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On the theory of moving dislocations

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Differential equations are derived for a continuous distribution of moving dislocations that describe the total displacement field and the elastic strain. These equations are based on the linear theory of elasticity and can be solved by the use of Green's dyadics. The Lagrange function for a moving dislocation is given such that the equation of motion for the propagation of plastic distortions can be obtained by an application of the principle of least action. All relations for the dislocation dynamics take on a formal similarity to the known relations of the statics of dislocations under a four-dimensional formulation of the problem.

1. Introduction

The properties of dislocations at rest have been examined very thoroughly up to now. Different procedures [**1**, **2**, **3**, **4**] have been given in the linear, as well as in the nonlinear, approximation for the theory of elasticity for ascertaining the internal stress state that is generated by a dislocation. By comparison, moving dislocations have not been examined much at all up to now, although they have great significance in various physical phenomena.

The first investigations of moving dislocations go back to FRENKEL and KONTOROVA [**5**, **6**, **7**, **8**, **9**], who could show with a one-dimensional dislocation model that the equation of motion could be brought into a form that was similar to that of a mass-point in the special theory of relativity. As in special relativity, a limiting velocity appears in that theory – viz., the speed of propagation of sound waves – and the measurements in the direction of motion will experience a Lorentz contraction. FRANK [**10**] also found the same behavior in the example of a moving straight screw dislocation in three dimensions. A complication arises for a moving edge dislocation due to the fact that the strain tensor includes a compression part, as well as a shear part. According to Eshelby [**11**], both parts experience a Lorentz contraction, in such a way that in one case the speed of sound for compression waves and in the other case, that of shear waves, takes on the role of limiting velocity. On the basis of the analogy between elastic and electromagnetic fields, as well as between dislocation density and electrical current density, HOLLÄNDER [**12**, **13**, **14**] developed a theory of moving dislocations that was four-dimensionally covariant, like Maxwell's theory. For the case of an isotropic "vacuum" – i.e., for an elastic medium with vanishing Poisson transverse contraction number ([1]) – the propagation of the strain that is generated by a moving distortion is given by just the speed of the shear wave. For that reason, as with electromagnetic waves, it is possible calculate the stress tensor from a potential that is itself determined by an inhomogeneous wave equation. The inhomogeneous part of that wave equation is the dislocation density tensor, which is a generalization of the dislocation density and velocity of the plastic distortion tensor. The procedure that HOLLÄNDER developed is applicable to a real crystal only with restrictions, since it offers no prescription for how the dilatation of the strain state should be calculated. It is therefore suitable to only a state of strain that is free of compression.

KOSEVICH [**15**, **16**] also derived fundamental equations for moving dislocations. In order to solve them, HOLLÄNDER introduced potential functions that are determined by a system of inhomogeneous wave equations. That system is linked with additional auxiliary conditions that are imposed upon the potential functions. The foregoing is problematic, since the plastic deformations are subject to additional (restricting) requirements.

MURA [**17**, **18**] solved the problem of moving dislocations in a completely different way. The elastic medium with dislocations is converted into a multiply-connected region in which the compatibility conditions are fulfilled everywhere by means of suitable cuts. The total displacements that are produced by dislocations can be expressed in terms of outer surface integrals of the plastic displacements over the separating surfaces by applying one of Green's methods. One can go from the expression that is obtained in that way to the case of continuously-distributed, moving dislocations most conveniently by passing to the limit.

Although no objections can be made against the procedure that MURA developed, in what follows, we would like to give a new path to solution that has the advantage that it is relatively simple and closely linked with the static treatment. In addition, we would like to derive a four-dimensional formulation of the problem that deviates from the one that HOLLÄNDER gave at several places. Our procedure allows one to ascertain the internal stress state that is produced by the moving dislocation by means of stress functions, and for that reason, it can also be employed in the nonlinear treatment.

2. The total displacement field of a moving dislocation

The essential difference between the statics and dynamics of dislocations consists of the fact that in the case of moving dislocations, an inertial force of magnitude:

$$-\rho \ddot{s}_i^G$$

will appear in every volume element, in which s_i^G denotes the i^{th} component of the total displacement field. ρ is the instantaneous density of the elastic medium. If no other volume forces are otherwise present then a volume element will be in equilibrium when one has ([2]):

([1]) In this case, Hooke's law will assume the simple form $\sigma_{ij} = 2\, G\, \varepsilon_{ij}$, where G is the shear modulus.
([2]) Indices that occur twice are summed over. We restrict ourselves to Cartesian coordinate systems.

$$\partial_k \, \sigma_{ik} - \rho \, \ddot{s}_i^{\,G} = 0. \tag{2.1}$$

The divergence of the stress tensor σ is referred to the instantaneous state of strain in that equation. div σ and also the density ρ can be referred to the initial state in the vicinity of the linear theory of elasticity, to which we would like to confine ourselves in what follows.

The stress tensor σ can be expressed in terms of the elastic strain tensor ε by way of Hooke's law:

$$\sigma_{ik} = C_{ijkl} \, \varepsilon_{kl} \equiv C_{ijkl} \, \beta_{kl} \, . \tag{2.2}$$

C_{ijkl} is the Hooke tensor of elastic moduli, and it has the symmetry properties:

$$C_{ijkl} = C_{jikl} = C_{ijlk} = C_{klij} \, . \tag{2.3}$$

Due to one of these symmetry conditions, the asymmetric elastic distortion tensor β can be written formally in place of the elastic strain tensor ε in equation (2.2). One has:

$$\varepsilon_{ij} = \tfrac{1}{2} (\beta_{ij} + \beta_{ji}). \tag{2.4}$$

The equilibrium conditions then read:

$$C_{ijkl} \, \partial_k \, \beta_{kl} - \rho \, \ddot{s}_i^{\,G} = 0 \, . \tag{2.5}$$

According to KRÖNER [**2**], the plastic distortion β^P and the elastic distortion β can be combined into the total distortion β^G of a volume element:

$$\beta^G = \beta + \beta^P. \tag{2.6}$$

In this, one must observe that the total distortion can be derived from the total displacement field:

$$\beta_{ij}^G = \partial_i \, s_j^G \, , \tag{2.7}$$

but that such a relation cannot be given for the plastic and elastic distortions individually. For that reason, it is also not possible to give an analytical expression for the elastic displacement field that is valid in all of space. In order to arrive at a differential equation for the total displacement field, we add the expression $C_{iklm} \, \partial_k \beta_{lm}^G$ to both sides of (2.5). That will lead to the following system of partial differential equations:

$$C_{iklm} \, \partial_l \, \partial_k s_m^G - \rho \, \ddot{s}_i^G = C_{iklm} \, \partial_k \beta_{lm}^G = C_{iklm} \, \partial_k \varepsilon_{lm}^G \, , \tag{2.8}$$

which allows one to calculate the total displacements when the plastic distortion is given. Due to the symmetry of the Hooke tensor the plastic strain ε^P can be written in place of the plastic distortion, which will show that the displacement field also depends upon only ε^P in the dynamics of dislocations.

In an isotropic medium, in which the Hooke tensor is given by:

$$C_{ijkl} = \lambda\, \delta_{ij}\, \delta_{kl} + \mu\, (\delta_{ik}\, \delta_{jl} + \delta_{il}\, \delta_{jk}), \tag{2.9}$$

with the Lamé constants λ and μ, (2.8) will assume the form of an inhomogeneous Lamé equation:

$$\mu \Delta s_i^G + (\lambda + \mu)\partial_i \partial_l s_l^G - \rho \ddot{s}_i^G = \lambda \partial_i \beta_{ll}^P + 2\mu \partial_l \varepsilon_{li}^P . \tag{2.10}$$

KOSEVICH [**15**] also found a similar relationship, but in which the inhomogeneous term depended upon only the dislocation density. That difference comes about as a result of additional assumptions that KOSEVICH made about the temporal evolution of the plastic distortion.

A particular integral of (2.8) can be given by using the Green dyadic G_{ij} ($\mathfrak{x}$, t):

$$s_i^G(\mathfrak{x},t) = -\frac{1}{4\pi}\int G_{ij}(\mathfrak{x}-\mathfrak{x}',t-t')\,C_{jklm}\,\partial_k'\,\varepsilon_{lm}^P(\mathfrak{x}',t')\,d\tau' dt' . \tag{2.11}$$

The Green dyadic satisfies the partial differential equation:

$$D_{im}(\partial)\, G_{mj}(\mathfrak{x}, t) = -4\pi\, \delta(\mathfrak{x})\, \delta(t)\, \delta_{ij} \tag{2.12a}$$

and the causality condition:

$$G_{ij}(\mathfrak{x}, t) = 0 \qquad \text{when} \quad t < 0. \tag{2.12b}$$

D_{im} is a tensorial differential operator with the components:

$$D_{im} = D_{mi} = C_{iklm}\, \partial_k\, \partial_l - \delta_{im}\, \rho \frac{\partial^2}{\partial t^2} . \tag{2.12c}$$

The Green dyadic G_{ij} ($\mathfrak{x}$, t) is the i^{th} component of the displacement at the location $\mathfrak{x}$ at time t that is produced by the j^{th} component of a temporal impulse of the force density with the strength – 4π at the location $\mathfrak{r}' = 0$ at the time $t' = 0$. Representations of the Green dyadic for an isotropic medium are given in the literature [**19**, **20**]. In the Appendix, it will be shown that the Green dyadic G_{ij} can also be expressed by differentiating the fundamental integral of a sixth-order differential equation. In an infinitely-extended medium, the particular integral (2.11) describes the total displacement that is produced by the plastic strains. In a finite region, solutions of the homogeneous (2.8) must be added to (2.11) in order to fulfill the outer surface conditions.

If can be easily proved with the help of a partial integration that the expression that is derived for the total displacement field in an infinitely-extended region is identical with the one that MURA [**17**, **18**] gave. Our consideration has the advantage of greater simplicity and brevity.

3. The elastic strain of a moving dislocation

In many problems that involve internal stresses, one is only interested in the strains and stresses that are produced by the dislocations, but not in the total displacements. The calculation of the strains from the total displacement represents a detour when one can give a determining equation for the strains themselves. To that end, we partially-differentiate equation (2.8) with respect to position coordinates. That will yield the following differential equation for the total distortion:

$$D_{jm}\,\beta^G_{im} = C_{jklm}\,\partial_i\,\partial_k\,\beta^P_{lm}\,. \tag{3.1}$$

On the left-hand side of this, we replace the total distortion β^G with the elastic distortion β and the plastic distortion β^P using equation (2.6), and move all expressions in the plastic distortion to the right-hand side:

$$D_{jm}\,\beta_{im} = \rho\,\ddot{\beta}^P_{ij} + C_{jklm}(\partial_i\,\partial_k\beta^P_{lm} - \partial_l\,\partial_k\beta^P_{im}) = \rho\,\ddot{\beta}^P_{ij} + C_{jklm}\,\varepsilon_{inl}\,\partial_k\alpha_{nm}\,. \tag{3.2}$$

α_{ij} is the dislocation density tensor that KRÖNER [**2**] defined by:

$$\alpha_{ij} = -\varepsilon_{ikl}\,\partial_k\beta^P_{lj} = \varepsilon_{ikl}\,\partial_k\beta_{lj}\,. \tag{3.3}$$

We can also give a particular integral for the elastic distortion in this case by means of the Green dyadic G_{ij} :

$$\beta_{ij}\,(\mathfrak{r},\,t) = -\frac{1}{4\pi}\int\left[\rho\,\ddot{\beta}^P_{in} + C_{nklm}\varepsilon_{ipl}\,\partial'_k\alpha_{pm}\right]G_{nj}(\mathfrak{r}-\mathfrak{r}',t-t')\,d\tau'\,dt'\,. \tag{3.4}$$

In the special case of the isotropic medium, the differential equation for the elastic displacements will assume an especially simple form. In order to do that, we introduce the representation of the Hooke tensor from (2.9) into (3.2). The following differential equation will imply the elastic distortion:

$$\mu\,\Delta\beta_{ij} + (\lambda+\mu)\,\partial_i\,\partial_j\,\beta_{kk} - \rho\,\ddot{\beta}_{ij} = \rho\,\ddot{\beta}^P_{ij} + \mu\,[H_{ij} - \delta_{ij}\,H_{kk} - \varepsilon_{ijk}\,\partial_m\,\alpha_{km}] \tag{3.5}$$

when one employs the identities:

$$\partial_j\,\partial_k\,\beta_{ik} - \partial_j\,\partial_i\,\beta_{kk} = \varepsilon_{ijk}\,\partial_j\,\alpha_{kl}\,, \tag{3.6}$$

$$\varepsilon_{ikl}\,\partial_l\,\alpha_{kj} - \varepsilon_{ikl}\,\partial_j\,\alpha_{kl} = H_{ij} - \delta_{ij}\,H_{kk}\,. \tag{3.7}$$

H_{ij} is the (asymmetric) incompatibility tensor that KRÖNER [**2**] defined by:

$$H_{ij} = \varepsilon_{ikl}\,\partial_l\,\alpha_{jk} = -\,\varepsilon_{ikl}\,\varepsilon_{jmn}\,\partial_l\,\partial_m\,\beta^P_{nk} = -\,(J\,n\,k\,\beta^P)_{ij}\,. \tag{3.8}$$

If one decomposes the elastic distortion into a symmetric and an antisymmetric tensor by way of:

$$\beta_{ij} = \varepsilon_{ij} + \omega_{ij} \tag{3.9}$$

then equation (3.5) will split into:

$$\mu\,\Delta\varepsilon_{ij} + (\lambda + \mu)\,\partial_i\,\partial_j\,\varepsilon_{kk} - \rho\,\ddot{\varepsilon}_{ij} = \rho\,\ddot{\varepsilon}_{ij}^{P} + \mu\,(\eta_{ij} - \delta_{ij}\,\eta_{kk}) \tag{3.10}$$

and

$$\mu\,\Delta\omega_k - \rho\,\ddot{\omega}_k = \rho\,\ddot{\omega}_k^{P} + \mu\left(\tfrac{1}{2}\partial_k\alpha_{mm} - \partial_m\alpha_{km}\right), \tag{3.11}$$

in which η is the symmetric part of H, and one has set:

$$\omega_{ij} = \varepsilon_{ijk}\,\omega_k \quad \text{and} \quad \omega_{ij}^{P} = \varepsilon_{ijk}\,\omega_k^{P}. \tag{3.12}$$

We will get the following system of partial differential equations for the stresses σ if we replace ε with σ using Hooke's law:

$$\Delta\sigma_{ij} + \frac{2\mu+2\lambda}{2\mu+3\lambda}\partial_i\,\partial_j\sigma_{kk} - \frac{\rho}{\mu}\ddot{\sigma}_{ij} - \frac{\lambda}{2\mu+3\lambda}\left[\Delta\sigma_{kk} - \frac{\rho}{\mu}\ddot{\sigma}_{kk}\right]\delta_{ij}$$
$$= 2\rho\,\ddot{\varepsilon}_{ij}^{P} + 2\mu\,[\eta_{ij} - \delta_{ij}\,\eta_{kk}]. \tag{3.13a}$$

The trace of the stress tensor is determined by the differential equation:

$$\Delta\sigma_I - \frac{\rho}{\lambda+2\mu}\ddot{\sigma}_I = \frac{2\mu+3\lambda}{2\mu+\lambda}[\rho\,\ddot{\varepsilon}_I^{P} - 2\mu\eta_I]. \tag{3.13b}$$

In the static case, equations (3.13a) and (3.13b) go to the generalized Beltrami equations that KRÖNER [**2**] gave. The differential equations for the stresses and strains in the dislocation-free state have been known for a long time [**21**, **22**].

4. The Lagrangian function of a moving dislocation

In many cases, the differential equations for the stresses and strains cannot be solved in closed form, so one turns to approximation procedures. Variational techniques are most useful for dynamical problems, for which the time integral of the Lagrangian function L – viz., the so-called *action functional* – must be extremized:

$$\delta W = \delta\int L\,dt = 0. \tag{4.1}$$

In an elastic medium, L can be computed as the volume integral of the Lagrangian density $\mathfrak{L}$:

$$L = \int \mathfrak{L}\,d\tau, \tag{4.2a}$$

in which $\mathfrak{L}$ is defined by:

$$\mathfrak{L} = \tfrac{1}{2}\rho\, \dot{s}_i^G \dot{s}_i^G - \tfrac{1}{2}\sigma_{ij} \cdot \varepsilon_{ij} . \tag{4.2b}$$

One can easily show that equation (2.1) is the Euler-Lagrange differential equation of the extremal problem that belongs to equations (4.1) and (4.2) for a given plastic distortion. One will come to another expression for the Lagrangian density by the use of the identity:

$$\dot{s}_i^G \cdot \dot{s}_i^G = \frac{\partial}{\partial t}(\dot{s}_i^G \cdot \dot{s}_i^G) - \ddot{s}_i^G \dot{s}_i^G . \tag{4.3}$$

If we can assume that the mass density ρ is constant (viz., the approximation of the linear theory of elasticity) and $\dot{s}_i^G$ are replaced with the stresses in the equilibrium condition then we will have:

$$L = -\tfrac{1}{2}\int (\partial_j \sigma_{ij})\, s_i^G d\tau - \tfrac{1}{2}\int \sigma_{ij}\varepsilon_{ij}\, d\tau + \frac{1}{2}\frac{\partial}{\partial t}\int \rho\, \dot{s}_i^G \cdot s_i^G\, d\tau . \tag{4.4}$$

We can convert the first expression partially into an outer surface integral using Gauss's theorem, when we perform the following conversion:

$$(\partial_j \sigma_{ij})\, s_i^G = \partial_j(\sigma_{ij} \cdot s_i^G) - \sigma_{ij} \cdot (\varepsilon_{ij} + \varepsilon_{ij}^P) . \tag{4.5}$$

The Lagrangian function is essentially composed of the energy of plastic deformation and an outer surface integral that describes the work done by external forces:

$$L = \tfrac{1}{2}\int \sigma_{ij}\varepsilon_{ij}^P\, d\tau - \tfrac{1}{2}\int \sigma_{ij}\, s_i^G dF_j + \frac{1}{2}\frac{\partial}{\partial t}\int \rho\, \dot{s}_i^G \cdot s_i^G\, d\tau . \tag{4.6}$$

The last term plays no role, since the functions to be varied in the action functional will be fixed at the lower and upper limits under variation. We get another expression for the Lagrangian function by appealing to (4.2b) and (4.6):

$$L = -\tfrac{1}{2}\int \rho\, \dot{s}_i^G \cdot \dot{s}_i^G\, d\tau + \tfrac{1}{2}\int (\varepsilon_{ij} + 2\varepsilon_{ij}^P)\, \sigma_{ij}\, d\tau - \tfrac{1}{2}\int \sigma_{ij}\, s_i^G dF_j + \frac{1}{2}\frac{\partial}{\partial t}\int \rho\, \dot{s}_i^G \cdot s_i^G\, d\tau . \tag{4.7}$$

(4.7) will go to an expression that COLONETTI [**23**] gave for a body with free outer surface (i.e., $\sigma_{ij}\, n_j = 0$ on the outer surface) and for static problems. As we mentioned before, for fixed plastic strains, the differential equations for the total displacement (2.8) will be the Euler-Lagrange equations for the variational problem. The total displacement field, the elastic strains, and the action functional are calculated for arbitrary plastic distortions. Under that law, we can choose any solution that extremizes the action. An equation of motion for the evolution of plastic distortion can be obtained by that double application of the action principle. Even in the case of an infinitely-extended medium, in which the integrals (2.11) and (3.4) describe the total displacement field and the elastic strains, the Euler-Lagrange equation of the extremal problem will lead to an integral equation for the plastic distortion that can be very hard to solve. For that reason, one solves that variational problem with the Ritz procedure, in which the plastic distortion is developed into a series of suitably-chosen comparison functions.

5. Four-dimensional formulation of the basic equations of dislocation dynamics

5.1 – *Fundamentals.*

In this section, our goal is to write the basic equations of the dynamics of dislocations that we derived in sections 2 to 4 in four-dimensional form. As in the special theory of relativity, the intrinsic connection between the various physical quantities will become visible when it had remained hidden in the three-dimensional formulation.

However, an essential difference between this and the special theory of relativity should be pointed out: In the theory of elasticity, just like in hydrodynamical problems, there is a special distinguished system of reference – namely, the one that is coupled with the matter. Due to that existence of an absolute space, the relativity principle does not have to be fulfilled for that class of problems in which all inertial systems follow the natural way of things in the same way. All displacements are initially defined in that absolute space. For that reason, it is not permitted, for example, to derive the displacement field of a dislocation that moves rectilinearly with constant speed v from the corresponding statics problem by means of a Lorentz transformation with the velocity v. The theory of moving dislocations will be further complicated by the fact that several speeds of sound are possible in an elastic medium that might take on the role of the limiting speed in the Lorentz transformation.

The advantage of the four-dimensional notation, which rests upon the introduction of the new coordinate:

$$x_4 = i\,c\,t, \tag{5.1}$$

in which c is a speed of sound, which we will establish more precisely below, consists of the fact that the asymmetry in the basic equation in the time coordinate is removed. With that, we will arrive at (among other things) the fact that the four-dimensional basic equations of the dynamics of dislocations are formally the same as the three-dimensional equations of the statics of dislocations, such that the same process as in the statics of dislocations can possibly be employed to solve the dynamical problem.

5.2 – *The energy-impulse tensor.*

The definition of the energy-impulse tensor is connected with the equilibrium conditions (2.1), which can be brought into the form:

$$\partial_\kappa \sigma_{\iota\kappa} + \partial_4(-i\,c\rho\,\dot{s}_i^G) = 0 \tag{5.2}$$

with the use of (5.1) ([3]). We shall now introduce a symmetric four-tensor $\boldsymbol{\sigma}$ whose purely-spatial components $\boldsymbol{\sigma}_{\iota\kappa}$ are identical with the stress tensor. If we define the space-time components by:

$$\boldsymbol{\sigma}_{\iota 4} = -\, i\, c \rho \dot{\mathbf{s}}_i^G , \tag{5.3}$$

then (5.2) will go to:

$$\partial_k \, \boldsymbol{\sigma}_{ik} = 0. \tag{5.4}$$

One gets the components $\boldsymbol{\sigma}_{4\iota}$ from the symmetry that is required of $\boldsymbol{\sigma}$. The purely-temporal component $\boldsymbol{\sigma}_{44}$ will be fixed by the demand that the (four-) divergence of the four-tensor $\boldsymbol{\sigma}$ should vanish:

$$\text{div}\, \boldsymbol{\sigma} = \partial_k \, \boldsymbol{\sigma}_{ik} = 0. \tag{5.5}$$

In that way, one will first obtain a relation for $\dot{\boldsymbol{\sigma}}_{44}$ that can be integrated directly. If the integration constant is established in such a way that $\boldsymbol{\sigma}_{44}$ is proportional to the momentary mass density $\rho\,(t)$ – i.e., such that one has:

$$\boldsymbol{\sigma}_{44} = -\, \rho\, c^2 \, (1 + \varepsilon_{\iota\iota}^G) = -\, c^2 \, \rho\,(t), \tag{5.6}$$

then the fourth component of (5.5) will express the conservation of mass. That will become especially clear when we represent it in integral form:

$$\int (\rho\, \dot{s}_\iota^G)\, dF_\iota + \frac{\partial}{\partial t} \int \rho(t)\, d\tau = 0, \tag{5.7}$$

in which the first term describes the matter that flows out of the outer surface.

In analogy with the theory of relativity, one can formally associate the density ρ of the inertial mass with the energy density $\rho\, c^2$. Equation (5.7) will then be a conservation law for the energy that is equivalent to the rest mass, and the four-tensor $\boldsymbol{\sigma}$ will have a meaning that is similar to that of the energy-impulse tensor in the theory of relativity.

5.3 – *The four-tensor of distortion*

In order to define the four-tensor of distortion, we must observe that the displacement field is defined by only a vector in three-dimensional space, such that the total displacement field cannot possess a temporal component of the form s_4^G. The four-tensor of total distortion, which we define in the usual way by the derivative of the total displacement field according to:

$$\boldsymbol{\beta}_{ij}^G = \partial_i \, \mathbf{s}_j^G , \tag{5.8}$$

([3]) In this section, we would like to denote all indices in the vector and tensor components that assume only the values 1, 2, 3 by Greek symbols; Latin symbols will assume the values 1, …, 4. Four-tensors will differ from the corresponding three-dimensional quantities by a semi-bold reproduction of them.

and for that reason, it can be represented by the following matrix:

$$\boldsymbol{\beta}^G = \left(\begin{array}{c|c} \beta^G_{\iota\kappa} & 0 \\ \hline \beta^G_{\iota\kappa} & 0 \end{array} \right). \tag{5.9}$$

We decompose the total distortion into an elastic and plastic part in a way that is completely analogous to what we do in the statics of dislocations:

$$\boldsymbol{\beta} = \boldsymbol{\beta} + \boldsymbol{\beta}^P. \tag{5.10}$$

As for the four-tensor of plastic distortion $\boldsymbol{\beta}^P$, we establish only that it should possess the components:

$$\boldsymbol{\beta}^P = \left(\begin{array}{c|c} \beta^P_{\iota\kappa} & 0 \\ \hline 0 & 0 \end{array} \right), \tag{5.11}$$

which agrees with the prescription that KRÖNER [**2**] gave for calculating the plastic distortion $\boldsymbol{\beta}^P_{\iota\kappa}$. $\boldsymbol{\beta}^P_{\iota\kappa}$ describes the κ^{th} component of the relative plastic displacement of the bounding surfaces of a volume element whose surface normals point in the ι^{th} direction and which are at a unit distance apart. Both indices of $\boldsymbol{\beta}^P_{\iota\kappa}$ then refer to the position space, such that $\boldsymbol{\beta}^P_{4\kappa}$ must be identically zero. That convention has the consequence that the purely-spatial components of the elastic strain tensor can be calculated from the elastic displacements, while the components $\boldsymbol{\beta}_{4\kappa}$ can be calculated from the total displacement field by:

$$\boldsymbol{\beta}_{4\kappa} = \partial_4 \, s^G_\kappa \, . \tag{5.12}$$

The four-tensor $\boldsymbol{\varepsilon}$ of the elastic strains will be defined as the symmetric part of the distortion tensor $\boldsymbol{\beta}$. One easily convinces oneself that on the basis of the aforementioned conventions, the purely-spatial components $\boldsymbol{\varepsilon}_{\iota\kappa}$ will coincide with the ordinary elastic strain tensor $\boldsymbol{\varepsilon}_{\iota\kappa}$ and the remaining components will be given by:

$$\boldsymbol{\varepsilon}_{4\iota} = \boldsymbol{\varepsilon}_{\iota 4} = -\frac{i}{2c}\frac{\partial s^G_i}{\partial t} = \frac{1}{2}\partial_4 \, s^G_i \tag{5.13}$$

and

$$\boldsymbol{\varepsilon}_{44} = 0. \tag{5.14}$$

The Lagrangian density for dynamical problem can be written in the very transparent form:

$$\mathfrak{L} = -\tfrac{1}{2}\boldsymbol{\varepsilon}_{ij}\,\boldsymbol{\varepsilon}_{ij} \tag{5.15}$$

with the use of (5.3) and (5.13). We can achieve a formal similarity between the dynamics of dislocations and the corresponding expressions in the dynamics of dislocations by means of the four-dimensional notation. One can guarantee that the

Lagrangian density $\mathfrak{L}$ is invariant under Lorentz transformations by means of equation (5.15).

The four-tensors of stress $\boldsymbol{\sigma}$ and the strain $\boldsymbol{\varepsilon}$ will be coupled by the generalized Hooke law:

$$\boldsymbol{\sigma}_{ik} = \boldsymbol{C}_{iklm}\, \boldsymbol{\varepsilon}_{lm}\,, \tag{5.16}$$

in which:

$$\left.\begin{aligned} \boldsymbol{C}_{iklm} &= C_{\iota\kappa\lambda\mu} \qquad \iota = i,\ \kappa = k,\ \lambda = l,\ \mu = m, \\ \boldsymbol{C}_{4klm} &= 2\rho c^2\, \delta_{km}\, \delta_{l4}\,, \\ \boldsymbol{C}_{44lm} &= -\rho c^2\, \delta_{lm}\,. \end{aligned}\right\} \tag{5.17}$$

$\boldsymbol{C}_{iklm}$ has the same symmetry properties as C_{iklm} [cf., (2.3)].

5.4 – *The four-tensors of dislocation density and incompatibility.*

We define the following third-rank four-tensor to be the dislocation density $\boldsymbol{\alpha}$:

$$\boldsymbol{\alpha}_{ijk} = -\, \boldsymbol{\varepsilon}_{ijlm}\, \partial_l\, \beta^P_{mk} = -\,(\mathrm{rot}\, \beta^P)_{ijk}\,. \tag{5.18}$$

$\boldsymbol{\varepsilon}_{ijlm}$ is the completely antisymmetric, four-dimensional unit tensor whose components will change sign under an exchange of two indices and will have non-zero components that are equal to ± 1; one has $\boldsymbol{\varepsilon}_{1234} = 1$. It follows directly from equation (5.18) that:

$$\partial_i\, \boldsymbol{\alpha}_{ijk} = -\, \partial_i\, \boldsymbol{\sigma}_{jik} = 0. \tag{5.19}$$

Since $\boldsymbol{\beta}^P_{l4} = 0$, the components $\boldsymbol{\alpha}_{ij4}$ will vanish in general. Since $\boldsymbol{\alpha}_{ijk}$ is antisymmetric in its first two indices, moreover, $\boldsymbol{\alpha}$ will have only eighteen non-zero components. With the help of the properties of the four-dimensional $\boldsymbol{\varepsilon}$-tensor, it can be shown that the nine components of $\boldsymbol{\alpha}$ are identical with the ordinary dislocation density. One has:

$$\boldsymbol{\alpha}_{4\iota\kappa} = \varepsilon_{\iota\mu\lambda}\, \partial_\mu\, \beta^P_{\lambda\kappa}\,. \tag{5.20}$$

The remaining nine components are:

$$\boldsymbol{\alpha}_{\iota\kappa l} = \boldsymbol{\varepsilon}_{\iota\kappa\mu 4}\, \partial_4\, \beta^P_{\mu l} = \varepsilon_{\iota\kappa\mu}\, \frac{1}{ic}\, \dot{\beta}^P_{\mu l}\,, \tag{5.21}$$

and they essentially describe the temporal change in the plastic distortion tensor. Along with the demand that one must have:

$$\mathrm{div}\, \alpha = \partial_\iota\, \alpha_{\iota\kappa} = 0, \tag{5.22}$$

equation (5.19) will also include the following statement about the temporal change in the dislocation density:

$$\frac{\partial \alpha_{\iota\kappa}}{\partial t} = -(\mathrm{rot}\,\dot{\beta}^P)_{\iota\kappa}. \tag{5.23}$$

In the statics of dislocations, one will get the incompatibility tensor H when one lets the operator rot act upon the dislocation density α on the right and the left. In a completely-analogous way, we next introduce the fourth-rank four-dimensional tensor:

$$\boldsymbol{H}_{ijkl} = \boldsymbol{\varepsilon}_{ijkl}\ \partial_n\, \boldsymbol{\alpha}_{klm} = -\, \boldsymbol{\varepsilon}_{ijmn}\ \boldsymbol{\varepsilon}_{klpq} \partial_n\, \partial_p\, \boldsymbol{\beta}^P_{qm}, \tag{5.24}$$

which is antisymmetric in the first and last two index-pairs. We define the four-dimensional incompatibility density $\boldsymbol{\eta}$ to be the second-rank tensor that arises by contracting the second and fourth indices of $\boldsymbol{H}$ and symmetrizing:

$$\boldsymbol{\eta}_{ij} = \mathrm{Sym}\ \boldsymbol{H}_{ikjk} = -\, \boldsymbol{\varepsilon}_{imnk}\ \boldsymbol{\varepsilon}_{jpqk} \partial_n\, \partial_p\, \boldsymbol{\beta}^P_{qm} = -\,(\mathrm{Ink}\ \boldsymbol{\varepsilon}^P)_{ij}. \tag{5.25}$$

In that way, the product of the two $\boldsymbol{\varepsilon}$-tensors can also be represented by the determinant ([4]):

$$\boldsymbol{\varepsilon}_{imnk}\ \boldsymbol{\varepsilon}_{jpqk} = \begin{vmatrix} \delta_{ij} & \delta_{ip} & \delta_{iq} \\ \delta_{mj} & \delta_{mp} & \delta_{mq} \\ \delta_{nj} & \delta_{np} & \delta_{nq} \end{vmatrix}. \tag{5.26}$$

We get the following representation for the components of $\boldsymbol{\eta}$:

$$\boldsymbol{\eta}_{\iota\kappa} = \eta_{\iota\kappa} + \frac{1}{c^2}\frac{\partial^2 \varepsilon^P_{\iota\kappa}}{\partial t^2} - \delta_{\iota\kappa}\frac{1}{c^2}\frac{\partial^2 \varepsilon^P_{\mu\mu}}{\partial t^2}, \tag{5.27a}$$

$$\boldsymbol{\eta}_{\iota 4} = \frac{1}{ic}\left(\partial_\kappa \dot{\varepsilon}^P_{\iota\kappa} - \partial_\iota\, \dot{\varepsilon}^P_{\kappa\kappa}\right), \qquad \boldsymbol{\eta}_{44} = \eta_{\kappa\kappa}. \tag{5.27b}$$

η denotes the ordinary incompatibility tensor that was introduced in (3.8) and (3.12). The purely-spatial incompatibility tensor η and the effect of inertial forces are united in the four-dimensional incompatibility tensor $\boldsymbol{\eta}$.

As a comparison with (3.10) [(2.10), resp.] will show, $\boldsymbol{\eta}_{\iota\kappa}$ essentially describes the inhomogeneous part of the differential equations for the stresses $\sigma_{\iota\kappa}$, while $\boldsymbol{\eta}_{\iota 4}$ is the inhomogeneous part for $\sigma_{\iota 4} = -\, i\, \rho\, c \cdot s^G_\iota$ in an isotropic medium when we set $c = c_T = \left(\frac{\mu}{\rho}\right)^{1/2}$.

It is easy to see from the representation (5.25) that the relation:

([4]) Some further relationships for multiply-contracted products of two $\boldsymbol{\varepsilon}$-tensors are:

$$\boldsymbol{\varepsilon}_{ijkl}\ \boldsymbol{\varepsilon}_{mnkl} = 2\,(\delta_{im}\ \delta_{jn} - \delta_{in}\ \delta_{jm}), \qquad \boldsymbol{\varepsilon}_{ijkl}\ \boldsymbol{\varepsilon}_{mjkl} = 3!\ \delta_{im}, \qquad \boldsymbol{\varepsilon}_{ijkl}\ \boldsymbol{\varepsilon}_{ijkl} = 4!$$

$$\operatorname{div} \boldsymbol{Ink} = 0 \tag{5.28}$$

is also true for the four-dimensional operator ***Ink***. For that reason, we can fulfill the generalized equilibrium conditions (5.3) in such a way that we represent $\boldsymbol{\sigma}$ as the incompatibility of a second-rank symmetric tensor $\boldsymbol{\chi}$:

$$\boldsymbol{\sigma} = \boldsymbol{Ink}\,\boldsymbol{\chi}\,, \tag{5.29a}$$

or in components:

$$\sigma_{ij} = \Box\chi_{ij} - (\partial_j\,\partial_k\chi_{ki} + \partial_i\,\partial_k\chi_{kj}) + \partial_i\,\partial_j\chi_{kk} + \delta_{ij}(\partial_l\,\partial_k\chi_{kl} - \Box\chi_{kk})\,. \tag{5.29b}$$

Since the incompatibility of the total strains vanishes, due to (5.15), we will have:

$$\boldsymbol{Ink}\,\boldsymbol{\varepsilon} = -\,\boldsymbol{Ink}\,\boldsymbol{\varepsilon}^P = \boldsymbol{\eta}\,. \tag{5.30}$$

Precisely as in the statics of dislocations [**2**], we will get a determining equation for the stress function $\boldsymbol{\sigma}$ when we replace $\boldsymbol{\varepsilon}$ with the stresses using Hooke's law and substitute (5.29) for it:

$$\boldsymbol{Ink}\,C^{-1}\,(\boldsymbol{Ink}\,\boldsymbol{\chi}) = \boldsymbol{\eta}, \tag{5.31}$$

in which C^{-1} is the inverse Hooke tensor. $\boldsymbol{\chi}$ is not determined completely, but it is possible to impose certain auxiliary conditions on the stress function $\boldsymbol{\chi}$. One might hope that those auxiliary conditions can be chosen in such a way that we will get a simple differential equation for $\boldsymbol{\chi}$.

5.5 – *Comparison with the investigations of HOLLÄNDER* [12, 13, 14].

Up to now, only HOLLÄNDER had given a four-dimensional formulation for the basic equations of the dynamics of dislocations, and his formulation deviated from our own, to some extent. Both representations coincide completely in the generalized equilibrium condition (5.3) and approximately in the definition of the four-dimensional stress tensor and strain tensor. The latter differ only in the purely-temporal $\boldsymbol{\sigma}_{44}$ and $\boldsymbol{\varepsilon}_{44}$. For HOLLÄNDER, $\boldsymbol{\sigma}_{44}$ depends (erroneously) upon on the elastic dilatation. HOLLÄNDER did not define the dislocation density $\boldsymbol{\alpha}'$ in terms of the asymmetric distortion tensor, but the elastic strain tensor, using the equation:

$$\alpha'_{ijk} = \varepsilon_{ijmn}\,\partial_m\,\varepsilon_{nk}\,. \tag{5.32}$$

This has the consequence that the dislocation density will also have non-zero components of the form α'_{ij4}, for which HOLLÄNDER himself could not find the correct interpretation. In order to calculate the stresses, the method of the first-order asymmetric stress function will be employed [**2**], which is based upon the idea that the equilibrium conditions are fulfilled identically by the Ansatz:

$$\sigma_{ij} = \varepsilon_{iklm} \, \partial_k \, \varphi_{lmj} \, . \tag{5.33}$$

In the case of an isotropic medium with vanishing Poisson number, that will yield the inhomogeneous wave equation for φ :

$$\Box \, \varphi_{ijk} = - \, \mu \, \alpha'_{ijk} \, , \tag{5.34}$$

when φ satisfies the auxiliary condition:

$$\partial_m \, \varphi_{mik} = 0.$$

For a real medium with non-vanishing Poisson number, our solution process seem to overlap with HOLLÄNDER's process, since it also allow one to determine the elastic dilatation.

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Appendix

In this section, we would like to show that the Green dyadic that is defined by equation (2.12) can be represented by the derivatives of a scalar function that is identical to the fundamental integral of a six-dimensional linear partial differential equation. Let D be the determinant of the tensor differential operator D_{ij} that is defined by (2.12c), and let D_{ij}^* be a subdeterminant of D such that one has:

$$D_{im} \cdot D_{mj}^* = D \, \delta_{ij} \, . \tag{A.1}$$

The Green dyadic can be represented as the derivative of a scalar function:

$$G_{ij} \, (\mathfrak{x}, t) = D_{ij}^* \, U \, (\mathfrak{x}, t). \tag{A.2}$$

Substituting (A.2) in (2.12a) will yield that U ($\mathfrak{x}$, t) satisfies the following sixth-order linear, inhomogeneous, partial differential equation:

$$D \, U \, (\mathfrak{x}, t) = - \, 4\pi \, \delta(\mathfrak{x}) \, \delta(t). \tag{A.3}$$

U ($\mathfrak{x}$, t) will be referred to as the fundamental integral of the linear differential equation $DU = 0$. Knowing U ($\mathfrak{x}$, t) implies knowing G ($\mathfrak{x}$, t) and therefore, knowing the particular integral of the inhomogeneous wave equation (2.8), as well. The latter can be represented in the following way by means of the fundamental integral U ($\mathfrak{x}$, t) :

$$s_i^G(\mathfrak{x},t) = -\frac{1}{4\pi} C_{jklm} \int \varepsilon_{lm}^P(\mathfrak{x}',t')\{D_{ij}^* \partial_k U(\mathfrak{x}-\mathfrak{x}',t-t')\}\, d\tau' dt'. \tag{A.4}$$

The differential equation for $U(\mathfrak{x}, t)$ assumes an especially simple form for an isotropic medium. With the help of the representation of Hooke's tensor in equation (2.9), one will have:

$$D_{ij} = (\lambda+\mu)\,\partial_i\,\partial_j + \left(\mu\Delta - \rho\frac{\partial^2}{\partial t^2}\right)\delta_{ij}, \tag{A.5}$$

$$D_{ij}^* = \left(\mu\Delta - \rho\frac{\partial^2}{\partial t^2}\right)^2 \left\{-(\lambda+2\mu)\partial_i\,\partial_j + \left[(\lambda+2\mu)\Delta - \rho\frac{\partial^2}{\partial t^2}\right]\delta_{ij}\right\}, \tag{A.6}$$

and

$$D = \left(\mu\Delta - \rho\frac{\partial^2}{\partial t^2}\right)^2 \left[(\lambda+2\mu)\Delta - \rho\frac{\partial^2}{\partial t^2}\right]. \tag{A.7}$$

D is then the product of differential operators, one of which describes the propagation of longitudinal waves, while the other one describes the propagation of transverse waves. Since D_{ij}^*, as well as D, includes the operator $\left(\mu\Delta - \rho\frac{\partial^2}{\partial t^2}\right)$, it will suffice to know the expression:

$$\tilde{U}(\mathfrak{x},t) = \left(\mu\Delta - \rho\frac{\partial^2}{\partial t^2}\right)U(\mathfrak{x},t), \tag{A.8}$$

instead of the fundamental integral $U(\mathfrak{x}, t)$ itself. One can derive the representation for $\tilde{U}(\mathfrak{x},t)$:

$$\tilde{U}(\mathfrak{x},t) = \begin{cases} -\dfrac{1}{\rho^2 c_L\, c_T\,(c_L + c_T)} & 0 < r < c_T t, \\ -\dfrac{1}{\rho^2 c_L\, c_T\,(c_L + c_T)}\left(\dfrac{c_L t}{r} - 1\right) & c_T t < r < c_L t,, \\ 0 & c_L t < r < \infty, \end{cases}$$

with:

$$c_T = \left(\frac{\mu}{\rho}\right)^{1/2}, \qquad c_L = \left(\frac{\lambda+2\mu}{\rho}\right)^{1/2}.$$

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