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On Hertz's mechanics ([1])

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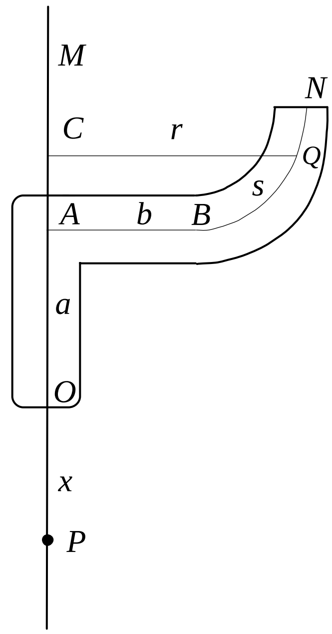
When in the year 1894, right after the death of the author, the book *Die Prinzipien der Mechanik* appeared by Heinrich Hertz, with a Foreword by H. von Helmholtz, physicists and mathematicians were gripped with the same enthusiasm for the book, which promised to be a brilliantly-written introduction to a completely new representation of mechanics that was envisioned and written in the spirit of mathematics. It was a mechanics that was built upon only one axiom and explained the controversy regarding the older concept of force in the spirit of the modern physics. However, many readers soon put the book down again, since it was precisely the fact that the new conceptual structures that were contained in it were founded upon physical origins, and were therefore hard to approach for the mathematicians, a fact that was made even more acute by the lack of examples.

Indeed, the difficulty that Helmholtz referred to more precisely in his Foreword still exists to this day that: "One must call upon a great degree of scientific imagination in order to explain even the simplest cases of physical forces in the sense of Hertz." In the meantime, however, the theory that Hertz might have originally had in mind when he presented the concept of "hidden mass" for the propagation of electrical and magnetic force effects through space has taken on more meaning and respectability in a broader context. In addition, many of the new concepts have already proved to be fruitful in other domains. That probably justifies the attempt to draw upon that theory (albeit one that is not free of contradictions) as an example for explaining Hertz's basic concept, and (as I have already tried to do in a lecture in the Winter of 1898-99) proceed from that example to the essence and concepts of the new mechanics without leaving the basis for the old. However, the book itself, whose study is indeed complicated by the peculiar form of the prose that often interrupts the train of thought, is most appealing for the rich, carefully-structured content and language that was chosen, so it might gain a few new friends from the following discussion.

Hertz (no. **469**) and Helmholtz (Foreword, pp. X, XX, etc.) used the terms *force-at-a-distance* and *action-at-a-distance* for the force effects between mass points that have *different* coordinates, and in particular for the force of attraction between gravitating or

([1]) The following is a transcription of an address that the author gave to the Plochinger Versammlung of the Society on 14 May 1899.

magnetic, electric, etc. masses. Hertz wished to eliminate such forces at a distance by introducing rigid (visible or hidden) constraints between the points. An example might show how one can understand that.



The apparatus that is drawn above, which is slightly altered from one that Boltzmann gave, allows the motion of a mass-point P, without the action of external forces, to be arranged such that it seems to move along the line AOP under a prescribed law of attraction (e.g., Newton's) with O as its center. The thin tube $OABQN$, a piece of which BQN is curved in a well-defined way, rotates about the axis OAM. An inextensible string that connects the material point Q with P runs through the tube, such that when $PO = x$, $OA = a$, $AB = b$, and BQ has arc-length s, the length of the string will be $l = x + a + b + s$. The tube is put into a rotational motion and left to itself. One must determine the rectilinear motion of P when one ignores friction and assumes that the apparatus is massless.

The *vis viva* principle and the law of areas imply that:

$$(P + Q)\left(\frac{ds}{dt}\right)^2 + Q\,r^2\,\omega^2 = h, \qquad r^2\,\omega = c,$$

in which r is the distance QC from the rotational axis, ω is the angular velocity, and h and c are constants.

Upon differentiating:

$$(P + Q)\left(\frac{ds}{dt}\right)^2 + \frac{Q\,c^2}{r^2} = h$$

with respect to time, one will get:

$$(P + Q)\frac{d^2 s}{dt^2} = \frac{Q\,c^2}{r^3}\frac{dr}{ds},$$

or:

$$(P + Q)\frac{d^2 x}{dt^2} = \frac{Q c^2}{r^3}\frac{dr}{dx}.$$

Now, if say $f(x)$ is the law of the force that seems to act upon P then one must set:

$$\frac{Q c^2}{r^3}\frac{dr}{dx} = -f(x)\frac{P+Q}{P},$$

so

$$\frac{1}{r^2} = -k\int f(\alpha - s)\,ds,$$

in which k, α are positive constants. r is expressed in terms of s in that way.

If z is the abscissa of the point Q, so:

$$ds^2 = dz^2 + dr^2,$$

then a second quadrature will also give z as a function of s, and with that the equation of the curve along which the tube BQN is bent in order to realize the prescribed motion $f(x)$ of P. The assumption that $f(x) = k / x^2$ will yield a hyperelliptic integral for z.

If one now imagines that the apparatus is invisible, except for the mass P, then the *rigidity constraint* that the string exhibits will effect that apparent force at a distance.

Certain curvilinear motions of a point that seem to result from the influence of forces, such as curves that roll on each other, can also be realized. I shall not go into that here.

The motion of a sphere that collides with another sphere and rebounds elastically can be represented by an apparatus that was described by Boltzmann and the author in the Jahresberichten der deutschen Mathematiker-Vereiningung for 1898 and 1899.

Moreover, Hertz's dynamical explanation for forces at a distance is not to be found in rigidity constraints of the type that was considered above, because the hidden motions that he assumed are not the motions of rods or other discrete masses, but they have a *cyclic* nature (no. **599**) and return to themselves in such a way that in place of each advancing mass-point, an equal one will enter immediately. One might do better to imagine the top that Helmholtz referred to when he spoke of forces that would be evoked by cyclic motion (J. f. Math. **100**, pp. 154; also his *Vorles. über Dynamik*, pub. by Krigar-Menzel, pp. 321). However, the Introduction to Hertz's *Mechanik* (pp. 31) suggests that one must take yet another step. In order to make the following attempt at an explanation understandable, permit me to first recall some things that are known.

A system of gravitating masses that are distributed in space in any way exerts a force on a unit mass whose direction and magnitude are known. If one follows the direction of the force that issues from the point to a neighboring point and then makes the same construction there, and proceeds similarly then one will obtain a *force-line* whose behavior, namely, in the vicinity of the attracting mass will be determined by potential theory. One can think of the distribution of such force-lines, which run through all of space, as being defined such that the measure of the intensity of the force at each location will be determined by the density of the lines. Electrical force-lines belong to electrical masses in an analogous way, and magnetic ones belong to magnetic masses. The latter, which one can, as is known, make visible in the neighborhood of a magnetic pole by iron filings, run through the magnetic masses and closes upon themselves ([1]). However, magnetic force-lines also fill up the neighborhood (i.e., the *field*) of *moving* electrical masses, and in particular, the field of an electric current that closed upon itself in the form of a ring, as one call also show with iron filings. Furthermore, the arrangement of these force-lines that are generated by a current exhibits an essential difference from the ones that are produced by magnetic masses. The latter can be contracted to a point, in the sense of *analysis situs*. However, for the former, the current (which one might think of as a tube that closes on itself) acts like a point of discontinuity for a function in the Gaussian plane when one takes its integral along a line that encloses the point: It makes space multiply-connected, and the potential becomes a multi-valued function of position.

Now, the new idea is that this system of force-lines, or rather their action along them, can be produced by the motion of a hidden mass *that fills all of space*, which one must imagine to be something that is intrinsic to the propagation of the outer surface of the mass (or a cavity in it).

The admissibility of that assumption might be shown by the example of the electromagnetic force-field that we would like to consider in what follows (but restricted to a dielectric that is filled with the invisible matter of "empty" space).

Since the time of Faraday ([2]), one of most distinguished problems in mathematical physics has been to replace the force-lines of the fields of gravitating, magnetic, etc., masses with identical *vector fields* ([3]) of a different type that define perhaps the displacements or velocities or stresses in a continuous mass (such as elastic or fluid bodies), whereby the potential energy of the force at a distance will go to the potential or kinetic energy of the medium.

As far as it concerns vectors of the sort that might come into question when they are applied to the outer surface of an *elastic* medium, one know that one can represent the pressures (i.e., stresses) that act upon a surface element that goes through an interior point by the radii of an ellipsoid, namely, the elasticity ellipsoid. Its *axes* at any point will yield *three* distinguished directions, so not a vector that would point in *one* direction. By contrast, displacements are vectors. In fact, **W. Thomson** (Lord Kelvin) represented the *electric* force-field in that way in his treatise "On a mechanical representation, etc." (1847, *Papers* I, pp. 76), whereas the magnetic one found a less intuitive interpretation.

Any small change of position – namely, a small cube – in the interior of an elastic body can be composed of:

1. A parallel displacement.
2. A rotation around an axis through the center.

([1]) See the beautiful Tables in Herger, Leipzig, 1844.

([2]) Faraday's ground-breaking work appeared in German in Ostwald's Klassiker-Bibliothek der exakt. Wiss; *ibidem*, see also the relevant work of Maxwell, translated into German by Boltzmann.

([3]) As is known, a vector is a spatial magnitude that is endowed with a direction.

The position of that rotational axis, along with the magnitude of the angle of rotation, for each point that is present will, in turn, give a vector field that was used to advantage by Thomson for the magnetic (electromagnetic, resp.) force field, which is an assumption that bears upon the behavior of the calculations for the boundaries of conductors and non-conductors, in particular. Although one might find it hard to imagine that a medium would not resist translation, but probably rotation, such as the magnetic force would experience under the assumption above, Lord Kelvin made that apparent by an ingenious apparatus that exists in a context where numerous rapidly-rotating tops have been placed.

Of course, a medium of that kind is no longer elastic, in the usual sense of the word. It has hidden cyclic motions, and for that reason, in order to distinguish it from the latter, the discoverer called it a "jelly' that was given the name of "ether." In essence, that "ether" comes down to the fluid that Maxwell considered, which will be discussed shortly. In a more recent paper (*Papers* III, pp. 436), Lord Kelvin returned to such matters and showed that when one calculates the energy stored inside and outside of a solenoid, on the one hand, for the elastic forces in the "jelly" and on the other, for the rotational forces in the "ether" that the latter will have the advantage.

He was associated with other researchers, such as Boltzmann and Heaviside. By contrast, Sommerfeld and Reiff postulated that the magnetic force would be assigned to a displacement of the ether-particle, while the *electric* one would be assigned to a *rotation*, because it is best for one to perform the actual calculations in such a way that the energy of the electric current in conductors will be converted into heat by friction (which opposes the rotation), which is a theory that Boltzmann had even more misgivings about.

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The experiments that were sketched out up to now, despite many gaps in the details, all refer to the fact that the electromagnetic force field can be represented by a vector field of *fluid* type. The fact that it must be incompressible was shown, *inter alia*, by Hertz's experiment on the propagation of the electric waves, which refer to absolutely transversal oscillations. Already in his 1858 treatise on vortex motions (Ostwald's Klassiker), Helmholtz had emphasized the analogy that exists between a line vortex in a fluid and an electric current, namely, the force that the vortex exerts on a particle in the surrounding mass of water is analogous to the force that a current exerts on a magnetic pole outside of it. Just as the latter moves perpendicular to the (rectilinearly-envisioned) conductor, so does a reaction on the water particle in the vicinity emanate from the outer surface of the line vortex, which one imagines to be closed, as one does for the current, and the one reaction is equal in magnitude and direction to the other.

However, whereas Helmholtz did not pursue that two-sided analogy any further, **Maxwell** arrived at a theory of the mutual dependency of electric and magnetic effects on the grounds of closely-related arguments that have been developed even further in a series of works, and today that theory defines the undisputed foundation for the entire study of electricity in the form of the "electromagnetic theory of light." Like Helmholtz, Maxwell (1861, 63, "Physikalische Kraftlinien," German trans. by Boltzmann, Ostwald's Klassiker) thought that the magneto-electric field was a fluid that was permeated by vortices, but magnetic force lines were arranged around the electric current, instead of line vortices, and the force itself prevailed along the vortex axis. Indeed, it is known that the pressure at a point in a fluid is the same in all directions, but on a surface element whose order of magnitude is that of a line vortex cross-section, the pressure in the direction of the axis will be smaller than it is on one that is perpendicular to it, such that there will be a suction in that direction. If one does not follow Maxwell in assuming that there are "particles of friction" between the vortices and if one overlooks the complication that arises for densely-packed rotating vortices with the same direction, or if one goes along with the hypotheses of the younger English physicists (cf., e.g., Lodge, *Electricity*) then one must once more represent the electric force by the displacement of particles (von Helmholtz, *Vorlesungen über elektromagn. Lichttheorie*, pub. by König and Runge, pp. 37).

Now, on the basis of those assumptions, one can exhibit a system of six differential equations (initially for the dielectric), by means of which one can conclude the spatial distribution of the magnetic force at a *given* moment from the change that the electric force at that location will experience at the *next* moment, and conversely, one can get the spatial change in the electric field from the temporal change in the magnetic field ([1]).

Maxwell's equations express a far-reaching duality between electric and magnetic forces. One derives an expression for the (combined electric and magnetic) energy in a spatial part of the dielectric from the,. Moreover, they also imply (as Helmholtz has also derived from *his own* assumption) the known laws of action at a distance with no further assumptions, and in particular, the Biot-Savart law for the action of a current element on a magnetic pole, under which, an increase or decrease in the kinetic energy of the medium will enter in place of the equally-large change in potential of the force-at-a-distance.

Later on, Maxwell derived his equations from other foundations. However, the representation that was suggested first here is especially worthwhile as an example of Hertz's mechanics, because it illustrates the cyclic motion of the intervening medium by means of vortices.

If none of the attempts to explain the electromagnetic force-at-a-distance in a dynamical way by means of motion are also unimpeachable, nonetheless, they collectively give a picture of what Hertz meant when he posed the problem in his mechanics (no. **596**) of "determining the motions of the visible masses in a system in advance, despite the ignorance that prevails in regard to the positions of the hidden masses," and when he consigned the future of the problem (Intro., pp. 49) to "reducing the alleged effect of forces at a distance to processes of motion in a medium that fills up space and whose smallest parts are subject to rigidity constraints [see below]."

Furthermore, in the foregoing, one must accept the reasons that Hertz gave for endowing his hidden masses and motions with the following properties:

([1]) One can find the details in the report that Planck presented to the Vers. des D. Math. Ver. in Düsseldorf in 1898.

1. The motion of the hidden mass is cyclic, and indeed the velocity of that cyclic motion is considerably larger than the change in the acyclic coordinates (viz., the "parameters"), or more precisely, it is so large that the terms in the expression for the energy of the cyclic system that include the rate of change of the parameters can be dropped in comparison to the ones that include cyclic velocities (**549**). The cyclic coordinates themselves do not enter into the expression for the energy at all.

2. The cyclic motion of the hidden masses is "adiabatic," insofar as it deals with the representation of conservative forces (i.e., ones that possess a force function); i.e., "a free-willed direct influence [of forces] on the cyclic coordinates is excluded" (**600**, **562**), such as, e.g., the rotational speed of a gyroscope that has been turned on can no longer be influenced directly afterwards. If follows from the property that was given (as one sees perhaps by appealing to the second form of Lagrange's differential equations) that the cyclic momentum will always keep the same value for an adiabatic motion.

3. What ordinary mechanics calls potential energy is nothing but the kinetic energy of hidden masses (**605**).

Helmholtz had already introduced the concepts of "cyclic" and "hidden motion," and right from the beginning, he had not merely top-like, vorticial motions in mind, but also ones of the type that one assumes in the theory of heat in a gas; hence, densely-packed colliding elastic molecules that zip through space irregularly at a detailed level, and which first suggest the earlier definition in their totality. For that reason, the theory of heat also yields examples of Hertz's mechanics (cf., Helmholtz, J. f. Math. **97**, pp. 111; *ibid*., **100**, pp. 147).

The focal point of Hertz's mechanics is the introduction of hidden masses in place of forces at a distance; the other innovations first flow out of that. The removal of the concept of force from the elements then changes the axioms; their reformulation implies new conceptual structures. The problem of the first book is to prepare and introduce them, which is entirely independent of the new theory, and can be added as an autonomous appendix to any other mechanics.

The fact that this view excludes everything strange from the book shall now be shown.

Above all, the concept of a *free system* must be imagined in such a way that everywhere the old mechanics assumes potential energy, the hidden masses that produce it in kinetic form must appear as a necessary component of the free system. Thus, Hertz required of a free system that the "connections" between its points should be "legitimate"; i.e., independent of time (**119**, **122**). Those "connections" will be given (**124**) by a system of equations in the coordinates of the mass-points into which their first differentials need to enter only linearly and homogeneously. However, one cannot express (e.g., the connection that is established between two gravitating mass-points by perhaps Newton's law) by equations of *that* type. For that reason, two such points do not define a "free system" in the sense of numbers **122**, **309**, etc., by themselves, but only when they are coupled by hidden masses (that are left undetermined).

Now, as far as the *axioms* of the new mechanics are concerned, Newton's third axiom, which demands the equality of action and reaction, becomes unnecessary, in any case. Indeed, Hertz did not by any means relinquish the concept of "force," since he always sought the connection with the usual representation of mechanics. However, force appeared to him only as the reciprocal influence of two "coupled" (i.e., continually-contacting) systems that pertained to just the location where the contact took place. Both of them together define a free system, and if one drops – say – the second one then its effect on the motion of the (now not free) first one can be replaced with certain terms with multipliers that appear in the equations of motion for the latter, which will take on the meaning of a force in the sense above in that way. The converse will be likewise true when one considers the motion of only the second one. The fact that force and counter-force must be equal when they are regarded in that way can be easily proved by combining both systems (**468**), so it is no longer an axiom.

The content of Newton's second axiom, which states the proportionality of force and acceleration, is a necessary consequence of the aforementioned formal definition of the force as a multiplier (**459**).

However, the axiom of inertia remains intact, and ultimately one more integral principle or its equivalent will be necessary for it. Hertz chose Gauss's principle of least constraint, which he combined with the law of inertia into his *fundamental law* (**309**).

One must now give that fundamental law a conception that is concise, as well as easy to understand. In order to do that, in addition to the aforementioned extension of the concept of free system, one also needs a convenient formulation of the concept of *constraint*. To that end, Hertz introduced some new terminology. Namely, he adapted certain concepts that would be common to the individual mass-points, such as path element, velocity, acceleration, and the curvature of the path to a system of such things, in which he referred to the mean values of the magnitudes and directions in question by those words, as he did in numbers **55**, **265**, **275**. In particular, the *path element ds* of a system of points $m_1, m_2, \ldots, m_n$ is defined by:

$$ds^2 \cdot \sum m_i = \sum m_i \, ds_i^2 ,$$

and the expression for the *curvature* of the path, which is the reciprocal value of the radius of principle curvature for the individual point, is defined by a quantity c, which is defined by the equation (**106**):

$$c^2 \cdot \sum m_i = \sum m_i \, (x_i''^2 + y_i''^2 + z_i''^2) ,$$

if x'', y'', z'' are the second differential quotients of the rectangular coordinates of a point with respect to the path-element of the system.

If connections between the system-points exist – i.e., if they are linked by condition equations in the coordinates – then a free system that is left to itself under the influence of those connections and its initial velocities will adopt a certain natural path. The principle of least constraint (in the absence of external forces) then says that the curvature *c* will have a smaller value for the natural path than it will for any other conceivable one that is compatible with the conditions on the system, or that it is a *straightest path*, in Hertz's terminology. The straightest path is usually, but not always, the shortest one. Namely, they will differ in the case of *non-holonomic* systems; i.e. a system for which non-integrable differential equations will enter into the condition equations (no. **132**; moreover, cf. Voss, Math. Ann. **25**, pp. 258, where such condition equations were treated previously). A sphere that rolls on a surface serves as an example of a non-holonomic system (Hölder, "Über die Prinzipien, etc.," Gött. Nach. 1896, pp. 150), which obviously does not generally describe a shortest path when it rolls from one position to another.

From what was said, the idea behind Hertz's fundamental law will now become understandable" "Any free system will remain in its state of rest or a state of uniform motion along a straightest path."

The law of the *conservation of* (kinetic) *energy* follows immediately from that fundamental law for a free system. If one considers a "conservative" system (**602**), in particular – i.e., one that is composed of two coupled subsystems (450), one of which contains all visible masses, while the other contains all hidden masses (with adiabatic cycles) – then the "parameters" (i.e., non-cyclic coordinates) of the *hidden* subsystem of the system (which vary slowly compared to its cyclic coordinates) are likewise coordinates of the visible ones. Now, the energy of the system (605) splits onto the energy of the visible masses and the energy of the hidden masses. Indeed, as was remarked above (pp. 7), the rates of change of the parameters will vanish in comparison to those of the cyclic coordinates in the expression for the energy of the hidden system. However, insofar as the parameters of the visible system are concerned, their velocities do not, in turn, vanish from the total energy. In order to resolve that contradiction, one might perhaps make the assumption that the hidden masses that come under consideration are very small compared to the visible masses.

It follows further from the fundamental law that the *time integral of the energy* is a minimum for the natural motion (**358**). If one defines it for a conservative system then if ($-U$) is the energy of the hidden masses and T is that of the visible ones then that condition can be written in the form (**626**):

$$\delta_{\mathrm{p}}\int(T-U)\,dt=0,$$

in which the variation δ_{p} refers to all coordinates, namely, the visible ones (which also appear as parameters in U), as well as the hidden ones. If one introduces the momenta of the hidden coordinates here in place of their velocities, which can be arranged by a well-known process (cf., say, **Jacobi**, *Vorlesungen über Dynamik*, ed. by Clebsch, Lecture 9, pp. 69), then the equation will take on the form:

$$\delta_{\mathrm{p}} \int (T - U)\, dt = 0,$$

in which one must now perform the variation as if the cyclic momenta were not varied at all, because the motion is adiabatic, by assumption (see above, pp. 7). However, that is just the assumption under which ordinary mechanics (which does not know about hidden masses) varies the integral, such that equation above will now represent *Hamilton's principle* (**628**, **629**).

The connection to ordinary mechanics is achieved with that, and the trivial examples, such as perhaps a falling stone, that can be addressed directly by Hertz's method, and which will present the difficulty that one does not know the type of coupling between visible and invisible masses and the motion of the latter, will lead back to Lagrange's equations. The aforementioned formal conversion of the variation of the integral will necessitate some preparations that fill up numerous numbers (such as **593**, **555**, **493**, **292**, **68**).

In conclusion, let us say a few words about the *condition equations* that define the connection between the system points. When Hertz demanded that they should contain only the coordinates of a point and its first differentials, he seemed to exclude continuous masses from the treatment from the outset, and yet, one must think of the hidden masses as being the matter that fills up space, which is already due to their cyclic motion. In that way, if one recalls the analogy with fluid vortices, then one must admit the condition equation that expressed incompressibility, or "the rigidity constraint of the smallest part" (Intro., pp. 49). **Lagrange** represented it with the help of the partial differential quotients of the coordinates x, y, z of a point in the fluid with respect to their initial values a, b, c, namely, by the equation:

$$\frac{d(x, y, z)}{d(a, b, c)} = 1. \tag{1}$$

However, one can give that equation the form:

$$\frac{1}{da \cdot db \cdot dc} \begin{vmatrix} x & y & z & 1 \\ x + \frac{\partial x}{\partial a} da & y + \frac{\partial y}{\partial a} da & z + \frac{\partial z}{\partial a} da & 1 \\ x + \frac{\partial x}{\partial b} db & y + \frac{\partial y}{\partial b} db & z + \frac{\partial z}{\partial b} db & 1 \\ x + \frac{\partial x}{\partial c} dc & y + \frac{\partial y}{\partial c} dc & z + \frac{\partial z}{\partial c} dc & 1 \end{vmatrix} = 1, \tag{2}$$

in which the expression on the left includes only the coordinates of the four (infinitely-close) points that the mass-particles will assume, which will define the vertices (*a*, *b*, *c*); (*a* + *da*, *b*, *c*); (*a*, *b* + *db*, *c*), (*a*, *b*, *c* + *dc*) of a tetrahedron, as well as the three sides *da*, *db*, *dc* of that tetrahedron, such that equation will no longer have the form that Hertz admitted.

One can make the transition from equation (1) to the more useful condition for incompressibility:

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z} = 0$$

in the way that was indicated in, say, Kirchhoff's *Mechanik*, Lect. 10, § 5, (pp. 107). That equation can also be just as well brought into the form of condition equation in the coordinates of neighboring points that is analogous to (2).

A continuation of Hertz's mechanics in the direction of introducing a space-filling mass instead of the discrete mass-points seems entirely possible, and for the sake of completeness, necessary, which is a step that even Hertz himself seemed to have had in mind (cf., no. **7**)([1]).

Tübingen, 13 October 1899.

([1]) As I saw later, the conception of "hidden masses" that was presented here differs essentially from the one that Boltzmann maintained in his speech to the Münchener Naturforscher-Versammlung in 1899. I have expounded in detail upon the basis for allowing me to persist in my opinion in a report to the Deutschen Mathematiker-Vereinigung in 1899.