – with the electric field strength $\mathfrak{E}$ [art. **46**, (6), (7)]. We regard the magnetic field strengths as an accompanying phenomenon that does not take the form of light, but still must not be left unconsidered, due to the conversion of energy that is coupled with it. We can also treat the particular integral [art. **43**, (9)] of equations [art. **44**, (1)], after a corresponding conversion, as integrals of MAXWELL's equations then. We will then get the following system of solutions for plane electromagnetic (or light) waves:

$$X = \frac{\rho\,\dot{u}}{a} = \dot{u}\cdot 2\sqrt{\frac{\pi\,\rho}{\varepsilon}} = A \sin 2\pi n\left(t - \frac{\alpha\,x + \beta\,y + \gamma\,z}{V}\right),$$

$$Y = \dot{v}\cdot 2\sqrt{\frac{\pi\,\rho}{\varepsilon}} = B \sin 2\pi n\left(t - \frac{\alpha\,x + \beta\,y + \gamma\,z}{V}\right), \tag{10}$$

$$Z = \dot{u} \cdot 2\sqrt{\frac{\pi \rho}{\varepsilon}} = A \sin 2\pi n \left(t - \frac{\alpha x + \beta y + \gamma z}{V} \right),$$

when one introduces the factor:

$$-\ 4\pi n \sqrt{\frac{\pi \rho}{\varepsilon}}$$

into the amplitudes *a*, *b*, *c*, which might make them go to *A*, *B*, *C*, resp. Furthermore:

$$L = -\ \frac{\mathsf{B}\,\xi}{b} = -\ \frac{\mathsf{B}}{b} \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} \right)$$

or

$$L = \sqrt{\frac{\varepsilon}{\mu}}\,(C\beta - B\gamma) \sin 2\pi n \left(t - \frac{\alpha x + \beta y + \gamma z}{V} \right), \quad \text{etc.} \tag{11}$$

Later, we will apply that formula to the case in which the plane of the incident wave is parallel to the *Z*-axis and the light vector (i.e., the electric field strength) falls in the plane of incidence. Equations (10) will then go to [art. **44**, (4)]:

$$\begin{aligned} X &= A \sin \varphi \sin 2\pi n \left(t - \frac{-x\cos\varphi + y \sin\varphi}{V} \right), \\ Y &= A \cos \varphi \sin 2\pi n \left(t - \frac{-x\cos\varphi + y \sin\varphi}{V} \right), \\ Z &= 0\,. \end{aligned} \tag{12}$$

The magnetic field strength will be (3):

$$L = M = 0\,,$$

$$N = \sqrt{\frac{\varepsilon}{\mu}} A \sin 2\pi n \left(t - \frac{-x\cos\varphi + y \sin\varphi}{V} \right). \tag{13}$$

§ 48. – Consequences. Poynting's radiation vector.

The system of equations (2.a), (3.a) in the previous article:

$$\begin{aligned} c \operatorname{rot} \mathfrak{H} &= \varepsilon \dot{\mathfrak{E}} + 4\pi \lambda \mathfrak{E}, \\ -c \operatorname{rot} \mathfrak{E} &= \mu \dot{\mathfrak{H}}, \end{aligned} \tag{1}$$

make it possible to ascertain the spatial distribution of the vectors $\mathfrak{E}$ and $\mathfrak{H}$, which are known at one time point, and which allow one to give the quantities rot $\mathfrak{E}$, λ $\mathfrak{E}$, rot $\mathfrak{H}$ at any point, at *the next time point*. If one now imagines that the calculation is continued from one time element to the next, so equations (1) are integrated (perhaps power series that converge on a certain neighborhood), then *the known arrangement* of magnetic and electric lines of force at any time-point will imply the form of the field *for a finite time interval*, as long as the individual media through which the motion propagates are defined by the functions of position ε, μ, and possibly λ ; c is a universal constant. However, that simple, as well as comprehensive, system of equations, which accounts for the reciprocity of electric and magnetic forces in an elegant way, will not account for certain phenomena that it should explain.

In the previous article, the speed of propagation of an electromagnetic state of motion in a non-conductor was found to be:

$$V' = \frac{c}{\sqrt{\varepsilon\mu}} ,$$

in which c is the one for empty space. Therefore, the ratio:

$$\frac{c}{V'} = \sqrt{\varepsilon\mu} = N \tag{2}$$

will be the speed of propagation of a light wave during the transition from empty space into a transparent medium, such as glass, which will then be a quantity that depends upon the material (ε, μ), while it is known that the index of refraction N has different values for different types of light (n). The quantities ε, μ cannot be merely constants of the media then. One seeks to avoid that contradiction by introducing the concepts of *electric* and *magnetic excitations* $\mathfrak{D}$ and $\mathfrak{B}$ (in place of the field strengths $\mathfrak{E}$ and $\mathfrak{H}$) and the *conduction current* $\mathfrak{J}$ (in place of the conductivity), which are also otherwise indispensable, and substituting those quantities in equations (1) by means of the relations:

$$\mathfrak{D} = \varepsilon\,\mathfrak{E} , \qquad \mathfrak{B} = \mu\,\mathfrak{H} , \qquad \mathfrak{J} = \lambda\,\mathfrak{E} . \tag{3}$$

The quantities ε, μ no longer appear in the system thus-obtained:

$$\begin{aligned} c \operatorname{rot} \mathfrak{H} &= \dot{\mathfrak{D}} + 4\pi\,\mathfrak{J} , \\ -c \operatorname{rot} \mathfrak{E} &= \dot{\mathfrak{B}} , \end{aligned} \tag{4}$$

and one leaves open, for the time being, whether and in which cases ([1]) the vectors $\mathfrak{D}$, $\mathfrak{B}$, $\mathfrak{J}$, and $\mathfrak{E}$, $\mathfrak{H}$ are coupled with each other by equations (3).

([1]) H. A. LORENTZ, *Enzykl. der math. Wissen.*, v. 2, pp. 88.

If one can encounter the aforementioned difficulty in that way then that will no longer be the case when one addresses the explanation of certain phenomena *in media that are themselves moving*. H. HERTZ undertook ([1]) the extension of MAXWELL's formulas to that case. We would not like to follow him in that quest, and only point out that his formulas left not only the DOPPLER phenomenon and the aberration of light unexplained, but also could not be made to agree with FIZEAU's experiment with light waves in a flowing fluid. As a result of the latter, the fluid *does not* communicate those light waves, which move in the direction of the flow, with their (full) velocity. In other words, the speed of the water $\mathfrak{v}$ does not simply add (subtract) to that of light c in order to give the speed c' of light relative to the medium that is itself moving, so one has:

$$c' \neq c + \mathfrak{v} .$$

However, since the HERTZ-MAXWELL equations for a uniformly-moving medium, when one rigidly couples that medium to a coordinate system, have exactly the same form with respect to it as they do for a coordinate system at rest, at least in their integral form (E. COHN, *Das elektromagnetische Feld, Vorlesungen über die Maxwellsche Theorie*, Leipzig, 1900, page 352), and indeed with the same constant c, and mind you, under the tacit assumption that the units of time and space are the same as they are for the medium at rest, one cannot represent the relationships for the aforementioned space correctly.

H. A. LORENTZ attempted to replace the MAXWELL-HERTZ differential equations with other ones (extend them, resp.) on those grounds, and at the same time with the intention of also including the evolution of individual electric processes. His theory of the electron is currently the best one for explaining multifaceted phenomena in the domain of the electromagnetic theory of light.

Before we go into LORENTZ's theory, we would like to derive a law from the foregoing equations that also remains valid in that theory: It is POYNTING's law for the propagation of electromagnetic energy.

Previously, in [art. **47**, (6)], we put the energy equation into the form:

$$\frac{d\mathsf{E}}{dt} - \frac{d'\mathsf{S}}{dt} = 0 , \tag{5}$$

in which the increase in volume-energy in the interior of a closed space T is represented by:

$$\frac{d\mathsf{E}}{dt} = \frac{d}{dt}\int \frac{1}{8\pi}(\varepsilon\,|\,\mathfrak{E}\,|^2 + \mu\,|\,\mathfrak{H}\,|^2)\,d\tau + \int \lambda\,|\,\mathfrak{E}\,|^2\,d\tau , \tag{6}$$

with:

$$|\,\mathfrak{E}\,|^2 = X^2 + Y^2 + Z^2, \qquad |\,\mathfrak{H}\,|^2 = L^2 + M^2 + N^2,$$

and the loss of energy that escapes through the outer surface of T is represented by:

([1]) H. HERTZ, *Untersuchungen über die Ausbreitung der elektrischer Kraft*, Leipzig, 1892, pp. 256.

$$-\frac{d'\mathsf{S}}{dt} = -\frac{1}{4\pi}\int c\,\mathsf{S}\{\cos(n,x)(Y\,N - Z\,M)\}\,d\sigma\ . \tag{7}$$

We convert that expression. From LAPLACE's determinant theory, one has:

$$\mathsf{S}\,(Y\,N - Z\,N)^2 = \mathsf{S}\begin{vmatrix} Y & Z \\ M & N \end{vmatrix}^2 = \begin{vmatrix} X^2+Y^2+Z^2 & XL+YM+ZN \\ LX+MY+NZ & L^2+M^2+N^2 \end{vmatrix}$$

$$= \mathsf{S}X^2\cdot\mathsf{S}L^2 - (\mathsf{S}X\,L)^2 = |\,\mathfrak{E}\,|^2\cdot|\,\mathfrak{H}\,|^2 - (|\,\mathfrak{E}\,|\,|\,\mathfrak{H}\,|\cos V)^2$$

$$= |\,\mathfrak{E}\,|^2\,|\,\mathfrak{H}\,|^2 \sin V,$$

when V is angle between $\mathfrak{E}$ and $\mathfrak{H}$, and $|\,\mathfrak{E}\,|$, $|\,\mathfrak{H}\,|$ are the absolute values (i.e., lengths) of the vectors $\mathfrak{E}$, $\mathfrak{H}$, as above.

Now, if α, β, γ is the direction s, which is perpendicular to the vector $\mathfrak{E}$, as well as $\mathfrak{H}$ (just like the vector product $[\mathfrak{E}, \mathfrak{H}]$, whose sense might agree with that of s), then since:

$$X\cos\alpha + Y\cos\beta + Z\cos\gamma = 0\ ,$$
$$L\cos\alpha + M\cos\beta + N\cos\gamma = 0\ ,$$

one will have:

$$\cos\alpha = (YN - ZM)\,\frac{1}{\sqrt{\mathsf{S}(Y\,N - Z\,M)^2}}\ , \qquad \text{etc.}$$

Therefore, one will have:

$$\mathsf{S}\,\{\cos(n,x)\,(YN - ZM)\} = \mathsf{S}\,\{\cos(n,x)\cos\alpha\}\,|\,\mathfrak{E}\,|\,|\,\mathfrak{H}\,|\sin V$$
$$= \cos(n,s)\cdot|\,\mathfrak{E}\,|\cdot|\,\mathfrak{H}\,|\sin V\ ,$$

and the expression for the energy that flows through the outer surface will read:

$$-\frac{d'\mathsf{S}}{dt} = -\frac{1}{4\pi}\int c\,|\,\mathfrak{E}\,|\cdot|\,\mathfrak{H}\,|\cdot\sin(\mathfrak{E},\mathfrak{H})\cos(n,s)\,d\sigma$$
$$= -\int\frac{1}{4\pi}\,|\,\mathfrak{S}\,|\cos(n,\mathfrak{S})\,d\sigma\ ,$$

when one introduces the "POYNTING radiation vector":

$$\mathfrak{S} = \frac{c}{4\pi}\,[\mathfrak{E}, \mathfrak{H}] \tag{8}$$

(whose magnitude is $|\,\mathfrak{S}\,|$), whose direction coincides with that of s.

The energy equation:

$$\frac{d\mathsf{E}}{dt} - \int |\,\mathfrak{S}\,| \cos(n, \mathfrak{S})\, d\sigma = 0\,, \tag{9}$$

which implies that the energy loss is equal to the current through the outer surface, can be put into the form:

$$\frac{d\mathsf{E}}{dt} = -\int \operatorname{div} \mathfrak{S}\, d\tau \tag{9.a}$$

with the help of GAUSS's law. When applied to an individual volume element with the magnitude:

$$d\tau = 1\,,$$

the equation:

$$\frac{d\mathsf{E}}{dt} + \operatorname{div} \mathfrak{S} = 0 \tag{9.b}$$

expresses the idea that in a non-conductor (for which $\lambda = 0$), the electromagnetic energy E can neither increase nor decrease at any location where div $\mathfrak{S}$ vanishes. A comparison between the formula (9.b) and the continuity equation:

$$\frac{d \log \rho}{dt} + \operatorname{div} \mathfrak{v} = 0$$

will show that the vector $\mathfrak{S}$ plays the same role in matter, as an "energy" of density e^{E}, that the vector $\mathfrak{v}$ plays in a fluid mass of density ρ. Wherever div $\mathfrak{S}$ is positive in an indestructible fluid that cannot be multiplied, there are energy sources, and wherever it is negative, there are sinks. From (8), the direction of the vector $\mathfrak{S}$ is perpendicular to $\mathfrak{E}$ and $\mathfrak{H}$, so in an isotropic medium, it is that of the light ray (viz., the wave normal). $\mathfrak{S}$ itself can then be regarded as the speed of the energy of the light in the case of high-frequency oscillations.

§ 49. – The Lorentz equations. System at rest.

From the first of equations (4) in the previous article:

$$c \operatorname{rot} \mathfrak{H} = \dot{\mathfrak{D}} + 4\pi \mathfrak{J}\,,$$

one will get (art. **22**, at the end):

$$\operatorname{div}(\dot{\mathfrak{D}} + 4\pi \mathfrak{J}) = 0\,,$$

in which $\mathfrak{D}$ is the electric excitation and $\mathfrak{I}$ is the conduction current.

If one imagines a region of space T inside of which an electromagnetic process plays out and which is surrounded by a closed surface that no current-carrying conductors enter then the integral that is extended over that space:

$$\int \operatorname{div}(\dot{\mathfrak{D}} + 4\pi\,\mathfrak{I})\,d\tau = \int \operatorname{div}\dot{\mathfrak{D}}\,d\tau = \frac{d}{dt}\int \operatorname{div}\mathfrak{D}\,d\tau = 0\,,$$

because the integral over div $\mathfrak{F}$ will vanish when one applies GAUSS's law (art. **24**). The integral:

$$4\pi\,e = \int \operatorname{div}\mathfrak{D}\,d\tau \tag{1}$$

will then have the character of a quantity that is unvarying in time, similar to mass. One calls the amount of electricity e that is contained in the space T its *electric charge*. It can have positive or negative values. The function of position:

$$\operatorname{div}\mathfrak{D} = \operatorname{div}\varepsilon\,\mathfrak{E} = 4\pi\,\rho\,, \tag{2}$$

so the electric charge per unit volume at the location x, y, z, is called the electric *density* ([1]).

The electron theory of H. A. LORENTZ is connected with that concept. Whereas all previous theories, like those of CL. MAXWELL, E. COHN, H. HERTZ, O. HEAVISIDE, had assumed that arbitrarily-large regions of space were filled continuously with electrical mass, LORENTZ imagined that only certain (perhaps spherical) spatial regions of smallest dimensions – called "electrons" – were filled with electricity of well-defined sign, and indeed the same electron would be filled with the same charge for all time. One can leave open the question of how the electrons relate to atomic bodies. However, one assumes that the positive and negative electrons in an electrically-uncharged (i.e., neutral) atomic body will combine to a total charge of zero.

An electromagnetic *excitation* will be transmitted from one atomic body to another and from one ponderable mass to another by means of the *ether*, which is likewise "excited" in that way. Just as the ether fills up empty space, so is it penetrated by electrons and ponderable masses as if it were not present, and it will transmit the state of excitation (oscillation) which will have shorter or longer wavelengths according to the state of oscillation that it exists in and be perceived as light or electromagnetic disturbances.

One must think of the state of excitation in the molecules of a non-conductor (which is often transparent) as being combined into a *separation* between the unperturbed states of two types of electricity (i.e., of electrons) that consists of a mutual displacement of the two (as MAXWELL also assumed). The separation of the combined positive and negative electrons in any such "dipole" can be justified only by a continued electrical excitation; in our way of looking at things, by continually repeating the impact:

([1]) The fact that the symbol ρ is given a different meaning than the one that it had in the previous articles cannot lead to any errors, since we shall no longer employ the previous equations.

$$\rho\,\dot{\mathfrak{u}} = \tfrac{1}{2}\mathfrak{D}\sqrt{\frac{\rho}{\varepsilon\pi}}\ .$$

One can imagine that the process in the ether is similar. However, the free ether includes no electric masses, so none that are in bound states, either (i.e., no dipoles).

In the molecules of a *conductor*, the subdivision of electricity into positive and negative will be opposed by much less resistance than in a non-conductor. The excitation (force) that causes the mutual distance between electrons of different type to increase will only be delayed by friction. When both of them move in opposite directions, that will define an electrical *current*. If an oscillatory state of excitation propagates through the conductor in addition to that "convection current" then it will overlap with that advancing motion ([1]).

That is perhaps the picture that one currently has of the electromagnetic processes in non-conductors and conductors. However, we would like to quickly add that only a few contours of that picture are actually employed when one deals with the presentation and further treatment of the *differential equations* that describe those processes. We shall point out the ideas that actually get used in each case expressly.

Now, as far as the equations for the state of excitation are concerned, H. A. LORENTZ initially just adapted MAXWELL's equations *for empty space* (ether). They are obtained from the ones (1), (4) in art. **48** when one sets:

$$\varepsilon = \mu = 1, \qquad \lambda = \mathfrak{J} = \rho = 0\ ,$$

and read:

$$c \text{ rot } \mathfrak{H} = \dot{\mathfrak{E}}\,, \tag{3}$$

$$-\,c \text{ rot } \mathfrak{E} = \dot{\mathfrak{H}}\,, \tag{4}$$

$$\text{div } \varepsilon\,\mathfrak{E} = \text{div } \mathfrak{D} = 4\pi\rho = 0 \tag{5}$$

[in which the last one is a repetition of (2)]. Here, $\mathfrak{E}$ is then the electric field strength, $\mathfrak{H}$ is the magnetic field strength, and c is the speed of light in empty space.

However, for the *interior of an electron*, as well as for any *region of space that is filled with electrons* more or less densely, on average (i.e., electrically charged), LORENTZ set:

$$\text{div } \mathfrak{E} = 4\pi\rho\,, \tag{6}$$

in which ρ is then a function of position and time. [That definition of ρ then differs from that of HERTZ, viz., (2)]

([1]) For *non-conductors* that are *opaque* (such as sulfur or rubber), which then let electromagnetic oscillations, but not light, go through undiminished, one theorizes that atomic bodies consist of dipoles that will indeed excite short-wavelength light oscillations, but not long-wavelength electromagnetic ones (or to a much lower degree), in such a way that the energy of a light wave can be converted into oscillations of electrons of opposite type in the dipole with respect to each other, and will thus disperse.

The condition for the conservation of electric mass in an electron (for a number of electrons that are enclosed by a surface that simultaneously moves and deforms, resp.) implies the *continuity equation* (art. **23**, footnote):

$$\frac{\partial \rho}{\partial t} + \mathrm{div}\,(\rho\,\mathfrak{v}) = 0\,, \tag{7}$$

when $\mathfrak{v}$ is the velocity of the volume element.

According to LORENTZ, an *electric current* consists of the motion of positive electrons in the direction of current and negative ones in the opposite direction. If their velocity is $\mathfrak{v}$ then the quantity:

$$\rho\,\mathfrak{v}$$

is the momentum (*impulse*) of the electric mass of the moving electrons, i.e., its *convection*. It overlaps with the *excitation* $\dot{\mathfrak{E}}$ (the "displacement current" according to MAXWELL) of the *ether* at the same location. The *total current* at that location is then:

$$\dot{\mathfrak{E}} + 4\pi\,\rho\,\mathfrak{v}\,.$$

Now, due to (6):

$$\mathrm{div}\,(\dot{\mathfrak{E}} + 4\pi\,\rho\,\mathfrak{v}) = 4\pi\frac{\partial \rho}{\partial t} + 4\pi\,\mathrm{div}\,(\rho\,\mathfrak{v})\,,$$

which is a quantity that vanishes, due to (7). The total current can then be represented by a rotation (art. **31**), which must go to c rot $\mathfrak{H}$ for $\rho = 0$, due to (3). For that reason, LORENTZ also posed the following equations ([1]) for an electron current that moves in the free ether (like, say, the current that emanated from the cathode in the experiments of W. KAUFFMANN), whose form is suggested by the analogy with the MAXWELL equations [art. **48**, (4)] for the current vector:

$$c\ \mathrm{rot}\ \mathfrak{H} = \dot{\mathfrak{E}} + 4\pi\,\rho\,\mathfrak{v}\,, \tag{8}$$

$$-\,c\ \mathrm{rot}\ \mathfrak{E} = \dot{\mathfrak{H}}\,, \tag{9}$$

which is why, since:

$$\mathrm{div}\ \dot{\mathfrak{H}} = 0\,,$$

one assumes that:

$$\mathrm{div}\ \mathfrak{H} = 0\,. \tag{10}$$

That equation says that there is no "free" (i.e., existing independently of polar opposites) magnetism. One has to pose equations (7)-(9) as ones that are fulfilled rigorously at each point in space, except that their use assumes that one must know $\rho\,\mathfrak{v}$, i.e., the position and motion of all electrons in the system. One will know those things, e.g., when one deals with a stationary current

([1]) See the oft-cited report by LORENTZ in *Enzykl. d. math. Wiss.*, V 2, pp. 155, *et seq.*

in a linear conductor. In other cases, one must be able to predict the influence of the uncharged bodies on the motion of the electrons, which is excluded. For that reason, LORENTZ derived equations for the mean values of the state quantities considered in ponderable bodies from those equations, but we cannot go into that here ([1]). Meanwhile, the equations that were presented are already capable of some applications and are entirely characteristic of the theory. For that reason, we can restrict ourselves to them here since we are dealing with only a sketch of the LORENTZ conception of things. Above all, their advantage consists of their utility for moving charges, as well, which we will address later on. In particular, we can explain the result of FIZEAU's experiment on the same basis, as we will see. We shall next infer a few consequences of equations (3)-(10).

§ 50. – The electric excitation and the magnetic force. Their potentials.

The components of the forces of the electric ([2]) and magnetic fields that are generated by electrons in given positions and motions satisfy eight equations that are obtained from the LORENTZ equations (3)-(10) of the previous in vector form:

$$c \text{ rot } \mathfrak{H} = \dot{\mathfrak{E}} + 4\pi \rho \mathfrak{v} , \tag{1}$$

$$- c \text{ rot } \mathfrak{E} = \dot{\mathfrak{H}} , \tag{2}$$

$$\text{div } \mathfrak{E} = 4\pi \rho , \tag{3}$$

$$\text{div } \mathfrak{H} = 0 , \tag{4}$$

$$\frac{\partial \rho}{\partial t} + \text{div } (\rho \mathfrak{v}) = 0 \tag{5}$$

by decomposing them into coordinate form:

$$\begin{aligned}
c\left(\frac{\partial N}{\partial y}-\frac{\partial M}{\partial z}\right) &= \dot{X}+4\pi \rho \mathfrak{v}_x , & - c\left(\frac{\partial N}{\partial y}-\frac{\partial M}{\partial z}\right) &= \dot{L} , \\
c\left(\frac{\partial L}{\partial z}-\frac{\partial N}{\partial x}\right) &= \dot{Y}+4\pi \rho \mathfrak{v}_y , & - c\left(\frac{\partial X}{\partial z}-\frac{\partial Z}{\partial x}\right) &= \dot{M} , \\
c\left(\frac{\partial M}{\partial x}-\frac{\partial L}{\partial y}\right) &= \dot{Z}+4\pi \rho \mathfrak{v}_z , & - c\left(\frac{\partial Y}{\partial x}-\frac{\partial X}{\partial y}\right) &= \dot{N} ,
\end{aligned} \tag{1.a),(2.a}$$

$$\frac{\partial X}{\partial x}+\frac{\partial Y}{\partial y}+\frac{\partial Z}{\partial z} = 4\pi \rho , \qquad \frac{\partial L}{\partial x}+\frac{\partial M}{\partial y}+\frac{\partial N}{\partial z} = 0 , \tag{3.a),(4.a}$$

([1]) See the multiply-cited Encyclopedia article by LORENTZ, V 2, pp. 200, *et seq.*

([2]) We essentially follow LORENTZ's notation, but we will often use the phrase "electric force" in the text instead of "electric field," which indeed coincides with "excitation" in empty space, since we would also like to preserve the symbol $\mathfrak{E}$ (instead of LORENTZ's $\mathfrak{D}$) for it.

to which, one adds the continuity equation:

$$\frac{\partial\rho}{\partial t}+\frac{\partial(\rho\mathfrak{v}_x)}{\partial x}+\frac{\partial(\rho\mathfrak{v}_y)}{\partial y}+\frac{\partial(\rho\mathfrak{v}_z)}{\partial z} = 0 . \tag{5}$$

Here, $\mathfrak{v}_x$, $\mathfrak{v}_y$, $\mathfrak{v}_z$ are the components of the velocity $\mathfrak{v}$ of the convection current $\rho\,\mathfrak{v}$. We can derive equations for the individual components from the eight equations (1.a)-(4.a) by eliminating five of the six quantities X, …, L, … We would like to carry out that elimination using vector calculus. If we apply the identity [art. **31**, (4.a)]:

$$\text{rot rot } \mathfrak{E} = -\Delta\mathfrak{E} + \text{grad div } \mathfrak{E} \tag{6}$$

to equation (2), by taking the rotation of both sides and substituting rot $\dot{\mathfrak{H}}$ in the equation (1), differentiated with respect to t, then we will get:

$$c \text{ rot } \dot{\mathfrak{H}} = -c^2 \text{ rot rot } \mathfrak{E} = \ddot{\mathfrak{E}} + 4\pi\frac{\partial}{\partial t}(\rho\mathfrak{v}),$$

or (6):

$$\Delta\mathfrak{E} - \frac{1}{c^2}\ddot{\mathfrak{E}} = 4\pi \text{ grad } \rho + \frac{4\pi}{c^2}\frac{\partial}{\partial t}(\rho\mathfrak{v}) . \tag{7}$$

In that way, we will get:

$$\Delta\mathfrak{H} - \frac{1}{c^2}\ddot{\mathfrak{H}} = -\frac{4\pi}{c} \text{ rot } (\rho\,\mathfrak{v}) . \tag{8}$$

The vectors $\mathfrak{E}$, $\mathfrak{H}$ (and therefore, their components, as well) are specified by equations (7), (8). It follows from (7) that:

$$\Delta X - \frac{1}{c^2}\ddot{X} = 4\pi\frac{\partial\rho}{\partial t}+\frac{4\pi}{c^2}\frac{\partial}{\partial t}(\rho\mathfrak{v}_x), \qquad \text{etc.,}$$

and from (8):

$$\Delta L - \frac{1}{c^2}\ddot{L} = -\frac{4\pi}{c}\left(\frac{\partial(\rho\mathfrak{v}_z)}{\partial y}-\frac{\partial(\rho\mathfrak{v}_y)}{\partial z}\right), \qquad \text{etc.}$$

However, instead of introducing the forces, it is better to introduce their *potentials*, and indeed initially one has the vector potential $\mathfrak{A}$ for the solenoidal vector $\mathfrak{H}$ (art. **31**):

$$\mathfrak{H} = \text{rot } \mathfrak{A} . \tag{9}$$

Equation (2):

$$0 = \text{rot } \mathfrak{E} + \frac{1}{c}\dot{\mathfrak{H}} \tag{2}$$

then goes to:

$$0 = \mathrm{rot}\left(\mathfrak{E} + \frac{1}{c}\dot{\mathfrak{A}}\right).$$

If one further introduces a scalar potential φ for the potential vector (art. **29**) in the bracket by the assumption that:

$$\mathfrak{E} = -\frac{1}{c}\dot{\mathfrak{A}} - \mathrm{grad}\,\varphi \tag{10}$$

and substitutes the values for $\mathfrak{E}$, $\mathfrak{H}$ in (9), (10) in the equations:

$$\mathrm{rot}\,\mathfrak{H} = \frac{1}{c}(\dot{\mathfrak{E}} + 4\pi\rho\mathfrak{v}), \tag{1}$$

$$\mathrm{div}\,\mathfrak{E} = 4\pi\rho \tag{3}$$

then one will get:

$$\begin{aligned}\mathrm{rot\,rot}\,\mathfrak{A} &= -\Delta\mathfrak{A} + \mathrm{grad\,div}\,\mathfrak{A}\\ &= \frac{1}{c}\left(-\frac{1}{c}\ddot{\mathfrak{A}} - \mathrm{grad}\,\dot{\varphi} + 4\pi\rho\mathfrak{v}\right),\end{aligned}$$

or

$$\mathrm{grad}\left(\mathrm{div}\,\mathfrak{A} + \frac{1}{c}\dot{\varphi}\right) = \Delta\mathfrak{A} - \frac{1}{c^2}\ddot{\mathfrak{A}} + \frac{4\pi}{c}\rho\mathfrak{v} \tag{11}$$

and

$$\mathrm{div}\left(-\frac{1}{c}\dot{\mathfrak{A}} - \mathrm{grad}\,\dot{\varphi}\right) = 4\pi\rho,$$

or, since div grad $\varphi = \Delta\varphi$:

$$-\frac{1}{c}\frac{\partial}{\partial t}\left(\mathrm{div}\,\mathfrak{A} + \frac{1}{c}\dot{\varphi}\right) = \Delta\varphi - \frac{1}{c^2}\ddot{\varphi} + 4\pi\rho. \tag{12}$$

Due to (9), the function $\mathfrak{A}$ can be determined only up to a gradient. One can think of it as being determined in such a way that the expression in parentheses on the left in (11), (12) will vanish. In fact: Let $\mathfrak{A}_0$, φ_0 be two functions that satisfy equations (11), (12). A function c can then be determined such that:

$$\varphi = \varphi_0 + \frac{1}{c}\dot{\chi},$$

$$\mathfrak{A} = \mathfrak{A}_0 - \mathrm{grad}\,\chi$$

are also satisfied. If one then substitutes those values in (11), (12) then since:

$$\mathrm{div\,grad}\,\chi = \Delta\chi,$$

one will get:

$$\operatorname{grad}\left(\operatorname{div}\mathfrak{A}_0 - \Delta\chi + \frac{1}{c}\dot{\varphi} + \frac{1}{c^2}\ddot{\chi}\right) = \Delta\mathfrak{A} - \frac{1}{c^2}\ddot{\mathfrak{A}} + \frac{4\pi}{c}\rho\,\mathfrak{v}\ ,$$

$$-\frac{1}{c}\frac{\partial}{\partial t}\left(\operatorname{div}\mathfrak{A}_0 - \Delta\chi + \frac{1}{c}\dot{\varphi} + \frac{1}{c^2}\ddot{\chi}\right) = \Delta\varphi - \frac{1}{c^2}\ddot{\varphi} + 4\pi\rho\ .$$

If one now determines the function χ such that each expression in parentheses on the left-hand sides vanishes then the potential functions $\mathfrak{A}$, φ that are employed in the calculation of $\mathfrak{E}$, $\mathfrak{H}$ must fulfill the equations:

$$\Delta\varphi - \frac{1}{c^2}\ddot{\varphi} = -\,4\pi\rho\,, \tag{13}$$

$$\Delta\mathfrak{A} - \frac{1}{c^2}\ddot{\mathfrak{A}} = -\,\frac{4\pi}{c}\rho\,\mathfrak{v}\ . \tag{14}$$

Now since conversely a function χ of the given type can always be found for any pair of solutions φ_0, $\mathfrak{A}_0$ of those equations, equations (13), (14) can serve to determine φ, $\mathfrak{A}$ instead of (11), (12). The relation exists between both functions that:

$$\operatorname{div}\mathfrak{A} + \frac{1}{c}\dot{\varphi} = 0 \tag{15}$$

[due to (11), (14); (12), (13)], which can replace the equation for χ.

The partial differential equations (13), (14) for the scalar potential φ and the vector potential $\mathfrak{A}$, which define the vector $\mathfrak{E}$ and the solenoidal vector $\mathfrak{H}$ by means of (9) and (10), will be equal to the ones in (9), (10) of art. **31**, which we presented there for the potential and solenoidal components of a general vector, in the case of a stationary motion. The density ρ gives the source distribution, and the convection current $\rho\,\mathfrak{v}$ gives the vortex distribution upon which $\mathfrak{E}$ and $\mathfrak{H}$ depend. By introducing a four-dimensional region that was determined by the three spatial coordinates *x*, *y*, *z*, and time *t*, MINKOWSKI (in an address that he presented to the Naturforschersammlung in Cologne and was edited shortly before his death that was entitled “Raum und Zeit,” Leipzig, 1909, pp. 12) had combined the potentials φ, $\mathfrak{A}$ into one that can be given an elegant interpretation in the conceptual picture of that space-time coordinate system, but that shall not be followed through here.

In the case of a stationary motion, equations (13), (14) will go to:

$$\Delta\varphi = -\,4\pi\rho\,, \tag{16}$$

$$\Delta\mathfrak{A} = -\,\frac{4\pi}{c}\rho\,\mathfrak{v}, \tag{17}$$

whose integral we have exhibited above [art. **30**, (7), (10)] and [art. **31**, (5.b), (6.a)]:

$$\varphi = \int \frac{\rho}{r}\,d\tau\,, \tag{18}$$

$$\mathfrak{A} = \frac{1}{c}\int \frac{\rho\mathfrak{v}}{r}\,d\tau\,, \tag{19}$$

where $\mathfrak{H}$ = – rot $\mathfrak{A}$, $\mathfrak{E}$ = – grad φ, and the components are:

$$L = \frac{\partial \mathfrak{A}_y}{\partial z} - \frac{\partial \mathfrak{A}_z}{\partial y}\,, \text{ etc.,} \qquad X = -\frac{\partial \varphi}{\partial x}, \qquad \text{etc.,}$$

when the indices x, y, z indicate the components of the vector upon which they depend.

§ 51. – Application to stationary states.

Some simple examples might show how the foregoing formulas are employed.

Example 1: Let the interior of a very small ball of radius α be charged with electrical mass.

If one investigates its effect on points of space whose distance r from the center of the ball is very large compared to a then one can set:

$$\varphi = \int \frac{\rho\,d\tau}{r} = \frac{1}{r}\int \rho\,d\tau = \frac{e}{r} \tag{1}$$

for every point in space outside of the ball [prev. art. (18)], when e – viz., the charge (art. **49**) – is a positive or negative constant that is independent of r. However, φ will then be nothing but the potential of a force-at-a-distance that acts in inverse proportion to the square of the distance, as COULOMB's law would attribute to an electrical mass e at a, b, c that acts upon the unit electrical mass at x, y, z.

If one has two small balls of radii α, α_1 at a, b, c and a_1, b_1, c_1 then their effect on the unit mass at x, y, z outside of those balls will again be represented by (art. **29**):

$$\varphi = \frac{e}{r} + \frac{e_1}{r_1}\,, \tag{2}$$

where e, e_1 are positive or negative constants. Since $\mathfrak{E}$ is the gradient of – φ (prev. art., at the end), when we assume that the electrical masses are at rest, so $\mathfrak{v}$ = 0 and $\mathfrak{A}$ = 0, the electromagnetic energy (art. **48**) will be expressed by:

$$\mathsf{E} = \int \frac{1}{8\pi} |\mathfrak{E}|^2 \, d\tau = \frac{1}{8\pi} \int \left[\left(\frac{\partial \varphi}{\partial x}\right)^2 + \left(\frac{\partial \varphi}{\partial y}\right)^2 + \left(\frac{\partial \varphi}{\partial z}\right)^2 \right] d\tau \, .$$

We calculated that integral expression above, if based upon an entirely different interpretation [art. **32**, (7)], and found that:

$$\mathsf{E} = \tfrac{1}{2}\left(\frac{e^2}{\alpha} + \frac{2e\,e_1}{R} + \frac{e_1^2}{\alpha_1}\right) = \frac{e\,e_1}{R} + \text{const.}, \tag{3}$$

where:

$$R^2 = (a - a_1)^2 + (b - b_1)^2 + (c - c_1)^2$$

is the square of the distance between the electrical mass-points α, α_1 .

The electromagnetic energy that is present in the medium is then (except for a positive constant) equal to the potential ee_1 / R of the force of attraction or repulsion that acts between the two charges.

Example 2: A constant, rectilinear current flows in the direction of the positive Z-axis through a very thin cylindrical conductor of unit cross-section. Let ρ be the amount of electricity that goes through it in a unit time (i.e., the current density). The components of the current velocity are then $\mathfrak{v}_x = \mathfrak{v}_y = 0$. We further set $\mathfrak{v}_z = 1$.

For one location of the current at a distance ζ from the origin (see Fig.), the volume element $d\tau = 1 \cdot d\zeta$, so (prev. art., at end):

$$\mathfrak{A}_x = \mathfrak{A}_y = 0 \, , \quad \mathfrak{A}_z = \frac{1}{c}\int \frac{\rho \, d\zeta}{r} \, .$$

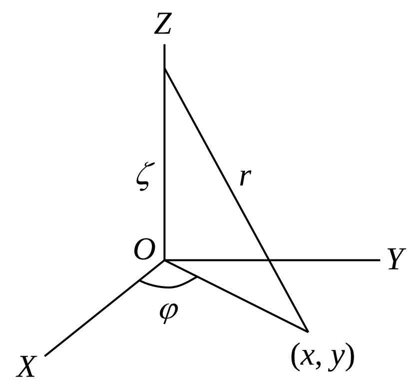

Figure 26.

It suffices to know the field in the XY-plane. Let R be the distance from a point in it to the origin:

$$R^2 = x^2 + y^2$$

and let:

$$r^2 = R^2 + \zeta^2 .$$

One will then have:

$$\frac{\partial}{\partial x}\mathfrak{A}_z = \frac{\rho}{c}\int_{+\infty}^{-\infty} \frac{\partial}{\partial x} \frac{d\zeta}{\sqrt{R^2 + \zeta^2}} \, .$$

Therefore:

$$\frac{\partial \mathfrak{A}_z}{\partial x} = -\frac{\rho\, x}{c}\int_{+\infty}^{-\infty} \frac{d\zeta}{\left(\sqrt{R^2 + \zeta^2}\right)^3} = -\frac{2\rho\, x}{R^2 c} \, .$$

One likewise finds:

$$\frac{\partial \mathfrak{A}_z}{\partial y} = -\frac{2\rho\, y}{R^2 c} .$$

Therefore, the components of the magnetic force are:

$$L = \frac{\partial \mathfrak{A}_z}{\partial y} = -\frac{2\rho}{c}\cdot\frac{1}{R}\sin\varphi\ , \quad M = -\frac{\partial \mathfrak{A}_z}{\partial x} = \frac{2\rho}{c}\cdot\frac{1}{R}\cos\varphi\,, \quad N = 0\ , \tag{4}$$

when φ is the angle between R and the X-axis.

That expression for the magnetic force that is directed perpendicular to the connecting line R represents precisely the BIOT-SAVART law for the magnetic effect of an infinitely-long rectilinear current ([1]).

We further define:

$$\varphi = \int_{-\infty}^{+\infty} \frac{\rho\, d\zeta}{r} = \int_{-\infty}^{+\infty} \frac{\rho\, d\zeta}{\sqrt{R^2 + \zeta^2}}$$

and get:

$$\frac{\partial\varphi}{\partial x} = -\frac{2\rho\, x}{R^2}\,, \qquad \frac{\partial\varphi}{\partial y} = -\frac{2\rho\, y}{R^2}\,, \qquad \frac{\partial\varphi}{\partial z} = 0\ .$$

Therefore, since:

$$\mathfrak{E} + \frac{1}{c}\dot{\mathfrak{A}} = -\operatorname{grad}\varphi\,, \qquad \dot{\mathfrak{A}} = 0\ ,$$

we have:

$$X = \frac{2\rho\, x}{R^2}\,, \qquad Y = \frac{2\rho\, y}{R^2}\,,$$

i.e., the electric force is perpendicular to the direction of current. That result contradicts the experiments. A force in the given direction does not exist. One must assume (art. **49**) that *two* currents of *opposite* direction and with opposite signs on their charges overlap and then cancel in their external action.

Example 3: The strengths of an electrical current can be measured in two different ways.

Let the amount of electricity j that flows through the cross-section of a current conductor of small cross-section q in a unit time (i.e., the current density) be known. The quantity $i = j \cdot q$, which is equal for all cross-sections (even unequal ones), will then be the *current strength*, as measured in *electrostatic* units. The *electromagnetic* unit J for the current strength is provided by the work

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([1]) Actually, the assumption of an infinitely-long rectilinear current contradicts the assumption that was made above (art. **50**, at the end) that the function $\mathfrak{A}$ and its derivatives will vanish at infinity. One might then imagine that the current ends at a correspondingly further, but finite, distance from the origin.

that is done on a magnetic pole of unit strength during a circuit of the conductor. That work is represented by the line integral:

$$4\pi J = \int (L\,dx + M\,dy + N\,dz) = \int (\mathfrak{H}, d\mathfrak{s})\,. \tag{5}$$

It can be converted into a surface integral using STOKES's theorem [art. **28**, (3.a)]:

$$\int (\mathfrak{H}, d\mathfrak{s}) = \int (\text{rot}\,\mathfrak{H})_n\,d\sigma = \frac{4\pi}{c}\int \rho\,\mathfrak{v}_n\,d\sigma \tag{6}$$

that is extended over a surface that is bounded by the path of integration J, or rather, over the part of it where the vector $\rho\,\mathfrak{v}$ is non-zero, i.e., where it cuts the current conductor. We can then let that surface coincide with the cross-section q, for which ρ and $|\,\mathfrak{v}\,| = \mathfrak{v}_n$ can prove to be constant, and get:

$$J = \frac{1}{c}\int \rho\,\mathfrak{v}_n\,d\sigma = \frac{\rho\,q\cdot|\,\mathfrak{v}\,|}{c}\,.$$

Now, the density ρ of the amount of electricity that passes through the cross-section q is connected with the current density j by the equation:

$$\rho\,|\,\mathfrak{v}\,| = j\,.$$

Therefore, the following relation exists between *the electrostatic or mechanical unit of current strength i and the electromagnetic unit J:*

$$c\,J = i\,, \tag{7}$$

where c is the speed of light in empty space.

As far as the *dimensions* of the quantities that were introduced in the foregoing are concerned, the potential for the force-at-a-distance that a small electrical mass e exerts upon another one e_1 that was calculated in example 1:

$$\frac{e\,e_1}{R}$$

will be an amount of work done, in the sense of the mechanics of ponderable masses, so it will have the dimension (see Introduction):

$$\left[\frac{e\,e_1}{R}\right] = [l^2\,m\,t^{-2}]\,.$$

Therefore, the dimension of an amount of electricity e, when measured mechanically, will be:

$$[e] = [l^{3/2}\,m^{1/2}\,t^{-1}]\,,$$

and that of the density of a spatially-distributed amount of electricity [cf., *supra*, (1)] will be:

$$[\rho] = [l^{-3/2}\, m^{1/2}\, t^{-1}] \ .$$

The pole strength of a magnet has the same dimension as an amount of electricity e because the dimensions for $|\,\mathfrak{H}\,|$ is the same as the one for $|\,\mathfrak{E}\,|$. Since:

$$\mathsf{E} = [l^2\, m\, t^{-2}] \,,$$

as one can infer from the energy formula (12) in art. **46**, where ε, μ are pure numbers (art. **45**), or also from (8.a) of art. **46**, the dimensions of $|\,\mathfrak{E}\,|$ will be those of energy (Introduction):

$$[|\,\mathfrak{E}\,|] = [|\,\mathfrak{H}\,|] = [l^{-1/2}\, m^{1/2}\, t^{-1}] \ ,$$

and furthermore:

$$[|\,\mathfrak{S}\,|] = [m\, t^{-3}] \ .$$

For the current strength [J] in electromagnetic units (5), one finds that:

$$[J] = [l^{1/2}\, m^{1/2}\, t^{-1}] \ ,$$

and therefore, one has for the current strength [i] in mechanical units:

$$[i] = [l^{3/2}\, m^{1/2}\, t^{-2}] \ .$$

§ 52. – Non-stationary states.

Although we shall not prove this fact here, the general integrals of the previously-found [art. **50**, (13), (14)] partial differential equations that were satisfied by the potentials φ, $\mathfrak{A}$ can be represented by the following spatial integral, in the event that those functions vanish at infinity. The integral of:

$$\Delta\varphi - \frac{1}{c^2}\frac{\partial^2\varphi}{\partial t^2} = -\,4\pi\,\rho \tag{1}$$

is:

$$\varphi = \int \frac{1}{r}\,\bar{\rho}\, d\tau \,, \tag{1.a}$$

when $\bar{\rho}$ means: The (given) function $\rho\,(x, y, z, t)$ is defined for $t - r \,/\, c$, instead of t, so:

$$\bar{\rho} = \rho\left(x, y, z, t - \frac{r}{c}\right),$$

in which ([1]):

$$r^2 = x^2 + y^2 + z^2 .$$

Moreover, the integral of:

$$\Delta\mathfrak{A} - \frac{1}{c^2}\frac{\partial^2\mathfrak{A}}{\partial t^2} = -4\pi\rho\,\mathfrak{v} \tag{2}$$

is:

$$\mathfrak{A} = \frac{1}{c}\int\frac{1}{r}\,\overline{\rho\mathfrak{v}}\,d\tau , \tag{2.a}$$

in which $\overline{\rho\mathfrak{v}}$ is understood to be similar to $\bar{\rho}$, and in which ρ (art. **49**) is the density at the location x, y, z, and at time t of the electron there that moves with a velocity $\mathfrak{v}$ relative to a coordinate system that is fixed in space. The following relation then exists between φ and $\mathfrak{A}$ [art. **50**, (15)]:

$$\operatorname{div}\mathfrak{A} + \frac{1}{c}\dot{\varphi} = 0 .$$

We would now like to construct the potentials φ, $\mathfrak{A}$, which one calls *delayed (retarded) potentials*, due to the quantities $\bar{\rho}$, $\overline{\rho\mathfrak{v}}$ that appear in (1.a), (2.a), and the associated forces for the case of non-stationary motion, as well.

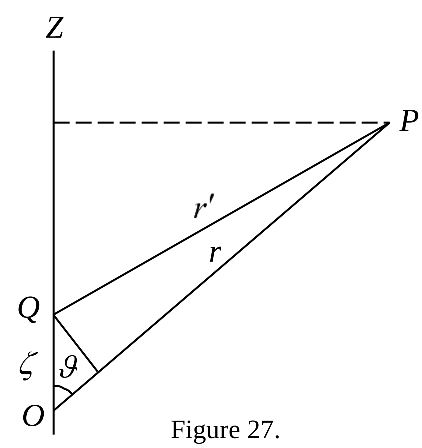

Figure 27.

Example: Oscillating dipole, according to HERTZ ([2]). Two oscillating electrons with equal and opposite charges $\pm e$ are found along the Z-axis of a rectangular coordinate system in the close vicinity of the origin: Determine the electric and magnetic fields.

We integrate equations (1), (2) under the assumption that when the electron A is found at a distance ζ from the origin ζ will be a very small quantity.

([1]) The solutions that are written in terms of $t + r / c$ (instead of $t - r / c$), which are satisfied formally, in any case, correspond to waves that advance towards the center of vibration, instead of away from it; for that reason, they will be suppressed. The outer surface integral, which generally does not appear, will vanish in the case that was assumed above. Tedone presented the general integral in 1896 in conjunction with the integration of the same integration by Kirchhoff for $\rho = 0$. See *Enzykl. d. math. Wiss*. IV 2, II, pp. 278.

([2]) H. HERTZ, “Die Kräfte elektrischer Schwingungen,” Wied. Ann. **36** (1889), *Ges. Abhandlungen*, II, pp. 147, *et seq*.., and H. A. LORENTZ, *Enzykl. d. math. Wiss*. V 2, pp. 177, *et seq*.

If O is the origin and P is a point x, y, z in the surrounding space then the distance r' from the point P to the electron Q can be expressed in terms of the distance $OP = r$ as follows:

If ϑ is the angle between OP and the Z-axis (see Fig.) then for small values of ζ, one will have:

$$r' = r - \zeta \cos\vartheta = r - \frac{z\zeta}{r}.$$

Moreover, one has:

$$\frac{1}{r'} = \frac{1}{r}\left(1 + \frac{z\zeta}{r^2}\right).$$

Now, one can replace the element $\bar{\rho}\,d\tau$ in (1.a), (2.a), which is non-zero at time $t - r'/c$ with the charge $\bar{e}$ of a point-like electron:

$$\varphi = \sum \frac{\bar{e}}{r'}, \qquad \mathfrak{A} = \frac{1}{c}\sum \frac{\overline{e\mathfrak{v}}}{r'}.$$

If one now develops the function $\bar{e}$ of ζ and $t - r'/c = t - r/c + \sigma$, where:

$$\sigma = \frac{z\zeta}{rc},$$

in powers of the small quantity σ then one will get:

$$\bar{e} = \bar{\bar{e}} + \sigma\dot{\bar{\bar{e}}} + \cdots,$$

in which double overbar might denote functions of $t - r / c$. One then has ([1]), in the first approximation:

$$\varphi = \sum\left(\bar{\bar{e}} + \dot{\bar{\bar{e}}}\,\sigma\right)\left(\frac{1}{r} + \frac{z}{r^2}\zeta\right) = \frac{1}{r}\sum\bar{\bar{e}} + \frac{z}{c^2r^2}\sum\dot{\overline{\overline{e\mathfrak{v}\zeta}}} + \frac{z}{c\,r^3}\sum\overline{\overline{e\mathfrak{v}\zeta}},$$

because ζ is a spatial coordinate that is independent of time. Analogously, one gets:

$$\mathfrak{A} = \frac{1}{c\,r}\sum\overline{\overline{e\mathfrak{v}}} + \frac{z}{c^2r^2}\sum\dot{\overline{\overline{e\mathfrak{v}\zeta}}} + \frac{z}{c\,r^3}\sum\overline{\overline{e\mathfrak{v}\zeta}}.$$

Now, if the oscillatory motion consists of, say, merely that of a negative electron with respect to a positive one, where both of them have equal charge and are placed symmetrically with respect to the center of oscillation O, then one will have:

([1]) **E. Wiechert**, Arch. Néerland. (2) **5** (1900), pp. 570.

$$\sum e = \sum \bar{e} = \sum \bar{\bar{e}} = 0, \qquad \text{and likewise} \qquad \sum \overline{\overline{e\mathfrak{v}\zeta}} = \sum \dot{\overline{\overline{e\mathfrak{v}\zeta}}} = 0\,,$$

and one will get:

$$\mathfrak{A} = \sum \frac{\overline{\overline{e\mathfrak{v}}}}{c\,r}\,,$$

$$\varphi = \frac{z}{c\,r^2}\sum \dot{\overline{\overline{e\zeta}}} + \frac{z}{r^3}\sum \overline{\overline{e\zeta}} = -\frac{\partial}{\partial z}\frac{\sum \overline{\overline{e\zeta}}}{r}\,,$$

because one has:

$$\frac{\partial}{\partial t}\left(t - \frac{r}{c}\right) = -\,c\,\frac{\partial}{\partial r}\left(t - \frac{r}{c}\right).$$

One will then have:

$$\varphi = -\frac{\partial}{\partial z}\frac{\gamma\cos\left\{n\left(t - \frac{r}{c}\right) + p\right\}}{r} = -\frac{z}{r}\frac{\partial}{\partial r}\frac{\gamma\cos\left\{n\left(t - \frac{r}{c}\right) + p\right\}}{r}\,,$$

and furthermore, since $\overline{\overline{e\mathfrak{v}}} = \dot{\overline{\overline{e\zeta}}}$:

$$\mathfrak{A} = \mathfrak{A}_z = -\frac{n\gamma}{c\,r}\sin\left\{n\left(t - \frac{r}{c}\right) + p\right\}, \quad \mathfrak{A}_x = \mathfrak{A}_y = 0\,.$$

The forces $\mathfrak{E}$ and $\mathfrak{H}$ can be derived from those expressions for the potential functions φ, $\mathfrak{A}$ by differentiating with respect to x, y, z, t. If one restricts oneself to points at such an appreciable distance r from the center of oscillation (i.e., the origin) that the quantity $1 / r$ vanishes in comparison to n / c then one will find that [art. **50**, (9), (10)]:

$$L = \frac{n^2\gamma}{c^2\,r}\cdot\frac{y}{r}\cos\Omega\,, \qquad X = \frac{n^2\gamma}{c^2\,r}\cdot\frac{x\,z}{r^2}\cos\Omega\,,$$

$$M = -\frac{n^2\gamma}{c^2\,r}\cdot\frac{x}{r}\cos\Omega\,, \qquad Y = -\frac{n^2\gamma}{c^2\,r}\cdot\frac{y\,z}{r^2}\cos\Omega\,,$$

$$N = \,, \qquad Z = -\frac{1}{c}\dot{\mathfrak{A}} - \frac{\partial\varphi}{\partial z} = \frac{n^2\gamma}{c^2\,r}\cdot\frac{r^2 - z^2}{r^2}\cos\Omega\,,$$

in which one sets:

$$\Omega = n\left(t - \frac{r}{c}\right) + p\,.$$

§ 53. – Moving systems.

Equations (1) to (5) of art. **50** were exhibited under the assumption of a coordinate system that is fixed in space. $\mathfrak{v}$ was the velocity of an amount of electricity with respect to that system. Under the transition to a comoving system, such as our Earth orbiting around the Sun, one must account for the fact that up to now no optical or electromagnetic process is known that would be useful in determining the absolute velocity of an observer in space. Rather, it would seem that it is the *relationships* that are described by those equations that continue to exist for any coordinate system that displaces parallel to it with uniform velocity, as long as one accepts that the *values* of the quantities that appear will change in ways that depend upon the motion. We would like to turn that remark into a *demand* in what follows, namely, the *postulate of relativity*, as H. POINCARÉ called it [“Sur la dynamique de l’électron,” Rend. circ. mat. Palermo **21** (1906)], and thus, with H. A. LORENTZ (see his report on MAXWELL’s electromagnetic theory in *Enzykl. d. math. Wiss.*, V 2, § 21), assume that the connections that are represented by the equations of art. **49** preserve their validity for any ponderable medium that moves rectilinearly with uniform velocity (for a coordinate system that is fixed in it, resp.).

Our representation is connected with the aforementioned treatises and the beautiful investigation of A. EINSTEIN [“Zur Elektrodynamik bewegter Körper,” Ann. Phys. (Leipzig) (4) **17** (1905)]. We will arrive at formulas for the propagation of disturbances in a comoving medium that will explain, on the one hand, the FIZEAU experiment (art. **48**), aberration, and the DOPPLER principle, but will, on the other hand, lead to remarkable and seemingly-paradoxical consequences in regard to the measurement of lengths, masses, and forces in a moving system.

Namely, it will be shown that any motion of a reference system is coupled with a simultaneous change in the units of time and length. In order to be able to express that situation more conveniently, we would like to introduce another unit of time besides the second, namely, one that makes the speed of light equal to unity, i.e., that light traverses a unit length (1 cm) in a unit of time in a medium at rest.

We shall once more denote the new time by t, which then emerges from the old one upon multiplying by the speed of light c, as measured in a medium at rest. When the differential equations for the scalar and vector potential from which the electric and magnetic forces $\mathfrak{E}$, $\mathfrak{H}$ are determined are referred to the coordinate system X, Y, Z [art. **50**, (13), (14)], which is fixed in space, they will read:

$$\begin{aligned} \Delta\varphi - \ddot{\varphi} &= -4\pi\,\rho, \\ \Delta\mathfrak{A} - \ddot{\mathfrak{A}} &= -4\pi\,\rho\,\mathfrak{v}. \end{aligned} \tag{1}$$

The electric and magnetic field strengths are calculated from φ and $\mathfrak{A}$ by means of:

$$\begin{aligned} \mathfrak{E} &= -\operatorname{grad}\varphi - \frac{\partial\mathfrak{A}}{\partial t}, \\ \mathfrak{H} &= \operatorname{rot}\mathfrak{A}. \end{aligned} \tag{2}$$

Now, that system of formulas shall retain its validity for medium that moves forth uniformly parallel to itself and with respect to the fixed system of axis X, Y, Z (a coordinate system X', Y', Z' that is fixed in it, resp., which means that a further change will enter into consideration, namely, a transition from t to perhaps the time t', as mentioned before). One then treats the presentation of equations of transformation between the quantities (x', y', z', t'), on the one hand, and (x, y, z, t), on the other, such that the form of the pair of equations (1), (2) is not changed after they have been applied. More concisely: that they remain *invariant* under (1), (2).

As far as the left-hand sides of those equations are concerned, since operation (art. **30**):

$$\Delta = \text{div grad}$$

is a vectorial one, an expression like $\Delta\varphi$ will already possess the character of invariance under rotations and displacements of the rectangular coordinate system (an orthogonal transformation with determinant 1). However, the role of the quantity t in the operator ([1]):

$$\Delta\varphi - \ddot{\varphi} = \frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} + \frac{\partial^2\varphi}{\partial z^2} - \frac{\partial^2\varphi}{\partial t^2} = \Box\,\varphi \tag{3}$$

is not essentially different from that of the coordinates x, y, z. We would not like to go into the general linear transformations under which $\Box\,\varphi$ and the relations (1), (2) remain invariant here ([2]), but only concern ourselves with the ones for which the motion of the coordinate system X', Y', Z' consists of a *displacement with uniform velocity q along the positive Z-axis*. When we let it coincide with the Z'-axis, we then make the Ansatz:

$$z' = k\,(z - q\,t)\,, \tag{4}$$

if k is a yet-to-be-determined factor (which is required by the demand of orthogonality). Now, in order for the transformation of the two variables z, t to be orthogonal with a determinant of 1, i.e., in order for the coefficients to be (hyperbolic) sines and cosines of the same angle (one imagines the formulas for the rotation of a planar rectangular coordinate system), (4) must be extended by:

$$t' = k\,(-\,q\,z + t)\,, \tag{4.a}$$

which then gives:

$$k = \frac{1}{\sqrt{1-q^2}}\,. \tag{4.b}$$

One sees that the factor k is indispensable for making the determinant equal to 1. We then set:

([1]) H. POINCARÉ's notation ["Sur la dynamique de l'électron," Rend. Circ. Mat. Palermo **21** (1906)].

([2]) H. MINKOWSKI, "Die Grundgleichungen für elektromagnetische Vorgänge in bewegten Körpern," Gött. Nachr. (1908).

$$\begin{aligned} x' &= x , \\ y' &= y , \\ z' &= k\,(\ z - q\,t) = \frac{z - q\,t}{\sqrt{1-q^2}} , \\ t' &= k\,(-\,q\,z + t) = \frac{-q\,z + t}{\sqrt{1-q^2}} , \end{aligned} \tag{5}$$

and regard q as a positive or negative constant, but with the condition that $|\,q\,| < 1$, so it is smaller than the speed of light. (One has $q = 0.0001$ for the motion of the Earth in an element of its orbit with respect to the fixed stars as the reference system.) The transformation (5) is then real. One will have:

$$\frac{\partial\varphi}{\partial x} = \frac{\partial\varphi'}{\partial x'} , \qquad \frac{\partial\varphi}{\partial y} = \frac{\partial\varphi'}{\partial y'} , \qquad \frac{\partial\varphi}{\partial z} = \frac{\partial\varphi'}{\partial z'}\,k - \frac{\partial\varphi'}{\partial t'}\,k\,q , \qquad \text{etc.,}$$

and thus, in fact, when one lets $\square'$ denote the operator:

$$\square' = \frac{\partial^2}{\partial x'^2} + \frac{\partial^2}{\partial y'^2} + \frac{\partial^2}{\partial z'^2} - \frac{\partial^2}{\partial t'^2} ,$$

one will have:

$$\square'\varphi' = \square\,\varphi , \quad \square'\mathfrak{A}' = \square\,\mathfrak{A} , \tag{6}$$

in which the notations φ', $\mathfrak{A}'$ imply the introduction of the primed coordinates.

When one demands that the transformation formulas (5) must fulfill equations (6), one assumes that the yardsticks that are used for space and time (so for z and t) coincide with the ones for z' and t', resp. In particular, one assumes that the speed of light, as measured in a moving system, keeps the same value that it has in the one at rest, namely, the value 1.

The left-hand sides of the partial differential equations (1) will go "to themselves" under the "LORENTZ transformation" (5), which is what POINCARÉ called it (*loc. cit.*) *That is not true for the right-hand sides.* That is because when one also assumes that with a suitable definition of the function ρ of x, y, z, t, it will go to ρ', nonetheless, the velocity $\mathfrak{v}$ will not go to $\mathfrak{v}'$, because only its Z-component $\mathfrak{v}_z$ will change. In order to see that, we would next like to examine the effect of the transformation (5) in a few examples ([1]).

A surface that appears to be a sphere to an observer who is at rest in the coordinate system X', Y', Z' that moves with velocity q:

$$(x' - \alpha)^2 + (y' - \beta)^2 + (z' - \gamma)^2 = a^2$$

will go to:

([1]) We take these examples from the aforementioned treatise by A. EINSTEIN.

$$(x-\alpha)^2+(y-\beta)^2+(k\,z-\gamma)^2 = a^2$$

under the transformation (5) for, e.g., the time-point $t = 0$, so it will appear to be a flattened ellipsoid of revolution to the observer that is at rest in the fixed coordinate system. For $q = 1$, i.e., when the system X', Y', Z' moves with the speed of light, it will appear to be completely flat (viz., the deformation of a spherical electron).

By contrast, the light wave:

$$(x'-\alpha)^2+(y'-\beta)^2+(z'-\gamma)^2 = (t'-\delta)^2 \tag{7}$$

will again go to another one:

$$(x-\alpha)^2+(y-\beta)^2+(z-k\,\gamma-\delta k\,q)^2 = (t-\delta k-\gamma k\,q)^2\,, \tag{7.a}$$

as a brief calculation will show. That explains, *inter alia*, the experimental fact that MICHELSON and MORLEY established that the speed of light in the direction of motion of the Earth does not differ from the speed that is measured perpendicular to that direction.

By contrast, lengths will be changed as a result of the motion. Indeed, they will be contracted for $t = 0$, i.e., at the moment when the origins of the two systems coincide, and the ratio of the length z' to z will be $k : 1 = 1 : \sqrt{1-q^2}$. At the same time, the time t' will be reduced in comparison to t for the location $z = 0$ by the ratio $k : 1$.; for other values of z, the ratio will be different.

One must then conclude that in a moving (rigid) medium, not only will the *lengths* be *contracted* in the direction of motion, but also the *clocks* that are coupled to the system will *work differently* ([1]). For that reason, one speaks of a "positional time" t' in the moving system, which differs from that t of the point in the fixed system that passes through O' precisely. For example, for $z' = 0$, one will have $t = k\,t'$, as one sees from the inverse of formulas (5):

$$\begin{aligned} z &= k\,z' + k\,q\,t', \\ t &= k\,q\,z' + k\,t'. \end{aligned} \tag{8}$$

The structure of this system of formulas is entirely similar to its inverse, and one concludes the invertibility of all relationships between moving and fixed systems, and above all, between any two different rapidly-moving systems: The transformation (5) has the "group property."

([1]) In order to sidestep that conclusion, M. ABRAHAM (*Theorie der Elektrizität*, II, 2nd ed., 1908, pp. 368) related the relativity principle to only changes of length by replacing the times t, t' with the lengths l'/c', l/c, where c' is the speed of light in the moving system (as measured at the terrestrial light source), which he (deviating from LORENTZ) assumed to be different from the c in the fixed system ($c' = c\sqrt{1-q^2}$), and where l', l are light-paths in the moving (fixed, resp.) coordinate system.

We further form the difference $z_2 - z_1$ between the z-coordinates of two fixed points A (z_1) and B ($z_2 > z_1$). If two points A', B' on the Z'-axis of a system that is comoving with any medium M and agree with A and B, resp., at the time-points t_1 and t_2, resp. (time is measured in the rest system), have the coordinates z_1', z_2', resp. then the distance between them is expressed in terms of the distance between A and B by (5):

$$\begin{aligned} z_2' - z_1' &= k\,(z_2 - q\,t_2) - k\,(z_1 - q\,t_1) \\ &= k\,(z_2 - z_1) - k\,q\,(t_2 - t_1)\,. \end{aligned}$$

We would now like to assume that the segment AB is traversed by a light ray that starts at A (A', resp.) at the time t_1 is reflected at B (B', resp.) and returns to the starting point A' ($z_3' = z_1'$), which now coincides with C in the fixed coordinate system (at a distance of z_3 from the origin), at time t_3 then it will need a time interval $t_2 - t_1$ to go from A' to B' and $t_3 - t_2$ to go from B' to A'. Now, if its velocity in the medium M when it is found in a state of *rest* is equal to V (≤ 1), so one has:

$$V\,(t_2 - t_1) = z_2 - z_1 \tag{9}$$

then

$$z_2' - z_1' = k\left(1 - \frac{q}{V}\right)(z_2 - z_1)\,. \tag{9.a}$$

One likewise finds that:

$$t_2' - t_1' = k\,(1 - q\,V)\,(t_2 - t_1)\,, \tag{10}$$

which is then the time that has elapsed for an observer in the moving coordinate system while the light went through M from A' to B'. On the other hand, the time that the light needed for the return path from B' to A':

$$z_3' - z_2' = z_2' - z_1' = k\,(z_3 - z_2) - k\,q\,(t_3 - t_2)$$

is equal to:

$$t_3' - t_2' = k\,(t_3 - t_2) - k\,q\,(z_3 - z_2)\,,$$

but one must now set:

$$V\,(t_3 - t_2) = -\,(z_3 - z_2) \tag{11}$$

for the returning ray. One then gets:

$$z_3' - z_2' = k\left(1 + \frac{q}{V}\right)(z_3 - z_2)\,, \tag{11.a}$$

$$t_3' - t_2' = k\,(1 + q\,V)(t_3 - t_2)\,. \tag{12}$$

If one divides formulas (9.a) and (11.a) then one will get:

$$\frac{V - q}{V + q} = -\,\frac{z_3 - z_2}{z_2 - z_1}\,, \tag{13}$$

and upon dividing (11.a) and (12) [(9.a) and (10), resp.], due to (9), (11), one will get:

$$-(t_3' - t_2')\frac{V+q}{1+qV} = (t_2' - t_1')\frac{V-q}{1-qV} = z_2' - z_1' \ . \tag{14}$$

The velocity relative to the moving system is then:

$$V' = \frac{V-q}{1-qV} \tag{14.a}$$

in the direction of motion and:

$$V'' = \frac{V+q}{1+qV} \tag{14.b}$$

in the opposite direction. With $V = 1$, that will again imply the fact (which was already verified above) that the light requires *the same* (relative) time to go between A' and B' along the forward and reverse paths, although the segments *AB*, *BC* that are traversed will be different when seen from the fixed system. *Time changes, along with length, under the transition to the moving system.*

A. EINSTEIN derived the basic formulas (5), (7) for relative motion based upon the demand that the condition that is expressed by formulas (14) for $V = 1$ must be fulfilled, while here they were obtained from the purely-formal demand that the "Lorentz transformation" must take the operator $\Box$ to itself above, i.e., it must be an orthogonal transformation.

The expression for the relative velocity V' will also give equations (5), differentiated with respect to time upon division. If $\mathfrak{v}_x$, … are the components of an arbitrarily-directed velocity $\mathfrak{v}$ then when one endows the velocity for the moving system with a prime, one will get:

$$\mathfrak{v}'_{x'} = \mathfrak{v}_x , \qquad \mathfrak{v}'_{y'} = \mathfrak{v}_y , \tag{15}$$

$$\mathfrak{v}'_{z'} = \frac{dz'}{dt'} = \frac{\dfrac{dz}{dt} - q}{-q\dfrac{dz}{dt} + 1} ,$$

or

$$\mathfrak{v}'_{z'} = \frac{\mathfrak{v}_z - q}{1 - q\,\mathfrak{v}_z} . \tag{15}$$

The parallelogram of velocities would correspond to $\mathfrak{v}'_{z'} = \mathfrak{v}_z - q$. One notes for later purposes, the formulas:

$$\frac{d^2x'}{dt'^2} = \frac{d}{dt}\frac{\dot{x}}{k - k\,q\,\dot{z}} \cdot \frac{1}{\dfrac{dt'}{dt}} = \frac{1}{k^2}\frac{\ddot{x}(1-q\,\dot{z}) + q\,\dot{x}\,\ddot{z}}{(1-q\,\dot{z})^3} , \tag{15.a}$$

$$\frac{d^2 z'}{dt'^2} = \frac{d\mathfrak{v}'_{z'}}{dt'} = \frac{\ddot{z}\left(\sqrt{1-q^2}\right)^3}{(1-q\,\dot{z})^3} \quad , \tag{15.b}$$

where the dots over x, z mean differential quotients with respect to t, as always.

We now return again to the differential equations for φ and $\mathfrak{A}$ in the transformed system. Formulas (15) confirm that the right-hand sides of equations (1) *do not* go to themselves under the transformation (5). However, it can be shown that this property befits certain *linear combinations of them* as long as one defines the transformed quantity ρ' suitably. The fact that the left-hand sides of the combined equations also possess that property is obvious since they have the individual summands.

One of those linear combinations $\mathfrak{A}$ must certainly be so arranged that it contains the factor $\mathfrak{v}'$, so $\mathfrak{A}'_{z'}$ will contain the factor $\mathfrak{v}'_{z'}$. For that reason, we assume that there are two undetermined coefficients α, β such that:

$$\alpha\,\rho + \beta\,\rho\,\mathfrak{v}_z = \rho'\,\mathfrak{v}'_{z'} = \rho'\,\frac{\mathfrak{v}_z - q}{1-q\,\mathfrak{v}_z} \quad ,$$

where ρ' can depend upon $\mathfrak{v}'_{z'}$ (or $\mathfrak{v}_z$), but α, β and ρ are independent of $\mathfrak{v}_z$. From:

$$(\alpha + \beta\,\mathfrak{v}_z)\,(1 - q\,\mathfrak{v}_z) = \frac{\rho'}{\rho}\,(\mathfrak{v}_z - q) \,,$$

one immediately gets:

$$\begin{gathered} \rho' = \rho\,(1-q\,\mathfrak{v}_z)\,\lambda\,, \\ \alpha = -q\,\lambda\,, \quad \beta = \lambda\,, \end{gathered} \tag{16}$$

where λ is a constant. We would now like to take the *group property* of the transformation into account in the determination λ. The same formulas that lead from the fixed to the moving coordinate system will give (8) for the return to the fixed system when one switches q with $-q$ and replaces the primed symbols with unprimed ones everywhere. One then gets:

$$\rho = \rho'\,(1+q\,\mathfrak{v}'_{z'})\,\lambda = \rho'\left(1+q\,\frac{\mathfrak{v}_z - q}{1-q\,\mathfrak{v}_z}\right)\lambda\,,$$

or

$$\rho = \rho'\,\frac{(1-q^2)\,\lambda}{1-q\,\mathfrak{v}_z} \;.$$

If one multiplies that equation by (16) then one will have:

$$\lambda^2(1-q^2) = 1$$

$$\lambda = k \, ,$$

which makes:

$$\rho' = \frac{\rho(1 - q\,\mathfrak{v}_z)}{\sqrt{1-q^2}} \tag{16.a}$$

and

$$\rho'\,\mathfrak{v}'_z = \frac{\rho(\mathfrak{v}_z - q)}{\sqrt{1-q^2}} \; . \tag{16.b}$$

Therefore, the *density of electric mass* in the moving system will be different from what it is in the fixed one.

The two linear combinations that led to (16.a), (16.b) are the now the following:

$$\left.\begin{aligned} \rho' &= \frac{1}{\sqrt{1-q^2}}\,(\varphi - q\,\mathfrak{A}_z)\,, \\ \mathfrak{A}'_{z'} &= \frac{1}{\sqrt{1-q^2}}\,(-q\,\varphi + \mathfrak{A}_z)\,, \end{aligned}\right\} \tag{17}$$

while:

$$\mathfrak{A}'_{x'} = \mathfrak{A}_x\,, \qquad \mathfrak{A}'_{y'} = \mathfrak{A}_y\,, \qquad \mathfrak{v}'_{x'} = \mathfrak{v}_x\,, \qquad \mathfrak{v}'_{y'} = \mathfrak{v}_y$$

remain unchanged. *The equations for the vector $\mathfrak{A}'$ and the scalar φ' that are composed of those components then go to:*

$$\left.\begin{aligned} \Box\,\varphi' &= -4\pi\,\rho'\,, \\ \Box\,\mathfrak{A}' &= -4\pi\,\rho'\,\mathfrak{v}' \end{aligned}\right\} \tag{18}$$

under the Lorentz transformation (5), and similarly for (1) to (5) in art. **50**. If one again ignores electrical mass, so one sets $\rho = \rho' = 0$, then one will again get equations (1) for $\rho = 0$ from equations (13), (6), i.e., the equations for the free ether, when referred to a fixed or moving coordinate system.

In conclusion, we repeat that the transformations (5), (8), (16.a), (16.b), (17) that were used in the present article possess the group property, and as a result, can be employed, not just for the transition from a system at rest to a moving one, but also *for any two systems that are in a state of relative motion with respect to each other*.

If one connects three Lorentz transformations like (5) that are applied to the direction *X*, *Y*, *Z* with a rotation of the spatial system then one will get the general space-time (i.e., Lorentz) transformation with sixteen real coefficients and a determinant of 1. The foregoing results will also be true for it. We shall come back to that later on.

§ 54. Fizeau's experiment. The Doppler principle and the aberration of light.

The formulas of the foregoing article were based upon the assumption that under the transition from the rest system to a uniformly-moving system, the lengths, times, and electromagnetic quantities that appeared would not actually keep the same values, but that the reciprocal relationships between them that are expressed by MAXWELL's formulas in the rest system would remain unchanged. We would like to illuminate that fact by a comparison with some experimental results.

One can next explain the aforementioned *Fizeau experiment*, which MICHELSON and MORLEY confirmed in 1886, with the use of the formulas in art. **53**. Namely, if one sends a light ray through one of two parallel tubes in which water flows in opposite directions, then makes the light ray go back through the second tube by means of a mirror, and then makes it interfere with a light ray that has traversed the reverse path then that will exhibit a difference between the paths. The light ray that proceeds in the direction of the current has a greater velocity than the opposite one.

The speed V' of light in water relative to the moving water is connected to its speed V relative to the observer that is fixed on the Earth by the relation (14.a) of the previous article. If one solves that for V then that will give:

$$V = \frac{V'+q}{1+qV'} = V' + q\,(1 - V'^2)\ , \tag{1}$$

up to second-order terms in the quantity q, which is small with respect to 1.

Let the index of refraction quotient for the transition from empty space to water at rest by N for the type of light employed. An observer that is moving with the water would find that $N = 1/V'$ when the light source also moves with it. However, since one is dealing with a light source that is either at rest with respect to moving water or moving with respect to matter at rest, one must replace the index of refraction N with $N' = 1/V'$ for a type of light that is displaced along the spectrum somewhat according to DOPPLER's law (see the following example).

One will then get ([1]):

$$V = V' + q\,(1 - V'^2)\,. \tag{2}$$

The factor in parentheses is the FRESNEL "dragging coefficient," which was confirmed by that physicist, and which H. A. LORENTZ derived from his theory in a different way (*Versuch einer Theorie der elektrischen und optischen Erscheinungen*, Leiden, 1895, pp. 97).

In order to compare that result with some other results of observation, we recall the particular solutions of the MAXWELL equations for empty space that were presented in [art. **47**, (10), (11)], which can be summarized in terms of the potentials (assuming that the speed of propagation is $V = c = 1$):

$$\begin{aligned} \varphi &= \varphi_0 \cos\Omega, \\ \mathfrak{A} &= \mathfrak{A}_0 \cos\Omega, \end{aligned} \tag{3}$$

([1]) Cf., M. Laue, Ann. Phys. (Leipzig) (4) **23** (1907).

where:

$$\Omega = 2\pi n\,[t - (\alpha x + \beta y + \gamma z)]\,, \qquad \alpha^2 + \beta^2 + \gamma^2 = 1\,, \tag{3.a}$$

and φ_0, $\mathfrak{A}_0$ are quantities that are independent of x, y, z, t. The expressions (3) satisfy equations (1) of the previous article for $\rho = 0$. We ask what sort of phenomena will be produced by those plane waves in a light medium that moves uniformly in the direction of the negative Z-axis. The answer is given by the introduction of x', y', z', t' into the expressions (3) by means of formulas (5), (17) of art. **53**. When one denotes the potentials in the primed coordinates by:

$$\begin{aligned} \varphi' &= \varphi'_0 \cos\,\Omega'\,, \\ \mathfrak{A}' &= \mathfrak{A}'_0 \cos\Omega'\,, \end{aligned} \tag{4}$$

one will get:

$$\begin{aligned} \varphi'_0 &= \quad k\,\varphi_0 - k\,q\,\mathfrak{A}_{0z}\,, \\ \mathfrak{A}'_{0z'} &= -k\,q\,\varphi_0 + k\,\mathfrak{A}_{0z}\,, \\ \mathfrak{A}'_{0x'} &= \mathfrak{A}_{0x}\,, \\ \mathfrak{A}'_{0y'} &= \mathfrak{A}_{0y}\,, \end{aligned} \tag{5}$$

and

$$\begin{aligned} \Omega' &= 2\pi n\,[t - \gamma z - (\alpha x + \beta y)] \\ &= 2\pi n(1-\gamma q)k\left\{t' - z'\frac{V-q}{1-\gamma q} - \frac{\alpha x + \beta y}{k(1-\gamma q)}\right\} \\ &= 2\pi n'[t' - (\alpha' x' + \beta' y' + \gamma' z')]\,, \end{aligned} \tag{6}$$

which will give the following relations between the frequency n' of the oscillations and the cosine γ' of the angle of inclination of the wave normal in the moving system:

$$\frac{n'}{n} = k\,(1 - \gamma q), \tag{7}$$

$$\gamma' = \frac{V-q}{1-\gamma q}\,. \tag{8}$$

From the remark that was made at the end of art. **53**, we can employ those formulas for any two systems that move with respect to each other, e.g., the Earth (n, γ) and a star (n', γ', q) that moves relative to it. The relation (7) then expresses *DOPPLER's principle*, namely, that a change in the relative velocity q of the observer with respect to a light source is coupled with a change in the oscillation number n' (viz., a displacement in the FRAUNHOFER lines in the moving light source with respect to the lines for a light source at rest).

The relation (8) explains the phenomenon of the *aberration of light from the fixed stars*, which do not appear to be at the locations that they are assigned by the annual averages of observations when observed through a telescope, but rather they seem to describe a small oval around that location every year, which is an image of the Earth's orbit. If, for the sake of simplicity, we take the case in which $\gamma' = 0$, i.e., a fixed star is found on a perpendicular to the ecliptic (i.e., the Earth's orbital plane) then formula (8) will give the cosine γ of the angle at which the light ray from the star seems to go through the telescope as $\gamma = q$, where q is the speed of the Earth in its orbit.

Astronomy confirms that result, and therefore, at the same time, the assumption that the theory is based upon, namely, that the light ray will experience no deflection when arrives in the region of the Earth, so the intermediary of the motion of light (i.e., the ether) does not move with the Earth (i.e., get carried along by it), but will behave like a rigid framework through which even the fastest-moving matter can move unimpeded.

§ 55. – Forces in moving systems.

The electric and magnetic field strengths $\mathfrak{E}'$, $\mathfrak{H}'$ in a moving coordinate system can be determined from formulas (17) when one calculates those quantities from the potentials φ', $\mathfrak{A}'$ in the same way that one calculates $\mathfrak{E}$, $\mathfrak{H}$ from φ, $\mathfrak{A}$ [art. **53**, (2)], namely, by:

$$\mathfrak{E}' = -\ \mathrm{grad}\,\varphi' - \frac{\partial \mathfrak{A}'}{\partial t},$$
$$\mathfrak{H}' = \mathrm{rot}\,\mathfrak{A}',$$

which are expressions that now, in turn, satisfy the LORENTZ equations (1) to (4) of art. **50**, since φ', $\mathfrak{A}'$ satisfy (13), (14) of the same article. For the components X', Y', Z' of $\mathfrak{E}'$ and L', M', N' of $\mathfrak{H}'$, with the use of the inverse (8) for formulas (5) of art **53**, one will then get:

$$\begin{aligned} z &= k\,z' + k\,q\,t', \\ t &= k\,q\,z' + k\,t', \\ x &= x', \; y = y', \end{aligned} \tag{1}$$

and after inverting the formulas (17) of art. **53**:

$$\begin{aligned} \varphi &= k\,\varphi' + k\,q\,\mathfrak{A}'_{z'}, \\ \mathfrak{A}_z &= k\,q\,\varphi' + k\,\mathfrak{A}'_{z'}, \\ \mathfrak{A}_x &= \mathfrak{A}'_{x'}, \; \mathfrak{A}_y = \mathfrak{A}'_{y'}, \end{aligned} \tag{2}$$

so (when we drop the primes on the indices x, y, z for the sake of simplicity):

$$N' = \frac{\partial \mathfrak{A}'_{y'}}{\partial x'} - \frac{\partial \mathfrak{A}'_{x'}}{\partial y'} = \frac{\partial \mathfrak{A}_y}{\partial x} - \frac{\partial \mathfrak{A}_x}{\partial y} = N,$$

$$
\begin{aligned}
Z' = -\frac{\partial \varphi'}{\partial z'} - \frac{\partial \mathfrak{A}'_z}{\partial t'} &= -\frac{\partial \varphi'}{\partial z} k - \frac{\partial \varphi'}{\partial t} k\, q - \frac{\partial \mathfrak{A}'_z}{\partial z} k\, q - \frac{\partial \mathfrak{A}'_z}{\partial t} k \\
&= -\frac{\partial}{\partial z}(\varphi' k + \mathfrak{A}'_z k\, q) - \frac{\partial}{\partial t}(\varphi' k\, q + \mathfrak{A}'_z k) \\
&= \frac{\partial \varphi}{\partial z} - \frac{\partial \mathfrak{A}_z}{\partial t} \\
&= Z ,
\end{aligned}
$$

$$
\begin{aligned}
L' = \frac{\partial \mathfrak{A}'_z}{\partial y'} - \frac{\partial \mathfrak{A}'_y}{\partial z'} &= \frac{\partial \mathfrak{A}'_z}{\partial y} - \frac{\partial \mathfrak{A}'_y}{\partial z} k - \frac{\partial \mathfrak{A}'_y}{\partial t} k\, q \\
&= -\, q\, k \left\{ \frac{\partial}{\partial y}(k\, \varphi' + k\, q\, \mathfrak{A}'_z) + \frac{\partial}{\partial y} \mathfrak{A}'_t \right\} + k \left\{ \frac{\partial}{\partial y}(k\, q\, \varphi' + k\, \mathfrak{A}'_z) - \frac{\partial}{\partial z} \mathfrak{A}'_y \right\} \\
&= q\, k \left\{ -\frac{\partial \varphi}{\partial y} + \frac{\partial \mathfrak{A}_y}{\partial t} \right\} + k \left\{ \frac{\partial \mathfrak{A}_z}{\partial y} - \frac{\partial \mathfrak{A}_y}{\partial z} \right\} \\
&= q\, k\, Y + k\, L .
\end{aligned}
$$

One finds the other values of the following table in the same way:

$$
\begin{aligned}
X' &= k\,(X - q\,M) , \qquad & L' &= k\,(q\,Y + L) , \\
Y' &= k\,(Y + q\,L) , \qquad & M' &= k\,(-\,q\,X + M) , \\
Z' &= Z , \qquad & N' &= N ,
\end{aligned}
\tag{3}
$$

where one again has:

$$
k = \frac{1}{\sqrt{1 - q^2}} .
$$

Based upon equations (3) and equations (15), (16.a) of art. **53**, one can confirm, by a quick calculation that we shall pass over here, a remark that MINKOWSKI (*loc. cit.*, pp. 67) had built his entire theory of electromagnetic processes in moving bodies upon. It relates to the formal behavior of the physical concept that was discussed here under the linear space-time transformation of determinant 1 with sixteen elements (the *general* LORENTZ transformation) that was mentioned at the end of art. **53** and to a certain unexpected connection between them that the relativity postulate implies. Namely, the four quantities $\rho\, \mathfrak{v}_x$, $\rho\, \mathfrak{v}_y$, $\rho\, \mathfrak{v}_z$, ρ (so the components of the "convection current" and the density) are connected with the primed quantities $\rho'\, \mathfrak{v}'_x$, etc., that emerge from a general LORENTZ transformation by linear equations [(15), (16.a), (16.b)] with *the same* coefficients as the ones that connect the space-time coordinates x, y, z, t with the primed ones [see (5) of art. **53**].

Moreover: If two sequences of variables x, y, z, t and ξ, η, ζ, τ go to the corresponding primed sequences by one and the same space-time transformation then two two-rowed determinants that are defined by them:

$$y\,\zeta - \eta\,z\,, \qquad z\,\xi - \xi\,y\,, \qquad x\,\eta - \xi\,y\,; \qquad x\,\tau - \xi\,t\,, \qquad y\,\tau - \eta\,t\,, \qquad z\,\tau - \zeta\,t\,,$$

will transform into the primed ones by equations with the same coefficients as the equations that connect the electric and magnetic field components:

$$X, Y, Z\,; \qquad L, M, N,$$

respectively, with the primed ones. That remarkable relationship between those four "space-time vectors of the first kind" and, on the other hand, those six "space-time vectors of second kind" among themselves brings about a sort of "mixing" [HILBERT, "Gedächtnisrede auf MINKOWSKI," Gött. Nachr. (1909), pp. 23] of current and density and of electric and magnetic field strengths, such that those concepts *individually no longer have any special meaning* with respect to a space-time coordinate system of a general type. *Rather, they are combined into vectors of a more-encompassing kind* that no longer depend upon the choice of reference system.

When one locates the velocity $q = |\,\mathfrak{v}\,|$ along the Z-axis and replaces it with a vector $\mathfrak{v}$, one can give equations (3), which express relations that are basically independent of the coordinate system, the form:

$$\begin{aligned} \mathfrak{E}' &= k\,(\mathfrak{E} + [\mathfrak{v}, \mathfrak{H}])\,, \\ \mathfrak{H}' &= k\,(\mathfrak{H} - [\mathfrak{v}, \mathfrak{E}])\,, \end{aligned} \tag{4}$$

or

$$\begin{aligned} \mathfrak{E}' &= \mathfrak{E} + [\mathfrak{v}, \mathfrak{H}], \\ \mathfrak{H}' &= \mathfrak{H} - [\mathfrak{v}, \mathfrak{E}], \end{aligned} \tag{4.a}$$

according to whether one component of the vectors $\mathfrak{E}'$, $\mathfrak{H}'$ is perpendicular to the direction of $\mathfrak{v}$ or points it its direction, respectively, and those formulas can also be combined (from a communication by R. GANS) into one:

$$\mathfrak{E}' = k\,(\mathfrak{E} + [\mathfrak{v}, \mathfrak{H}]) + \frac{1-k}{q^2}(\mathfrak{E}, \mathfrak{v})\,\mathfrak{v}\,,$$

$$\mathfrak{H}' = k\,(\mathfrak{H} - [\mathfrak{v}, \mathfrak{E}]) + \frac{1-k}{q^2}(\mathfrak{H}, \mathfrak{v})\,\mathfrak{v}\,.$$

Formulas (4), (4.a) will coincide when the speed $|\,\mathfrak{v}\,| = q$ is small compared to 1 (i.e., the speed of light). As a result, in LORENTZ's theory:

$$\mathfrak{f} = \mathfrak{E} + [\mathfrak{v}, \mathfrak{H}]\,, \tag{5}$$

or when we again introduce the second as the unit of time and denote the speed of light by c :

$$\mathfrak{f} = \mathfrak{E} + \left[\frac{\mathfrak{v}}{c}, \mathfrak{H}\right] \tag{5.a}$$

expresses the *ponderomotive (i.e., electromotive) force on a unit charge* that moves rectilinearly and uniformly with a velocity $\mathfrak{v}$ that is small compared to the speed of light relative to a (fixed) coordinate system when the electric and magnetic fields that it passes through are once more denoted by $\mathfrak{E}$ and $\mathfrak{H}$, resp.

§ 56. – The electromagnetic mass of the electron.

By means of the formulas that we found in the previous article for the forces $\mathfrak{E}'$, $\mathfrak{H}'$ in the moving coordinate system X', Y', Z', we can (and we are once more following EINSTEIN) exhibit the equations of motion for an electron relative to that system when we base them upon the following assumptions (definitions, resp.), which are connected with the usual representation of mechanics, moreover:

1. When the electron is accelerated by a force *from a state of (relative) rest*, the force will differ from the acceleration that it relates to by a scalar factor (even in the moving system) that one calls its "mass" (in the narrow sense), and which is a measure of the inertia by which it opposes the force vector under the transition from *rest* to motion, regardless of whether the resistance is of a ponderable nature or originates in an electric charge.

2. That factor μ is independent of the velocity of the (uniform) advancing motion of the coordinate system.

Now, if one finds an electron of charge *e*, say, at the origin of a coordinate system X', Y', Z' that moves unfirmly and rectilinearly parallel to itself along the *Z*-axis of a rest system *X*, *Y*, *Z*, such that the *Z* and Z' axes coincide, and that electron is at rest relative to the moving system at time *t*, and the forces $\mathfrak{E}'$, $\mathfrak{H}'$ of an electromagnetic field act upon it then with the assumptions that were made, its equations of motion at the initial moment will be ([1]):

$$\begin{aligned} \mu \frac{d^2 x'}{dt'^2} &= e X', \\ \mu \frac{d^2 y'}{dt'^2} &= e Y', \\ \mu \frac{d^2 z'}{dt'^2} &= e Z'. \end{aligned} \tag{1}$$

([1]) Having the same notation for the components of $\mathfrak{E}$ and $\mathfrak{E}'$ and the coordinate axes is probably harmless.

We would like to transform those equations back to the rest system X, Y, Z. From art. **53**, (15.a), (15.b), when one sets:

$$\dot{x} = \dot{y} = 0 , \quad \dot{z} = q \tag{2}$$

in them, one will have:

$$\frac{d^2 x'}{dt'^2} = k^2 \frac{d^2 x}{dt^2}, \qquad \frac{d^2 y'}{dt'^2} = k^2 \frac{d^2 y}{dt^2}, \qquad \frac{d^2 z'}{dt'^2} = k^2 \frac{d^2 z}{dt^2} . \tag{3}$$

Furthermore, from [art. **55**, (3)]:

$$\begin{aligned} X' &= k\,(X - q\,M) , \\ Y' &= k\,(Y + q\,L) , \\ Z' &= Z . \end{aligned} \tag{4}$$

If one substitutes those values in (1) then one will get:

$$\begin{aligned} \mu\,k \frac{d^2 x}{dt^2} &= e\,(X - q\,M) , \\ \mu\,k \frac{d^2 y}{dt^2} &= e\,(Y + q\,L) , \\ \mu\,k^3 \frac{d^2 z}{dt^2} &= e\,Z \end{aligned} \tag{5}$$

for $t = 0$, in which the components of the electromagnetic force, as measured in the rest system, are now on the right-hand sides. Therefore, the force and acceleration *in the rest system* X, Y, Z of an electron that *moves* with a velocity q no longer differ by a scalar factor, but by one whose components along the axes X, Y (the direction of the axis that agrees with the velocity q, resp.) are:

$$\mu\,k, \qquad \mu\,k, \qquad \mu\,k^3 ,$$

resp. Hence, the mass, which measures the resistance (i.e., inertia) by which the *moving* electron opposes the force vector is *different* according to whether one is dealing with an acceleration that is perpendicular to the direction of motion or in the same direction as it. The *transverse electromagnetic mass* (in the first case) is equal to $\mu\,k$; the *longitudinal* one is equal to $\mu\,k^3$. Naturally, the factors will prove to be different when one defines "force" differently. If one represents it by the vector $\mathfrak{E}'$ [prev. art., (4)], as EINSTEIN did, then they will be equal to $\mu\,k^2$ ($\mu\,k^3$, resp.).

> "A force that acts upon a moving electron in the direction of the velocity q will produce a different acceleration from an equally-large one that acts perpendicular to it. If the force is oriented skew to the direction of motion then the direction of the acceleration will not coincide with that of the force… The electromagnetic mass

is not a scalar like the mass of ordinary mechanics, but a tensor (in the terminology of W. VOIGT) with the symmetric of an ellipsoid of revolution."

[ABRAHAM, "Dynamik des Elektrons," Gött. Nachr. (1902), where the concepts of "longitudinal" and "transverse" mass first appeared.]

Now, it is very remarkable that the differential equations (5) do not, by any means, forfeit their simple form when they are referred to a coordinate system in *general* position (i.e., inclined with respect to *X*, *Y*, *Z*), [M. PLANCK, "Das Prinzip der Relativität und die Grundgleichungen der Mechanik," Ber. der Deutschen Phys. Ges. Berlin **4** (1906)]. In order to show that, we transform the system (5) to a coordinate system that possesses the same origin as *X*, *Y*, *Z*, but whose axes are inclined with respect to them and therefore with respect to the direction of the velocity q. We effect that rotation by means of the formulas (1), (2) of art. **5**:

$$\begin{aligned} \xi &= \alpha_1 x + \alpha_2 y + \alpha_3 z, & x &= \alpha_1 \xi + \alpha_2 \eta + \alpha_3 \zeta, \\ \eta &= \beta_1 x + \beta_2 y + \beta_3 z, & y &= \beta_1 \xi + \beta_2 \eta + \beta_3 \zeta, \\ \zeta &= \gamma_1 x + \gamma_2 y + \gamma_3 z, & z &= \gamma_1 \xi + \gamma_2 \eta + \gamma_3 \zeta, \end{aligned} \tag{6}$$

where known relations exist between the α, β, γ that result from:

$$\begin{aligned} \alpha_3 &= \beta_1 \gamma_2 - \gamma_1 \beta_2, \\ \beta_3 &= \gamma_1 \alpha_2 - \alpha_1 \gamma_2, \\ \gamma_3 &= \alpha_1 \beta_2 - \beta_1 \alpha_2, \end{aligned} \tag{7}$$

inter alia. Differentiating them will give:

$$\dot{\xi} = \alpha_1 \dot{x} + \alpha_2 \dot{y} + \alpha_3 \dot{z}, \qquad \text{etc.,} \tag{7.a}$$

$$\ddot{\xi} = \alpha_1 \ddot{x} + \alpha_2 \ddot{y} + \alpha_3 \ddot{z}, \qquad \text{etc.} \tag{7.b}$$

Thus, at the time-point when the electron leaves the origin, so when $\dot{x} = \dot{y} = 0$, $\dot{z} = q$, due to (7.a), one will have:

$$\alpha_3 = \frac{\dot{\xi}}{q}, \qquad \beta_3 = \frac{\dot{\eta}}{q}, \qquad \gamma_3 = \frac{\dot{\zeta}}{q}. \tag{8}$$

If one multiplies equations (5) by α_1, α_2, α_3, resp., and adds them then when one recalls the meaning of k :

$$k = \frac{1}{\sqrt{1-q^2}},$$

the left-hand side will become:

$$\begin{aligned}\mu k(\alpha_1\ddot{x}+\alpha_2\ddot{y})+\mu k^3\alpha_3\ddot{z} &= \mu k(\alpha_1\ddot{x}+\alpha_2\ddot{y}+\alpha_3\ddot{z})+\mu k(k^2-1)\alpha_3\ddot{z}\\ &= \mu k\ddot{\xi}+\mu k(k^2-1)\alpha_3\frac{d}{dt}\dot{z}\\ &= \mu k\ddot{\xi}+\mu k(k^2-1)\alpha_3\frac{\dot{\xi}}{q}\dot{q}\\ &= \frac{\mu}{\sqrt{1-q^2}}\ddot{\xi}+\frac{\mu q}{\left(\sqrt{1-q^2}\right)^3}\dot{\xi}\dot{q}\\ &= \frac{d}{dt}\left(\frac{\mu\dot{\xi}}{\sqrt{1-q^2}}\right) = \frac{d}{dt}(\mu k\dot{\xi}) .\end{aligned} \tag{9}$$

Finally, if one sets:

$$\alpha_1 X + \alpha_2 Y + \alpha_3 Z = \Xi , \qquad \text{etc.,} \tag{10}$$

and

$$\begin{aligned}\alpha_1 L + \alpha_2 M + \alpha_3 N &= \Lambda ,\\ \beta_1 L + \beta_2 M + \beta_3 N &= \mathrm{M} ,\\ \gamma_1 L + \gamma_2 M + \gamma_3 N &= \mathrm{N}\end{aligned} \tag{11}$$

then one will have:

$$\begin{aligned}L &= \alpha_1 \Lambda + \beta_1 \mathrm{M} + \gamma_1 \mathrm{N} ,\\ M &= \alpha_2 \Lambda + \beta_2 \mathrm{M} + \gamma_2 \mathrm{N} ,\\ N &= \alpha_3 \Lambda + \beta_3 \mathrm{M} + \gamma_3 \mathrm{N} ,\end{aligned} \tag{11.a}$$

and therefore, the right-hand side of the equation that is derived from (5) [due to (11.a), or (7), (8), resp.]:

$$\begin{aligned}e(\alpha_1 X + \alpha_2 Y + \alpha_3 Z) + e\,q\,(-\alpha_1 M + \alpha_2 L)\\ &= e\,\Xi + eq\,[-(\alpha_1\beta_2 - \beta_1\alpha_2)\,M + (\alpha_2\gamma_1 - \gamma_2\alpha_1)\,N]\\ &= e\,\Xi + e\,q\,[\beta_3 N - \gamma_3 M]\\ &= e\,\Xi + e\,(\dot{\eta}N - \dot{\zeta}M) .\end{aligned}$$

One then gets the following equations, which include the ones in (5):

$$\begin{aligned}\frac{d}{dt}\left(\frac{\mu\dot{\xi}}{\sqrt{1-q^2}}\right) &= e[\Xi + (\dot{\eta}\,\mathrm{N} - \dot{\zeta}\,\mathrm{M})] ,\\ \frac{d}{dt}\left(\frac{\mu\dot{\eta}}{\sqrt{1-q^2}}\right) &= e[\mathrm{H} + (\dot{\zeta}\,\Lambda - \dot{\xi}\,\mathrm{N})] ,\\ \frac{d}{dt}\left(\frac{\mu\dot{\zeta}}{\sqrt{1-q^2}}\right) &= e[\mathrm{Z} + (\dot{\xi}\,\mathrm{M} - \dot{\eta}\,\Lambda)] ,\end{aligned} \tag{12}$$

or in vector notation, when $\mathfrak{f}$ is once more the pondermotive force that acts upon an electron of unit charge (art. **55**) and $\mathfrak{v} = (\dot{\xi}, \dot{\eta}, \dot{\zeta})$ is the velocity of the electron relative to the coordinate system at rest (Ξ, H, Z):

$$\frac{d}{dt}\left(\frac{\mu\,\mathfrak{v}}{\sqrt{1-q^2}}\right) = e\,\mathfrak{f}\,. \tag{12.a}$$

§ 57. – The equations of motion when referred to a uniformly-moving system.

M. PLANCK made a far-reaching observation about the foregoing elegant formulation of the differential equations of moving electron. If one replaces the electromotive force $e\,\mathfrak{f}$ with any other accelerating force $\mathfrak{P}$ with the components $\mathfrak{P}_\xi$, $\mathfrak{P}_\eta$, $\mathfrak{P}_\zeta$ *then one can demand that equations* (12) *must also be valid for that force*. When one again introduces the *second as a unit*, so *t*, $\mathfrak{v}$, and *q* are replaced by $t\,c$, $\mathfrak{v} / c$, q / c, resp., they will then take the form:

$$\begin{aligned}
\frac{1}{c}\frac{d}{dt}\left(\frac{\mu\,\dot{\xi}}{\sqrt{c^2-q^2}}\right) &= \mathfrak{P}_\xi\,, \\
\frac{1}{c}\frac{d}{dt}\left(\frac{\mu\,\dot{\eta}}{\sqrt{c^2-q^2}}\right) &= \mathfrak{P}_\eta\,, \\
\frac{1}{c}\frac{d}{dt}\left(\frac{\mu\,\dot{\zeta}}{\sqrt{c^2-q^2}}\right) &= \mathfrak{P}_\zeta\,.
\end{aligned} \tag{1}$$

Vectorially:

$$\frac{1}{c}\frac{d}{dt}\left(\frac{\mu\,\mathfrak{v}}{\sqrt{c^2-q^2}}\right) = \mathfrak{P}\,. \tag{1.a}$$

They are then the equations of motion for the *moving* mass-point μ relative to a *fixed* coordinate system, as long as one assumes that its motion has the *usual* LAGRANGIAN form when viewed from the rest state relative to a coordinate system that *is itself moving*.

Due to the reciprocity that is guaranteed for relative motion by the group property of the transformation, one can, however, also *regard* (1) *as the differential equation of a moving* ($q < c$) *point* μ *relative to a uniformly-moving coordinate system when the ordinary differential equations* $\mu\,\dot{\mathfrak{v}} = \mathfrak{P}$ *are true for them relative to a system with respect to which it is at rest at the initial moment*. They then define *the generalization of the LAGRANGIAN equations for the case of an extremely fast-moving mass-point* and go back to them when one replaces $c^2\,\mathfrak{P}$ with $\mathfrak{P}$ and sets *c*

$= \infty$. τ will then be equal to t. That is because when one introduces the usual unit of time into equations (5) of art. **53**, one will get the formulas:

$$z' = \frac{c\,(z - qt)}{\sqrt{c^2 - q^2}}, \qquad t' = \frac{-q\,z + c^2 t}{c\sqrt{c^2 - q^2}}, \tag{1.b}$$

which go to:

$$z' = z - q\,t\,, \quad t' = t \tag{1.c}$$

for $c = \infty$ ([1]). In fact, the differential equations of classical mechanics are invariant under the transformation (1.c), to the extent that they relate to the reciprocal action of point systems.

We again set $c = 1$ and would now like to construct an integral from the system of equations (1.b) by multiplying the components by:

$$\frac{\mathfrak{v}_x\,dt}{\sqrt{1-q^2}} = \frac{d\xi}{\sqrt{1-q^2}}, \text{ etc.,}$$

resp., and adding them. One will get:

$$\frac{\mu}{2}\,d\left(\frac{\dot{\xi}^2 + \dot{\eta}^2 + \dot{\zeta}^2}{1-q^2}\right) = \frac{1}{\sqrt{1-q^2}}\,(\mathfrak{P}_\xi\,d\xi + \mathfrak{P}_\eta\,d\eta + \mathfrak{P}_\zeta\,d\zeta)\,. \tag{2}$$

The increment on the left is:

$$d\left(\frac{\dot{\xi}^2 + \dot{\eta}^2 + \dot{\zeta}^2}{1-q^2}\right) = d\,\frac{q^2}{1-q^2} = d\left(\frac{q^2}{1-q^2} - 1\right) = d\,\frac{1}{1-q^2}\,. \tag{3}$$

Now, instead of time t, one might now introduce the *proper* (positional) *time* $t' = \tau$ of the origin of the moving system of the point μ that is found at $\xi = \eta = \zeta = 0$ at time t *as an independent variable*. Equation (8) of art. **53** then gives the following relation for $z' = 0$:

$$t = k\,t' = k\tau = \frac{\tau}{\sqrt{1-q^2}},$$

so one also has:

$$\frac{dt}{d\tau} = \frac{1}{\sqrt{1-q^2}}, \tag{4}$$

with which, one has, e.g.:

([1]) One will find a comparison between classical and electromagnetic mechanics from the standpoint of group theory in PH. FRANK, Wiener Akad. Ber. **118** (1909).

$$\frac{\dot{\xi}}{\sqrt{1-q^2}} = \frac{d\xi}{d\tau}. \tag{5}$$

If one introduces (4) into (2) then that will give, with the help of (3):

$$(\mathfrak{P}_\xi\, d\xi + \mathfrak{P}_\eta\, d\eta + \mathfrak{P}_\zeta\, d\zeta)\frac{dt}{d\tau} = \frac{\mu}{2} d\left(\frac{dt}{d\tau}\right)^2 = \mu \frac{dt}{d\tau} d\,\frac{dt}{d\tau}$$

or

$$\mathfrak{P}_\xi\, d\xi + \mathfrak{P}_\eta\, d\eta + \mathfrak{P}_\zeta\, d\zeta = \mu\, d\,\frac{dt}{d\tau}. \tag{6}$$

Thus, the increase in kinetic energy is:

$$\mu\, d\,\frac{dt}{d\tau} = \mu\, d\left[\frac{dt - d\tau}{d\tau}\right],$$

and the kinetic energy itself is equal, up to a constant, *to the product of the mass* μ of the point with the *increment of time t with respect to its proper time* τ.

If one once more regards the *time and space coordinates* t, ξ, η, ζ of the mass-point μ *as equally-justified defining data*, as a consequence of the remark that was made in art. **55**, and accordingly extends equations (1), which take the form:

$$\begin{aligned} \mu \frac{d}{dt}\frac{d\xi}{d\tau} &= \mathfrak{P}_\xi, \\ \mu \frac{d}{dt}\frac{d\eta}{d\tau} &= \mathfrak{P}_\eta, \\ \mu \frac{d}{dt}\frac{d\zeta}{d\tau} &= \mathfrak{P}_\zeta, \end{aligned} \tag{7}$$

with the help of (5), by way of:

$$\mu \frac{d}{dt}\frac{dt}{d\tau} = \mathfrak{P}_t,$$

then [due to (6)] the component $\mathfrak{P}_t$ of the vector $\mathfrak{P}$ in the direction of the t-axis will possess the value:

$$\mathfrak{P}_t = \mathfrak{P}_\xi \frac{d\xi}{dt} + \mathfrak{P}_\eta \frac{d\eta}{dt} + \mathfrak{P}_\zeta \frac{d\zeta}{dt}, \tag{8}$$

while one has:

$$\left(\frac{d\xi}{d\tau}\right)^2 + \left(\frac{d\eta}{d\tau}\right)^2 + \left(\frac{d\zeta}{d\tau}\right)^2 = \frac{q^2}{1-q^2} = \left(\frac{dt}{d\tau}\right)^2 - 1. \tag{9}$$

Therefore, the last equation to be added to the system (7) has *the meaning of the energy equation.*

Equations (7) will take on the known form that is invariant under orthogonal transformations upon multiplying by $dt / d\tau$. First, the vector $\mathfrak{P}$, multiplied by that factor, is appealed to as a force vector in the system that is at rest relative to the mass-point.

When one then takes the viewpoint of the general space-time coordinate system and, with MINKOWSKI, then speaks of the "world postulate," instead of the "relativity postulate," the energy equation will appear to be a *necessary extension of the LAGRANGIAN equations of motion*. Since the kinetic energy changes with the inclination of the time-axis t with respect to the "proper time" τ (as one then says instead of "positional time"), the one energy equation, when formulated for an undetermined axis t, will also subsume the system of three equations of motion, so the entire mechanics of material points [MINKOWSKI, Gött. Nachr. (1908), pp. 108]. Therefore, in a certain sense, energetics, which wishes to build all of mechanics upon the principle of the conservation of energy, is correct: Namely, *when the law of energy keeps its validity under any space-time transformation*. The equations of motion and the law of energy then merge, like current and density, and the electric and magnetic vectors (art. **55**), into a greater whole that subsumes the relations that exist between their components.

We would like to emphasize one more thing about *invariants* under a general space-time transformation. From art. **53**, at the end, the general LORENTZ transformation can be composed of transformations of the form by which we took (x, y, z) to (ξ, η, ζ) above, and ones of the form (5) in art. **53**. Now, since the quantity:

$$\rho'^2(1-\mathfrak{v}_z'^2) = \frac{\rho^2}{1-q^2}[(1 - q\,\mathfrak{v}_z)^2 - (\mathfrak{v}_z - q)^2] = \rho^2(1-\mathfrak{v}_z^2) \tag{10}$$

goes to itself under the latter, $\mathfrak{v}_z^2$ goes to $|\,\mathfrak{v}\,|^2 = \dot{\xi}^2 + \dot{\eta}^2 + \dot{\zeta}^2$ under the former, and at the same time ρ goes to itself, the expression:

$$\rho^2\,(1 - |\,\mathfrak{v}\,|^2) = \rho^2\left(\frac{d\tau}{dt}\right)^2$$

will not change at all under a space-time transformation (MINKOWSKI, *loc. cit.*, pp. 63). Hence, it is not the density itself that is an invariant, but the product of the density and the rate at which proper time τ changes in in term t.

§ 58. – Relationship to the principle of action and reaction.

In conclusion, we would like to relate this sketch to the remarkable fact that LORENTZ's theory contradicts the basic law of the equality of action and reaction in the usual sense, namely, between ponderable masses. In order to gain some insight into the point of departure, we would like to use the foregoing equations to calculate the total force that acts upon the region of space T, inside of which electromagnetic processes play out in ponderable matter.

From art. **55**, that force is represented by the integral:

$$\mathfrak{F} = \int \rho \, \mathfrak{f} \, d\tau \,, \tag{1}$$

which is extended over the space T (only those locations in T where the charge ρ is non-zero contribute to it). Therefore, one has:

$$\mathfrak{f} = \mathfrak{E} + \left[\frac{\mathfrak{v}}{c}, \mathfrak{H} \right] . \tag{2}$$

We restrict ourselves (LORENTZ, *Enzykl.* V 2, pp. 161) to calculating the *X*-component $\mathfrak{F}_x$ of that force:

$$\mathfrak{F}_x = \int \rho \, \mathfrak{f}_x \, d\tau = \int \rho \, X \, d\tau + \int \left(\frac{\rho}{c} \mathfrak{v}_y N - \frac{\rho}{c} \mathfrak{v}_y M \right) d\tau \,, \tag{3}$$

when we enter the values for ρ and $\rho \, \mathfrak{v}$ from (3) and (1) in art. **50**. One next gets by partial integration [art. **24**, (4)]:

$$4\pi \int \rho X \, d\tau = \int X \operatorname{div} \mathfrak{E} \, d\tau = - \int (\mathfrak{E}, \operatorname{grad} X) \, d\tau - \int X \, \mathfrak{E}_n \, d\sigma \,,$$

in which the last integral is extended over the outer surface Σ of the space T. Due to equations (2.a) of art. **50**, one will further have:

$$\begin{aligned} (\mathfrak{E}, \operatorname{grad} X) &= X \frac{\partial X}{\partial x} + Y \frac{\partial X}{\partial y} + Z \frac{\partial X}{\partial z} \\ &= X \frac{\partial X}{\partial x} + Y \left(\frac{\partial Y}{\partial y} + \frac{1}{c} \dot{N} \right) + Z \left(\frac{\partial Z}{\partial x} - \frac{1}{c} \dot{M} \right) \\ &= \frac{1}{2} \frac{\partial}{\partial x} (X^2 + Y^2 + Z^2) + \frac{1}{c} (Y \dot{N} - Z \dot{M}) \,, \end{aligned}$$

so

$$4\pi \int \rho X \, d\tau = - \int X \, \mathfrak{E}_n \, d\tau - \tfrac{1}{2} \int \frac{\partial}{\partial x} (X^2 + Y^2 + Z^2) \, d\tau - \frac{1}{c} \int (Y \dot{N} - Z \dot{M}) \, d\tau \,, \tag{4}$$

in which one has [art. **24**, (2)]:

$$- \int \frac{\partial}{\partial x} (X^2 + Y^2 + Z^2) \, d\tau = \int (X^2 + Y^2 + Z^2) \cos(n, x) \, d\sigma \,, \tag{4.a}$$

when one again introduces *dy dz* = ± $d\sigma$ cos (*n*, *x*) (where *n* is the inward-pointing normal).

On the other hand, equations (1.a), *loc. cit.*, imply that:

$$4\pi\int\left(\frac{\rho}{c}\mathfrak{v}_y\,N-\frac{\rho}{c}\mathfrak{v}_z\,M\right)d\tau = \int\left\{N\left(\frac{\partial L}{\partial z}-\frac{\partial N}{\partial x}\right)-M\left(\frac{\partial M}{\partial x}-\frac{\partial L}{\partial y}\right)\right\}d\tau-\frac{1}{c}\int(\dot{Y}\,N-\dot{Z}\,M)\,d\tau$$

$$= -\int\left(L\frac{\partial L}{\partial x}+M\frac{\partial M}{\partial x}+N\frac{\partial N}{\partial x}\right)d\tau+\int\left(L\frac{\partial L}{\partial x}+M\frac{\partial M}{\partial y}+N\frac{\partial N}{\partial z}\right)d\tau-\frac{1}{c}\int(\dot{Y}\,N-\dot{Z}\,M)\,d\tau\,, \tag{5}$$

where one again has:

$$-\int\left(L\frac{\partial L}{\partial x}+M\frac{\partial M}{\partial x}+N\frac{\partial N}{\partial x}\right)d\tau = -\int(L^2+M^2+N^2)\cos(n,x)\,d\sigma\,, \tag{5.a}$$

and upon partial integration [(4) of art. **24**], one will get:

$$\int\left(L\frac{\partial L}{\partial x}+M\frac{\partial M}{\partial x}+N\frac{\partial N}{\partial x}\right)d\tau$$

$$= -\int L[L\cos(n,x)+M\cos(n,y)+N\cos(n,x)]\,d\sigma-\int\left(\frac{\partial L}{\partial x}+\frac{\partial M}{\partial y}+\frac{\partial N}{\partial z}\right)L\,d\tau$$

$$= -\int L\,\mathfrak{H}_n\,d\sigma\,, \tag{5.b}$$

since:

$$\operatorname{div}\mathfrak{H} = 0\,.$$

If one substitutes (4.a) in (4), (5.a), (5.b) in (5) and adds (4) and (5) then one will get the following expression for the X-component of the force $\mathfrak{F}$ (LORENTZ, *Versuch einer Theorie der elektr. und opt. Erscheinungen*, etc., Leiden, 1895, pp. 26):

$$\mathfrak{F}_x = -\frac{1}{4\pi}\int X\,\mathfrak{E}_n\,d\sigma+\frac{1}{8\pi}\int(X^2+Y^2+Z^2)\cos(n,x)\,d\sigma+\frac{1}{8\pi}\int(L^2+M^2+N^2)\cos(n,x)\,d\sigma$$

$$-\frac{1}{4\pi}\int L\,\mathfrak{H}_n\,d\sigma-\frac{1}{4\pi c}\frac{d}{dt}\int(Y\,N-Z\,M)\,d\tau\,. \tag{6}$$

The force $\mathfrak{F}$ then splits into two parts:

$$\mathfrak{F} = \mathfrak{F}_1+\mathfrak{F}_2\,, \tag{7}$$

namely, an outer surface integral $\mathfrak{F}_1$ and a spatial integral $\mathfrak{F}_2$.

$\mathfrak{F}_1$ has the X-component:

$$\mathfrak{F}_{1x} = -\frac{1}{4\pi}\int[X\,\mathfrak{E}_n\,d\sigma+L\,\mathfrak{H}_n-\tfrac{1}{2}(X^2+Y^2+Z^2)\cos(n,x)-\tfrac{1}{2}(L^2+M^2+N^2)\cos(n,x)]\,d\sigma\,, \tag{8}$$

while $\mathfrak{F}_2$ has the X-component:

$$\mathfrak{F}_{2x} = -\frac{1}{4\pi c}\frac{d}{dt}\int (Y\,N - Z\,M)\,d\tau = -\frac{1}{c^2}\frac{d}{dt}\int \mathfrak{S}_x\,d\tau\,, \tag{9}$$

where $\mathfrak{S}_x$ is the X-component of the previously-introduced [art. **48**, (8)] POYNTING radiation vector:

$$\mathfrak{S} = \frac{c}{4\pi}[\mathfrak{E}, \mathfrak{H}]\,.$$

The force $\mathfrak{F}_1$ can be regarded as an *ether stress* that acts upon both sides of the outer surface element in opposite directions. MAXWELL and HERTZ developed its representation. LORENTZ rejected it. Indeed, he assumed that the ether acted upon matter [by way of the force $\mathfrak{f}$, cf., art. **66**, (5.a)], but not that matter acted upon the ether, which was only empty space for him, and he imagined that it was only the carrier of electromagnetic disturbances. On the other hand, the volume force $\mathfrak{F}_2$ appears under any change in the POYNTING radiation vector, so it will be, e.g., non-zero when an electromagnetic wave passes through space (regardless of whether or not ponderable matter is found there). One sees that $\mathfrak{F}_1$ is also non-zero on the latter case from the fact that when no matter is found in T, one will necessarily have $\mathfrak{F} = 0$, so one must have $\mathfrak{F}_1 = -\mathfrak{F}_2$.

The surface traction $\mathfrak{F}_1$ will then vanish when the space T is extended to an infinite space, while the forces $\mathfrak{E}$, $\mathfrak{H}$ are arranged so that $R\,\mathfrak{E}$ and $R\,\mathfrak{H}$ approach the value zero when $R = \sqrt{a^2 + b^2 + c^2}$ becomes infinitely large. If an electromagnetic perturbation acts in this case on, say, an electron at rest in space, then the force:

$$\mathfrak{F} = \mathfrak{F}_2 = -\frac{1}{c^2}\frac{d}{dt}\int \mathfrak{S}\,d\tau$$

that acts upon the electron *will not* confront any force of reaction, as with, say, the gravitational action between *two* mass-particles, because only one such thing is present, while the expression on the right enters in place of the other. Now, in order to justify the law of the equality of pressure and counter-pressure in a certain sense, one has the expression:

$$\mathfrak{G} = \frac{1}{c^2}\int \mathfrak{S}\,d\tau$$

itself, which is a vector quantity that appears freely in empty space and has the dimension $[l\,m\,t^{-1}]$ of a quantity of motion (art. **51** at the end) – as if it were coupled with a mass – and which is called the *electromagnetic quantity of motion* (of the ether) (H. A. LORENTZ, *loc. cit.*, pp. 162), and it is assigned the role of a reaction to the quantity of motion of the electron.

Now, it is very remarkable that experiments have confirmed the appearance of such a pressure on a body in empty space. P. LEBEDEV [Ann. Phys. (Leipzig) (4) **6** (1901)] has determined the "MAXWELL radiation pressure" that a congruence of rays that is incident upon a metal mirror exerts by measurements made with a radiometer and found that it was in agreement with the result of the theory.

We regard a metal as a substance with infinitely-large conductivity in whose interior the electric field strength $\mathfrak{E}$ is always infinitely small everywhere. As in art. **47**, at the end, we introduce the plane $x = 0$ as the boundary plane of the mirror. Let the plane of incidence of the ray be the plane $z = 0$, and assume that polarized light oscillates *in* the plane of incidence, as it did before. The vectors $\mathfrak{E}_e$, $\mathfrak{H}_e$ [art. **47**, (12), (13)] for the incident wave are then:

$$X_e = A \sin \varphi \sin 2\pi n \left(t - \frac{-x \cos \varphi + y \sin \varphi}{V} \right) ,$$

$$Y_e = A \cos \varphi \sin 2\pi n \left(t - \frac{-x \cos \varphi + y \sin \varphi}{V} \right) ,$$

$$Z_e = 0 ,$$

$$L_e = M_e = 0 ,$$

$$N_e = A \sin 2\pi n \left(t - \frac{-x \cos \varphi + y \sin \varphi}{V} \right) ,$$

since $\varepsilon = \mu = 1$ for the ether. V (= c) is the speed of light. For the reflected ray, one has [art. **44**, (4.a)]:

$$X_r = \quad A_r \sin \varphi \sin 2\pi n \left(t - \frac{x \cos \varphi + y \sin \varphi}{V} \right) ,$$

$$Y_r = - A_r \cos \varphi \sin 2\pi n \left(t - \frac{x \cos \varphi + y \sin \varphi}{V} \right) ,$$

$$Z_r = 0 ,$$

and for the refracted ray:

$$X_b = Y_b = Z_b = 0 ,$$

because one has $\mathfrak{E} = 0$ in the interior of the conductor.

Furthermore, one has $A_r = A$, since [art. **44**, (5)] the components of $\mathfrak{E}$ that fall in the plane of separation $Y_e + Y_r$, $Z_e + Z_r$ must be zero, since $\mathfrak{E}$ is zero in a conductor.

Furthermore, the components L_r, M_r, N_r of the magnetic field strength $\mathfrak{H}_r$ for the reflected way are obtained from formulas (13) of art. **47**:

$$L_r = M_r = 0\,, \qquad N_r = A \sin 2\pi n \left(t - \frac{x\cos\varphi + y\sin\varphi}{V} \right) .$$

In particular, if the ray falls perpendicular to the mirror, so $\varphi = 0$, then one will have:

$$\begin{aligned}
X_e &= Z_e = 0\,, & Y_e &= A \sin 2\pi n \left(t + \frac{x}{V} \right) , \\
X_r &= Z_r = 0\,, & Y_r &= -A \sin 2\pi n \left(t - \frac{x}{V} \right) , \\
L_e &= M_e = 0\,, & N_e &= A \sin 2\pi n \left(t + \frac{x}{V} \right) , \\
L_r &= M_r = 0\,, & N_r &= A \sin 2\pi n \left(t - \frac{x}{V} \right) .
\end{aligned}$$

If one now substitutes the values $\mathfrak{E} = \mathfrak{E}_e + \mathfrak{E}_r$, $\mathfrak{H} = \mathfrak{H}_e + \mathfrak{H}_r$, namely:

$$\begin{aligned}
X &= 0, \quad Z = 0, \quad Y = A\left\{ \sin 2\pi n \left(t + \frac{x}{V} \right) - \sin 2\pi n \left(t - \frac{x}{V} \right) \right\}, \\
L &= 0, \quad M = 0, \quad N = A\left\{ \sin 2\pi n \left(t + \frac{x}{V} \right) + \sin 2\pi n \left(t - \frac{x}{V} \right) \right\},
\end{aligned}$$

in the expression for the component $\mathfrak{f}_{1x}$ of the surface traction $\mathfrak{f}_1$ that is under the integral sign in the formula (8) for $\mathfrak{F}_{1x}$, so the pressure per unit area, and likewise in the expressions for $\mathfrak{f}_{1y}$, $\mathfrak{f}_{1z}$, then when one recalls that for $x = 0$, one has:

$$\mathfrak{E}_n = X = 0\,, \qquad \mathfrak{H}_n = L = 0\,,$$

and furthermore that $\cos(n, y) = 0$, $\cos(n, z) = 0$, $\cos(n, x) = 1$, as well as the fact that one also has $Y = 0$ for $x = 0$, then one will get:

$$\begin{aligned}
\mathfrak{f}_{1x} &= -\frac{1}{4\pi}[X\,\mathfrak{E}_n + L\,\mathfrak{H}_n - \tfrac{1}{2}(X^2 + Y^2 + Z^2)\cos(n,x) - \tfrac{1}{2}(L^2 + M^2 + N^2)\cos(n,x)] \\
&= \frac{1}{8\pi} N^2 = \frac{1}{2\pi} A^2 \sin^2 2\pi n t\,,
\end{aligned}$$

$$\mathfrak{f}_{1y} = \mathfrak{f}_{1z} = 0\,.$$

Therefore, the pressure $\mathfrak{f}_1$ per unit area is:

$$\mathfrak{f}_1 = \frac{1}{2\pi} A^2 \sin^2 2\pi n t ,$$

and the mean value of the pressure for waves of the oscillation number n will be:

$$n \int_0^{1/n} \mathfrak{f}_1 \, dt = \frac{A^2}{4\pi} .$$

That relationship can be tested experimentally; one finds that it is confirmed. Hence, in this way of looking at things, it is also necessary that in the experiment one must assume a quantity of motion in empty space that is neither bound to ponderable matter nor to an electron, so one must once more impose the idea of a *current* of electromagnetic *energy* that starts from the location of the vibration and flows through empty space. W. RITZ developed that picture, which differs from NEWTON's emanation theory, which is believed to have long since been surpassed, only in regard to moving matter, into a theory ["Recherches critiques sur l'électrodynamique générale," Ann. Chim. Phys. (8) **13** (1908)] that was based upon the criticism that POINCARÉ made in the second edition of his *Électricité et Optique* (Paris 1901, IV partie) of the theories up to that point in time and posed the problem of completely eliminating the concept of ether based upon the concept of the electromagnetic quantity of motion and replacing its effect with the ejection of energy from the center of electromagnetic perturbation. Moreover, the LORENTZ-EINSTEIN theory that was presented in the foregoing (when one overlooks the problem of establishing the fundamental equations) does not make use of any specific properties of the ether at any point.

A. H. BUCHERER [Phys. Zeit. 9 Jahrg. (1908), pp. 755, Ann. Phys. (Leipzig) (4) **28**, pp. 513] has recently found that the change in mass that follows from the principle of relativity for a rapidly-moving (more than half the speed of light) electron was confirmed by experiments with Becquerel rays. So did E. HUPKA ["Die träge Masse bewegter Elektronen," Ber. d. Deutschen Phys. Ges. Berlin (1909), pp. 249].