ÉLIE CARTAN

Lessons on Integral Invariants

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INTRODUCTION

This work is the reproduction of a professional course that was held during the summer semester of 1920-1921 at the Faculté des Sciences in Paris.

The theory of integral invariants was founded by H. Poincaré and examined by him in tome III of his *Méthodes nouvelles de la Méchanique céleste*.

In his notes to the Comptes rendus de l'Académie des Sciences (16 and 30 June, 1902) the author was led, in the study of differential equations that admit given transformations, to consider certain differential forms that are called *integral forms.* They are the ones that are characterized by the property that they are expressible in terms of only first integrals of the given differential equations and their differentials. It was in contemplating research along the same lines that the author arrived, on the one hand, to found his method of integration of systems of partial differential equations that admit characteristics that depend only on arbitrary constants (Cauchy characteristics), and, on the other hand, to found his theory of the structure of continuous groups of transformations, both finite and infinite.

Now, one finds that the notion of integral form does not differ essentially from that of integral invariant. It is the confrontation of these two notions that forms the basis for this work.

For example, consider a system of three first-order differential equations in the three unknown functions x, y, z of the independent variable t. One may regard them as defining an infinitude of trajectories of a moving point. A differential form, such as $P\,dx + Q\,dy + R\,dz + H\,dt$, for example, may be envisioned as a quantity that is attached to a *state* (x, y, z, t) of the moving point and an infinitely close *state* $(x + dx, y + dy, z + dz, t + dt)$. We say that this form is *integral* (or *invariant*, following the terminology that will be adopted in these Lessons), which obviously signifies that this quantity depends only on the trajectory that contains the first state and the infinitely close trajectory that contains the second state. In other words, an invariant form does not change its value if one displaces the two states (x, y, z, t) and $(x + dx, y + dy, z + dz, t + dt)$ along their trajectories in an arbitrary manner. If one then considers a continuous, linear collection of trajectories, and if one takes the integral $\int P\,dx + Q\,dy + R\,dz$ along the arc of the curve of positions taken by the moving point on its trajectories *at the same instant t* then this integral will be independent of t; it is an *integral invariant* in the sense of H. Poincaré. Conversely, by a method that is quite simple to exhibit, there exists an invariant form $P\,dx + Q\,dy + R\,dz + H\,dt$ that corresponds to the integral invariant $\int P\,dx + Q\,dy + R\,dz$ of H. Poincaré.

These considerations are not limited to linear differential forms. Any invariant differential form that is susceptible to being placed under an integration sign – simple or multiple – gives rise to an integral invariant, in the sense of H. Poincaré, if one suppresses the terms that contain linear differential forms or the differentials of the independent variables ([1]).

By definition, *the quantity under the integration sign in an integral invariant, in the sense of H. Poincaré, is nothing but a truncated invariant differential form.* The invariant character of the *complete* integral is preserved if it is taken over an arbitrary set of states, *whether simultaneous or not.*

([1]) R. Hargreaves, in a paper in the *Transactions of the Cambridge Philosophical Society* (v. XXI, 1912), has already considered integrals that contain differentials of the independent variables. However, his point of view is completely different from that of this book, and it will always play a role on the part of the independent variable.

The consequences of this agreement between the two notions of integral invariant and invariant differential form are numerous. In the first place, all of the properties that relate to the formation of these integral invariants and the derivation of one from the other are obviously important in their own right. Similarly, the same is true for the applications to the integration of differential equations.

Another consequence has been pointed out that relates to the principles of Mechanics. H. Poincaré has shown that the general equations of dynamics possess the property that they admit a (relative) linear integral invariant, namely:

$$(1) \qquad \int p_1\,\delta q_1 + p_2\,\delta q_2 + \ldots + p_n\,\delta q_n\,,$$

in which the q_i and p_i denote Hamilton's canonical variables. If one completes the differential form under the $\int$ sign then the integral invariant will take the form:

$$(2) \qquad \int p_1\,\delta q_1 + p_2\,\delta q_2 + \ldots + p_n\,\delta q_n - H\,\delta t,$$

in which H denotes Hamilton's function. One thus sees the *energy H* of the material system considered along with its *quantity of motion* ($p_1, \ldots, p_n$). The form under the $\int$ sign thus acquires an extremely important mechanical significance. One may give it the name of the *"quantity of motion-energy" tensor* ([1]). The elementary Hamilton *action* is nothing but this tensor, when considered along a trajectory. The notion of *action* is therefore related to the notion of the quantity of motion and energy.

There is more: Not only do the differential equations of motion admit the integral invariant (2), but they are, moreover, the *only* differential equations that enjoy this property. One may then place the following principle at the basis of mechanics, which one may give the name of "the principle of conservation of motion and energy:"

The motion of a material system (with perfect holonomic constraints and subject to forces that are derived from a function of position) is regulated by first-order differential equations in time, the parameters of position, and the parameters of velocity, and these differential equations are characterized by the property that when the integral of the "quantity of motion-energy" tensor is taken over an arbitrary closed, continuous, linear collection of states of the system it will not change value when one displaces these states in an arbitrary manner along their respective trajectories.

In this statement, the word *state* refers to the set of quantities that define the position of the system in space, the instant where it is considered, and its velocity at that instant.

The preceding statement is more abstract and less intuitive than Hamilton's principle of least action, for example. It nevertheless has an advantage that is important to point out. Lagrange's equations permit us to give the laws of mechanics a form that is *independent of the framing adopted for space,* and this is what makes them important; however, time occupies a privileged

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([1]) The indicated form presents itself in a completely natural way when one calculates the variation of Hamilton's action integral. This point of view has already been pointed out, which is why it is introduced in these Lessons.

position in them. On the contrary, the principle of the conservation of the quantity of motion and energy gives the laws of mechanics a form that is independent of the framing adopted for the universe (space-time). If one effects a change of variables that involves *both* the parameters of position and time for the system then it suffices to know the form that will be taken by the "quantity of motion energy" tensor in the new system of coordinates in order to deduce the equations of motion. One thus obtains a schema to which all of the mechanical theories will be subordinate, and to which, indeed, relativistic mechanics itself is subordinate.

It is important to remark that this schema applies only to material systems that depend on a finite number of parameters.

The present work omits a great number of applications of the theory of integral invariants. In particular, some that are extremely important to celestial mechanics, and are related to the theory of periodic solutions of the three-body problem and the theory of Poisson stability are systematically omitted. One is principally limited to applications that relate to the integration of differential equations, although the problem is only begun along those lines.

Meanwhile, one is forced to show that this problem cannot be considered in isolation. One only makes it narrow in scope if one does not regard it as a particular aspect of a more general problem in which not only the consideration of integral invariants enter into it, but also that of invariant Pfaff equations for the given differential equations, as well as the infinitesimal transformations that preserve these differential equations. A complete exposition of the problem is beyond the scope of these Lessons and would demand some knowledge of the theory of continuous groups. On several occasions, one is limited to showing the fundamental role that is played in the final analysis by the group G of transformations that, when applied to the integrals of the given differential equations, leave all of the information that is known *a priori* about these integrals invariant ([1]). Any system of differential equations may be converted into a system of this type with a corresponding a *simple* group G. If this simple group is *finite* then one will obtain systems of differential equations that have been studied especially by S. Lie and Vessiot, who gave them the name of *Lie systems.* They are attached to the theory of integral invariants, in the sense that they admit as many linear integral invariants as they have unknown functions, if necessary by the adjunction of auxiliary unknown functions. One will find some general indications of why one would take this latter viewpoint in chapter XV of these Lessons.

If the group G is *infinite*, and if one abstracts to the case where this is the most general group in n variables – a case in which one knows nothing about the corresponding system of differential equations – then it admits either an integral invariant of maximum degree (the theory of the Jacobi multiplier), a relative linear integral invariant (the theory of equations that are reducible to canonical form), or an invariant Pfaff equation (equations that amount to a first-order partial differential equation). Chapters XI-XIV are dedicated to these classical theories.

The notion of an integral invariant may be envisioned from a viewpoint that is slightly different from the usual one, which is that of H. Poincaré, and which is, in summary, the one that is used in these Lessons. Instead of considering a system of differential equations as being attached to a multiple integral, relative to which they enjoy the property of invariance, one may consider the system as being attached to a group of transformations with respect to which they are invariant. These viewpoints are, moreover, related. The latter is the one that S. Lie took and the one that he has wagered on several occasions to be the only truth. There again, the notion of

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([1]) Cf. É. Cartan. - *Les sous-groups des groups continus de transformations;* Ann. Ec. Norm. (3), t. XXV (1908), pp. 57-194 (Chap. I).

integral invariant plays an important role since, as the author has shown ([1]), any group of transformations may be defined as a set of transformations that admit a certain number of linear integral invariants, by the adjunction of auxiliary variables, if necessary. This aspect of the notion of integral invariant is completely omitted in these Lessons.

Several chapters are dedicated to the rules of the calculus of the differential forms that present themselves under multiple integration signs. Goursat has given these forms the name of *symbolic expressions*. I propose to call them differential forms with exterior multiplication, or, more briefly, *exterior differential forms,* because they obey the rules of exterior multiplication of H. Grassmann. Similarly, I propose to call the operation that permits us to pass from a multiple integral of degree $p - 1$ that is taken over a manifold of dimension $p - 1$ to a multiple integral of degree p that is taken over a manifold of dimension p that is bounded by the latter, *exterior derivation* ([2]). This operation, which reduces to the classical operations when the coefficients of the differential form under the $\int$ sign admit first-order partial derivatives, may be preserved in a sense when this is no longer true. In this regard, it poses interesting problems that have not been systematically studied, although they deserve to be.

This work terminates in two chapters, which are very sketchy moreover, on the relations between the theory of integral invariants, the calculus of variations, and the principles of optics.

One will find a list at the end of this volume, which makes no pretense of being complete, of the principal works that relate to the theory of integral invariants. Papers that relate to the classical theory of Jacobi multipliers, canonical equations, and first-order partial differential equations are cited only when they are directly concerned with the theory of integral invariants.

Le Chesnay, 24 November 1921.

([1]) É. Cartan. - *Sur la structure des groupes infinis de transformations;* Ann. Norm. (3), t. XXI (1904), pp. 153-206; t. XXII (1905), pp. 219-308.

([2]) This is the "*D* operation" of Goursat.

CHAPTER I

HAMILTON'S LEAST-ACTION PRINCIPLE AND THE "QUANTITY OF MOTION-ENERGY" TENSOR

I. - *Case of the free material point.*

1. One may base all of analytical mechanics on a principle that reduces the determination of the motion of a material system to the solution of a problem in the calculus of variations; this principle is Hamilton's *least-action principle.* We shall first discuss it in the case of a free material point that is subject to a force that derives from a force function U that is a given function of the rectangular coordinates x, y, z of the point and time t.

In this simple case, Hamilton's least-action principle is stated as:

Amongst all of the possible motions that take a material point with a given position (x_0, y_0, z_0) *at the instant* t_0 *to another position* (x_1, y_1, z_1) *at the instant* t_1 *the true motion is the one that minimizes the definite integral:*

$$W = \int_{t_0}^{t_1} \left[\frac{1}{2} m(x'^2 + y'^2 + z'^2) + U \right] dt .$$

In this expression, m denotes the mass of the point and x', y', z' denote the components of its velocity. The quantity under the integral sign is called the *elementary action,* and the integral W is the *action* over the interval of time (t_0, t_1).

In order to prove this principle, one must regard x, y, z as functions of t and an arbitrary parameter α, and calculate the *variation* of W when one gives α an increase $\delta\alpha$, while supposing that x, y, z reduce to x_0, y_0, z_0 for $t = t_0$, and to x_1, y_1, z_1 for $t = t_1$, and that this is true *for any* α. One has:

$$\delta W = \int_{t_0}^{t_1} \left[m(x'\delta x' + y'\delta y' + z'\delta z') + \frac{\partial U}{\partial x}\delta x + \frac{\partial U}{\partial y}\delta y + \frac{\partial U}{\partial z}\delta z \right] dt .$$

Now, one has:

$$\delta x' = \frac{\partial}{\partial \alpha}\left(\frac{\partial x}{\partial t}\right)\delta\alpha = \frac{\partial}{\partial t}\left(\frac{\partial x}{\partial \alpha}\delta\alpha\right) = \frac{\partial(\delta x)}{\partial t},$$

so an integration by parts then gives, if one remarks that δx, δy, δz vanish at the limits:

$$\delta W = \int_{t_0}^{t_1} \left[\left(\frac{\partial U}{\partial x} - m\frac{d^2 x}{dt^2}\right)\delta x + \left(\frac{\partial U}{\partial y} - m\frac{d^2 y}{dt^2}\right)\delta y + \left(\frac{\partial U}{\partial z} - m\frac{d^2 z}{dt^2}\right)\delta z \right] dt .$$

If one makes δW zero for $\alpha = 0$ and all of the functions δx, δy, δz of t zero at the limits then, by a classical argument, it is necessary and sufficient that one have for $\alpha = 0$:

$$\text{(3)} \qquad \begin{cases} m\dfrac{d^2x}{dt^2} = \dfrac{\partial U}{\partial x}, \\ m\dfrac{d^2y}{dt^2} = \dfrac{\partial U}{\partial y}, \\ m\dfrac{d^2z}{dt^2} = \dfrac{\partial U}{\partial z}. \end{cases}$$

It then results from this that *the motions that the material point makes under the action of the given force realize the extremum for the integral W with respect to all of the infinitely close possible motions that correspond to the same initial and final positions of the point, and, moreover, these motions are the only ones that enjoy this property.*

To be rigorous, one may speak only of the *extremum* of the action and not of the *minimum*, because the condition that the first variation δW vanish is a necessary, but not sufficient, condition for a minimum.

2. The elementary action:

$$\left[\tfrac{1}{2}m(x'^2 + y'^2 + z'^2) + U\right]dt$$

seems to have been introduced here in a purely artificial way in order to state the laws of motion in a condensed form. We shall see that one may replace Hamilton's principle with another principle that is equivalent to it, in which a linear expression in dx, dy, dz, dt also appears, but where all of the coefficients have a simple mechanical significance.

Indeed, continue to use the same action W, but now suppose that t_0 and t_1 are themselves functions of the parameter α, while the corresponding values x_0, y_0, z_0, x_1, y_1, z_1 are also functions of α. If one applies the preceding derivation to the definite integral then the calculation of δW would give:

$$\begin{aligned} \delta W = {} & \left[\tfrac{1}{2}m(x'^2 + y'^2 + z'^2) + U\right]_{t=t_1} \delta t_1 - \left[\tfrac{1}{2}m(x'^2 + y'^2 + z'^2) + U\right]_{t=t_0} \delta t_0 \\ & + [mx'\delta x + my'\delta y + mz'\delta z]_{t=t_1} - [mx'\delta x + my'\delta y + mz'\delta z]_{t=t_0} \\ & + \int_{t_0}^{t_1}\left[\left(\frac{\partial U}{\partial x} - m\frac{d^2x}{dt^2}\right)\delta x + \left(\frac{\partial U}{\partial y} - m\frac{d^2y}{dt^2}\right)\delta y + \left(\frac{\partial U}{\partial z} - m\frac{d^2z}{dt^2}\right)\delta z\right]dt\,. \end{aligned}$$

We now remark that one has:

$$[\delta x]_{t=t_1} = \left[\frac{\partial x}{\partial \alpha}\right]_{t=t_1} \delta\alpha , \qquad \delta x_1 = \left[\frac{\partial x}{\partial t}\right]_{t=t_1} \delta t_1 + \left[\frac{\partial x}{\partial \alpha}\right]_{t=t_1} \delta\alpha ,$$

and, as a result:

$$[\delta x]_{t=t_1} = \delta x_1 - x_1' \, \delta t_1 .$$

The formula that gives δW is thus:

$$(4) \quad \begin{cases} \delta W = \quad mx_1'(\delta x_i - x_i'\delta t_1) + my_1'(\delta y_1 - y_1'\delta t_1) + mz_1'(\delta z_i - z_1'\delta t_1) \\ \qquad + \left[\tfrac{1}{2} m(x_1'^2 + y_1'^2 + z_1'^2) + U_1\right]\delta t_1 \\ \qquad - \{mx_0'(\delta x_0 - x_0'\delta t_0) + my_0'(\delta y_0 - y_0'\delta t_0) + mz_0'(\delta z_0 - z_0'\delta t_0) \\ \qquad + \left[\tfrac{1}{2} m(x_0'^2 + y_0'^2 + z_0'^2) + U_0\right]\delta t_0\} \\ + \int_{t_0}^{t_1} \left[\left(\frac{\partial U}{\partial x} - m\frac{d^2 x}{dt^2}\right)\delta x + \left(\frac{\partial U}{\partial y} - m\frac{d^2 y}{dt^2}\right)\delta y + \left(\frac{\partial U}{\partial z} - m\frac{d^2 z}{dt^2}\right)\delta z\right] dt. \end{cases}$$

Set:

$$(5) \quad \begin{cases} \omega_\delta = mx'(\delta x - x'\delta t) + my'(\delta y - y'\delta t) + mz'(\delta z - z'\delta t) \\ \qquad + \left[\tfrac{1}{2} m(x'^2 + y'^2 + z'^2) + U\right]\delta t \\ = mx'\delta x + my'\delta y + mz'\delta z - \left[\tfrac{1}{2} m(x'^2 + y'^2 + z'^2) - U\right]\delta t. \end{cases}$$

The differential expression that is thus introduced has for its components, first:

$$mx', my', mz' ,$$

i.e., the components of the *quantity of motion* of the moving point, and then:

$$\tfrac{1}{2} m(x'^2 + y'^2 + z'^2) - U ,$$

i.e., the *energy E*.

Thanks to this notation, one may write:

$$\delta W = [\omega_\delta]_0^1 + \int_{t_0}^{t_1} \left[\left(\frac{\partial U}{\partial x} - m\frac{d^2 x}{dt^2}\right)\delta x + \left(\frac{\partial U}{\partial y} - m\frac{d^2 y}{dt^2}\right)\delta y + \left(\frac{\partial U}{\partial z} - m\frac{d^2 z}{dt^2}\right)\delta z\right] dt .$$

Now suppose that one considers a collection of real trajectories that depend on a parameter α, and that one limits each trajectory to an interval of time (t_0, t_1) that varies with α. The formula that gives the variation of the action along these variable trajectories reduces to

$$\delta W = (\omega_\delta)_1 - (\omega_\delta)_0 .$$

Finally, suppose that we consider a *tube of trajectories*, i.e., a *closed* continuous linear collection of trajectories, each of which is limited to a time interval (t_0, t_1). When one returns to the initial trajectory the total variation of the action is obviously zero, in such a way that, if one integrates with respect to α then one will have:

$$\int (\omega_\delta)_1 = \int (\omega_\delta)_0 .$$

3. In order to interpret the result we obtained, we agree to call the seven quantities:

$$x, y, z, x', y', z', t$$

the *state* of the material point, in which the first three quantities define the position of the point, the next three define its velocity, and the last one defines the instant when the point is considered. One may regard a *state* as a point in a seven-dimensional space: *the state space.* A *trajectory* may be defined as the collection of all states that correspond to exactly one real motion of the point, i.e., in summation, it is a solution of the system of differential equations:

$$(6) \qquad \begin{cases} \dfrac{dx}{dt} = x', & m\dfrac{dx'}{dt} = \dfrac{\partial U}{\partial x}, \\ \dfrac{dy}{dt} = y', & m\dfrac{dy'}{dt} = \dfrac{\partial U}{\partial y}, \\ \dfrac{dz}{dt} = z', & m\dfrac{dz'}{dt} = \dfrac{\partial U}{\partial z}. \end{cases}$$

From this, when the *curvilinear integral*:

$$\int \omega_\delta = \int \quad mx'\, dx + my'\, \delta y + mz'\, \delta z - E\, \delta t$$

is taken along an arbitrary closed curve in the state space it does not vary if one displaces each of the states that comprise it along the trajectory that corresponds to that state in an arbitrary manner.

One may then say that, *given an arbitrary tube of trajectories, if the integral* $\int \omega_\delta$ *is taken along a closed curve that makes a circuit of the tube then that integral will be independent of that curve and will depend only upon the tube.*

One can remark that the expression ω_δ may be regarded as the elementary work done on a vector in a universe of four dimensions (x, y, z, t). This vector will have the three ordinary components of the quantity of motion for its spatial components and energy for the component that corresponds to time.

4. If one considers a *simultaneous* collection of states – i.e., if one supposes that $\delta t = 0$ – then the integral $\int \omega_\delta$ will reduce to:

$$\int mx' \, dx + my' \, \delta y + mz' \, \delta z.$$

If we assume the latter viewpoint then we will obtain the following theorem:

If one considers a closed collection of trajectories, and if one takes the states that correspond to an arbitrary fixed instant t then if the integral $\int mx' \, dx + my' \, \delta y + mz' \, \delta z$ *is taken over the closed collection of states thus obtained then that integral will be independent of t.*

This theorem is due to H. Poincaré, who characterized the property thus obtained by giving the name of *integral invariant* to the integral:

$$\int mx' \, dx + my' \, \delta y + mz' \, \delta z$$

when it is taken over a closed contour.

In Poincaré's way of thinking, the notion of energy is not involved. It necessarily appears if, instead of considering a closed collection of *simultaneous* states, one considers a closed collection of *arbitrary* states.

We say that the integral $\int \omega_\delta$ of the "quantity of motion-energy" is a *complete integral invariant* – or, more simply, an *integral invariant,* when there is no danger of confusion – for the differential equations of motion. Poincaré's integral invariant is thus the complete integral invariant of the "quantity of motion-energy" when it is viewed in a particular light.

It is remarkable that if, instead of considering a collection of simultaneous states, one considers a collection of states that satisfy the relations:

$$\delta x = x' \, \delta t, \qquad \delta y = y' \, \delta t, \qquad \delta z = z' \, \delta t$$

then the tensor ω_δ will reduce to Hamilton's elementary action:

$$\left[\tfrac{1}{2} m(x'^2 + y'^2 + z'^2) + U\right] \delta t \, .$$

As a result, *H. Poincaré's integral invariant and Hamilton's action are different aspects of the integral of the "quantity of motion-energy,"* although on first glance there is no relation between these two notions.

5. In the preceding section, we simply deduced a *property* of the "quantity of motion-energy" tensor from Hamilton's principle, namely, that the integral of this tensor along a closed curve of states does not change when one deforms this closed curve without changing the trajectories over which it is taken. We shall now show that *this property may replace Hamilton's principle, i.e., the differential equations of motion are the ones that admit the integral* $\int \omega_\delta$ *as an integral invariant when it is taken over an arbitrary closed contour.*

Let:

$$\frac{dx}{X} = \frac{dy}{Y} = \frac{dz}{Z} = \frac{dx'}{X'} = \frac{dy'}{Y'} = \frac{dz}{Z'} = \frac{dt}{T}$$

be an arbitrary system of differential equations whose denominators are particular functions of the seven variables x, y, z, x', y', z', t. Imagine a tube of integral curves of this system that depend upon one parameter α. This parameter will vary, for example, from 0 to l, such that the integral curve that corresponds to $\alpha = 1$ coincides with the one that corresponds to $\alpha = 0$. In order to express the idea that when the integral $\int \omega_\delta$ is taken over a closed curve that makes a circuit around this tube it does not depend on the chosen closed curve, we imagine that the coordinates x, y, z, x', y', z', t of an arbitrary state of the tube are functions of the parameter α and another parameter u. If one gives a fixed value to u then one will have a closed curve that makes a circuit of this tube. If one displaces that closed curve along an integral curve of the tube then one will have:

$$\rho\, du = \frac{dx}{X} = \frac{dy}{Y} = \ldots = \frac{dt}{T},$$

in which ρ denotes an arbitrary factor *that one may always choose in such a manner as to obtain an arbitrary succession, given in advance, of closed contours that go around the tube for u =* const.

Having said this, when one gives a definite value to u the integral $I = \int_{(C)} \omega_\delta$ as a function of u and, if one reserves the notation d for a displacement that makes only u vary, then one will have:

$$dI = \int_{(C)} m dx' \delta x + m\, dy'\, \delta y + m\, dz'\, \delta z - dE\, \delta t + m\, x'\, d(\delta x) + m\, y'\, d(\delta y) + m\, z'\, d(\delta z) - E\, d(\delta t),$$

or, upon changing the order of differentiations of d and δ and integrating by parts:

$$\begin{aligned} dI = {} & [\, mx'\, dx + my'\, dy + mz'\, dz - E\, dt]_C \\ & + \int_{(C)} (\, mdx'\, \delta x + mdy'\, \delta y + mdz'\, \delta z - dE\, \delta t \\ & \qquad - mdx\, \delta x' - mdy\, \delta y' - mdz\, \delta z' + dt\, \delta E). \end{aligned}$$

The total integral part is obviously zero *since the integration contour is closed.* As for the integral that remains in the right-hand side, in order for $\int \omega_\delta$ to be an integral invariant for the differential system considered, it is necessary and sufficient that the integral must vanish when one replaces:

$$dx,\ dy,\ dz,\ dx',\ dy',\ dz',\ dt$$

with:

$$\rho X,\ \rho Y,\ \rho Z,\ \rho\, X',\ \rho\, Y',\ \rho\, Z',\ \rho\, T,$$

respectively, and that this must be true *for any closed contour* (C) *and any function* ρ. One easily deduces from this that the coefficients of:

$$dx,\ dy,\ dz,\ dx',\ dy',\ dz',\ dt$$

become identically zero. As a result, *in order for a system of differential equations to admit the integral invariant* $\int \omega_\delta$, *it is necessary and sufficient that the equations:*

$$(6) \qquad \begin{cases} m\,dx' + \dfrac{\partial E}{\partial x}dt = 0, \\ m\,dy' + \dfrac{\partial E}{\partial y}dt = 0, \\ m\,dz' + \dfrac{\partial E}{\partial z}dt = 0, \\ -m\,dx + \dfrac{\partial E}{\partial x'}dt = 0, \\ -m\,dy + \dfrac{\partial E}{\partial y'}dt = 0, \\ -m\,dz + \dfrac{\partial E}{\partial z'}dt = 0, \\ -dE + \dfrac{\partial E}{\partial t}dt = 0, \end{cases} \qquad \text{or} \qquad \begin{cases} m\,dx' - \dfrac{\partial U}{\partial x}dt = 0, \\ m\,dy' - \dfrac{\partial U}{\partial y}dt = 0, \\ m\,dz' - \dfrac{\partial U}{\partial z}dt = 0, \\ -m\,dx + m x' dt = 0, \\ -m\,dy + m y' dt = 0, \\ -m\,dz + m z' dt = 0, \\ -m(x'dx' + y'dy' + z'dz') + dU - \dfrac{\partial U}{\partial t}dt = 0 \end{cases}$$

be consequences of the differential equations of the system.

The first six of these equations are nothing but the classical differential equations of motion. As for the seventh one, it has the *vis viva* theorem as a consequence.

6. In the preceding, one sees the fundamental role that is played by the "quantity of motion-energy" tensor. *If one assumes that a trajectory is defined to be a succession of states that constitute a solution for a system of ordinary differential equations then this system will be characterized by the property that, among all of the imaginable systems of differential equations, it will admit the curvilinear integral of the "quantity of motion-energy" tensor, when taken over an arbitrary closed contour of states, as an integral invariant.*

One thus obtains a new principle that may be called *the principle of the conservation of the quantity of motion and energy.*

II. - *General case.*

7. All of the foregoing may be extended to the material systems that one habitually considers in analytical mechanics. We suppose that these systems satisfy three conditions:

1. *The constraints to which they are subject are perfect*, i.e., at each instant *t*, the sum of the elementary works done by the constraint forces is zero for any virtual displacement that is

compatible with the constraints that exist at that the instant t. Under these conditions, d'Alembert's principle is valid and may be stated in the form:

d'ALEMBERT'S PRINCIPLE. – *If one considers the motion of a material system that is subject to perfect constraints under the action of given forces then at each instant the sum of the elementary works done by the given forces and the forces of inertia is zero for any virtual displacement of the system that is compatible with the constraints that exist at the instant t.*

d'Alembert's principle translates into the formula:

$$\sum\left[\left(X-m\frac{d^2x}{dt^2}\right)\delta x+\left(Y-m\frac{d^2y}{dt^2}\right)\delta y+\left(Z-m\frac{d^2z}{dt^2}\right)\delta z\right]=0, \tag{7}$$

in which X, Y, Z denote the components of the given force that is applied to the point (x, y, z) of mass m and δx, δy, δz denote the components of the most general elementary displacement that is compatible with the constraints.

Amongst all of the systems with perfect constraints, we now consider the ones that have *holonomic* constraints, i.e.:

2. *We suppose that the constraints can be translated into a finite number of equations between the coordinates of the points of the system and time t.* Again, this amounts to saying that it is possible to express the coordinates of the different points of the system by formulas such as

$$\begin{aligned} x_i &= f_i(q_1, \ldots, q_n, t), \\ y_i &= g_i(q_1, \ldots, q_n, t), \qquad\qquad (i = 1, 2, \ldots) \\ z_i &= h_i(q_1, \ldots, q_n, t), \end{aligned}$$

with n arbitrary parameters q. To each system of values of q and t there corresponds one and only one position of the system that is compatible with the constraints that exist at the instant t. Any virtual displacement that is compatible with the constraints that exist at the instant t may be obtained by giving arbitrary increments $\delta q_1, \ldots, \delta q_n$ to $q_1, \ldots, q_n$.

We make one last hypothesis:

3. *For any arbitrary virtual displacement that is compatible with the constraints that exist at the instant t, the sum of the elementary works that are done by the given forces is the total differential of a certain function U of the q and t*, i.e.:

$$\sum(X\,\delta x+Y\,\delta y+Z\,\delta z)=\frac{\partial U}{\partial q_1}\delta q_1+\ldots+\frac{\partial U}{\partial q_n}\delta q_n.$$

The term $\frac{\partial U}{\partial t}\delta t$ does not appear in the right-hand side because the virtual displacements to which d'Alembert's principle refer suppose that t remains constant.

8. Hamilton's least-action principle may be extended without difficulty to the preceding systems.

Set:

$$W = \int_{t_0}^{t_1} \left[\tfrac{1}{2} \sum m(x'^2 + y'^2 + z'^2) + U \right] dt .$$

Regard the parameters $q_1, ..., q_n$, as functions of t and a parameter α; while the lower and upper limits of the integral may depend on α. A calculation that is identical to the one that was given above (sec. **2**) gives us the variation δW of the *action* when one gives a variation $\delta\alpha$ to α. One obtains:

$$(8) \qquad \delta W = [\omega_\delta]_1 - [\omega_\delta]_0 + \int_{t_0}^{t_1} \left[\delta U - \sum m \left(\frac{d^2 x}{dt^2} \delta x + \frac{d^2 y}{dt^2} \delta y + \frac{d^2 z}{dt^2} \delta z \right) \right] dt,$$

if one sets:

$$(9) \qquad \begin{cases} \omega_\delta = \sum m(x'\delta x + y'\delta y + z'\delta z) - \left[\tfrac{1}{2} \sum m(x'^2 + y'^2 + z'^2) - U \right] \delta t, \\ [\omega_\delta]_1 = \sum m(x_1'\delta x_1 + y_1'\delta y_1 + z_1'\delta z_1) - \left[\tfrac{1}{2} \sum m(x_1'^2 + y_1'^2 + z_1'^2) - U_1 \right] \delta t_1, \\ [\omega_\delta]_0 = \sum m(x_0'\delta x_0 + y_0'\delta y_0 + z_0'\delta z_0) - \left[\tfrac{1}{2} \sum m(x_0'^2 + y_0'^2 + z_0'^2) - U_0 \right] \delta t_0. \end{cases}$$

Having said this, d'Alembert's principle immediately shows that when one is given a real motion of the system, and one considers this motion in an arbitrary interval of time (t_0, t_1) then this motion will realize an *extremum* of the action W with respect to all of the imaginable infinitely close motions that correspond to the same initial *position* and the same final *position* of the system. Conversely, the only motions that enjoy this property are the real motions of the system. *This is Hamilton's least-action principle.*

Formula (8) shows, moreover, that when the integral $\int \omega_\delta$ is taken over a closed contour of *states* of the system (that is compatible with the constraints), it does not change if one deforms this closed contour by displacing each of the states that constitute it in an arbitrary manner along the corresponding system trajectory. In other words, *the integral* $\int \omega_\delta$ *is an integral invariant for the differential equations of motion.*

If one supposes that one considers only states of the system that are compatible with the constraints then the differential form ω_δ may again be called the "quantity of motion-energy" tensor of the system.

9. The differentials δx, δy, δz, δt that enter into the expression ω_δ are not arbitrary, in general, because they must satisfy the equations that are obtained by totally differentiating the constraint equations of the system. One may also express them in terms of

$$\delta q_1 , \delta q_2 , ..., \delta q_n , \delta t$$

if one has introduced the n position parameters of the system. We shall take this viewpoint and determine, on the one hand, the differential equations of motion, and, on the other, the "quantity of motion-energy” tensor. It will suffice for us to calculate δW while supposing that the elementary action is expressed by means of the parameters of q and time t. Set:

$$T = \sum \tfrac{1}{2} m(x'^2 + y'^2 + z'^2).$$

T, the kinetic energy, is a function of the second degree in the derivatives $\dfrac{dq_i}{dt}$, which we write as q_i', and which we regard as independent arguments of q_i and t. We temporarily set:

$$F = T + U, \qquad W = \int_{t_0}^{t_1} F dt.$$

A simple calculation gives:

$$\delta W = F_1 \delta t_1 - F_0 \delta t_0 + \int_{t_0}^{t_1} \sum \left(\frac{\partial F}{\partial q_i} \delta q_i + \frac{\partial F}{\partial q_i'} \delta q_i' \right) dt .$$

However:

$$\delta q_i' dt = \delta \frac{\partial q_i}{\partial t} dt = \frac{\partial}{\partial t} (\delta q_i) dt = d(\delta q_i),$$

so one has, after integrating by parts:

$$\delta W = F_1 \delta t_1 - F_0 \delta t_0 + \left[\sum \frac{\partial F}{\partial q_i'} \delta q_i' \right]_{t_0}^{t_1} + \int_{t_0}^{t_1} \sum \left[\frac{\partial F}{\partial q_i} - \frac{d}{dt} \left(\frac{\partial F}{\partial q_i'} \right) \right] dt.$$

In conclusion, we remark that one has:

$$[\delta q_i]_{t_0} = \frac{\partial}{\partial \alpha} q_i(t_0, \alpha) \delta\alpha \qquad \text{and} \qquad \delta(q_i^0) = \frac{\partial q_i(t_0, \alpha)}{\partial t} \delta t_0 + \frac{\partial q_i(t_0, \alpha)}{\partial \alpha} \delta\alpha ,$$

hence:

$$[\delta q_i]_{t_0} = \delta(q_i^0) - q_i' \delta t_0 \qquad \text{and} \qquad [\delta q_i]_{t_1} = \delta(q_i^{(1)}) - q_i'^{(1)} \delta t .$$

Finally, one has

$$(10) \qquad \left\{ \begin{aligned} \delta W = {} & \sum \left(\frac{\partial F}{\partial q_i'} \right)_1 \delta(q_i^{(1)}) - \left(\sum q_i' \frac{\partial F}{\partial q_i'} - F \right)_1 \delta t_1 \\ & - \left[\sum \left(\frac{\partial F}{\partial q_i'} \right)_0 \delta(q_i^0) - \left(\sum q_i' \frac{\partial F}{\partial q_i'} - F \right)_0 \delta t_0 \right] \\ & + \int_{t_0}^{t_1} \sum \left[\frac{\partial F}{\partial q_i} - \frac{d}{dt} \left(\frac{\partial F}{\partial q_i'} \right) \right] dt. \end{aligned} \right.$$

Hamilton's principle then leads us to the following equations of motion, which are nothing but the *Lagrange equations:*

(11) $$\frac{d}{dt}\left(\frac{\partial T}{\partial q_i'}\right)-\frac{\partial T}{\partial q_i}-\frac{\partial U}{\partial q_i}=0 \qquad (i = 1, 2, ..., n).$$

Comparing the two expressions (8) and (10) that we found for δW leads to the following expression for the tensor ω_δ:

(12) $$\omega_\delta=\sum\frac{\partial T}{\partial q_i'}\delta q_i - H\,\delta t,$$

in which we set:

(13) $$H=\sum q_i'\frac{\partial T}{\partial q_i'}-T-U.$$

The quantities $\frac{\partial T}{\partial q_i'}$ are the quantities of generalized motion (with respect to the chosen system of coordinates); the quantity H is the *generalized energy*.

10. A simple remark permits us to simplify the calculation of the generalized energy H in practice. In general, the kinetic energy T may contain terms of second degree, first degree, and zero degree in $q_1', q_2', ..., q_n'$, namely:

$$T = T_2 + T_1 + T_0 .$$

An application of Euler's formula concerning homogeneous functions then immediately gives:

$$H = T_2 - T_0 - U.$$

In the generalized energy, the term T_2 may be regarded as of kinetic origin since the term $-T_0 - U$ is of dynamic origin.

For example, take the case of a free material point whose axes rotate around Oz with angular velocity r. One has:

$$2T = m\,[(x' - ry)^2 + (y' + rx)^2 + z'^2],$$

and, as a result, the energy, *when referred to the chosen reference system,* is:

$$H = \tfrac{1}{2}\,m\,(x'^2 + y'^2 + z'^2) - \tfrac{1}{2}\,mr^2(x^2 + y^2) - U.$$

The part of the energy that is dynamical in origin may be decomposed into two terms, one of which provides the given forces and the other provides the centrifugal forces. As for the components of the quantity of motion, they are:

$$m(x' - ry), \qquad m(y' + rx), \qquad m\,z',$$

i.e., they are the projections of the quantity of *absolute* motion onto the chosen coordinate axes

11. *Hamilton's canonical variables.* – When the equations of motion are regarded as first-order differential equations in q_i, q_i', t they take an extremely simple form if one introduces the variables:

$$p_i = \frac{\partial T}{\partial q_i'}. \tag{14}$$

The new variables, which one substitutes for q_i', are all simply the components of the quantity of motion of the system. The tensor ω_δ then takes the simple form:

$$\omega_\delta = \sum p_i\,\delta q_i - H\,\delta t, \tag{15}$$

in which H may be regarded as a function of the q_i, the p_i, and t.

We shall search for the equations of motion directly *after expressing the idea that they must admit the integral* $\int \omega_\delta$ *as an integral invariant when it is taken over an arbitrary closed curve of states of the system.*

Let:

$$\frac{dq_1}{Q_1} = \frac{dq_2}{Q_2} = \dots = \frac{dp_n}{P_n} = \frac{dt}{T} \tag{16}$$

be an arbitrary system of differential equations. In order to express the idea that it admits the integral invariant $\int \omega_\delta$, we need only to repeat the argument of sec. 5, word for word. We consider a tube of integral curves of system (16) and express the $2n + 1$ coordinates p_i, q_i, t of a state of the tube as functions of the two parameters α and u, the first of which is constant on an integral curve and varies in the interval (0, l) in such a manner that the integral curve that is defined by $\alpha = 1$ coincides with the integral curve that is defined by $\alpha = 0$. We let d denote the symbol for differentiation with respect to the variable u and set:

$$I = \int_{(C)} \omega_\delta$$

We then have, by an immediate integration by parts:

$$dI = \int_{(C)} (dp_i\,\delta q_i - dq_i\,\delta p_i) - dH\,\delta t + dt\,\delta H\,.$$

In order for the system (16) to admit the integral invariant $\int \omega_\delta$, it is necessary and sufficient that the coefficients of:

$$\delta q_1,\ \delta q_2,\ \dots,\ \delta q_n,\ \delta t$$

in the quantity under the $\int$ sign vanish everywhere, if one takes the equations of the system into account. Now, if one annuls these coefficients then one will obtain the $2n + 1$ equations:

(17)
$$\begin{cases} dp_i + \dfrac{\partial H}{\partial q_i} dt = 0, \\ -dq_i + \dfrac{\partial H}{\partial p_i} dt = 0, \\ -dH + \dfrac{\partial H}{\partial t} dt = 0. \end{cases}$$

This shows that *there is only one system of differential equations that admits the integral invariant* $\int \omega_\delta$, and that would be the system that gives us both of the equations of motion in the *canonical* form of Hamilton:

(18)
$$\begin{cases} \dfrac{dq_i}{dt} = \dfrac{\partial H}{\partial p_i}, \\ \dfrac{dp_i}{dt} = -\dfrac{\partial H}{\partial q_i}. \end{cases}$$

The last equation:

$$dH - \frac{\partial H}{\partial t} dt = 0$$

is the analytical translation of the *vis viva* theorem; it is a consequence of the first $2n$ equations.

12. In the general case of the material systems in analytical mechanics, we thus arrive at the *generalized principle of the conservation of the quantity of motion and energy:*

If one assumes that any motion of a system that is subject to given forces is a continuous succession of states that satisfy a system of first-order differential equations then these differential equations will be characterized by the property that they admit the integral of the "quantity of motion-energy" tensor as an integral invariant when it is taken over an arbitrary closed contour of states.

The "quantity of motion-energy" tensor can arbitrarily take any one of the forms:

$$\omega_\delta = \sum m(x'\delta x + y'\delta y + z'\delta z) - \left[\sum \tfrac{1}{2} m(x'^2 + y'^2 + z'^2) - U\right]\delta t,$$

$$\omega_\delta = \sum \frac{\partial T}{\partial q_i'} \delta q_i - H\delta t \qquad \left(H = \sum q_i' \frac{\partial T}{\partial q_i'} - T - U\right),$$

$$\omega_\delta = \sum p_i \delta q_i - H\delta t.$$

If one displaces oneself in the space of states in such a way as to satisfy the relations:

$$\delta q_i = q_i' \, dt$$

then the expression ω_δ will reduce to Hamilton's elementary action $(T + U)\delta t$. On the contrary, if one considers only a collection of simultaneous states ($\delta t = 0$) then one will obtain the expression:

$$\sum p_i\,\delta q_i$$

that constitutes the element under the $\int$ sign in the integral invariant that was proposed by H. Poincaré.

13. The principle of conservation of the quantity of motion and energy permits us to form equations of motion no matter how we choose the parameters $q_1, \ldots, q_n$, t that serve to localize the system in space and time. In other words, it gives us the laws of mechanics, since that fact rests *implicitly* upon Hamilton's principle in a form that is *independent of any particular mode of spacetime framing.* This property becomes analytically obvious if, instead of introducing the derivatives $q_1', \ldots, q_n'$ of the spatial parameters with respect to the temporal parameter, one introduces $n + 1$ quantities:

$$\dot{q}_1, \dot{q}_2, \ldots, \dot{q}_n, \dot{t},$$

whose mutual relations are defined by the equalities:

$$\frac{\dot{q}_1}{q_1'} = \frac{\dot{q}_2}{q_2'} = \ldots = \frac{\dot{q}_n}{q_n'} = \frac{\dot{t}}{1}.$$

If one sets:

$$F = \dot{t}(T + U),$$

the right-hand side of which is homogeneous of the first degree in $\dot{q}_1, \ldots, \dot{q}_n, \dot{t}$ and is expressed in terms of the q_i, t, $\dot{q}_i$, $\dot{t}$, then the "quantity of motion-energy" tensor will take the form:

$$\omega_\delta = \frac{\partial F}{\partial \dot{q}_1}\delta q_1 + \ldots + \frac{\partial F}{\partial \dot{q}_n}\delta q_n + \frac{\partial F}{\partial \dot{t}}\delta t.$$

In the general theory of relativity, the motion of a point that is subject to gravitational forces obeys the preceding principle. The function F will then be of the form:

$$F = \sqrt{\sum a_{ik}\dot{q}_i\dot{q}_k},$$

in which the four variables q_i serve to localize the point in space and time.

III. - *Transformation of the canonical equations. Jacobi's theorem.*

14. An important application of the preceding considerations relates to the transformation of the canonical equations and the method for integrating the dynamical equations that is due to Jacobi.

When the integral $\int \omega_\delta$ is taken over a closed contour, it obviously does not change if one adds an exact differential to ω_δ. Conversely, if another linear differential form ϖ_δ enjoys the same property of giving the same integral as ω_δ when one takes it over an arbitrary closed contour then ϖ_δ will differ from ω_δ only by an exact differential.

Suppose then that one must find $2n$ new variables r_i, s_i, and a function K such that the two expressions:

$$\omega_\delta = \sum p_i\,\delta q_i - H\delta t,$$
$$\varpi_\delta = \sum r_i\,\delta s_i - K\delta t$$

differ only by an exact differential. The differential equations of motion may be characterized by the property that they admit the integral invariant $\int \varpi_\delta$, and, as a result, they may be written:

$$\frac{ds_i}{dt} = \frac{\partial K}{\partial r_i}, \qquad \frac{dr_i}{dt} = -\frac{\partial K}{\partial s_i},$$

so the canonical form of these equations is preserved.

The hypothesis may be translated by an identity into the form:

$$\text{(19)} \qquad \sum p_i\,\delta q_i - \sum r_i\,\delta s_i - (H - K)\,\delta t = \delta V.$$

It is easy to realize such an identity. Indeed, start with an arbitrary function V of $2n + 1$ arguments q_i, s_i, t, and set:

$$\text{(20)} \qquad p_i = \frac{\partial V}{\partial q_i}, \qquad r_i = -\frac{\partial V}{\partial s_i}, \qquad K = \frac{\partial V}{\partial t} + H.$$

If these equations define a change of variables – i.e., if the first n are solvable with respect to the $s_1, s_2, \ldots, s_n$ – then the following n will give $r_1, \ldots, r_n$, the latter will give the function K, and the new variables that are thus obtained will preserve the canonical form of the dynamical equations. It is important to remark that if equations (20) are solvable in terms of the r_i and the s_i then, conversely, they will be solvable in terms of the p_i and q_i. In both cases, the possibility condition is that the determinant:

$$\left| \frac{\partial^2 V}{\partial q_i\,\partial s_j} \right|$$

must not be identically zero.

The solution thus obtained from identity (19) is not the most general solution. It leaves aside the case in which the $2n + 1$ quantities q_i , p_i , t are related by one or more relations. However, this singular case is easy to treat directly by giving the relations that exist between q_i , p_i , and t *a priori*.

15. The applications of the preceding general theory become particularly interesting in two cases.

The first one is the case in which the function K is identically zero. The canonical equations become:

$$\frac{ds_i}{dt} = 0\,, \qquad \frac{dr_i}{dt} = 0\,.$$

The equations of the trajectories reduce to:

$$s_i = a_i\,, \qquad\qquad r_i = b_i\,,$$

in which a_i and b_i are $2n$ arbitrary constants. From (20), in order for this to be the case, it is sufficient to find a function $V(t;\, q_1, \dots, q_n\,;\, a_1\,, \dots, a_n)$ that satisfies the partial differential equations:

$$\frac{\partial V}{\partial t} + H\left(t, q_i, \frac{\partial V}{\partial q_i}\right) = 0\,. \tag{21}$$

If this function V, into which n arbitrary constants a_1, ..., a_n enter, is such that the determinant:

$$\left| \frac{\partial^2 V}{\partial q_i \partial a_j} \right|,$$

is not identically zero then the equations of motion will be:

$$p_i = \frac{\partial V}{\partial q_i}, \qquad b_i = -\frac{\partial V}{\partial a_i};$$

this is *Jacobi's theorem.* The condition on the determinant amounts to saying that the function V is a *complete integral* of Jacobi's first-order partial differential equation (21).

The second application appears in the context of the theory of *perturbations.* Suppose that the function H is the sum of two terms H_1 and H_2, the second of which is very small with respect to the first one. This amounts to dividing the given forces into two groups, one of which is of little importance with respect to the other and is composed of what one calls *perturbing forces*. The method that is employed in celestial mechanics in this case consists of searching for a complete integral V of the Jacobi equation:

$$\frac{\partial V}{\partial t} + H_1 = 0$$

that involves only the principal term of the function *H*. The 2*n* new variables r_i , s_i thus introduced will be constant *if the perturbing forces do not exist.* They are thus the unperturbed trajectory parameters. *The introduction of these new variables will preserve the canonical form of the equations with the new function* $K = H_2$ *, i.e., the part of H that relates only to the perturbing forces.*

Nonetheless, we do not insist – at least for the moment – upon the canonical equations and Jacobi's theorem. In particular, the relation that exists between the integration of the dynamical equations and the integration of a first-order partial differential equation *that does not explicitly contain the unknown function* will be clarified some other time when we have shown that *one may associate a linear integral invariant* to any partial differential equation of that type – or, more generally, to any first-order partial differential equation that admits a known infinitesimal transformation.

CHAPTER II

THE TWO-DIMENSIONAL INTEGRAL INVARIANTS OF DYNAMICS

I. - *Formation of the two-dimensional integral invariant in dynamics*.

16. We have seen that Hamilton's elementary action can be obtained by supposing that in the expression:

$$\omega_\delta = \sum p_i\, \delta q_i - H\, \delta t$$

one has

$$\delta q_i = q_i'\, dt.$$

It is remarkable that *the trajectories of a material system will again realize an extremum for the integral:*

$$W = \int_{t_0}^{t_1} \sum p_i\, \delta q_i - H\, dt$$

if we simply suppose that the q_i and q_i' are arbitrary functions of t that are subject only to the condition that the q_i take given values at the limits. As in Hamilton's principle, one thus supposes only that the q_i' are the derivatives of the q_i with respect to time. Similarly, one may suppose, more generally, that the q_i and q_i' are functions of the same parameter u, which varies from 0 to 1, in such a way that the quantities q_i and t take given values at the limits.

An easy calculation gives:

$$\delta W = \left[\sum p_i\, \delta q_i - H\delta t\right]_{u=0}^{u=1} + \int_{u=1}^{u=1} \left(\sum (\delta p_i\, dq_i - \delta q_i\, dp_i) - \delta H\, dt + \delta t\, dH\right)$$

The total integral part is zero, by hypothesis. The extremal equations are thus obtained by annulling the coefficients of:

$$\delta q_1,\ \delta q_2,\ \ldots,\ \delta p_n\,,\ \delta t$$

in the quantity under the summation sign, but this calculation was done in sec. **11**, and we correctly gave the equations of motion in their canonical form.

17. The expression:

$$\sum (dp_i\, \delta q_i - dq_i\, \delta p_i) - dH\, \delta t + dt\, \delta H,$$

that we have encountered twice is linear with respect to both types of differentials. One may write it in the simpler form:

$$d\omega_\delta - \delta\omega_d$$

if one assumes that the two differentiation symbols are interchangeable. The expression that we denote by $\omega'(d, \delta)$ enjoys the property of being zero whenever the symbol d defines an elementary displacement in the direction of a trajectory in the state space, and the symbol δ defines an *arbitrary* elementary displacement. Moreover, upon expressing that property, we obtain relations between $dq_1, dq_2, ..., dp_n$, dt that define the differential equations of the trajectories, or from another point of view, the differential equations that admit the integral invariant $\int \omega_\delta$.

More generally, we now consider two *arbitrary* elementary displacements that are defined by the differentiation symbols δ and δ', and we propose to look for the significance of the bilinear form $\omega'(\delta, \delta')$. In order to do this, imagine a continuous two-dimensional set of states. One will realize such a set by taking functions of two parameters α and β for the q_i, p_i, and t. Each state of the set can be represented in the plane by a point with coordinates (α, β), and the set will be represented by an area in the plane. The symbols δ and δ' refer to increments in α alone and β alone, respectively. Therefore, consider four states A, B, C, D in the state space that correspond to the parameter values:

$\alpha,$	$\beta,$
$\alpha + \delta\alpha,$	$\beta,$
$\alpha,$	$\beta + \delta'\beta,$
$\alpha + \delta\alpha,$	$\beta + \delta'\beta,$

respectively, and form the integral $\int \omega$, which is taken over the closed contour $ABCD$. One obviously has:

$$\int_{AB} = \omega_\delta, \qquad \int_{AC} = \omega_{\delta'}, \qquad \int_{CD} = \omega_\delta + \delta'\omega_\delta, \qquad \int_{AC} = \omega_{\delta'} + \delta\omega_{\delta'},$$

and, as a result

$$\int_{ABCD} = \delta\omega_{\delta'} - \delta'\omega_\delta = \omega'(\delta, \delta').$$

18. The bilinear form $\omega'(\delta, \delta')$ that appears when summing an arbitrary state and two infinitely close states, and represents, from the foregoing, the value of the integral $\int \omega$ when it is taken over a closed contour, is an *invariant* of the system of differential equations of the corresponding trajectories, in the sense that it will not change value if one displaces each of the states along the corresponding trajectory. This form is also a double integral element. Moreover, if one regards, for example, p_1 and q_1 as the coordinates (which depend on two parameters α and β) of a point in the plane then the expression $\delta p_1 \delta' q_1 - \delta q_1 \delta' p_1$ will be the area element of this plane with respect to the curvilinear coordinates α and β, which one habitually writes as:

$$dp_1\, dq_1 \qquad \text{or} \qquad \delta p_1\, \delta q_1.$$

This leads us to the notion of a new integral invariant:

$$\iint \omega' = \iint \sum \delta p_i \, \delta q_i - \delta' H \, \delta t. \tag{1}$$

This double integral, when taken over a two-dimensional area in the state space, is reproduced if one displaces each of the states of that area along the corresponding trajectory. This double integral is then obtained by the generalized Stokes formula as the expression of the curvilinear integral:

$$\int \omega = \int \left(\sum p_i \delta q_i - H \delta t \right),$$

when it is taken over the closed contour that bounds the area.

In Poincaré's way of thinking, one considers only areas that are composed of *simultaneous* states. One may then state the result obtained in the following form:

Given a two-dimensional set of trajectories, if one takes the set of states on each trajectory that correspond to a given instant t then the double integral:

$$\iint \sum \delta p_i \, \delta q_i$$

will be independent of t when it is taken over these states.

As one sees, this theorem expresses a particular aspect of the property that was proved above.

19. Poincaré calls the two-dimensional integral invariant $\iint \omega'$ *absolute,* as opposed to the invariant $\int \omega$, which he calls *relative.* This signifies that the double integral $\iint \omega'$ possesses an invariant character for any domain of integration – whether *open* or *closed,* whereas the integral $\int \omega$ possesses an invariant character only if it is taken over a *closed* contour.

Since the integral $\iint \omega'$ is nothing but the integral $\int \omega$ when it is taken over a closed contour, one may confirm that the *differential equations of motion are the only ones that admit the integral invariant* $\iint \omega'$. The invariance of the integral $\iint \omega'$ is the analytic translation of a new generalized principle in the form of the conservation of the quantity of motion and energy.

II. - *Applications to vortex theory.*

20. Up until now, we have considered only sets of trajectories that are realized in our imagination. There is one case where such sets have a concrete existence. It is the case of a perfect fluid that is subject to forces that are derived from a force function *U*. In hydrodynamics, one proves the following equations:

(2)
$$\begin{cases} \gamma_x = \dfrac{\partial U}{\partial x} - \dfrac{1}{\rho}\dfrac{\partial p}{\partial x}, \\ \gamma_y = \dfrac{\partial U}{\partial y} - \dfrac{1}{\rho}\dfrac{\partial p}{\partial y}, \\ \gamma_z = \dfrac{\partial U}{\partial z} - \dfrac{1}{\rho}\dfrac{\partial p}{\partial z}, \end{cases}$$

in which γ_x, γ_y, γ_z denote the components of the acceleration of the molecule that occupies the position (x, y, z), and p and ρ denote the pressure and density at that point, respectively.

We add to these hypotheses that there is a relation between p and ρ that is given in advance, which is certainly true if the motion is isothermal

If we direct our attention to a given motion of the fluid then we may regard p as a given function of x, y, z, t, and by setting:

$$q = \int \frac{dp}{\rho}$$

we will see that *each molecule moves like a material point of mass* 1 *that has been placed in a force field that is derived from a force function* $U - q$.

We thus have a concrete realization of an infinitude of trajectories of a moving point subject to given forces. We remark that the $-q$ part of the force function represents the action that is exerted on the molecule by the molecules in its environment.

21. The trajectory of each molecule can be regarded as a particular solution of the system of differential equations:

(3)
$$\begin{aligned} \frac{dx}{dt} &= u, & \frac{du}{dt} &= \frac{\partial(U-q)}{\partial x}, \\ \frac{dy}{dt} &= v, & \frac{dv}{dt} &= \frac{\partial(U-q)}{\partial y}, \\ \frac{dz}{dt} &= w & \frac{dw}{dt} &= \frac{\partial(U-q)}{\partial z}. \end{aligned}$$

Thus, if one considers a closed collection of molecules in the fluid (each taken at an arbitrary instant) then the integral:

(4) $$\int u\,\delta x + v\,\delta y + w\,\delta z - E\,\delta t$$

will not change in value if one displaces each molecule along its trajectory when it is taken over that closed collection. In this expression, one has set:

(5) $$E = \tfrac{1}{2}(u^2 + v^2 + w^2) - U + q.$$

E is the energy (per unit mass) of the fluid. This energy is the sum of the kinetic energy $\frac{1}{2}(u^2 + v^2 + w^2)$, the potential energy $-U$, and the internal hydrodynamic energy q.

In particular, if one considers a closed collection of molecules, all of which are considered at the same instant t – i.e., a closed streamline – then the integral $\int u\,\delta x + v\,\delta y + w\,\delta z$ *will keep the same value if one takes the same fluid line (i.e., the streamline that is composed of the same molecules) at different instants of its motion.* This is the classical theorem of the *conservation of circulation.* One gives the name of *circulation* to the integral $\int u\,\delta x + v\,\delta y + w\,\delta z$.

22. Now let us take a different point of view. Let us always consider a particular motion of the fluid mass. In this motion, the components u, v, w of the velocity are given functions of x, y, z, t, and the trajectories of the different molecules may be regarded as solutions of the system of differential equations:

$$\text{(6)} \qquad \begin{cases} \dfrac{dx}{dt} = u, \\ \dfrac{dy}{dt} = v, \\ \dfrac{dz}{dt} = w, \end{cases}$$

on the right-hand side of which u, v, w are assumed to be replaced by their values as functions of x, y, z, t. The integral:

$$\int u\,\delta x + v\,\delta y + w\,\delta z - E\,\delta t$$

is again obviously a relative integral invariant for these new differential equations. Upon transforming it into a double integral, we obtain an absolute integral for the system (6).

If we form the expression $\delta\omega_{\delta'} - \delta'\omega_{\delta}$ then we will obtain:

$$\omega'(\delta, \delta') = \delta u\,\delta' x - \delta x\,\delta' u + \delta v\,\delta' y - \delta y\,\delta' v + \delta w\,\delta' z - \delta z\,\delta' w - \delta E\,\delta' t + \delta t\,\delta' E\,.$$

The right-hand side is linear with respect to the 6 combinations:

$$\begin{array}{lll} \delta y\,\delta' z - \delta z\,\delta' y\,, & \delta z\,\delta' x - \delta x\,\delta' z\,, & \delta x\,\delta' y - \delta y\,\delta' x\,, \\ \delta x\,\delta' t - \delta t\,\delta' x\,, & \delta y\,\delta' t - \delta t\,\delta' y\,, & \delta z\,\delta' t - \delta t\,\delta' z\,. \end{array}$$

A simple calculation, which is nothing but an application of Stokes's formula, gives the following coefficients for the first three terms:

$$\xi = \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}, \qquad \eta = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}, \qquad \zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y};$$

these are the components of the *vorticity* vector. In order to calculate the other three coefficients, we can use the fact that since the expression ω' is invariant for equations (6), the equations that are obtained by annulling the coefficients of δx, δy, δz, δt in $\omega(d,\delta')$ can be consequences of equations (6). Therefore, set:

$$\omega'(d,\delta) = \xi\,(dy\ \delta z - dz\ \delta y) + \eta\,(dz\ \delta x - dx\ \delta z) + \zeta\,(dx\ \delta y - dx\ \delta y)$$
$$+\ P\,(dx\ \delta t - dt\ \delta x) + Q\,(dy\ \delta t - dt\ \delta y) + R\,(dz\ \delta t - dt\ \delta z).$$

The equations considered are:

(7) $$\begin{cases} \eta dz - \zeta\, dy - Pdt = 0, \\ \zeta\, dx - \xi dz - Qdt = 0, \\ \xi dy - \eta dx - Rdt = 0, \\ Pdx + Qdy + Rdz = 0. \end{cases}$$

If we say that they are a consequence of equations (6) then we will obtain:

$$\begin{aligned} P &= \eta\, w - \zeta\, v \\ Q &= \zeta\, u - \xi\, w, \\ R &= \xi\, v - \eta\, u. \end{aligned}$$

As a result, the desired double integral invariant is:

$$\iint\ \xi\ \delta y\ \delta z + \ \eta\ \delta x\ \delta y + \zeta\ \delta x\ \delta y + (\eta\, w - \zeta\, v)\ \delta x\ \delta t + (\zeta\, u - \xi\, w)\ \delta y\ \delta t + (\xi\, v - \eta\, u)\ \delta z\ \delta t.$$

When this integral is taken over a closed area of molecules that are all taken at the same instant t we obtain the *vorticity flux* across that area. From this, we recover the theorem of the conservation of vorticity flux through a fluid surface.

23. We will now calculate the expression $\omega'(d,\delta)$ directly. In particular, the coefficient P of $dx\delta t$ is obviously:

$$P = -\frac{\partial u}{\partial t} - \frac{\partial E}{\partial x} = -\frac{\partial u}{\partial t} - u\frac{\partial u}{\partial x} - v\frac{\partial v}{\partial x} - w\frac{\partial w}{\partial x} + \frac{\partial U}{\partial x} - \frac{1}{\rho}\frac{\partial p}{\partial x},$$

and if one sets it equal to the preceding value:

$$P = \eta w - \zeta v = w\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) - v\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$

then one will obtain the equation:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = \frac{\partial U}{\partial x} - \frac{1}{\rho}\frac{\partial p}{\partial x},$$

which is nothing but the main equation of hydrodynamics. Indeed, the left-hand side is the expression that we developed for γ_x.

This result reminds us that the integral $\int u\ \delta x + v\ \delta y + w\ \delta z - E\ \delta t$ is invariant for the differential equations (6) only if *u, v, w* are the components of the velocity of a molecule of a perfect fluid that is subject to a force that is derived from a force function, or again, if there is an *acceleration potential.*

24. Equations (7), which may also be written:

$$(7)\qquad \begin{cases} \eta(dz - wdt) - \zeta(dz - vdt) = 0, \\ \varsigma(dx - udt) - \xi(dz - wdt) = 0, \\ \xi(dy - vdt) - \eta(dx - udt) = 0, \end{cases}$$

are a consequence of the differential equations (6), but they are not *equivalent* to those differential equations. In other words, *the equations* (6) *of the trajectories are not the only ones that admit the integral invariant* $\int \omega$. In particular, this is also the case for the equations:

$$(9)\qquad \frac{dx}{\xi} = \frac{dy}{\eta} = \frac{dz}{\varsigma} = \frac{dt}{0};$$

equations (7) are obviously a consequence of this. The solutions of these equations are what one calls *vortex lines. Imposing the requirement that the differential equations of the trajectories and the differential equations of the vortex lines admit the same integral invariant leads us to the fundamental theorem of vortex theory.*

Indeed, one may characterize an elementary displacement (*dx*, *dy*, *dz*, 0) (in the four-dimensional universe of *x*, *y*, *z*, *t*) in the direction of a vortex line by the property that the bilinear form $\omega'(d, \delta)$ is zero *for any displacement* δ; this results immediately from equations (7). Having said this, consider a vortex line (Γ) at an instant *t*. The molecules that comprise it form a line (Γ') at another instant t'. *We shall show that* (Γ') *is a vortex line for the instant* t'. Indeed, let $(dx', dy', dz', 0)$ be an elementary displacement along (Γ') and associate it with an *arbitrary* displacement $(\delta x', \delta y', \delta z', \delta t')$. If we displace the three states

$$(x', y', z', t'), \quad (x' + dx', y' + dy', z' + dz', t'), \ (x' + \delta x', y' + \delta y', z' + \delta z', t' + \delta t')$$

along their respective trajectories – the first two, up to the instant *t* and the second two, up to the instant $t + \delta t$ – then we will obtain a two-dimensional element for which (*dx*, *dy*, *dz*, 0) represents a displacement along the *vortex line* (Γ). The form $\omega'(\delta, \delta')$ will thus have the value zero. It is therefore zero for the original element as well, and as a result (Γ') is a vortex line. This is the celebrated theorem of Helmholtz.

25. Consider a vortex tube at the instant t and two closed curves (C) and (C') that encircle the tube. The *circulation* around these two closed curves is the same since $\int \omega$ is an integral invariant for the differential equations (9) of the vortex lines. At another instant t', the vortex tube will take another position in space, but the circulation around any closed line that encircles the new tube will not have changed since $\int \omega$ is an integral invariant for the differential equations of the trajectories. We recover the notion of what one calls the *moment* or the *intensity* of a vortex tube in hydrodynamics, a quantity that is conserved through the duration of the motion. This property is only a particular aspect of the invariance of the integral:

$$\int u\,\delta x + v\,\delta y + w\,\delta z - E\,\delta t,$$

for the differential equations of the trajectories and for those of the vortex lines.

Furthermore, we recover all of these results as a particular case of a general theorem that concerns differential forms that are simultaneously invariant for several systems of differential equations.

It is trivial to remark that in all of the preceding we have essentially supposed that not all three of ξ, η, ζ were zero; i.e., that the motion of the fluids was *rotational*.

CHAPTER III

INTEGRAL INVARIANTS AND INVARIANT DIFFERENTIAL FORMS

I. - *General notion of an integral invariant.*

26. The preceding chapters have shown us the importance of the notion of integral invariant in mechanics. We shall now discuss this notion in its full generality.

Consider an arbitrary system of first-order ordinary differential equations (one knows that one may always reduce to this case), which we write:

$$(1) \qquad \begin{cases} \dfrac{dx_1}{dt} = X_1, \\ \dfrac{dx_2}{dt} = X_2, \\ \quad \dots \\ \dfrac{dx_n}{dt} = X_n. \end{cases}$$

We have distinguished the independent variable t from the dependent variables $x_1, x_2, ..., x_n$, but, as one can verify, that distinction is not essential, so we continue to say that t represents time. The set of values of $x_1, ..., x_n, t$ that correspond to a solution will be said to constitute a *trajectory*, which we may regard as a curve in the $n + 1$-dimensional space of $(x_1, ..., x_n, t)$.

Having said this, H. Poincaré gave the name of *integral invariant* to an integral (simple or multiple) that, when taken over an arbitrary set of *simultaneous* (i.e., ones that correspond to the same value of t) points, does not change value when one displaces the points of that set along the corresponding trajectories up to another arbitrary instant t'. An integral invariant is called *absolute* if its invariance property is true for any integration domain. It is called *relative* if the invariance property is only true for a *closed* integration domain. The linear integral invariant of mechanics:

$$\int \sum p_i \, \delta q_i$$

is relative; the double integral invariant of mechanics:

$$\iint \sum \delta p_i \, \delta q_i$$

is absolute.

The simplest forms for integral invariants are:

$$\int a_1 \delta x_1 + a_2 \delta x_2 + \dots + a_n \delta x_n \, ,$$
$$\int \sqrt{a_{11} \delta x_1^2 + a_{22} \delta x_2^2 + \dots + 2a_{12} \delta x_1 \delta x_2 + \dots} \, ,$$

$$\iint a_{12}\delta x_1\delta x_2 + a_{12}\delta x_1\delta x_3 + \ldots + a_{n-1,n}\delta x_{n-1}\delta x_n ,$$
$$\iiint a_{123}\delta x_1\delta x_2\delta x_3 + \ldots$$

27. The quantity under the summation sign in an integral invariant is a differential form into which the variables – both dependent and independent – and their differentials (or, similarly, several sets of differentials) enter. That form F may be considered by itself, and it enjoys the property that when it is calculated for an arbitrary point and one or more infinitely close but *simultaneous* points it will not change values if one displaces these points along their respective trajectories *while keeping all of the points simultaneous.* It is quite clear that from this viewpoint one may consider more general forms F than the ones that are susceptible to appearing under an integration sign; for example, an arbitrary rational (homogenous) function of $\delta x_1, \ldots, \delta x_n$.

As we showed in the examples that were treated in the first two chapters, *there is no point in restricting ourselves to the consideration of simultaneous points.* We shall see that any elementary integral invariant, in the sense of H. Poincaré, may be regarded as resulting from suppressing all of the terms that contain the differential or differentials of the independent variable t in a more complete elementary integral invariant.

However, in order to arrive at this essential result, which will give us the key to almost all of the properties of integral invariants, it is necessary to briefly recall the classical properties of first integrals of a system of differential equations.

II. - *First integrals.*

28. As one knows, one calls a function $u(x_1, \ldots, x_n , t)$ a *first integral* of system (1) if it enjoys the property that if one replaces the $x_1, \ldots, x_n$ with their values as functions of t along an *arbitrary* trajectory then the function u of t thus obtained can be reduced to a constant. These first integrals are solutions of the first-order linear partial differential equation:

$$\frac{\partial u}{\partial t} + X_1\frac{\partial u}{\partial x_1} + X_2\frac{\partial u}{\partial x_2} + \ldots + X_n\frac{\partial u}{\partial x_n} = 0. \tag{2}$$

Imagine that one has integrated equations (1), and that one has expressed the dependent variables $x_1, \ldots, x_n$ as functions of time t and their initial values $x_1^0, x_2^0, \ldots, x_n^0$ for $t = 0$, namely

$$\begin{cases} x_1 = f_1(t; x_1^0, \ldots, x_n^0), \\ \quad\ldots \\ x_n = f_n(t; x_1^0, \ldots, x_n^0). \end{cases}$$

When these equations are solved for $x_1^0, \ldots, x_n^0$ they will give functions of $x_1, \ldots, x_n , t$ that are obviously first integrals for these n quantities. One thus obtains a system of n first integrals that are obviously *independent*; i.e., ones that are not related by any identity relation in $x_1, \ldots, x_n , t$.

It is clear that any function of the first integrals x_1^0, ..., x_n^0 will be a first integral, *and, conversely,* if u is an arbitrary first integral then its numerical value for an *arbitrary* trajectory will be, by the same property, equal to $u(x_1^0, ..., x_n^0, 0)$.

The total differential of any function u of $x_1, x_2, ..., x_n, t$ may obviously be put into the form:

$$du = \lambda_1 (dx_1 - X_1\, dt) + \lambda_2 (dx_2 - X_2\, dt) + ... + \lambda_n (dx_n - X_n dt) + \lambda\, dt.$$

The necessary and sufficient condition for this to be a first integral is that the coefficient λ must be identically zero. One may easily account for this by a direct argument. One may also verify this by remarking that λ is nothing but the left-hand side of equation (2). Thus, *the differential of any first integral is a linear combination of n linear differential forms:*

$$dx_1 - X_1\, dt,\ \ dx_2 - X_2\, dt, ..., dx_n - X_n dt,$$

and, conversely, each of these forms is a linear combination of the differentials of n given independent first integrals.

III. - *Absolute integral invariants and invariant differential forms.*

29. Having said this, we first occupy ourselves with the *absolute* integral invariants. The element of any absolute integral invariant is a differential form $F(x_1, ..., x_n, t; \delta x_1, ..., \delta x_n)$ that does not change value if one displaces the point $(x_1, ..., x_n, t)$ and the infinitely close point $(x + \delta x_1, ..., x_n + \delta x_n, t)$ along their respective trajectories while always considering them at the same instant. In particular, consider them at the instant $t = 0$. We have:

$$F(x_1, ..., x_n, t; \delta x_1, ..., \delta x_n) = F(x_1^0, ..., x_n^0, 0; \delta x_1^0, ..., \delta x_n^0)\,.$$

Now regard the x_i^0 in the right-hand side as first integrals of the system (1) and replace them with their values as functions of $x_1, ..., x_n, t$. We obtain a new identity:

$$F(x_1^0, ..., x_n^0, 0; \delta x_1^0, ..., \delta x_n^0) = \Phi(x_1, ..., x_n, t; \delta x_1, ..., \delta x_n, \delta t)\,.$$

The left-hand side of this identity is obviously a quantity whose numerical value is of interest to the trajectory that is defined by the initial values $x_1^0, ..., x_n^0$ and the infinitely close trajectory. Its value is thus independent of the particular point $(x_1, ..., x_n, t)$ that was taken from the first trajectory and the particular point $(x_1 + \delta x_1, ..., x_n + \delta x_n, t + \delta t)$ that was taken from an infinitely close trajectory. It is therefore also an element of an integral invariant, but of a more complete integral invariant than the one that served as our point of departure, *since now we are no longer obliged to restrict ourselves to the consideration of simultaneous points.*

We now remark that it is easy to pass from the initial form F to the final form Φ. Indeed, if one regards t as a constant in the calculation of $x_1^0, \ldots, x_n^0, \delta x_1^0, \ldots, \delta x_n^0$ then one will obviously revert to the form F. One thus has:

$$\Phi(x_1, \ldots, x_n, t; \delta x_1, \ldots, \delta x_n, 0) = F(x_1, \ldots, x_n, t; \delta x_1, \ldots, \delta x_n).$$

Now, δt appears only by the intermediary of the $\delta x_1^0, \ldots, \delta x_n^0$, and these n differentials are linear combinations of

$$\delta x_1 - X_1\, \delta t, \quad \delta x_2 - X_2\, \delta t, \quad \delta x_3 - X_3\, \delta t.$$

As a result, Φ depends only on these n linear combinations, and when one has its expression for $\delta t = 0$ one immediately has its expression for arbitrary δt by replacing δx_1 with $\delta x_1 - X_1\, \delta t$, etc.

Finally, one has:

(3) $$\Phi(x_1, \ldots, x_n, t; \delta x_1, \ldots, \delta x_n, \delta t) = F(x_1, \ldots, x_n, t; \delta x_1 - X_1 \delta t, \ldots, \delta x_n - X_n\, \delta t).$$

30. We summarize the results that we just obtained; they are two in number.

1. The form F, which constitutes the *element* of an absolute integral invariant, in the sense of H. Poincaré, and in which only the differentials of the dependent variables appear, is associated with a more complete form Φ, in which the differential (or differentials) of the independent variable t is also involved. One passes from the form Φ to the form F by suppressing the terms that contain δt, and, conversely, one passes from the form F to the form Φ by replacing:

$$\delta x_1, \delta x_2, \ldots, \delta x_n$$

with:

$$\delta x_1 - X_1\, \delta t, \quad \delta x_2 - X_2\, \delta t, \quad \delta x_3 - X_3\, \delta t.$$

2. The form Φ may be expressed in terms of first integrals of the system (1) and their differentials.

This latter property makes the invariant character of the form Φ self-evident.

A simple example will make the relation between the two forms F and Φ comprehensible. If one starts with an arbitrary integral u then the total differential δu will obviously be a form F. The corresponding form Φ is:

$$F = \frac{\partial u}{\partial x_1}\delta x_1 + \frac{\partial u}{\partial x_2}\delta x_2 + \ldots + \frac{\partial u}{\partial x_n}\delta x_x,$$

and one has indeed:

$$\Phi = \delta u = \frac{\partial u}{\partial x_1}(\delta x_1 - X_1 \delta t) + \frac{\partial u}{\partial x_2}(\delta x_2 - X_2 \delta t) + \ldots + \frac{\partial u}{\partial x_n}(\delta x_n - X_n \delta t).$$

31. We agree to say that a differential form that can be expressed in terms of first integrals of the system (1) and their differentials is an *invariant* form for the system (1). The quantity under the integration sign in an absolute integral invariant is obtained by annulling δt in an invariant form. This is the double integral invariant of mechanics that corresponds to the invariant form:

$$\iint \sum \delta p_i\, \delta q_i - \delta H\, \delta t,$$

or, if one prefers, after introducing two types of differentials:

$$\Phi = \iint \sum\, (\delta p_i\, \delta q_i' - \delta q_i\, \delta p_i') - \delta H\, \delta' t + \delta t\, \delta' H\,.$$

Its expression in terms of the first integrals p_i^0, q_i^0 is obviously:

$$\Phi = \sum (\delta\, p_i^0 \delta' q_i^0 - \delta q_i^0 \delta'\, p_i^0).$$

IV. - *Relative integral invariants. Hamilton's function.*

32. One part of these results can be extended to the theory of relative integral invariants. It is the fact that the linear integral invariant of dynamics that was considered by H. Poincaré:

$$\int \sum p_i\, \delta q_i$$

does not change value when one displaces each state along its trajectory at the instant t up to an arbitrary instant t' and equals the integral:

$$\int \sum p_i^0 \delta q_i^0\,.$$

Any relative integral invariant may thus be given an expression in terms of only first integrals and their differentials, and, in that form, it may taken over any closed domain of simultaneous or non-simultaneous states without losing its invariant character.

However, if one replaces the first integrals by their expressions as functions of the dependent and independent variables in the new expression then one will obtain a form Φ inside the summation sign *that is not derived from the initial form F by the same process* as in the case of an absolute invariant.

The equality:

$$\int F(x_1, ..., x_n, t;\, \delta x_1, ..., \delta x_n) = \int \Phi(x_1, ..., x_n\,, t;\, \delta x_1, ..., \delta x_n\,, 0)$$

holds for any *closed* integration domain that is formed from simultaneous points, but the term-by-term equality of the two sums does not result from this, and one necessarily has only the identity:

$$F(x_1, ..., x_n, t; \delta x_1, ..., \delta x_n) = \Phi(x_1, ..., x_n, t; \delta x_1, ..., \delta x_n, 0),$$

which will be necessary in order for one to deduce, conforming to the formula (3), that:

$$\Phi(x_1, ..., x_n, t; \delta x_1, ..., \delta x_n, \delta t) = F(x_1, ..., x_n, t; \delta x_1 - X_1 \delta t, ..., \delta x_n - X_n \delta t).$$

This is true in the simple case of a free material point. The element:

$$F = m\,(x' \,\delta x + y' \,\delta y + z' \,\delta z)$$

that appears inside the summation sign in the expression of the linear integral invariant of H. Poincaré leads to the form:

$$m(x' \,\delta x + y' \,\delta y + z' \,\delta z) - m\,(x'^2 + y'^2 + z'^2)\,\delta t,$$

which is nothing but a complete integral invariant, and differs from the form:

$$\omega_\delta = m(x' \,\delta x + y' \,\delta y + z' \,\delta z) - [\tfrac{1}{2}\, m\,(x'^2 + y'^2 + z'^2) - U]\,\delta t$$

by only a simple exact differential, if necessary.

We should remark that the difficulty that presents itself here in passing from a relative integral invariant, in the sense of H. Poincaré, to a complete integral invariant does not have great practical importance, because any relative integral invariant may be reduced to an absolute integral invariant. Indeed, one knows that an integral taken over a closed contour, closed surface, etc., may be reduced to an integral over an area bounded by the closed contour, a volume bounded by the closed surface, etc.

33. It is useful to illustrate the preceding considerations with some examples.

Recall the (complete) linear integral invariant of dynamics, i.e., the "quantity of motion-energy" tensor:

$$\omega_\delta = \sum p_i \,\delta q_i - H \delta t.$$

One has the equality:

$$\int_{(C)} \sum p_i \delta q_i - H \delta t = \int_{(C_0)} \sum p_i^0 \delta q_i^0,$$

in which one supposes that the closed contour (C_0) is composed of the states that constitute (C), but are displaced along their trajectories up to the instant $t = 0$. One may furthermore consider the integral of the right-hand side as being extended over the same contour (C) as the integral on the left-hand side, on the condition that one regards the p_i^0 and the q_i^0 as functions of the p_i, the q_i, and t. From that point of view, the two expressions:

$$\sum p_i \delta q_i - H \delta t \qquad \text{and} \qquad \sum p_i^0 \delta q_i^0$$

give the same integral along an arbitrary closed contour, or one that differs only by an exact form, and one has:

$$\sum p_i \delta q_i - H\,\delta t = \delta S + \sum p_i^0 \delta q_i^0. \tag{4}$$

The function S is what one calls the *Hamilton function*, and it has a simple concrete interpretation. If we refer to formula (10) of chapter I, which gives the variation of the action along a variable trajectory, then we will see that *S may be interpreted as representing the action between the instant* 0 *and the instant* 1 *along the trajectory that leads up to the state* (p, q, t).

The function S was considered by Hamilton, and it has a certain importance from a historical perspective because it was the remarks that Hamilton made on that subject that led Jacobi along the path of his discoveries that relate to the equations of mechanics. Indeed, Hamilton remarked that if one knows how to express the function S, not as a function of the p_i , q_i , and t, but as a function of the p_i^0, q_i^0, and t, then one will get the equations of motion by the same integration. Indeed, when the identity (4) is put into the form:

$$\delta S = \sum p_i \delta q_i - H\,\delta t - \sum p_i^0 \delta q_i^0 ,$$

it will give:

$$p_i = \frac{\partial S}{\partial q_i}, \qquad -p_i^0 = \frac{\partial S}{\partial q_i^0}, \qquad \frac{\partial S}{\partial t} + H = 0. \tag{5}$$

The second set of equations will give the p_i as functions of t and the $2n$ initial values. The first one will give the quantities of motion p_i . Finally, the last one shows that the function S is a solution of the partial differential equation:

$$\frac{\partial S}{\partial t} + H\left(t, q_i, \frac{\partial S}{\partial q_i}\right) = 0. \tag{6}$$

The difficulty with this way of looking at things is not merely in integrating that partial differential equation, but in finding a solution for which *the arbitrary constants q_i^0 agree precisely with the initial values of the q_i* . Jacobi resolved this difficulty by showing that this condition was totally unnecessary for the purpose of making the integration of the partial differential equation (6) serve as the integration of the equations of motion; this is what we briefly discussed already in sec. **14**.

34. It is quite instructive to effect the calculation of the Hamilton function S in a simple case; for example, the case of a free point of mass 1 that is not subject to any force. Here the equations of motion are:

$$\begin{aligned} x &= x_0' t + x_0, & x' &= x_0', \\ y &= y_0' t + y_0, & y' &= y_0', \\ z &= z_0' t + z_0, & z' &= z_0'. \end{aligned}$$

The difference:

$$\delta S = \omega_\delta - (\omega_\delta)_0 = x'\,\delta x + y'\,\delta y + z'\,\delta z - \tfrac{1}{2}(x'^2 + y'^2 + z'^2)\,\delta t - (x'_0\,\delta x_0 + y'_0\,\delta y_0 + z'_0\,\delta z_0)$$

is equal to:

$$\begin{aligned}\delta S &= x'\,\delta x + y'\,\delta y + z'\,\delta z - \tfrac{1}{2}(x'^2 + y'^2 + z'^2)\,\delta t - x'\,\delta(x - t\,x') - y'\,\delta(y - t\,y') - z'\,\delta(z - t\,z')\\ &= \tfrac{1}{2}(x'^2 + y'^2 + z'^2)\,\delta t + t\,(x'\delta x' + y'\delta y' + z'\delta z'),\end{aligned}$$

from which, upon taking into account that S must be annulled with t, we will get:

$$S = \tfrac{1}{2}(x'^2 + y'^2 + z'^2)\,t.$$

Upon expressing S by means of $x, y, z, x_0, y_0, z_0, t$, one obtains:

$$S = \frac{1}{2}\,\frac{(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2}{t}.$$

Hamilton's formulas (5) permit us to deduce the equations of motion from this function:

$$\begin{aligned} x' &= \frac{\partial S}{\partial x} = \frac{x - x_0}{t}, & -x'_0 &= \frac{\partial S}{\partial x_0} = -\frac{x - x_0}{t},\\ y' &= \frac{\partial S}{\partial y} = \frac{y - y_0}{t}, & -y'_0 &= \frac{\partial S}{\partial y_0} = -\frac{y - y_0}{t},\\ z' &= \frac{\partial S}{\partial z} = \frac{z - z_0}{t} & -z'_0 &= \frac{\partial S}{\partial z_0} = -\frac{z - z_0}{t}.\end{aligned}$$

V. - *Examples. The "element of matter" form.*

35. Now that we have made the preceding parenthetical remarks, we return to *absolute* integral invariants.

In the simplest case, it is good to directly account for the invariant character of the differential forms Φ that are deduced, as was said above, from forces F by replacing δx_i with $\delta x_i - X_i\,\delta t$ in them.

To simplify matters, take a system of two differential equations in two unknown functions:

$$\frac{dx}{dt} = X, \qquad\qquad \frac{dy}{dt} = Y,$$

and start with an absolute linear integral invariant:

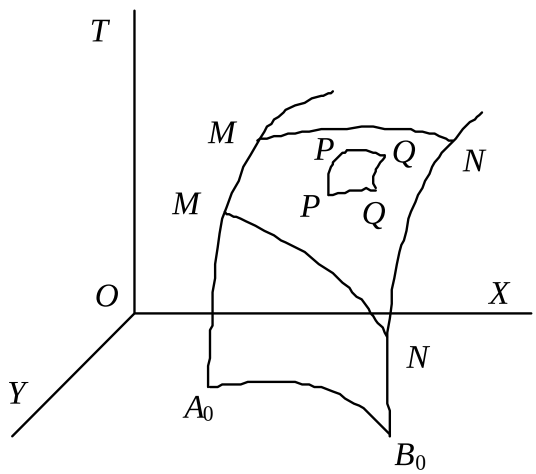
$$I = \int a(x, y, z)\,\delta x + b(x, y, t)\,\delta y.$$

The complete integral invariant that is associated with it is:

$$J = \int a(x, y, z)(\delta x - X\,\delta t) + b(x, y, t)\,(\delta y - Y\delta t).$$

Start with an arc of a curve $A_0\,B_0$ in the xy-plane and guide the different points of that arc along the corresponding trajectories. One thus obtains a type of cylindrical surface whose (non-rectilinear) generators will be the trajectories. Trace two curved arcs MN and $M'N'$ on the surface that connect the trajectory that issues from A_0 to the trajectory that issues from B_0. We shall show that one has:

$$J_{MN} = J_{M'N'}.$$



The two curved arcs MN and $M'N'$, together with the trajectory arcs MM' and NN', bound a closed area of the surface. On the other hand, when the integral J is taken over each of these latter two arcs the result is obviously zero since upon displacing along one of these arcs one constantly has:

$$\delta x = X\,\delta t, \qquad \delta y = Y\,\delta t.$$

As a result, when the integral J is taken over the closed contour $MNN'M'$, one gets:

$$J_{MNN'M'} = J_{MN} - J_{M'N'},$$

and everything comes down to showing that this integral is zero. Now, from Stokes's formula this integral amounts to a surface integral that is taken over the area $MNN'M'$. *We shall show that this surface integral has a zero element.* Indeed, in order to do this, we decompose the surface into surface elements that are formed from small parallelograms that are defined, on the one hand, by arcs of trajectories, and, on the other hand, *by sections by the planes* t = const. Let $PQQ'P'$ be one of these surface elements. The element of the corresponding surface integral is equal to

$$J_{PQ} - J_{P'Q'},$$

but, *since the points of PQ are simultaneous*, as well as those of $P'Q'$, J_{PQ} reduces to I_{PQ}, and $J_{P'Q'}$ to $I_{P'Q'}$. Now, from the fact that I is an integral invariant, the two integrals I_{PQ} and $I_{P'Q'}$ are equal.

Hence, any integral element of the surface is precisely zero, and the theorem is proved.

36. An analogous argument will be made in the case of a double integral invariant:

$$I = \iint a(x, y, t)\,\delta x\,\delta y\,.$$

Here, the passage from the form F to the form Φ is a little more difficult than it was in the preceding case.

One arrives at it by associating the surface element $\delta x\ \delta y$ with a bilinear fom $\delta x\,\delta' y - \delta y\,\delta' x$. In order to do this, it suffices to imagine an indeterminate system of curvilinear coordinates (α, β), and regard δx, δy as the elementary displacement relative to an increment $\delta\alpha$ in the first coordinate α, and regard $\delta' x$, $\delta' y$ as the elementary displacement relative to an increment $\delta'\beta$ in the second one β. One then has:

$$F = a\begin{vmatrix} \delta x & \delta y \\ \delta' y & \delta' y \end{vmatrix}.$$

From this, one deduces that:

$$\Phi = a\begin{vmatrix} \delta x - X\delta t & \delta y - Y\delta t \\ \delta' x - X\delta' t & \delta' y - Y\delta' t \end{vmatrix} = a\begin{vmatrix} \delta x & \delta y \\ \delta' x & \delta' y \end{vmatrix} + aX\begin{vmatrix} \delta y & \delta t \\ \delta' y & \delta' t \end{vmatrix} + aY\begin{vmatrix} \delta t & \delta x \\ \delta' t & \delta' x \end{vmatrix},$$

or, upon returning to the notations in the theory of surface integrals:

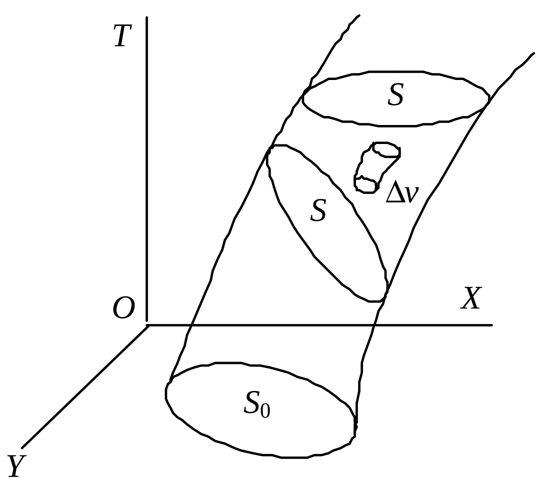
$$\Phi = a\ \delta x\ \delta y + aX\ \delta y\ \delta t + aY\ \delta t\ \delta x.$$

Therefore, consider a surface element:

$$J = \iint a\,\delta x\,\delta y + aX\ \delta y\ \delta t + aY\ \delta t\ \delta x,$$

and let us try to account for its invariant character directly. In order to do this, imagine an arbitrary area S_0 in the xy-plane and construct the trajectories that issue from different points of that area. We thus obtain an indefinite volume bounded by a type of cylindrical lateral surface that is generated by the trajectories that begin on the contour of S_0. Cut this volume with two arbitrary surfaces. We thus obtain two (plane or curved) areas S and S' that are in the interior of the volume, but extend up to the lateral surface. We would like to show that one has:

$$J_S = J_{S'}\,.$$



The areas S and S', along with a portion of the lateral surface of the cylinder, define a volume V. On the other hand, when the integral J is taken over the area that bounds this volume, the result is obviously zero, since, if we call the area element $d\sigma$ and the direction cosines of the normal α, β, γ then we will have:

$$J = \iint a(\gamma + X\alpha + Y\beta)\, d\sigma,$$

and the fact that the direction (X, Y, 1) is normal to the direction (α, β, γ), since it is the direction of the tangents to the trajectories that generate the lateral surface considered. It results from this that the difference $J_{S'} - J_S$ can be regarded as the surface integral J taken over the closed area that bounds the volume V. Everything amounts to showing that the volume integral is obviously zero. In order to account for this, it suffices to take the elementary volume to be the volume that is laterally bounded by small arcs of the trajectory and two plane areas *that are parallel to the xy-plane* at the extremities, because then when the surface integral J is taken over each of the bases it will reduce to the integral I, and, by hypothesis, the value of the integral I is the same for both bases.

37. The kinematics of continuous media provides us with a concrete illustration of the considerations that were developed in this chapter.

In a continuous medium in motion, the trajectory of each molecule can be regarded as a solution to the system of differential equations:

$$\frac{dx}{dt} = u, \qquad \frac{dy}{dt} = v, \qquad \frac{dz}{dt} = w,$$

in which the components of velocity u, v, w are assumed to be expressed as functions of x, y, z, t. On the other hand, let $\rho(x, y, z, t)$ be the density at the instant t at the point (x, y, z). The mass that occupies an arbitrary volume V at the instant t will be given by the triple integral:

$$\iiint_V \rho\, \delta x\, \delta y\, \delta z.$$

This integral obviously constitutes an absolute integral invariant, in the sense of H. Poincaré. It was the first example of an integral invariant that was given by H. Poincaré. If the molecules that occupy the volume V at the instant t occupy the volume V' at another instant t' then one will obviously have:

$$\iiint_V \rho(x, y, z, t)\, \delta x\, \delta y\, \delta z = \iiint_{V'} \rho'(x', y', z', t')\, \delta x'\, \delta y'\, \delta z'.$$

The form Φ that is associated with the form $F = \rho\, \delta x\, \delta y\, \delta z$ can be calculated, as in the preceding example, by writing F in the form:

$$F = \rho \begin{vmatrix} \delta x & \delta y & \delta z \\ \delta' x & \delta' y & \delta' z \\ \delta'' x & \delta'' y & \delta'' z \end{vmatrix}.$$

One deduces from this that:

$$\Phi = \rho \begin{vmatrix} \delta x - u\delta t & \delta y - v\delta t & \delta z - w\delta t \\ \delta' x - u\delta' t & \delta' y - v\delta' t & \delta' z - w\delta' t \\ \delta'' x - u\delta'' t & \delta'' y - v\delta'' t & \delta' z - w\delta'' t \end{vmatrix},$$

from which, by an easy calculation one obtains:

$$\Phi = \rho(\delta x\ \delta y\ \delta z - u\ \delta y\ \delta z\ \delta t - v\ \delta z\ \delta x\ \delta t - w\ \delta x\ \delta y\ \delta t).$$

The form Φ represents the *element of matter*, when envisioned in its complete kinematical aspect. If one considers an arbitrary three-dimensional set of molecules, and if one takes each molecule of the set at an arbitrary instant t of its motion, then one will obtain a three-dimensional domain in a four-dimensional universe (x, y, z, t). The triple integral of Φ taken over this domain will be equal to the total mass of the set of molecules considered. If the molecules are all taken at the same instant t then they will occupy a certain volume V at that instant, and the integral of Φ will reduce to the integral $\iiint_V \rho\ \delta x\ \delta y\ \delta z$. However, this is entirely peculiar to the case at hand.

Consider, to be specific, an area S in space, for example, and the set of all of molecules that cross that area between an instant t_0 and an instant t_1. Take each of these molecules *at the moment when it traverses the area S*. We will then get a three-dimensional domain in the universe of (x, y, z, t). The states of this domain are easily expressed by means of the three parameters α, β, γ. In order to do this, it suffices to express the coordinates of a point of S as functions of the two parameters α, β, and to take $t = \gamma$. One then will have formulas such as:

$$\begin{aligned} x &= f(\alpha, \beta), \\ y &= g(\alpha, \beta), \\ z &= h(\alpha, \beta), \\ t &= \gamma, \end{aligned}$$

in which the parameters α and β take all of the values that correspond to the different points of the area S and the parameter γ takes all of the possible values in the interval (t_0, t_1). When the integral F is taken over this domain, the result will obviously be, ignoring the sign:

$$\int_{t_0}^{t_1} \delta t \left[\iint_{(S)} \rho u\, \delta y\, \delta z - \rho v\, \delta z\, \delta x + \rho w\, \delta x\, \delta v \right].$$

The surface integral between brackets represents the *matter flux* that traverses the surface S at the instant t. When it is multiplied by δt, it will represent the quantity of matter that traverses the surface S during the interval $(t, t + \delta t)$. The total integral thus represents what we have been waiting for: viz., the total mass that crosses S in the interval (t_0, t_1).

38. Analogous remarks apply to the double integral invariant that we encountered in hydrodynamics (Chap. II, formula (8)), viz.:

$$J = \iint \xi\,\delta y\,\delta z + \eta\,\delta z\,\delta x + \zeta\,\delta x\,\delta y + (\eta w - \zeta v)\,\delta x\,\delta t + (\zeta u - \xi w)\,\delta y\,\delta t + (\xi v - \eta u)\,\delta z\,\delta t.$$

We saw (sec. **25**) that when this integral is taken over a two-dimensional set of molecules that are taken at the same instant t, the result will represent the moment or the intensity of the vortex tube that is formed from the vortex lines that start at these molecules. Therefore, consider the set of molecules that traverse an arc of the curve C in an interval of time (t_0, t_1). Instead of taking these molecules at the same instant t, take each of them at the instant when it crosses the arc of the curve C. At an arbitrary instant t, the moment of the vortex tube that originates from it will be equal to the integral:

$$\int_{t_0}^{t_1} \delta t \int_{(C)} \begin{vmatrix} \delta x & \delta y & \delta z \\ \xi & \eta & \zeta \\ u & v & w \end{vmatrix}.$$

CHAPTER IV

THE CHARACTERISTIC SYSTEM OF A DIFFERENTIAL FORM

I. - *The class of a differential form.*

39. In all of this chapter, we will consider systems of differential equations in n variables x_1, x_2, ..., x_n without distinguishing the independent variable by a special notation; it will be any of the variables x_1, x_2, ..., x_n, arbitrarily. In other words, we will consider systems of differential equations of the form:

(1) $$\frac{dx_1}{X_1} = \frac{dx_2}{X_2} = \dots = \frac{dx_n}{X_n}.$$

One of the first problems that presents itself in the theory of integral invariants is the following one: *to recognize whether a given differential form is invariant for a given system of differential equations, and, more generally, to determine all of the systems of equations that admit a given differential form as an invariant form.*

Before commencing the solution of this problem for the differential forms that habitually present themselves in applications we make several general remarks that lead us to an extremely important theorem.

In order for a form Φ to be an invariant of the system (1) it is necessary and sufficient that it be expressible in terms of first integrals of (1) and their differentials. Thus, a *necessary* condition for a given form Φ to be invariant for a conveniently chosen system of differential equations is that this form can be expressed by means of *at most* $n - 1$ *quantities* and their differentials.

40. Suppose then that the form Φ can be expressed by means of $r < n$ quantities y_1, ..., y_r (i.e., functions of x_i) and their differentials. Suppose, in addition, that they *may not* be expressed in an analogous manner by means of *less than* r quantities. With these conditions, we shall prove the following theorem:

In order for such a system of differential equations to admit Φ *as an invariant form it is necessary and sufficient that* y_1, ..., y_r *be first integrals of the system.*

The condition is obviously sufficient. To prove that it is necessary, consider a system of differential equations that admit Φ as an invariant form, and write the equations of that system by taking y_1, ..., y_r, and $n - r$ other independent quantities y_{r+1}, ..., y_n to be the new variables. Let:

(2) $$\frac{dy_1}{Y_1} = \frac{dy_2}{Y_2} = \dots = \frac{dy_r}{Y_r} = \frac{dy_{r+1}}{Y_{r+1}} = \dots = \frac{dy_n}{Y_n}$$

be the equations of the system. If $y_1, ..., y_r$ are not all first integrals then the first r denominators $Y_1, ..., Y_r$ will not all be zero. Suppose, for example, that $Y_r \neq 0$. One may then take y_r to be the independent variable, and the form Φ will not change value if one replaces y_r and δy_r by 0 everywhere. One then replaces:

$$y_1, ..., y_{r-1}, y_{r+1}, ..., y_n$$

with their initial values:

$$y_1^0, ..., y_{r-1}^0, y_{r+1}^0, ..., y_n^0,$$

which are regarded as first integrals of system (2), and finally one replaces the differentials:

$$\delta y_1, ..., \delta y_n,$$

with:

$$\delta y_1^0, ..., \delta y_n^0.$$

But then, as Φ contains neither $y_{r+1}, ..., y_n$ nor their differentials, the new form Ψ thus obtained depends upon only the $y_1^0, ..., y_{r-1}^0$ and their differentials. In other words, one can find $r - 1$ functions $z_1, ..., z_{r-1}$ of the x_i such that Φ can be expressed in terms of these $r - 1$ functions and their differentials. This result is contrary to the hypothesis. The number r will be called the *class* of the form Φ.

II. - *The characteristic system of a differential form.*

41. This extremely general theorem leads to some important consequences that help us to better understand its scope.

From the foregoing, the most general system of differential equations that admit the form Φ, when written in terms of the variables $y_1, ..., y_n$, as an invariant form is:

$$\frac{dy_1}{0} = \frac{dy_2}{0} = ... = \frac{dy_r}{0} = \frac{dy_{r+1}}{Y_{r+1}} = ... = \frac{dy_n}{Y_n}, \tag{3}$$

where $Y_{r+1}, ..., Y_n$ are *arbitrary* functions. We immediately deduce that any first integral that is common to these systems is a function of the $y_1, ..., y_r$. *As a result, if the form Φ can be expressed in a second manner by means of the r quantities $z_1, ..., z_r$ and their differentials then the z_i will be functions of the y_i and conversely,* since the z_i are first integrals that are common to all of the differential systems that admit Φ as an invariant form. This amounts to saying that *there is essentially only one manner of expressing the form Φ in terms of a minimum number of variables and their differentials,* in the sense that if one has an expression that involves the minimum number r of quantities $y_1, ..., y_r$ then all of the others can be obtained by perfomring an arbitrary change of variables on the y. This conclusion will obviously be invalid if r is not the minimum number of variables.

42. Another consequence is the following one: agree to say that a certain number – three, perhaps – of the systems of differential equations in n variables:

$$\frac{dx_1}{X_1} = \frac{dx_2}{X_2} = \ldots = \frac{dx_n}{X_n},$$
$$\frac{dx_1}{X_1'} = \frac{dx_2}{X_2'} = \ldots = \frac{dx_n}{X_n'},$$
$$\frac{dx_1}{X_1''} = \frac{dx_2}{X_2''} = \ldots = \frac{dx_n}{X_n''}$$

are *linearly independent* if it is impossible to find three coefficients $\lambda, \lambda', \lambda''$ *that are not all zero* such that one has:

$$\lambda X_1 + \lambda' X_1 + \lambda'' X_1 = 0,$$
$$\lambda X_2 + \lambda' X_2 + \lambda'' X_2 = 0,$$
$$\ldots$$
$$\lambda X_n + \lambda' X_n + \lambda'' X_n = 0.$$

In the contrary case, we say that they are *linearly dependent.*

The property of several systems being linearly independent or not obviously persists under an arbitrary change of variables.

Among the systems (3) that admit Φ as an invariant form, one can obviously find $n - r$ linearly independent systems, namely, the ones that one obtains by setting all of the denominators $Y_{r+1}, \ldots, Y_n$ equal to zero, *except for one.* Furthermore, all of the systems (3) depend linearly upon these $n - r$ particular systems.

We thus see that *if a form Φ is invariant for $n - r$ and only $n - r$ linearly independent systems of differential equations then it will be invariant for any system that depends linearly upon it, and, moreover, all of these systems will have r independent first integrals in common.*

43. For example, suppose $n - r = 2$. There exist two systems of differential equations, namely:

$$\frac{dx_1}{X_1} = \ldots = \frac{dx_n}{X_n},$$

that admit Φ as an invariant form, and any other system that enjoys this property will depend linearly upon these two. Call the trajectories of the first system (C) and those of the second system (Γ). For any arbitrary point M in n-dimensional space, take the trajectories (C) and (Γ) that pass through that point. Take an arbitrary point P on (C) and an arbitrary point Q on (Γ). Finally, construct the trajectory (Γ') that passes through P and the trajectory (C') that passes through it. *These two new trajectories intersect.* If $y_1, \ldots, y_{n-2}$ are first integrals that are common to the two systems considered, and if $a_1, \ldots, a_{n-2}$ are the numerical values of these integrals at the point M then their numerical values at the point P and at the point Q will again be the same. As a result, the curves (C), (Γ), (Γ'), (C') will be all situated on the same two-dimensional manifold:

$$y_1 = a_1, \qquad y_2 = a_2, \qquad \ldots, y_{n-2} = a_{n-2};$$

hence, the last two intersect each other.

44. The preceding case presents itself precisely in the case of the double integral invariant of vortex theory, which corresponds to the differential form:

$$\Phi = \xi\, \delta y\, \delta z + \eta\, \delta z\, \delta y + \zeta\, \delta x\, \delta y + (\eta\, w - \zeta\, v)\, \delta x \delta t + (\zeta\, u - \xi\, w)\, \delta y\, \delta t + (\xi\, v - \eta\, u)\, \delta z\, \delta t. \tag{5}$$

We have seen (sec. **24**) that the systems of differential equations that admit Φ as an invariant form are the ones that imply, as a consequence, the three equations:

$$\begin{cases} \eta(dz - w dt) - \zeta(dy - v dt) = 0, \\ \zeta(dx - u dt) - \xi(dz - w dt) = 0, \\ \xi(dy - v dt) - \eta(dx - u dt) = 0. \end{cases} \tag{6}$$

The most general of these systems may be written:

$$\frac{dx}{\lambda u + \mu\xi} = \frac{dy}{\lambda v + \mu\eta} = \frac{dz}{\lambda w + \mu\zeta} = \frac{dt}{\lambda},$$

and from this we linearly derive the two systems:

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w} = \frac{dt}{1},$$
$$\frac{dx}{\xi} = \frac{dy}{\eta} = \frac{dz}{\zeta} = \frac{dt}{0},$$

that define the trajectories of the molecules of fluid and the vortex lines. The first system defines the curves (C), and the second system defines the curves (Γ) at any time, and the properties that one obtains in the general case may be expressed here by saying that *the molecules that form a vortex line* (Γ) *at the instant t again form a vortex line* (Γ') *at the instant* t'. Helmholtz's theorem is thus a very special consequence of the theorem that was proved at the beginning of this chapter.

45. In the preceding two sections we supposed that $n - r = 2$. Analogous geometric considerations may be developed for any values of n and r. They are based upon the existence of manifolds that are defined by r equations of the form:

$$y_1 = a_1, \qquad y_2 = a_2, \qquad \ldots, \qquad y_r = a_r,$$

and are such that any trajectory of a differential system (3) that contains one point of the manifold will be contained completely in the manifold. Each of these manifolds, which are $n - r$-dimensional, can be obtained by starting with an arbitrary point M and passing a trajectory of one of the systems that admit Φ as an invariant form through that point, and then passing a trajectory of either system through an arbitrary point P of this trajectory, and so on. One can generate any $n - r$-dimensional manifold by these operations and never escape it.

We give such a manifold the name of *characteristic manifold* for the form Φ.

Characteristic manifolds may be regarded as resulting from the equations:

$$dy_1 = 0, dy_2 = 0, ..., dy_r = 0.$$

However, if one returns to the original variables $x_1, ..., x_n$ *then these equations will be composed of the set of linear relations in* $dx_1, ..., dx_n$ *that are consequences of the equations of any differential system that admits* Φ *as an invariant form.*

More simply, one may say: *the necessary and sufficient condition for the elementary displacement* $(dx_1, ..., dx_n)$ *to be performed in the direction of a trajectory of a differential system that admits* Φ *as an invariant form translates analytically into a certain number of linear equations in* $dx_1, ..., dx_n$. *These equations, r of which are assumed to be independent, define* $n - r$*-dimensional manifolds that depend upon r arbitrary constants, such that one and only one of them passes through any point of space; these are the characteristic manifolds. The linear system of total differentials itself is called the characteristic system of the form* Φ.

46. To abbreviate, call an equation that is linear in $dx_1, ..., dx_n$ a *Pfaff equation,* and a system of Pfaff equations, a *Pfaff system.* A system of r Pfaff equations in n variables can always be regarded as defining r variables, which are considered to be dependent variables, that are functions of the other $n - r$, which are considered to be independent variables. *In general, such a system is impossible.* For example, a classical result is that a Pfaff equation in three variables:

$$P\,dx + Q\,dy + R\,dz = 0,$$

in which one regards z as an unknown function of x and y, admits a solution that corresponds to arbitrary given initial values only if a certain integrability condition, namely:

$$P\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) + Q\left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right) + R\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) = 0,$$

is satisfied. In this case, one says that the system is *completely integrable.*

Similarly, one says that a Pfaff system of r equations with r unknown functions in $n - r$ variables is *completely integrable* if it always admits a solution that corresponds to arbitrary given initial values of these variables. *That is what happens for the characteristic Pfaff system of a form* Φ.

The fundamental theorem of this chapter may be stated as follows:

The characteristic Pfaff system of an arbitrary differential form Φ *is always completely integrable.*

47. Let us return one last time to the form Φ of vortex theory. The characteristic Pfaff system of that form is defined by equations (5) or, what amounts to the same thing:

$$\frac{dx-udt}{\xi}=\frac{dy-vdt}{\eta}=\frac{dz-wdt}{\zeta}.$$

If we know how to express the idea that such a system is completely integrable then we will necessarily arrive at the analytical translation of Helmholtz's theorem. As for the characteristic manifolds, they are composed of the set of all *states* of the molecules that constitute such a vortex line.

The characteristic Pfaff system for the double integral invariant of dynamics reduces to the equations of motion, and the characteristic manifolds reduce to the trajectories.

The situation could be otherwise if, as we did in vortex theory, we were to consider only one type of trajectory; for example, trajectories that satisfy some system of relations between the variables.

CHAPTER V

SYSTEMS OF PFAFF INVARIANTS AND THEIR CHARACTERISTIC SYSTEMS

I. - *The notion of an invariant Pfaff system*

48. Instead of invariant *forms* for a system of differential equations one may also consider invariant *equations.* In particular, H. Poincaré used a *finite* system of invariant equations. They enjoy the property that if a point satisfies such a system then all of the points that can be obtained by displacing along the corresponding trajectory will also satisfy such a system. To use geometric language: *The manifold that is represented by a system of invariant equations is generated by the trajectories.*

One may also consider invariant differential equations. First of all, we restrict our point of view to the simple case of two differential equations:

(1) $$\frac{dx}{dt} = X, \qquad \frac{dy}{dt} = Y.$$

The equation:

(2) $$\delta y - m(x, y, t)\,\delta x = 0,$$

will be called *invariant in the sense of H. Poincaré* if, given two arbitrary infinitely close *simultaneous* points (x, y, t) and $(x + \delta x, y + \delta y, t)$ that satisfy relation (2), the points (x', y', z', t') and $(x' + \delta x, y' + \delta y, z' + \delta z, t')$ that are obtained by displacing them along their respective trajectories up to another arbitrary instant t' also satisfy relation (2), i.e., one has once more that:

$$\delta y' - m(x', y', t')\,\delta x' = 0.$$

If equation (2) is invariant, in the sense that we just made precise, then it will be equivalent to the equation:

(3) $$\delta y_0 - m(x_0, y_0, 0)\;\delta x_0 = 0,$$

in which one denotes the initial values of x, y along the trajectory that passes through the point (x, y, t) by x_0, y_0. Now, if one regards x_0, y_0 in equation (3) as functions of (x, y, t), and if one replaces x_0, y_0 with their numerical values then this equation will obviously take the form:

(4) $$\delta y - Y\,\delta t - m(\delta x - X\,\delta t) = 0.$$

This new equation (4), on account of its origin, has an *invariant* significance, in the complete sense of the term, because it expresses an *intrinsic* property of the two trajectories that correspond to the points (x, y, t) and $(x + \delta x, y + \delta y, t + \delta t)$.

Geometrically, equation (4) makes a plane (P) that passes through any point $M = (x, y, t)$ correspond to that point. The invariance property signifies that the line MM_1 that joins a point M to an infinitely neighboring point M_1 is situated in the plane (P) that corresponds to the point M, and if one displaces M and M_1 along their respective trajectories (while always keeping them infinitely close to each other) into MN and M_1N then the line MNM_1N will be situated in the plane (PN) that corresponds to the point MN. We remark that *the plane* (P) *is tangent to the trajectory that passes through the point M*.

From the preceding, it is obvious that if a curve (C) satisfies equation (4) at each of its points, i.e., if it is an *integral curve*, then the surface that is generated by the trajectories that pass through the different points of (C) will also be an integral surface of equation (4). This also results analytically from the form (3) of equation (4).

49. The preceding considerations are easily generalized. Given a system of differential equations:

$$\frac{dx_i}{dt} = X_i \qquad (i = 1, 2, ..., n), \tag{5}$$

a system of Pfaff equations:

$$\begin{cases} a_{11}\delta x_1 + \ldots + a_{1n}\delta x_n + a_1\delta t = 0, \\ \qquad\qquad \ldots\ldots. \\ a_{h1}\delta x_1 + \ldots + a_{hn}\delta x_n + a_h\delta t = 0, \end{cases} \tag{6}$$

will be called *invariant* for system (5) if equations (6) can be expressed uniquely in terms of first integrals of (5) and their differentials; for example, ones that have the form:

$$\begin{cases} a_{11}^0\delta x_1^0 + \ldots + a_{1n}^0\delta x_n^0 = 0, \\ \qquad\qquad \ldots \\ a_{h1}^0\delta x_1^0 + \ldots + a_{hn}^0\delta x_n^0 = 0. \end{cases} \tag{7}$$

This demands that equations (6) must be verified identically when one replaces δx_i by $X_i\delta t$. However, this condition is obviously not sufficient. Be that as it may, if the Pfaff system (6) is invariant then it will enjoy the important geometric property that *given an arbitrary integral manifold of the system* (6), *the manifold that is obtained by guiding each point of the given manifold along the trajectory that corresponds to equations* (7) *will again be integral.* Indeed, this results if one displaces it onto this new manifold at an arbitrary point while equations (7) do not cease to be verified.

(It should be understood that one calls a manifold an integral manifold if it is such that equations (6) are verified if one displaces *on that manifold* in an arbitrary direction $(\delta x_1, ..., \delta t)$.)

From this, it results that *any integral manifold of an invariant Pfaff system* (6) *enjoys either the property of being generated by the trajectories of the given equations* (5) *or the property of* *being a part of an integral manifold of very large dimension that is itself generated by the trajectories.*

II. - *The characteristic system of a Pfaff system.*

50. Given an arbitrary Pfaff system of n variables ($x_1, ..., x_n$):

$$\text{(8)} \qquad \begin{cases} a_{11}\delta x_1 + \ldots + a_{1n}\delta x_n = 0, \\ \ldots \\ a_{h1}\delta x_1 + \ldots + a_{hn}\delta x_n = 0, \end{cases}$$

one may propose to determine all systems of differential equations:

$$\text{(9)} \qquad \frac{dx_1}{X_1} = \frac{dx_2}{X_2} = \ldots = \frac{dx_n}{X_n}$$

for which system (8) is invariant. This is a problem that we will solve later on, but, without solving it, one may still prove an important theorem that concerns all of these systems, a theorem that is identical to one that was proved in the preceding chapter in the context of a given differential form.

Suppose that the given equations (8) can be written in terms of the r quantities $y_1, ..., y_r$, which are functions of x and its derivatives, in the form:

$$\text{(10)} \qquad \begin{cases} b_{11}(y)\delta y_1 + \ldots + b_{1r}\delta y_r = 0, \\ \ldots \\ b_{h1}(y)\delta y_1 + \ldots + b_{hr}(y)\delta y_r = 0, \end{cases}$$

and suppose moreover that they cannot be written in terms of less than r quantities and their differentials. The number r will be called the *class* of the system. *The necessary and sufficient condition for the Pfaff system* (8) *to be invariant for the equations* (9) *is that these equations* (9) *must admit* $y_1, y_2, ..., y_r$ *as first integrals.*

The proof is exactly the same as in the preceding chapter, and the consequences that one deduces are also the same. In particular, the equations that express that the given Pfaff system (8) is invariant for the system of differential equations (9) can be reduced to r linear equations in $X_1, ..., X_n$, or – what amounts to the same thing – to r linear equations in $dx_1, ..., dx_n$, and these r equations form a Pfaff system that is equivalent to:

$$dy_1 = 0, ..., dy_r = 0;$$

i.e., they are *completely integrable.* This Pfaff system is called the *characteristic system* of the given Pfaff system (8). The equations of the characteristic system can be ultimately obtained by adjoining $r - h$ other equations to the given system of h equations (8).

The necessary and sufficient condition for such a Pfaff system (8) *to be completely integrable is obviously that it must coincide with its own characteristic system,* in such a way that if one knows how to form the characteristic system of an arbitrary Pfaff system then one will know how to express that it is completely integrable by the same means.

51. It is obvious that a Pfaff system (8) can be regarded as invariant for its characteristic system. *Any integral manifold of the system* (8) *is either generated by its characteristic manifolds or defines a subset of an integral manifold of much larger dimension that is itself generated by characteristic manifolds.*

If one considers an arbitrary differential form, and if this form is invariant for a certain system of differential equations then the characteristic Pfaff system of the form will be invariant for the same system of differential equations.

It follows that in hydrodynamics the Pfaff system:

$$\frac{\delta x - u\delta t}{\xi} = \frac{\delta y - v\delta t}{\eta} = \frac{\delta z - w\delta t}{\zeta},$$

is invariant for the differential equations of the trajectories of the fluid molecules (and also for the differential equations of the vortex lines).

All of these theorems, and some others that one may easily imagine, are immediate consequences of the characteristic property of an invariant system that it involves only first integrals of the differential equations for which it is invariant.

52. Consider either a differential form or a Pfaff system, or likewise, a set of several differential forms and a Pfaff system, and let $y_1, \ldots, y_r$ denote either the first integrals of the characteristic Pfaff system, the given differential form, or the given Pfaff system, etc. It is obvious that *if one directs one's attention uniquely to the manner by which the differentials* $\delta x_1, \ldots, \delta x_n$ *figure* in the differential form, or the in the Pfaff system, etc., without being preoccupied with the coefficients then *these differentials will enter only in combinations of* $\delta y_1, \ldots, \delta y_r$. However, it might also be the case that they enter only as linear combinations of a number less than r. In any case, if one knows the linear combinations in a minimum number of δx_i by means of which the form (or Pfaff system, etc.) can be expressed then *the equations that are obtained by annulling these linear combinations will be a subset of the characteristic system.*

III. - *The rank of an algebraic form and its associated system.*

53. The preceding considerations increase in clarity if we prove a theorem for algebraic forms that is analogous to the one that led to the notion of a characteristic system:

If an algebraic form in n variables $u_1, \ldots, u_n$ *can be expressed in terms of r linearly independent combinations* $v_1, \ldots, v_r$ *of variables without being expressible in terms of a smaller number, and if, moreover, one has found another expression for the form by means of r other* *linear combinations* $w_1, ..., w_r$ *of variables then the* w_i *will be independent linear combinations of the* v_i.

Indeed, consider the $2r$ linear forms:

$$v_1, ..., v_r; \qquad w_1, ..., w_r$$

of the given variables. Suppose that among these forms there are $2r - \rho$ independent ones ($0 \leq \rho \leq r$). This amounts to saying that there exist ρ linearly independent combinations of the v that are, at the same time, linear combinations of the w; call them $t_1, ..., t_\rho$. Suppose, moreover – as is legitimate – that the $t_1, ..., t_\rho$ are independent linear combinations of both the $v_1, ..., v_\rho$ and the $w_1, ..., w_\rho$. One then has a double equality of the form:

$$F(x_1, ..., x_n) = \Phi(t_1, ..., t_\rho; v_{\rho+1}, ..., v_r) = \Psi(t_1, ..., t_\rho; w_{\rho+1}, ..., w_r).$$

Since the quantities $t_1, ..., t_\rho, v_{\rho+1}, ..., v_r, w_{\rho+1}, ..., w_r$ are independent, this is possible only if F, for example, does not depend upon the $v_{\rho+1}, ..., v_r$. This is not compatible with the hypothesis that $\rho = r$, and the theorem is thus proved.

The system of linear equations:

$$v_1 = v_2 = ... = v_r = 0$$

will be called the *associated* system of the given form. The notion of associated system obviously extends to a set of forms, or again to a system of algebraic equations. We say that the integer r is the *rank* of the form.

From this, we infer that the characteristic system of a differential form always contains the *associated* system of that form, which is considered to be an algebraic form in $\delta x_1, ..., \delta x_n$. *However, it may contain other equations besides the equations of the associated system.*

CHAPTER VI

FORMS WITH EXTERIOR MULTIPLICATION

I. - *The associated system of a quadratic form.*

54. We must say a word about ordinary algebraic, quadratic, cubic, forms, etc.

A quadratic form:

$$F(x) = \sum_{i,j}^{1,\cdots,n} a_{ij} u_i u_j = a_{11}u_1^2 + a_{22}u_2^2 + \cdots + 2a_{12}u_1u_2 + \cdots \tag{1}$$

is, as one knows, reducible to a sum of squares. If the discriminant of the form is different from zero then there will be as many squares as the number n of independent variables. We propose to determine the minimum number of variables by means of which the form may be expressed (by a convenient substitution). In order to obtain these variables it will suffice to consider the system of linear equations:

$$\frac{\partial F}{\partial u_1} = 0, \qquad \frac{\partial F}{\partial u_2} \ldots, \ \frac{\partial F}{\partial u_n} = 0. \tag{2}$$

It is immediately obvious that this system is independent of the choice of variables. Suppose that it reduces to r independent equations, which one may always suppose to be:

$$x_1 = 0, \ x_2 = 0, \ldots, x_r = 0.$$

Having said this, *the form F can be expressed in terms of r variables* $x_1, \ldots, x_r$ *and cannot be expressed in terms of less than r variables.*

Indeed, express F by means of $x_1, \ldots, x_r$ and of $n - r$ other independent forms $x_{r+1}, \ldots, x_n$. The variable x_{r+1}, for example, does not enter in F because if it did enter into a term such as $A\, x_{r+1}\, x_\alpha$ then the equation:

$$\frac{\partial F}{\partial x_\alpha} = 0,$$

would contain x_{r+1}, which is contrary to the hypothesis.

Conversely, suppose that the form F can be expressed by means of $\rho \leq r$ variables $y_1, y_2, \ldots, y_r$. The system (2) that is formed by starting with the variables $y_1, \ldots, y_\rho, \ldots, y_n$ manifestly contains only the variables $y_1, \ldots, y_r$. It is therefore necessary that $\rho = r$, and system (2) will then reduce to:

$$y_1 = y_2 = \ldots = y_r = 0.$$

The $y_1, \ldots, y_r$ are thus independent linear combinations of the $x_1, \ldots, x_r$.

The last part of the proof shows, as we already know, that expressing F in terms of a minimum number of variables is possible in *essentially* one manner, up to a linear substitution of a minimum number of these variables.

System (2) is the *associated* system of the form F.

The foregoing can be extended to a form that is integer and homogeneous of arbitrary degree. For example, if F is a cubic form then the associated system of linear equations will be obtained by annulling all of the *second* derivatives of F:

$$\frac{\partial^2 F}{\partial u_i \partial u_j} = 0.$$

This system gives the minimum number of variables by means of which F may be expressed.

II. - *Alternating bilinear forms and quadratic exterior forms.*

55. The forms with which we shall now occupy ourselves are the ones that appear under a multiple integral sign when one considers the differentials to be variables. These are forms that have special rules of calculation, upon which it is not pointless to insist.

We start with a bilinear form:

$$f(u, v) = \sum a_{ij}\, u_i\, u_j\,,$$

in two series of variables:

$$u_1, \ldots, u_n\,; \quad v_1, \ldots, v_n\,.$$

Such a form is called *symmetric* if it is preserved when one exchanges the two series of variables:

$$f(u, v) = f(v, u)$$

and *alternating* if it is preserved *with a sign change* under the same conditions:

$$f(u, v) = -f(v, u).$$

The conditions that the coefficients of the symmetric form must satisfy are:

$$a_{ij} = a_{ji}\,.$$

The conditions for it to be alternating are:

$$a_{ij} + a_{ji} = 0, \quad a_{ii} = 0.$$

If one subjects the two series of variables u_i and v_i to the *same* linear substitution then the form
$f(u, v)$ will be changed into a new bilinear form $F(U, V)$ in the new variables U_i, V_i, and it is obvious that the form $F(U, V)$ will again be symmetric if f is, and alternating if f is. This says that the exchange of the two series of new variables U and V amounts to the exchange of the original two series of variables u and v.

One may make a quadratic form, namely $f(u, u)$, correspond to any symmetric bilinear function
$f(u, v)$, and the correspondence is invertible. If one sets:

$$f(u, u) = F(u)$$

then one will have:

$$f(u, v) = \frac{1}{2}\left(v_1 \frac{\partial F}{\partial u_1} + v_2 \frac{\partial F}{\partial u_2} + \cdots + v_n \frac{\partial F}{\partial u_n} \right).$$

An analogous correspondence can no longer be established for the alternating forms because $f(u, u)$ becomes identically zero in that case. This is an inconvenience that one may obviate in a manner that we shall now describe.

56. We first remark that the coefficients of the terms $u_i\, u_i$ are all zero for an alternating bilinear form, and the coefficients of the terms $u_i\, u_j$ and $u_j\, u_i$ will have opposite signs. One may then write:

$$f(u, v) = \sum_{(ij)} a_{ij}\,(u_i u_j - u_j u_i),$$

in which the summation on the right-hand side is taken over all *combinations* of n index pairs, in such a way that there are $\frac{n(n-1)}{2}$ terms in this summation. Since the expression $u_i\, v_j - u_j\, v_i$ is nothing but the determinant:

$$\begin{vmatrix} u_i & u_j \\ v_i & v_j \end{vmatrix},$$

one may, *by a notational convention*, denote it by the notation (†):

$$u_i\, v_j - u_j\, v_i = [u_i\, u_j]$$

by writing first the one and then the other of the two elements of the first row of the determinant and then putting them between brackets. With this notation, one has:

(†) [DHD] This notation is misleading, compared to the more modern notation of $u \wedge v$, insofar as it makes no mention of v; however, since Cartan uses this notation throughout, we shall remain faithful to the original.

$$f(u, v) = \sum a_{ij}\,[u_i\,u_j].$$

Similarly, we agree to denote the alternating bilinear form that is defined by the determinant:

$$\begin{vmatrix} f(u) & f'(u) \\ f(v) & f'(v) \end{vmatrix}$$

by the notation $[f(u)\,f'(u)]$, in which f and f' denote two arbitrary linear forms:

$$\begin{aligned} f(u) &= a_1\,u_1 + a_2\,u_2 + \ldots + a_n\,u_n\,, \\ f'(u) &= a'_1\,u_1 + a'_2\,u_2 + \ldots + a'_n\,u_n\,. \end{aligned}$$

If one develops the preceding determinant then one will immediately find that:

$$[f(u)\,f'(u)] = \begin{vmatrix} f(u) & f'(u) \\ f(v) & f'(v) \end{vmatrix} = \sum_i \sum_j a_i\,a'_j \begin{vmatrix} u_i & u_j \\ v_i & v_j \end{vmatrix} = \sum_i \sum_j a_i\,a'_j\,[u_i\,u_j].$$

Comparing the left and right-hand sides shows that *the development of* $[f(u)\,f'(u)]$ *can be obtained by regarding that expression as a product and then developing that product according to the ordinary rules of algebra – but without changing the order of the factors in the partial products – and agreeing that any partial product that contains two identical variables will be zero, and that any partial product of two different variables will change sign when one changes the order of the factors.*

The multiplication whose rules that we have just stated is due to H. Grassmann, who gave it the name of *exterior multiplication.*

Upon using that operation, one sees that *one can make a form of second degree in only one series of variables – but with exterior multiplication – correspond to any alternating bilinear form, and conversely, to any quadratic form with exterior multiplication, there corresponds an alternating bilinear form.*

To abbreviate, we say "exterior form" instead of "form with exterior multiplication."

57. If one performs a linear substitution of the variables in an exterior form $F(u)$ then the new form will be obtained simply by developing each partial product $[u_i\,u_j]$ as a function of the new variables.

The partial derivative $\partial F / \partial u_1$ of an exterior quadratic form is defined simply as the sum of the partial derivatives of its terms. A term that does not contain u_1 will naturally have a zero derivative. As for a term that does contain u_1, one can always suppose that u_1 appears as the first term in the partial product. The derivative of $A[u_1\,u_j]$ will then be $A\,u_i$. For example, one has:

$$\frac{\partial [u_1 u_2]}{\partial u_1} = u_2, \qquad \frac{\partial [u_1 u_2]}{\partial u_2} = -\,u_1, \qquad \frac{\partial [u_1 u_2]}{\partial u_3} = 0, \cdots, \frac{\partial [u_1 u_2]}{\partial u_n} = 0\,.$$

With these conventions:

$$2F(u) = \left[u_1 \frac{\partial F}{\partial u_1} \right] + \left[u_2 \frac{\partial F}{\partial u_2} \right] + \cdots + \left[u_n \frac{\partial F}{\partial u_n} \right],$$

in which the partial products of the right-hand side are exterior products.

If $F(u)$ corresponds to the alternating form $f(u, v)$ then one will obviously have:

$$f(u, v) = -\left(v_1 \frac{\partial F}{\partial u_1} + v_2 \frac{\partial F}{\partial u_2} + \cdots + v_n \frac{\partial F}{\partial u_n} \right),$$

in which the partial products will be formed according to the rules of ordinary multiplication.

Finally, we remark that if one performs a linear substitution:

$$u_i = h_{i1} U_1 + \ldots + h_{in} U_n \qquad (i = 1, \ldots, n)$$

on the u_i, and if $F(u)$ becomes $\Phi(U)$ by that substitution then one will have:

$$\frac{\partial \Phi}{\partial U_k} = h_{1k} \frac{\partial F}{\partial u_1} + h_{2k} \frac{\partial F}{\partial u_2} + \cdots + h_{nk} \frac{\partial F}{\partial u_n},$$

since F is an ordinary algebraic form.

58. The system of linear equations:

(3) $$\frac{\partial F}{\partial u_1} = 0 \qquad \frac{\partial F}{\partial u_2} = 0, \cdots, \frac{\partial F}{\partial u_n} = 0,$$

in which F is a given exterior quadratic form, is obviously independent of the choice of variables. One may then suppose that it reduces to the equations:

$$u_1 = 0, \quad u_2 = 0, \ldots, u_r = 0 \qquad (\rho \leq n).$$

It is then the case that *the form F will depend upon only $u_{r+1}, \ldots, u_n$*. Indeed, if it contains a term such as $A[u_{r+1}\, u_\alpha]$ then the equation:

$$\frac{\partial F}{\partial u_\alpha} = 0$$

will not be a consequence of equations (3). The form F can thus be expressed uniquely by means of the right-hand sides of equations (3).

Conversely, suppose that the form F can be expressed by means of $\rho \leq r$ variables $v_1, v_2, \ldots, v_\rho$. The left-hand sides of the equations of the system:

$$\frac{\partial F}{\partial v_1} = 0, \cdots, \frac{\partial F}{\partial v_n} = 0$$

will depend upon only $v_1, ..., v_\rho$. This system will thus contain at most ρ independent equations. As a result, one will have $\rho = r$, and the v_i will be linear combinations of the u_i.

The associated system of an exterior quadratic form is therefore obtained by annulling all of its first-order partial derivatives.

59. This result can made be made more precise. We shall show that *the rank r is necessarily even*, and at the same time, find a reduced form for the exterior quadratic form that will play the same role that the sum of the squares does for ordinary quadratic forms.

Suppose, to fix ideas, that the coefficient a_{12} of $F(u)$ are not zero, and consider the form:

$$\frac{1}{a_{12}}\left[\frac{\partial F}{\partial u_1}\frac{\partial F}{\partial u_2}\right] = \frac{1}{a_{12}}[(a_{12}u_2 + a_{13}u_3 + \cdots + a_{1n}u_n)(a_{21}u_1 + a_{23}u_3 + \cdots + a_{2n}u_n)].$$

This form has the same coefficients as F for the terms in:

$$[u_1\, u_2],\ [u_1\, u_3],\ ...,\ [u_1\, u_n],\ [u_2\, u_3],\ ...,\ [u_2\, u_n],$$

i.e., for the terms that contain at least one of the variables u_1 or u_2. As a result, the form:

$$F(u) - \frac{1}{a_{12}}\left[\frac{\partial F}{\partial u_1}\frac{\partial F}{\partial u_2}\right] = F'(u)$$

will contain only the variables $u_3, u_4, ..., u_n$. Suppose then that the coefficient a''_{34} of that form is not zero. Similarly, one will verify that the form:

$$F'(u) - \frac{1}{a'_{34}}\left[\frac{\partial F'}{\partial u_3}\frac{\partial F'}{\partial u_4}\right] = F'(u)$$

contains only the variables $u_5, u_6, ..., u_n$. One may thus continue, step by step, until one arrives at a form that is identically zero. For example, suppose that one has:

$$F(u) = \frac{1}{a_{12}}\left[\frac{\partial F}{\partial u_1}\frac{\partial F}{\partial u_2}\right] + \frac{1}{a'_{34}}\left[\frac{\partial F'}{\partial u_3}\frac{\partial F'}{\partial u_4}\right] + \frac{1}{a''_{36}}\left[\frac{\partial F''}{\partial u_5}\frac{\partial F''}{\partial u_6}\right].$$

The six linear forms:

$$\frac{\partial F}{\partial u_1}, \quad \frac{\partial F}{\partial u_2}, \quad \frac{\partial F'}{\partial u_3}, \quad \frac{\partial F'}{\partial u_4}, \quad \frac{\partial F''}{\partial u_5}, \quad \frac{\partial F''}{\partial u_6},$$

will obviously be independent. Upon setting:

$$\begin{aligned} \frac{\partial F}{\partial u_1} &= U_1, & \frac{\partial F}{\partial u_2} &= a_{12} U_2, \\ \frac{\partial F'}{\partial u_3} &= U_3, & \frac{\partial F'}{\partial u_4} &= a'_{34} U_4, \\ \frac{\partial F''}{\partial u_5} &= U_5, & \frac{\partial F''}{\partial u_6} &= a''_{36} U_6, \end{aligned}$$

the form F will be reduced to desired canonical form:

$$F(U) = [U_1U_2] + [U_3U_4] + [U_5U_6].$$

This argument is obviously general and leads to the canonical form:

$$F(U) = [U_1U_2] + \ldots + [U_{2s-1}U_{2s}] \qquad (2s \leq n).$$

The associated system is obviously:

$$U_1 = U_2 = \ldots = U_{2s} = 0.$$

This result will have great significance later on.

60. The reduction of an exterior quadratic form to its canonical form is obviously possible in an infinitude of ways. The set of linear substitutions that make a canonical form pass into another constitute an important group that depends on s $(2s + 1)$ arbitrary parameters. If $s = 1$ then these substitutions of two variables will be characterized by the condition that their determinant must be equal to unity.

III. - *Exterior forms of degree greater than two.*

61. One may imagine exterior forms of arbitrary degree. One arrives at them most naturally upon starting with a linear form in p series of variables u_i, v_i, ..., w_i :

$$f(u, v, \ldots, w)$$

that satisfies the condition that exchanging two series of variables will reproduce the form, but with a sign change. In the case $p = 3$, for example, this hypothesis entails the consequence that any term in which the same index appears twice will have a zero coefficient and that the set of terms in which three distinct indices appear – for example, 1, 2, 3 – will be of the form:

$$a_{123}\begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}.$$

The same notational convention as above leads to a distributive, but not *commutative*, multiplication law, by which each product changes sign if one exchanges two of the variables that appear in it. Consequently, one will have:

$$[u_1\, u_2\, u_3] = -\,[u_2\, u_1\, u_3] = -\,[u_1\, u_3\, u_2] = -\,[u_3\, u_2\, u_1] = [u_2\, u_3\, u_1] = [u_3\, u_1\, u_2].$$

From this, one may define an exterior product such as:

$$[F\, \Phi\, \Psi],$$

in which F, Φ, Ψ, are exterior forms of arbitrary degree. The degree of the product is the sum of the degrees of the factors. The product will necessarily be zero if the sum of the degrees exceeds n. One easily confirms that if one exchanges two factors in the product then the product will not change sign if at least one of these factors is of even degree, and it will undergo a sign change if both of the degrees are odd. Similarly, one may define a sum of products of this nature.

In particular, the product of a form with itself will be zero if this form is of odd degree, but it will not necessarily be zero if it is of even degree. For example, take a quadratic form F that has been reduced to its canonical form:

$$F = [u_1\, u_2] + [u_3\, u_4] + \ldots + [u_{2s-1}\, u_{2s}].$$

One has:

$$\frac{1}{2}[F^2] = [u_1 u_2 u_3 u_4] + [u_1 u_2 u_5 u_6] + \cdots + [u_{2s-3} u_{2s-2} u_{2s-1} u_{2s}],$$

$$\frac{1}{3!}[F^3] = [u_1 u_2 u_3 u_4 u_5 u_6] + \cdots,$$

$$\frac{1}{s!}[F^s] = [u_1 u_2 u_3 u_4 \cdots u_{2s-1} u_{2s}],$$

$$\frac{1}{(s+1)!}[F^{s+1}] = 0.$$

The rank $2s$ of a quadratic form F is therefore twice the largest power to which one may raise F without annihilating it.

A simple application of the theory of determinants is the following one. Let:

$$F = a_{12}[u_1\, u_2] + a_{13}[u_1\, u_3] + a_{14}[u_1\, u_4] + a_{23}[u_2\, u_3] + a_{24}[u_2\, u_4] + a_{34}[u_3\, u_4]$$

be a form in four variables. One has:

$$\frac{1}{2}[F^2] = (a_{12}a_{34} - a_{13}a_{24} + a_{14}a_{23})[u_1u_2u_3u_4].$$

On the other hand, the associated system of F is:

$$\begin{array}{lllll}
a_{12}u_2 & + a_{13}u_3 + a_{14}u_4 & & & = 0, \\
a_{21}u_1 + & + a_{23}u_3 + a_{24}u_4 & & & = 0, \\
a_{31}u_1 & + a_{32}u_2 & + & + a_{34}u_4 & = 0, \\
a_{41}u_1 & + a_{42}u_2 & + a_{43}u_3 & + & = 0.
\end{array}$$

The condition for the form to be expressed by means of at least four variables is, on the one hand, that one must have $[F^2] = 0$, i.e.:

$$a_{12}\,a_{34} - a_{13}\,a_{24} + a_{14}\,a_{23} = 0,$$

and, on the other hand, that the determinant of the associated system must be zero, i.e.:

$$\begin{vmatrix} 0 & a_{12} & a_{13} & a_{14} \\ a_{21} & 0 & a_{23} & a_{24} \\ a_{31} & a_{32} & 0 & a_{34} \\ a_{41} & a_{42} & a_{43} & 0 \end{vmatrix} = 0.$$

Despite appearances, these two equations are equivalent. Indeed, one may prove that the determinant, which is anti-symmetric of even degree, is the square of the expression:

$$a_{12}\,a_{34} - a_{13}\,a_{24} + a_{14}\,a_{23}\,.$$

62. Any exterior form of degree n (which is equal to the number of variables) is of the form:

$$A[u_1\,u_2\,\ldots\,u_n].$$

One may obtain canonical forms when the degree is $n - 1$ or $n - 2$. One then easily arrives at the notion of the *adjoint form* to a given form.

Consider a form F of degree p, and let $\xi, \xi', \ldots, \xi^{(n-p-1)}$ denote the linear forms with indeterminate coefficients:

$$\begin{aligned}
\xi &= \xi_1\,u_1 + \ldots + \xi_n\,u_n\,, \\
\xi' &= \xi_1'\,u_1 + \ldots + \xi_n'\,u_n\,, \\
&\ldots
\end{aligned}$$

The exterior product $[F\,\xi'\,\ldots\,\xi^{(n-p-1)}]$ is of degree n, and so, as a result, is the form:

$$[\Phi\, u_1\, u_2 \ldots u_n].$$

The coefficient F is linear with respect to each series of coefficients ξ. Furthermore, it is *alternating*. Therefore it corresponds to an exterior form of degree $n - p$ in the variables $\xi_1, \ldots, \xi_n$; this is the definition of the form that is adjoint to F.

If one performs a linear substitution on the u and if, at the same time, one performs a linear substitution on the ξ that preserves the expression $\xi_1 u_1 + \ldots + \xi_n u_n$ then the expression $\Phi[u_1 \ldots u_n]$ will obviously be preserved. In other words, *the adjoint form is reproduced, but multiplied by the determinant of the substitution that was performed on the variables u*.

The adjoint form to a form $F = F_1 + F_2$ is obviously the sum of the adjoint forms Φ_1 and Φ_2. Similarly, the adjoint form to $a\Phi$, where a is a numerical coefficient, is aF. From this, in order to calculate the adjoint form to an arbitrary form it will suffice to know how to calculate the adjoint form to a monomial form such as:

$$F = [u_{\alpha_1} u_{\alpha_2} \cdots u_{\alpha_n}].$$

Upon applying the given definition, one finds that:

$$\Phi = [\xi_{\alpha_{p+1}}, \cdots, \xi_{\alpha_n}],$$

since the indices $\alpha_{p+1}, \ldots, \alpha_n$ are the ones among the indices $1, 2, \ldots, n$ that do not appear in the set of $\alpha_1, \alpha_2, \ldots, \alpha_p$. These indices are supposed to range in such an order that the total sequence:

$$\alpha_1, \ldots, \alpha_p, \alpha_{p+1}, \ldots, \alpha_n$$

is even.

63. From this, suppose that F is a form of degree $n-1$. The adjoint form will be of first degree. One may thus suppose that it is reduced to ξ_n ; for example, in such a way that F *may always be converted into an expression of the form:*

$$F = [u_1\, u_2 \ldots u_{n-1}].$$

Now suppose that F is of degree $n - 2$. The form F will be of second degree. Therefore, one may always suppose that it is given by formula:

$$\Phi = [\xi_1\, \xi_2] + \ldots + [\xi_{2s-1} \xi_{2s}].$$

As a result, one will have:

$$F = [u_3\, u_4\, u_5 \ldots u_n] + [u_1\, u_2\, u_5\, u_6 \ldots u_n] + \ldots + [u_1\, u_2 \ldots u_{2s-2}\, u_{2s+1} \ldots u_n].$$

If $s = 1$ then F will be reduced to a monomial form.

For example, if $n = 5$ then any form F of degree $5 - 2 = 3$ will be reducible to one of the canonical forms:

$$F = [u_3\, u_4\, u_5],$$
$$F = [u_1\, u_2\, u_5] + [u_3\, u_4\, u_5].$$

If $n = 6$ then any form F of degree 4 will be reducible to one of the forms:

$$F = [u_3\, u_4\, u_5\, u_6],$$
$$F = [u_3\, u_4\, u_5\, u_6] + [u_1\, u_2\, u_5\, u_6] = [([u_1\, u_2] + [u_3\, u_4])\, u_5\, u_6],$$
$$F = [u_3\, u_4\, u_5\, u_6] + [u_3\, u_4\, u_5\, u_6] + [u_3\, u_4\, u_5\, u_6] = \tfrac{1}{2}\,[[u_1\, u_2] + [u_3\, u_4] + [u_5\, u_6]]^2.$$

The notion of adjoint form permits us to define the product of two forms when the sum of their degrees exceeds n. This is the operation that H. Grassmann called *regressive exterior multiplication,* but we shall not use it.

64. Again, we point out several applications of exterior multiplication. Suppose that f_1, f_2, ..., f_h are h *independent* linear forms. *The equation:*

$$[F\, f_1\, f_2 \,...\, f_h] = 0,$$

in which F is an arbitrary exterior form, gives the necessary and sufficient condition for F to be annulled when one establishes relations:

$$f_1 = 0, \quad f_2 = 0, ..., f_h = 0$$

between the variables.

Indeed, one may consider the case in which one has $f_i = u_i$. In that case, if every term of F contains one or more of the variables u_1, .., u_h then it will be obvious that the product $[F\, u_1 \,...\, u_h]$ must be zero. Conversely, if this product is zero then an arbitrary term of F will contain one or more of the variables u_1, ..., u_h as a factor, except that the multiplication of this term by $[u_1...u_h]$ will give a non-zero product that cannot be reduced to any other.

IV. - *The associated system of an exterior form.*

65. The determination of the associated system may performed just as easily for an exterior form of arbitrary degree as it is for a quadratic form. If the form is of degree p then the associated system will be obtained by annulling all of the partial derivatives of F of order $p - 1$. One will define a first order derivative, such as $\frac{\partial F}{\partial u_1}$, to be the coefficient of u_1 in the set of terms of F that contain that variable, after first taking the precaution of insuring that u_1 has rank one in each of these terms. We remark that this derivative $\frac{\partial F}{\partial u_1}$ no longer depends on u_1. By definition, the derivative $\frac{\partial^2 F}{\partial u_1 \partial u_2}$ will be the derivative of $\frac{\partial F}{\partial u_1}$ with respect to u_2 . One obtains it as a result of taking the set of terms of F that contain both the variables u_1 and u_2, while making sure that u_1 is of first rank and u_2 is of second rank in each of these terms and finally suppressing the variables u_1 and u_2 from all of these terms. From this, one has:

$$\frac{\partial^2 F}{\partial u_1 \partial u_2} = -\frac{\partial^2 F}{\partial u_2 \partial u_1}.$$

The higher-order partial derivatives are defined in the same fashion; they are necessarily taken with respect to variables that are all different.

The rank of a form of degree n that is not identically zero is obviously equal to n. The rank of a form of degree $n - 1$ is equal to $n - 1$. The rank of a form of degree $n - 2$ is equal to $n - 2$ if it is reducible to a monomial form and to n in any other case. If the degree is less than $n - 2$ then one can say nothing *a priori* about the rank.

V. - *Formulas that relate to exterior quadratic forms.*

66. We return to the case of an exterior quadratic form F in n variables $u_1, u_2, ..., u_n$. It can happen that one supposes that the variables are coupled by a linear relation:

$$f \equiv a_1 u_1 + a_2 u_2 + ... + a_n u_n = 0.$$

The form F, in which we will suppose, for example, that u_n is expressed as a function of u_1, ..., u_{n-1} by means of the given relation, will have a certain rank that corresponds to the number of linearly independent equations in its associated system. The latter obviously has:

$$\frac{\partial F}{\partial u_1} - \frac{a_1}{a_n}\frac{\partial F}{\partial u_n} = 0, \cdots, \frac{\partial F}{\partial u_{n-1}} - \frac{a_{n-1}}{a_n}\frac{\partial F}{\partial u_n} = 0, \quad f = 0$$

for its equations, or again:

$$\frac{\dfrac{\partial F}{\partial u_1}}{a_1} = \frac{\dfrac{\partial F}{\partial u_2}}{a_2} = \cdots = \frac{\dfrac{\partial F}{\partial u_n}}{a_n}, \quad f = 0.$$

More generally, one may suppose that these variables are coupled by an arbitrary number of relations:

$$\begin{aligned} f &\equiv a_1 u_1 + a_2 u_2 + ... + a_n u_n = 0, \\ g &\equiv b_1 u_1 + b_2 u_2 + ... + b_n u_n = 0, \\ &\quad ... \\ h &\equiv l_1 u_1 + l_2 u_2 + ... + l_n u_n = 0. \end{aligned}$$

The associated system of F will then be defined by the formulas:

$$\left\| \begin{matrix} \frac{\partial F}{\partial u_1} & \frac{\partial F}{\partial u_2} & \cdots & \frac{\partial F}{\partial u_n} \\ a_1 & a_2 & \cdots & a_n \\ b_1 & b_2 & \cdots & b_n \\ \cdots & \cdots & \cdots & \cdots \\ l_1 & l_2 & \cdots & l_n \end{matrix} \right\| = 0, \qquad f = 0, \quad g = 0, \ldots, h = 0.$$

Equating the matrix above means annulling all of the determinants that are formed from the rows of that matrix and the same number of columns.

One may remark that *when one supposes that the variables are coupled by given relations, the rank* $2s$ *of the form* F *will be twice the largest exponent such that the form:*

$$[f\, g \ldots h\, F^{s'}]$$

is not zero.

67. In particular, suppose that $n = 2s$ and that the form F is of rank n. If one couples the variables by only one relation then it will be obvious that the rank of the form cannot exceed $n - 1 = 2s - 1$, and *since this rank is even* it will be equal to at most $2s - 2$. Furthermore, it is easy to see that it cannot descend below that limit.

From this, it results that if one couples the variables by p independent linear relations then the rank of F will be diminished by *at most* $2p$ units. We look for the case in which the maximum reduction will be obtained. If the relations are:

$$f_1 = 0, \quad f_2 = 0, \ldots, f_p = 0$$

then it will be necessary and sufficient that one must have:

$$[f_1 f_2 \ldots f_p F^{s-p+1}] = 0. \tag{4}$$

This condition can be replaced by other simpler conditions. Indeed, we remark that if one takes any two of the p given relations then these two relations will necessarily diminish the rank of F by 4 units. Therefore, one will have:

$$[f_i f_j F^{s-1}] = 0 \qquad (i, j = 1, 2, \ldots, p). \tag{5}$$

We shall prove that *these* $\frac{p(p-1)}{2}$ *necessary equations are also sufficient.*

Indeed, suppose that these conditions are satisfied and make a change of variables in such a manner as to take f_i to u_i. One will thus have

$$[u_i u_j F^{s-1}] = 0,$$

which shows that there is no term in $[\xi_i\ \xi_j]$ *in the form* $\Phi(\xi)$ *that is adjoint to* $F^{s-1}(u)$. Now, the form that is adjoint to F^{s-q} is Φ^q; this is easily recognized by supposing that F is reduced to its canonical form. As a result, each term of the adjoint form to F^{s-p+1}, which is F^{p-1}, will contain *at most* $p - 1$ of the variables $\xi_1, ..., \xi_p$. As a result, the form that is adjoint to F^{p-1} will contain *at least* one of the variables $u_1, ..., u_p$. This amounts to saying that one will have:

$$[u_1 u_2 \dots u_p F^{s-p+1}] = 0.$$

Q.E.D.

The significance of the preceding theorem is easy to point out. Since the forms $[f_i f_j F^{s-1}]$ are of degree n, the equations to be written will be $\frac{p(p-1)}{2}$ in number, whereas, since the form $[f_1 f_2 \dots f_p F^{s-p+1}]$ is of degree $n - p + 2$, the number of equations that express that it is zero will be C_n^{p-2}, and furthermore, each of them will contain the coefficients of *all* the given relations.

For example, if one has:

$$F = [u_1 u_2] + [u_3 u_4] + [u_5 u_6],$$
$$f_1 = a_1 u_1 + \dots + a_6 u_6,$$
$$f_2 = b_1 u_1 + \dots + b_6 u_6,$$
$$f_3 = c_1 u_1 + \dots + c_6 u_6$$

then, on account of the three relations $f_1 = f_2 = f_3 = 0$, the condition that F be of rank $6 - 6 = 0$ will be, by the first procedure:

$$[f_1 f_2 f_3 F] = 0,$$

which gives:

$$\begin{vmatrix} a_1 & a_3 & a_4 \\ b_1 & b_3 & b_4 \\ c_1 & c_3 & c_4 \end{vmatrix} + \begin{vmatrix} a_1 & a_5 & a_6 \\ b_1 & b_5 & b_6 \\ c_1 & c_5 & c_6 \end{vmatrix} = 0, \qquad \begin{vmatrix} a_2 & a_3 & a_4 \\ b_2 & b_3 & b_4 \\ c_2 & c_3 & c_4 \end{vmatrix} + \begin{vmatrix} a_2 & a_5 & a_6 \\ b_2 & b_5 & b_6 \\ c_2 & c_5 & c_6 \end{vmatrix} = 0,$$

$$\begin{vmatrix} a_3 & a_5 & a_6 \\ b_3 & b_5 & b_6 \\ c_3 & c_5 & c_6 \end{vmatrix} + \begin{vmatrix} a_3 & a_1 & a_2 \\ b_3 & b_1 & b_2 \\ c_3 & c_1 & c_2 \end{vmatrix} = 0, \qquad \begin{vmatrix} a_4 & a_5 & a_6 \\ b_4 & b_5 & b_6 \\ c_4 & c_5 & c_6 \end{vmatrix} + \begin{vmatrix} a_4 & a_1 & a_2 \\ b_4 & b_1 & b_2 \\ c_4 & c_1 & c_2 \end{vmatrix} = 0,$$

$$\begin{vmatrix} a_5 & a_1 & a_2 \\ b_5 & b_1 & b_2 \\ c_5 & c_1 & c_2 \end{vmatrix} + \begin{vmatrix} a_5 & a_3 & a_4 \\ b_5 & b_3 & b_4 \\ c_5 & c_3 & c_4 \end{vmatrix} = 0, \qquad \begin{vmatrix} a_6 & a_1 & a_2 \\ b_6 & b_1 & b_2 \\ c_6 & c_1 & c_2 \end{vmatrix} + \begin{vmatrix} a_6 & a_3 & a_4 \\ b_6 & b_3 & b_4 \\ c_6 & c_3 & c_4 \end{vmatrix} = 0.$$

On the contrary, the theorem above puts the required conditions into the very simple form:

$$b_1 c_2 - c_1 b_2 + b_3 c_4 - c_3 b_4 + b_5 c_6 - c_5 b_6 = 0,$$
$$c_1 a_2 - a_1 c_2 + c_3 a_4 - a_3 c_4 + c_5 a_6 - a_5 c_6 = 0,$$

$$a_1 b_2 - b_1 a_2 + a_3 b_4 - b_3 a_4 + a_5 b_6 - b_5 a_6 = 0.$$

68. One may point out a theorem that is much more precise than the preceding one, and which permits us to find, in the simplest fashion, the rank of the form to which F reduces when one supposes that the variables are coupled by p given relations. In order to do this, we define the alternating bilinear form:

$$\Phi(\xi, \xi') = \sum a_{ij} \xi_i \xi'_j$$

by the equality:

$$s[F^{s-1}(\xi_1 u_1 + \ldots + \xi_n u_n)(\xi'_1 u_1 + \ldots + \xi'_n u_n)] = \Phi(\xi, \xi')[\Phi'].$$

The exterior quadratic form:

$$\Phi(\xi) = \sum a_{ij} [\xi_i \xi_j]$$

is (up to a factor) the adjoint form to $\dfrac{F^{s-1}}{(s-1)!}$. It is the *absolute covariant* of F, in the sense that if one performs an arbitrary linear substitution on the variables $u_1, \ldots, u_n$, and the linear substitution that preserves $\xi_1 u_1 + \ldots + \xi_n u_n$ on the variables $\xi_1, \ldots, \xi_n$, and if these two substitutions take the two forms $F(u)$ and $\Phi(\xi)$ to $\bar{F}(\bar{u})$ and $\bar{\Phi}(\bar{\xi})$ then one will also have:

$$s[\bar{F}^{s-1}(\bar{\xi}_1\bar{u}_1 + \cdots + \bar{\xi}_n\bar{u}_n)(\bar{\xi}'_1\bar{u}_1 + \cdots + \bar{\xi}'_n\bar{u}_n)] = \bar{\Phi}(\bar{\xi}, \bar{\xi}')[\bar{F}'].$$

In particular, if F has been reduced to its canonical form:

$$F = [u_1 u_2] + \ldots + [u_{2s-1} u_s]$$

then one will immediately find for the canonical form of Φ:

$$\Phi = [\xi_1 \xi_2] + \ldots + [\xi_{2s-1} \xi_s] .$$

From this, there easily results the general identity:

$$(6) \qquad \begin{cases} \left[\dfrac{F^{s-p}}{(s-p)!}(\xi_1 u_1 + \cdots + \xi_n u_n)(\xi'_1 u_1 + \cdots + \xi'_n u_n)\cdots(\xi_1^{(2p-1)} u_1 + \cdots + \xi_n^{(2p-1)} u_n\right] \\ \qquad = \dfrac{\Phi^{(p)}(\xi, \xi', \cdots, \xi^{(2p-1)}}{p!}\left[\dfrac{F^s}{s!}\right], \end{cases}$$

in which the exterior form of degree p that corresponds to the alternating multilinear form $\Phi^{(p)}$ is equal to $[\Phi^p(\xi)]$. This identity is obvious when F has been reduced to its canonical form and is therefore true in the general case. This basically amounts to the property that was invoked in the preceding section that the adjoint form to $[F^{s-p}]$ is equal to $[\Phi^p]$, up to a scalar factor.

In particular, upon setting p = 2, and upon taking the terms in $[\xi_i, \xi'_j, \xi''_k, \xi'''_l]$ in identity (6) one will obtain:

$$\left[\frac{F^{s-2}}{(s-2)!}u_i u_j u_k u_l\right] = (a_{ij}a_{kl} + a_{ik}a_{lj} + a_{il}a_{jk})\left[\frac{F^s}{s!}\right], \tag{7}$$

in which the coefficients a_{ij} are defined by:

$$\left[\frac{F^{s-2}}{(s-2)!}u_i u_j\right] = a_{ij}\left[\frac{F^s}{s!}\right].$$

Finally, one may deduce another identity that we will use later on. Consider the form:

$$\left[\frac{F^{s-2}}{(s-2)!}u_i u_j u_k\right] - \left[\frac{F^{s-1}}{(s-1)!}(a_{ij}u_k + a_{jk}u_i + a_{ki}u_j\right];$$

it is of degree $2s - 1$. If one exterior multiplies it by any of the variables $u_1, ..., u_{2s}$ – say u_l – then one will immediately confirm from (7) that the product is zero. As a result, the form itself will be identically zero. Since u_i , u_j , u_k can be replaced by three arbitrary linear forms in these variables, we arrive at the following theorem:

If one considers an arbitrary number of linear forms $f_1, f_2, ..., f_p$, *and if one sets:*

$$\left[\frac{F^{s-1}}{(s-1)!}f_i f_j\right] = a_{ij}\left[\frac{F^s}{s!}\right] \qquad (i, j = 1, 2, ..., p)$$

then one will have the identities:

$$\left[\frac{F^{s-2}}{(s-2)!}f_i f_j f_k\right] = \left[\frac{F^{s-1}}{(s-1)!}(a_{ij}f_k + a_{jk}f_i + a_{ki}f_j)\right]. \tag{8}$$

69. We shall now go on to the problem that was stated above, which consists of finding the rank of the form to which F is reduced when one supposes that the variables are coupled by p independent linear relations:

$$f_1 = 0, f_2 = 0, ..., f_p = 0.$$

We may suppose that these relations are:

$$u_1 = 0, u_2 = 0, ..., u_p = 0.$$

They permit us to perform an arbitrary linear substitution on the u, with the one condition that the first p variables $u_1, \ldots, u_p$ must be exchanged amongst themselves. It results from this that we can perform an arbitrary linear substitution on the variables ξ, with the one condition that the *last* $2s - p$ *variables* $\xi_{p+1}, \ldots, \xi_{2s}$ must be exchanged amongst themselves. We then set:

$$\left[\frac{F^{s-1}}{(s-1)!}u_i u_j\right] = a_{ij}\left[\frac{F^s}{s!}\right], \qquad (i, j = 1, 2, \ldots, p).$$

If one suppresses the terms in $\xi_{p+1}, \ldots, \xi_{2s}$ from Φ then one will obviously obtain:

$$\bar{\Phi} = \sum_{(ij)}^{1,\cdots,p} a_{ij}[\xi_i \xi_j].$$

Let $2q$ be the rank of the form $\bar{\Phi}$. One may, by a convenient linear substitution that is performed on the $\xi_1, \ldots, \xi_p$, reduce $\bar{\Phi}$ to:

$$\bar{\Phi} = [\xi_1 \xi_2] + \cdots + [\xi_{2q-1}\xi_{2q}].$$

As a result, one may, by removing linear combinations of the $\xi_{p+1}, \ldots, \xi_{2s}$ from the $\xi_1, \ldots, \xi_p$, if necessary – *and this is permissible* – reduce Φ to:

$$\Phi = [\xi_1\,\xi_2] + \ldots + [\xi_{2q-1}\,\xi_{2q}] + [\xi_{2q+1}\,\xi_{p+1}] + \ldots + [\xi_p\,\xi_{2p-2q}] \\ + [\xi_{2p-2q+1}\,\xi_{2p-2q+2}] + \ldots + [\xi_{2s-1}\,\xi_{2s}],$$

but then the form F will be changed into:

$$F = [u_1\,u_2] + \ldots + [u_{2q-1}\,u_{2q}] + [u_{2q+1}\,u_{p+1}] + \ldots + [u_p\,u_{2p-2q}] + \ldots + [u_{2s-1}\,u_{2s}]\,.$$

One sees that if one now takes the relations:

$$u_1 = 0, \quad u_2 = 0, \ldots, u_p = 0$$

into account then the rank of F will be reduced by $2p - 2q$ units.

We therefore arrive at the following theorem:

Consider the independent linear forms $f_1, f_2, \ldots, f_p$, *the* $\frac{p(p-1)}{2}$ *quantities* a_{ij} *that are defined by the equalities*:

$$\left[\frac{F^{s-1}}{(s-1)!} f_i f_j\right] = a_{ij}\left[\frac{F^s}{s!}\right],$$

and the exterior quadratic form in p *variables* $\xi_1, \ldots, \xi_p$:

$$\Phi(\xi) = \sum_{(ij)}^{1,\cdots,p} a_{ij} [\xi_i \xi_j].$$

If that form is of rank $2q$ *then the rank of the form F will be reduced by* $2p - 2q$ *units when one supposes that the variables are coupled by p relations:*

$$f_1 = 0, f_2 = 0, \ldots, f_p = 0.$$

Furthermore, if one performs a linear substitution on the p given linear forms such that Φ *is reduced to its canonical form:*

$$\Phi = [\xi_1\, \xi_2] + \ldots + [\xi_{2q-1}\, \xi_{2q}]$$

then the form F will be reduced to the canonical form:

$$F = [f_1 f_2] + \ldots + [f_{2q-1} f_{2q}] + [f_{2q+1} f_{p+1}] + \ldots + [f_p f_{2p-2q}] + \ldots + [f_{2s-1} f_{2s}],$$

in which $f_{p+1}, \ldots, f_{2s}$ *denote new forms that are conveniently chosen to be mutually independent and independent of the given forms.*

In particular, if $q = 0$ then one recovers the preceding theorem that was stated and proved in sec. **67**.

CHAPTER VII

EXTERIOR DIFFERENTIAL FORMS AND THEIR DERIVED FORMS

I. - *The bilinear covariant of a Pfaff form.*

70. Now consider a linear differential form (Pfaff form):

$$\omega_\delta = a_1 \delta x_1 + a_2 \delta x_2 + \ldots + a_n\, \delta x_n\,.$$

One can derive an alternating bilinear form in two types of differentials from that form, namely:

$$\delta\omega_{\delta'} - \delta'\omega_\delta = \sum a_i\,(\delta\delta' x_i - \delta'\delta x_i) + \sum (\delta a_i \delta' x_i - \delta x_i \delta' a_i)\,.$$

Suppose that the two differentiation symbols are interchangeable, i.e., that one has:

$$\delta\delta' x_i = \delta'\delta x_i\,.$$

As for the right-hand side, which one calls the *bilinear covariant* of the form ω, one may make it correspond to an exterior quadratic differential form that we write, with the conventions made above:

$$\omega'_\delta = \sum_i [\delta a_i \delta x_i] = \sum_{(ij)} \left(\frac{\partial a_j}{\partial x_i} - \frac{\partial a_i}{\partial x_j}\right)[\delta x_i \delta x_j].$$

This form will be called the *exterior derivative* of the form ω.

This derivation procedure has a significance that is *independent of the choice of variables.* Furthermore, it is the one that takes place when one passes from a curvilinear integral taken over a closed contour to a double integral taken over a surface bounded by the contour.

For example, if one has three variables x, y, z and one sets:

$$\omega_\delta = P\delta x + Q\delta y + R\delta z$$

then one will have:

$$\begin{aligned}\omega_\delta = [\delta P\, \delta x] + [\delta Q\ \delta y] + [\delta R\ \delta z] = & \frac{\partial P}{\partial x}[\delta x\, \delta x] + \frac{\partial P}{\partial y}[\delta y\, \delta x] + \frac{\partial P}{\partial z}[\delta z\, \delta x] \\ & + \frac{\partial Q}{\partial x}[\delta x\, \delta y] + \frac{\partial Q}{\partial y}[\delta y\, \delta y] + \frac{\partial Q}{\partial z}[\delta z\, \delta y] \\ & + \frac{\partial R}{\partial x}[\delta x\, \delta z] + \frac{\partial R}{\partial y}[\delta y\, \delta z] + \frac{\partial R}{\partial z}[\delta z\, \delta z]\end{aligned}$$

$$= \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)[\delta y\, \delta z] + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)[\delta z\, \delta x] + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)[\delta x\, \delta y],$$

and Stokes's formula can be written:

$$\int_{\mathrm{C}} \omega = \iint_{\mathrm{S}} \omega',$$

in which S denotes a surface that is bounded by the contour C.

The necessary and sufficient condition for ω' to be annihilated is that the form ω must be an exact differential.

REMARK. – The permutability of the two differentiation symbols δ and δ' must take place in the case where the differentiations are applied to an arbitrary function of independent variables *or else the differentiations would not have a covariant character.* This is easy to verify. If one sets:

$$\omega_\delta = \delta y = \frac{\partial y}{\partial x_1}\delta x_1 + \cdots + \frac{\partial y}{\partial x_n}\delta x_n$$

then ω_δ will be an exact differential, and one will have:

$$\delta\omega_{\delta'} - \delta'\omega_\delta = 0,$$

i.e.:

$$\delta\delta' y = \delta'\delta y .$$

II. - *Exterior derivation.*

71. The same derivation procedure can be applied to an exterior differential form of any degree. For example, let:

$$\Omega = \sum a_{ij}\,[\delta x_i\, \delta x_j]$$

be a quadratic form, and consider the alternating bilinear form:

$$\Omega(\delta, \delta') = \sum a_{ij}\,(\delta x_i\, \delta' x_j - \delta x_j\, \delta' x_i)$$

that corresponds to it, and introduce *three* differentiation symbols $\delta, \delta', \delta''$ that are mutually interchangeable. Finally, consider the expression:

$$\delta\Omega(\delta', \delta'') - \delta'\,\Omega(\delta, \delta'') + \delta''\,\Omega(\delta, \delta'),$$

which obviously has an intrinsic significance that is independent of the choice of variables. Upon doing this calculation, one easily confirms that it is reduced to an alternating trilinear expression:

$$\Omega'(\delta, \delta', \delta'') = \sum \; [\delta a_{ij}(\delta' x_i \delta'' x_j - \delta' x_j \delta'' x_i) - \delta' a_{ij}(\delta x_i \delta'' x_j - \delta x_j \delta'' x_i) \\ + \delta'' a_{ij}(\delta x_i \delta' x_j - \delta x_j \delta' x_i).$$

Now, a cubic exterior differential form:

$$\Omega'_\delta = \sum [\delta a_{ij} \delta x_i \delta x_j] = \sum \left(\frac{\partial a_{ij}}{\partial x_k} + \frac{\partial a_{jk}}{\partial x_i} + \frac{\partial a_{ki}}{\partial x_j} \right) [\delta x_i \delta x_j \delta x_k],$$

which we shall call the *derived* form of Ω, corresponds to that trilinear form.

72. In the case examined, it is important to account for the relation that exists between the derivation of a quadratic exterior form and the operation that consists of passing from a double integral taken over a *closed* surface to the triple integral taken over the volume bounded by surface.

In order to do this, imagine that $x_1, ..., x_n$ are functions of three parameters α, β, γ, and consider an elementary parallelepiped in n-dimensional space whose edges are portions of the coordinate lines, and whose vertices A, B, C, D, E, F, G, H correspond to the curvilinear coordinates:

$$(\alpha, \beta, \gamma), \quad (\alpha+\delta\alpha, \beta, \gamma), \quad (\alpha, \beta+\delta'\beta, \gamma), \quad (\alpha, \beta, \gamma' + \delta''\gamma),$$
$$(\alpha+\delta\alpha, \beta+\delta'\beta, \gamma), (\alpha+\delta\alpha, \beta, \gamma+\delta''\gamma), \quad (\alpha, \beta+\delta'\beta, \gamma+\delta''\gamma), \quad (\alpha+\delta\alpha, \beta+\delta\beta, \gamma+\delta''\gamma).$$

As one knows, the symbols $\delta, \delta', \delta''$ refer to differentiations with respect to the three parameters α, β, γ, respectively.

Now consider the curvilinear integral $\iint \Omega$, which is taken over the surface that bounds this parallelepiped.

The integrals that are taken over the three faces that contain A are, up to a sign:

$$\Omega(\delta', \delta''), \quad \Omega(\delta'', \delta), \quad \Omega(\delta, \delta'),$$

and, in order for these integrals to be taken over all of the internal faces or all of the external faces, it is necessary to take them to be either equal to the three preceding expressions or equal and opposite. If we take them to be equal and opposite then the sum of the integrals taken over the six faces will be:

$$- \Omega(\delta', \delta'') - \Omega(\delta'', \delta) - \Omega(\delta, \delta') + [\Omega(\delta', \delta'') + \delta\Omega(\delta', \delta'')] \\ + [\Omega(\delta'', \delta) + \delta' \Omega(\delta'', \delta)] + [\Omega(\delta, \delta') + \delta'' \Omega(\delta, \delta')]$$

$$= \delta\Omega(\delta', \delta'') + \delta'\Omega(\delta'', \delta) + \delta''\Omega(\delta, \delta') = \Omega'(\delta, \delta', \delta'').$$

The surface integral $\iint \Omega$ is thus transformed into the volume integral $\iiint \Omega'$.

In the simple case of three variables, if one sets:

$$\Omega = P\,[\delta y\ \delta z] + Q\,[\delta z\ \delta x] + R\,[\delta x\ \delta y]$$

then one will have:

$$\Omega' = [\delta P\ \delta y\ \delta z] + [\delta Q\ \delta z\ \delta x] + [\delta R\ \delta x\ \delta y] = \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\right)[\delta x\ \delta y\ \delta z].$$

73. These considerations can be extended to exterior forms of arbitrary degree. Any exterior form admits a derived form whose degree is greater by one unit and whose calculation is extremely easy, since each term of the form:

$$A\,[\delta x_i\ \delta x_j \ldots \delta x_l]$$

gives rise to the derived term:

$$[\delta A\ \delta x_i\ \delta x_j \ldots \delta x_l].$$

We note several formulas that are useful and easy to prove. *If m is a coefficient that is a finite function of the variables and* Ω *is an arbitrary exterior form then one will have*:

$$(m\Omega)' = [dm\ \Omega] + m\,\Omega'.$$

If Ω *and* Π *are two arbitrary exterior differential forms then one will have*:

$$[\Omega\ \Pi]' = [\Omega'\ \Pi] \pm [\Omega\ \Pi'];$$

the + sign refers to the case in which Ω *is of even degree and the − sign, to the case in which* Ω *is of odd degree.* In particular, if Ω is of even degree then the derived form of $[\Omega^p]$ will be given by the ordinary formula:

$$[\Omega^p]' = p\,[\Omega^{p-1}\,\Omega'].$$

74. In the preceding, we supposed that the coefficients of the forms under consideration were continuous functions that admitted partial derivatives of the first order. *However, it is also the case that one may still define an exterior derivative* Ω' *when the coefficients of a form* Ω *do not admit derivatives*. A classic example is provided by potential theory.

We consider a material volume V that is bounded by a surface S. Let ρ be the density at a point of V. We suppose that the function ρ is continuous. The potential U of that mass is a continuous function on all of space that everywhere admits continuous first order derivatives.

Regarding that function, there exists a theorem (Gauss's theorem), which is expressed by the formula:

$$\iint \frac{\partial U}{\partial x} dy\,dx + \frac{\partial U}{\partial y} dz\,dx + \frac{\partial U}{\partial z} dx\,dy = \iint -4\pi\rho\,dx\,dy\,dz\ ;$$

the integral on the left-hand side is taken over an *arbitrary* closed surface and the one on the right-hand side is taken over the volume that is bounded by that surface. From this, it results that upon setting:

$$\Omega = \frac{\partial U}{\partial x}[dy\ dz] + \frac{\partial U}{\partial y}[dz\,dx] + \frac{\partial U}{\partial z}[dx\,dy]$$

one can define the exterior derivative Ω' of Ω by:

$$\Omega' = -4\pi\rho\,[dx\ dy\ dz].$$

If the function U admits second-order partial derivatives then this will amount to the classical Poisson formula, because the derivation procedure defined above will immediately give:

$$\Omega' = \left(\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2}\right)[dx\,dy\ dz].$$

However, one can still define the derivative Ω' if the function U does not admit second-order partial derivatives, *which is the general case when one does not make supplementary hypotheses on the function* ρ.

One thus concedes the possibility of defining the exterior derivative as an *autonomous* operation that is independent of classical derivation. There is then a direct proof of the formula from the preceding section:

(1) $$[\Omega\,\Pi]' = [\Omega'\,\Pi] \pm [\Omega\,\Pi'],$$

in which one simply supposes that Ω and Π are exterior-differentiable.

75. Take the simplest case of a linear form in two variables:

$$\omega = P\ \delta x + Q\ \delta y$$

that admits an exterior derivative:

$$\omega' = R[\delta x\ \delta y]\ .$$

Suppose that the functions P and Q are continuous, and consider a function m that admits continuous first-order partial derivatives. Here, the formula:

$$(m\omega)' = m\,\omega' + [\delta m\,\omega]$$

amounts to:

$$\int m(P\,\delta x + Q\,\delta y) = \iint \left(mR + Q\frac{\partial m}{\partial x} - P\frac{\partial m}{\partial y} \right) \delta x\,\delta y.$$

The proof of this formula can be carried out very simply. Let A be the integration area, and let C be the contour that bounds it. Divide the area A into a large number of partial areas; for example, areas parallel to the axes. Take a point (x_0, y_0) in each of the partial areas, and call the values of the functions m, P, Q, $\frac{\partial m}{\partial x}, \frac{\partial m}{\partial y}$ at that point:

$$m_0, P_0, Q_0, \left(\frac{\partial m}{\partial x}\right)_0, \left(\frac{\partial m}{\partial y}\right)_0.$$

In the interior, or on the contour of that area, one may set:

$$m = m_0 + (x - x_0)\left[\left(\frac{\partial m}{\partial x}\right)_0 + \varepsilon_1\right] + (y - y_0)\left[\left(\frac{\partial m}{\partial y}\right)_0 + \varepsilon_2\right],$$

$$P = P_0 + \varepsilon_3, \qquad Q = Q_0 + \varepsilon_4.$$

When the integral $\int m(P\ \delta x + Q\ \delta y)$ is taken over the contour of that partial area, it will be equal to:

$$\int m_0(P\delta x + Q\delta y) + \int \left[(x - x_0)\left(\frac{\partial m}{\partial x}\right)_0 + (y - y_0)\left(\frac{\partial m}{\partial y}\right)_0 \right](P_0\delta x + Q_0\delta y),$$

plus a quantity that is less than $\varepsilon M\,\Delta l$, upon denoting the upper limit of ε_1, ε_2, ε_3, ε_4, by ε, a fixed number by M, the diameter of the area by Δ, and the length of its contour by l. The sum of all these supplementary quantities can obviously be made as small as one pleases, because $\sum \Delta l$ is of order of the total area. As for the sum of the two integrals written above, it is equal to:

$$\sum \iint \left[m_0 R + Q_0\left(\frac{\partial m}{\partial x}\right)_0 - P_0\left(\frac{\partial m}{\partial y}\right)_0 \right] \delta x\,\delta y.$$

One easily derives the proof of the formula in question from this.

With analogous hypotheses, this proof may be extended to the case of a quadratic form:

$$\Omega = P\,[\delta y\ \delta z] + Q\,[\delta z\ \delta x] + R\,[\delta x\ \delta y].$$

The existence of the equality:

$$\iint\ P\ \delta y\ \delta z + Q\ \delta z\ \delta x + R\ \delta x\ \delta y = \iiint\ H\ \delta x\ \delta y\ \delta z$$

implies that:

$$\iint m(P\,\delta y\,\delta z + Q\,\delta z\,\delta x + R\,\delta x\,\delta y) = \iiint \left(mH + P\frac{\partial m}{\partial x} + Q\frac{\partial m}{\partial y} + R\frac{\partial m}{\partial z} \right) \delta x\,\delta y\,\delta z.$$

The proof seems very difficult in the case of two linear forms in three variables:

$$\omega = A\,\delta x + B\,\delta y + C\,\delta z,$$
$$\omega_1 = A'\,\delta x + B'\,\delta y + C'\,\delta z.$$

Suppose that these two forms are differentiable and that one has, for example:

$$\int A\,\delta x + B\,\delta y + C\,\delta z = \iint P\,\delta y\,\delta z + Q\,\delta z\,\delta x + R\,\delta x\,\delta y,$$
$$\int A'\,\delta x + B'\,\delta y + C'\,\delta z = \iint P'\,\delta y\,\delta z + Q'\,\delta z\,\delta x + R'\,\delta x\,\delta y.$$

Formula (1) then becomes:

$$\iint (BC' - CB')\,\delta y\,\delta z + (CA' - AC')\,\delta z\,\delta x + (AB' - BA')\,\delta x\,\delta y$$
$$= \iiint (PA' + QB' + RC' - P'A - Q'B - R'C)\,\delta x\,\delta y\,\delta z.$$

It does not seem possible to prove this by the same procedure as in the preceding case without adding supplementary hypotheses in order to this; for example, the hypothesis that the functions A, B, C, A', B', C' must satisfy a condition that is analogous to the Lipshitz condition. It is interesting to study this question and see if it is really true that *the differentiability of an exterior product always results in the differentiability of its factors.*

As for the question of knowing the conditions under which an exterior differentiable form is differentiable, it is related, at least for forms of degree $n - 1$ in n variables, to the theory of additive functions on the set of C. de la Valée-Poussin ([1]). For example, the form $\Omega = P\,[\delta y\,\delta z] + Q\,[\delta z\,\delta x] + R[\delta x\,\delta y]$ is differentiable if the sum of the integrals $\iint \Omega$, when it is taken over the surfaces that bound a finite number of cubes that are composed of planes that are parallel to the coordinate planes, tends to zero when the sum of the volumes of these cubes tends to zero. The function H that appears in this expression is the derivative:

$$\Omega' = H\,[\delta x\,\delta y\,\delta z],$$

which is not naturally continuous, in general.

In what follows, we shall always assume the legitimacy of the operations performed.

([1]) See the book entitled: *Integrales de Lebesgue, fonctions d'ensemble, classes de Baire;* Paris, Gauthier-Villars, 1916.

III. - *Exact exterior differential forms.*

76. Now, here is an important theorem:

The derivative of the derivative Ω' of an arbitrary exterior differential form Ω is identically zero.

Indeed, take an arbitrary term in Ω, say:

$$a\,[\delta x_1\,\delta x_2\,\ldots\,\delta x_p].$$

The corresponding term in Ω' is:

$$[\delta a\;\delta x_1\,\delta x_2\,\ldots\,\delta x_p].$$

If a depends upon only $x_1, \ldots, x_p$ then the latter term will be zero, and its derivative as well. If, on the contrary, a is independent of $x_1, \ldots, x_p$ then one can make a change of variables such that a becomes equal to x_{p+1}. The derivative of the term:

$$[\delta x_{p+1}\;\delta x_1\;\delta x_2\,\ldots\,\delta x_p]$$

will then be zero, since the coefficient of that term is unity, so its derivative will contribute nothing to the exterior derivative.

This theorem has a converse, namely:

If the derivative of a differential form Ω is zero then the form Ω can be regarded as the derivative of a form Π whose degree is less than that of Ω by one unit.

In order to prove this theorem, we shall appeal to the following lemma, which we will use later on, moreover:

If the derivative of a form Ω is zero, and if that form does not contain the differential δx_n then its coefficients will be independent of x_n.

Indeed, take a term in Ω such as:

$$A\,[\delta x_1\,\delta x_2\,\ldots\,\delta x_p].$$

In the derivation, it will furnish the term:

$$[\delta A\;\delta x_1\,\delta x_2\,\ldots\,\delta x_p],$$

which, upon developing, will give several terms, such as:

$$\frac{\partial A}{\partial x_n}[\delta x_n \delta x_1 \delta x_2 \cdots \delta x_p].$$

Obviously, the last term cannot be reduced any further, since no other term of Ω contains δx_n. Since $\Omega' = 0$, one necessarily has:

$$\frac{\partial A}{\partial x_n} = 0.$$

Having thus proved the lemma, we return to our theorem. Let us call Ω_0 what the form Ω becomes when one makes $x_n = x_n^0$ and $\delta x_n = 0$. The derivative of Ω_0 will obviously be zero if that of Ω is. Suppose then that the theorem has been proved for $n - 1$ variables. It will be possible to find a form Π_0 that is constructed from the variables $x_1, ..., x_{n-1}$, such that Ω_0 is its derivative:

$$\Pi_0' = \Omega_0 .$$

If this is the case then separate the terms in the given form Ω and the unknown form Π that do not contain δx_n from those that do. One may write:

$$\Omega = \Omega_1 + [\delta x_n \, \Omega_2], \qquad \Pi = \Pi_1 + [\delta x_n \, \Pi_2].$$

If we calculate the terms in Π' that contain δx_n then we will find that:

$$\Pi' = \left[\delta x_n \frac{\partial \Pi_1}{\partial x_n} \right] - [\delta x_n \, \Pi_2'] + \cdots$$

Choose the form Π_2 *arbitrarily*, and determine Π_1 by the conditions:

1) Π_1 reduces to Π_0 for $x_n = x_n^0$.
2) One has:

$$\frac{\partial \Pi_1}{\partial x_n} = \Pi_2' + \Omega_2 .$$

One thus obtains Π_1 by quadratures.

When the form Π is chosen in the manner that we just described, it will enjoy the following properties:

1) The difference $\Pi' - \Omega$ will not contain δx_n when one reduces the similar terms.
2) It will reduce to zero when one makes $x_n = x_n^0$ in its coefficients.

We now remark that the derivative of that form is zero; consequently, from the lemma, all of its coefficients will have values that are independent of x_n. It is therefore identically zero, and the theorem is proved.

The same proof shows that one may arbitrarily choose the terms in the form Π that contain δx_n , arbitrarily choose the values for $x_n = x_n^0$ in the terms that do not contain δx_n , but do contain δx_{n-1} , arbitrarily choose the values for $x_n = x_n^0$, $x_{n-1} = x_{n-1}^0$ in the terms that contain neither δx_n nor δx_{n-1} , but do contain δx_{n-2} , and so on.

Furthermore, it is quite clear that if one has a solution of the problem then all of the others can be deduced from it by adding the derivative of an arbitrary form (whose degree is two units less than that of Ω) to Π.

77. If Ω is a linear form then, from the preceding theorem, the hypothesis that its exterior derivative is zero will thus lead to the conclusion that was pointed out before that Ω is an exact differential. If Ω is a quadratic form in three variables:

$$\Omega = P\,[\delta y\;\delta z] + Q\,[\delta z\;\delta x] + R\,[\delta x\;\delta y]$$

then the condition:

$$\frac{\partial P}{\partial x_1} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} = 0$$

will be necessary and sufficient for Ω to be regarded as the derivative of a linear form, i.e., for one to find three functions *A*, *B*, *C* that satisfy:

$$\frac{\partial C}{\partial y} - \frac{\partial B}{\partial z} = P\,,$$
$$\frac{\partial A}{\partial z} - \frac{\partial C}{\partial x} = Q,$$
$$\frac{\partial B}{\partial x} - \frac{\partial A}{\partial y} = R.$$

Remark. – If the coefficients of the form Ω are uniform in a certain domain then the condition $\Omega' = 0$ will not always be sufficient to assure the existence of a form Π that is *uniform in that domain* and has Ω for its exterior derivative. For example, consider the two-dimensional domain (closed and without boundary) that is composed of the points of a sphere Σ, and let Ω be a form of degree 2 that is uniform in that domain (and has coefficients that admit continuous partial derivatives of the first order). The derivative Ω' will obviously be zero. Nevertheless, if there exists a linear form ω whose derivative ω' is equal to Ω then when one integrates $\int \omega$ twice around the same great circle of the sphere in the two different senses one will obtain:

$$\iint \Omega = 0,$$

in which the integral is taken over the entire surface of the sphere. *The preceding equation gives a supplementary condition for* Ω *to be regarded as the exact derivative of a form* ω *that is uniform over all of the sphere.*

CHAPTER VIII

THE CHARACTERISTIC SYSTEM OF AN EXTERIOR DIFFERENTIAL FORM. FORMATION OF INTEGRAL INVARIANTS

I. - *The characteristic system of an exterior differential form.*

78. The results of the preceding chapter permit us to easily form the *characteristic* Pfaff system of a given exterior differential form Ω.

In order to do this, we remark that if Ω is invariant for the system of differential equations:

$$\frac{dx_1}{X_1} = \frac{dx_2}{X_2} = \ldots = \frac{dx_n}{X_n} \tag{1}$$

then Ω can be expressed in terms of $n - 1$ independent first integrals $y_1, \ldots, y_{n-1}$, and their differentials; *as a result, this will also be true for its derivative* Ω'. As a consequence, the system of linear equations (in total differentials) that is associated with the two exterior forms Ω and Ω' will be a consequence of the equations:

$$dy_1 = 0, \ldots, dy_{n-1} = 0, \tag{2}$$

and, as a result, of equations (1).

In other words, *in order for system* (1) *to admit* $\Omega(\delta)$ *as an invariant form it is necessary that the associated system of* Ω *and* Ω' $\mu\upsilon\sigma\tau$ *be verified when one replaces the variables* $\delta x_1, \ldots, \delta x_n$ *with* $X_1, \ldots, X_n$.

Conversely, suppose this condition is true. Since the associated system of $\Omega(\delta)$ is satisfied on account of equations (1) it will also be satisfied on account of the equivalent equations (2). Hence, when $\Omega(\delta)$ is considered as an exterior form in the quantities dx_i, it can be uniquely expressed in terms of $dy_1, \ldots, dy_{n-1}$. Since the coefficients are functions of x, one may always suppose that they are expressed by means of $y_1, \ldots, y_{n-1}$, and x_n (if $X_n \neq 0$). One will thus have:

$$\Omega = \sum \mathrm{A}_{i_1 \ldots i_p} [dy_{i_1} \ldots dy_{i_p}].$$

When one forms Ω', one will find that the only term in $[dx_n dy_{i_1} \ldots dy_{i_p}]$ has $\dfrac{\partial A_{i_1 \ldots i_p}}{\partial x_n}$ for its coefficient. Now, by hypothesis, Ω' can also be expressed by means of only the quantities dy_i. One thus has

$$\frac{\partial A_{i_1 \ldots i_p}}{\partial x_n} = 0.$$

As a result, Ω can be expressed by means of first integrals of the given system and their differentials; it is therefore an invariant form.

It results immediately from this that *the equations of the characteristic system of* Ω *reduce to the equations of the associated system of* Ω, *combined with the equations of the associated system of* Ω'.

79. We shall now examine some important particular cases.

Suppose that Ω is an exact form; i.e., $\Omega' = 0$. In this case, *the characteristic system of the differential form* Ω *coincides with the associated system of the form* Ω.

As an application, we look for the characteristic system of the (complete) *relative* integral invariant $\int\Omega$. That relative invariant reduces to the absolute invariant $\int\Omega'$. However Ω' is an exact derivative. Hence, *the characteristic system of a relative integral invariant* $\int\Omega$ *coincides with the associated system of the derived form* Ω'.

This is just what happens with the linear integral invariant of dynamics:

$$\int \omega_\delta = \int \sum p_i\,\delta q_i - H\,\delta t.$$

Here one has:

$$\omega'_\delta = \sum [\delta p_i\,\delta q_i] - [\delta H\,\delta t].$$

The associated system of the form ω' is:

$$\frac{\partial\omega'}{\partial(\delta q_i)} = 0, \quad \frac{\partial\omega'}{\partial(\delta p_i)} = 0, \quad \frac{\partial\omega'}{\partial(\delta t)} = 0\,;$$

i.e., if we use the symbol d instead of δ:

$$-dp_i - \frac{\partial H}{\partial q_i}dt = 0,$$

$$dq_i - \frac{\partial H}{\partial p_i}dt = 0,$$

$$dH - \frac{\partial H}{\partial t}dt = 0;$$

this is the same as what we found in chapter I (sec. **11**).

Here, the differential form ω' is quadratic. As a result, we know in advance that the number of linearly independent equations of the associated system will be *even;* this explains why the $2n + 1$ equations of the characteristic system reduce to $2n$. Similarly, we will have an explanation for what happens in hydrodynamics in relation to the invariant form:

$$\xi\,[\delta y\,\delta z] + \eta\,[\delta z\,\delta y] + \zeta\,[\delta x\,\delta y] + (\eta\,w - \zeta\,v)[\delta x\,\delta t] + (\zeta\,u - \xi\,w)[\delta y\,\delta t] + (\xi\,v - \eta\,u)[\delta z\,\delta t].$$

Here $n = 4$; the characteristic system thus contains 4 or 2 or 0 independent equations. Now, there cannot be 4, since the form is invariant for the differential equations of the trajectories of the molecules. Hence, there are 2 or 0. There are 0 if $\xi = \eta = \zeta = 0$, i.e., if the motion is irrotational. In the contrary case, one may expect, *a priori,* that the trajectories will not be the only characteristic curves of the form.

80. One last important case is the one in which the form Ω is of degree $n - 1$. If it is invariant for a system of differential equations then this system will necessarily be unique, because the associated Pfaff system of Ω will be composed of $n - 1$ independent equations. In order for the associated system of Ω' to contain no more than $n - 1$ independent equations, it is obviously necessary that Ω' must be zero. As a result, *in order for a form Ω of degree $n - 1$ to be invariant for a system of differential equations, it is necessary and sufficient that its derivative must be identically zero.*

A simple example is furnished by the integral invariant of the kinematics of continuous media:

$$\iiint \rho\,(\delta x - u\,\delta t)(\delta y - v\,\delta t)(\delta z - w\,\delta t).$$

Here, the form Ω is:

$$\Omega = \rho\,[\delta x\;\delta y\;\delta z] - \rho u\,[\delta y\;\delta z\;\delta t] - \rho v\,[\delta z\;\delta x\;\delta t] - \rho w\,[\delta x\;\delta y\;\delta t].$$

The condition $\Omega' = 0$ gives:

$$\Omega' \equiv \left[\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z}\right][\delta t\,\delta x\,\delta y\,\delta z] = 0\,.$$

This is the condition of continuity, or the law of conservation of matter:

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} = 0.$$

One sees that *this law of conservation of matter translates into the simple condition that* the derivative Ω' *of the form that defines the quantity of elementary matter must be zero.*

81. The conservation laws of physics can often be translated into analogous conditions. The law of conservation of flux for a force field X, Y, Z translates into the condition that *divergence* of that force field be zero, i.e.:

$$\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} + \frac{\partial Z}{\partial z} = 0.$$

This simply expresses the idea that the derivative of the elementary force flux:

$$\Omega = X\,[\delta y\;\delta z] + Y[\delta z\;\delta x] + Z[\delta x\;\delta y]$$

is zero.

Any (static) magnetic field satisfies this condition. The electromagnetic field, defined by means of the exterior form:

$$\Omega = H_x\,[\delta y\;\delta z] + H_y\,[\delta z\;\delta x] + H_z\,[\delta x\;\delta y] + E_x\,[\delta x\;\delta t] + E_y\,[\delta y\;\delta t] + E_z\,[\delta z\;\delta t]$$

also satisfies the condition that Ω' is zero. One has:

$$\begin{aligned}\Omega' \quad &= \left(\frac{\partial H_x}{\partial x}+\frac{\partial H_y}{\partial y}+\frac{\partial H_z}{\partial z}\right)[\delta x\,\delta y\,\delta z]+\left(\frac{\partial H_x}{\partial t}+\frac{\partial E_z}{\partial y}-\frac{\partial E_y}{\partial z}\right)[\delta y\,\delta z\,\delta t]\\ &+\left(\frac{\partial H_y}{\partial t}+\frac{\partial E_x}{\partial z}-\frac{\partial E_z}{\partial x}\right)[\delta z\,\delta x\,\delta t]+\left(\frac{\partial H_z}{\partial t}+\frac{\partial E_y}{\partial x}-\frac{\partial E_x}{\partial y}\right)[\delta x\,\delta y\,\delta t].\end{aligned}$$

Upon annulling the four coefficients of Ω', one obtains the four classical equations, which one may write, in vector notation:

$$\operatorname{div}\mathbf{H} = 0,$$

$$\frac{\partial \mathbf{H}}{\partial t} + \operatorname{curl}\mathbf{E} = 0.$$

The form from hydrodynamics, which we have often considered already:

$$\Omega = \xi\,[\delta y\;dz] + \eta\,[\delta z\;\delta y] + \zeta\,[\delta x\;\delta y] + (\eta\,w - \zeta\,v)[\delta x\;\delta t] + (\zeta\,u - \xi\,w)[\delta y\;\delta t] + (\xi\,v - \eta\,u)[\delta z\;\delta t],$$

also has a zero derivative, since Ω is the derivative of the linear form of the "quantity of motion-energy," and the vectors (ξ, η, ζ) and $(\eta\,w - \zeta\,v,\ \zeta\,u - \xi\,v,\ \xi\,v - \eta\,u)$ satisfy the same relations as the magnetic field and the electric field. These two vectors are the vorticity, which plays the role of magnetic field, and the vector product of vorticity with velocity, which plays the role of electric force.

We remark that the electromagnetic field (or rather, the form Ω that represents it) might not be invariant for any system of differential equations, since $[\Omega^2]$ is not zero, in general. The exception is when the magnetic field is perpendicular to the electric field. The characteristic system will then be defined by the equations:

$$\begin{aligned} H_z\,dy - H_y\,dz + E_x\,dt &= 0,\\ H_x\,dz - H_z\,dx + E_y\,dt &= 0,\\ H_y\,dx - H_x\,dy + E_z\,dt &= 0,\\ -E_x\,dx - E_y\,dy - E_z\,dz &= 0,\end{aligned}$$

which reduce to three. The system of differential equations that admits Ω as invariant form will then be:

$$\frac{dx}{H_x} = \frac{dy}{H_y} = \frac{dz}{H_z} = \frac{dt}{0}.$$

At any instant, it defines the lines of force of the magnetic field. Another is:

$$\frac{dx}{E_y H_z - E_z H_y} = \frac{dy}{E_z H_x - E_z H_x} = \frac{dz}{E_x H_y - E_y H_x} = \frac{dt}{H_x^2 + H_y^2 + H_z^2}.$$

If the magnetic field is zero then the characteristic manifolds will be defined by the equations:

$$dt = 0, \quad E_x\, dx + E_y\, dy + E_z\, dz = 0;$$

they will be the equipotential surfaces when considered at each instant t.

II. - *Formation of integral invariants.*

82. It is obvious that the exterior product of two invariant exterior forms is also an invariant form. From this, the knowledge of an invariant exterior form Ω implies the knowledge of any other series of invariant forms, namely Ω' and all of the forms that are deduced from Ω and Ω' by exterior multiplication.

First, suppose that Ω is an *absolute* invariant form of *even* degree. One will then have two series of absolute invariant forms:

$$[\Omega^p], \quad [\Omega^{p-1}\,\Omega'] \qquad (p = 1, 2, \ldots).$$

The derivative of a form of the first series will be a form of the second series. The derivative of a form of the second series will be zero.

Now, suppose that one has a *relative* integral invariant $\int \Omega$, and suppose first that Ω is of even degree. One can deduce only one new invariant from that: viz., the absolute invariant $\int \Omega'$.

On the contrary, if Ω is of odd degree then one will have a series of relative integral invariants $\int \Omega\,\Omega'^{p-1}$ and a series of absolute integral invariants $\int \Omega'^{p}$. Moreover, the relative integral invariant $\int \Omega\,\Omega'^{p-1}$ will reduce to the absolute invariant $\int \Omega'^{p}$ by differentiation.

For example, this is true for the relative invariant of dynamics:

$$\int \omega = \int \sum_{i=1}^{i=n} p_i \delta q_i - H\,\delta t.$$

The relative integral invariants that one deduces from it are:

$$\int \omega\omega'^{p-1} \qquad (p = 1, ..., n).$$

The absolute integral invariants are:

$$\int \omega'^{p} \qquad (p = 1, ..., n).$$

There thus exists a (relative or absolute) invariant of an arbitrary given degree that is less than or equal to $2n$.

83. One must not assume that the new invariants whose existence that we have just pointed out are always the only ones that can be deduced (without integration) from a given invariant. For example, suppose that one knows an invariant form Ω that is reducible to the form:

$$\Omega = [\omega_1 \; \omega_2 \; \omega_3] + [\omega_4 \; \omega_5 \; \omega_6],$$

in which $\omega_1, \omega_2, ..., \omega_6$ are six linearly independent (Pfaff) forms. Introduce six indeterminates $\xi_1, ..., \xi_6$, and consider the auxiliary quadratic form:

$$\Pi = \xi_1 \frac{\partial\Omega}{\partial\omega_1} + \xi_2 \frac{\partial\Omega}{\partial\omega_2} + ... + \xi_6 \frac{\partial\Omega}{\partial\omega_6}.$$

It is obvious that if one regards the ξ as quantities that are covariant in the ω then the form Π will be covariant in Ω. We say that this form is of rank 2. We obtain the conditions:

$$\xi_i \, \xi_\alpha = 0, \quad (i = 1, 2, 3; \alpha = 4, 5, 6).$$

This gives two possible solutions: one for:

$$\xi_1 = \xi_2 = \xi_3 = 0,$$

and one for:

$$\xi_4 = \xi_5 = \xi_6 = 0.$$

The existence of two systems of three equations in Pfaff covariants results from this, namely:

$$\omega_1 = \omega_2 = \omega_3 = 0,$$
$$\omega_1 = \omega_2 = \omega_3 = 0.$$

As a result, the form $[\omega_1 \; \omega_2 \; \omega_3]$ and the form $[\omega_4 \; \omega_5 \; \omega_6]$ will themselves also be covariant. The first one is obtained by taking Ω into account in the equations of the second covariant system, and the second one is obtained by taking the equations of the first covariant system into account.

Now, suppose that Ω is expressed by means of first integrals of the system of equations, as well as their differentials, and that Ω is an invariant form. The formation of the two systems of

Pfaff covariants will also work for the reduced form, and each of them will contain only first integrals and their differentials. This will also be true for the two forms $[\omega_1\ \omega_2\ \omega_3]$ and $[\omega_4\ \omega_5\ \omega_6]$, *which are, as a result, invariant forms.*

The existence of the integral invariant $\int \omega$ *therefore implies the existence of each of the integral invariants* $\int \omega_1\ \omega_2\ \omega_3$ and $\int \omega_4\ \omega_5\ \omega_6$.

One verifies by an analogous argument that *the existence of an invariant form of degree p* > 2 *that is reducible to a sum of h monomial terms such that the hp factors that enter into these terms are linearly independent implies the property that each of these monomial terms must be an invariant form.*

This theorem is not true if $p = 2$.

84. In certain cases, the existence of an invariant form implies the existence of an invariant *equation.* For example, consider the form:

$$\Omega = [\omega_1\ \omega_2\ \omega_5] + [\omega_3\ \omega_4\ \omega_5],$$

in which $\omega_1, ..., \omega_5$ denote five independent Pfaff forms. The only linear relation between these forms that annihilates Ω is obviously:

$$\omega_5 = 0.$$

This latter equation is thus *invariant.* It may be expressed by means of first integrals of differential equations for which Ω is an invariant form.

In a general manner, if Ω is an invariant form, and *if the associated system of* Ω *is not identical to its characteristic system* then the associated system will be a system of Pfaff invariants.

One may alter these considerations in various ways.

85. Once more, take the case of two quadratic invariant forms Ω_1 and Ω_2 that have the same associated system. Let $2s$ be their common rank. The equation of degree s in λ:

$$[(\Omega_1 - \lambda\Omega_2)] = 0,$$

which expresses the idea that the rank of the form $\Omega_1 - \lambda\Omega_2$ is less than $2s$, obviously has an invariant significance. *The roots of the equation in* λ *are thus first integrals of the differential equations that admit* Ω_1 *and* Ω_2 *as first integrals.* One may show that, *in the general case,* Ω_1 and Ω_2 are reducible to the forms:

$$\Omega_1 = \lambda_1 [\omega_1\ \omega_2] + \lambda_2 [\omega_3\ \omega_4] + ... + \lambda_s [\omega_{2s-1}\ \omega_{2s}],$$
$$\Omega_2 = [\omega_1\ \omega_2] + [\omega_3\ \omega_4] + ... + [\omega_{2s-1}\ \omega_{2s}].$$

Each of the monomial forms $[\omega_1\ \omega_2]$, $[\omega_3\ \omega_4]$, ..., $[\omega_{2s-1}\ \omega_{2s}]$ *is invariant.*

CHAPTER IX

DIFFERENTIAL SYSTEMS THAT ADMIT AN INFINITESIMAL TRANSFORMATION

I. - *The notion of an infinitesimal transformation.*

85. A *transformation* in n variables is defined by a system of equations:

$$x_i' = f_i(x_1, ..., x_n) \qquad (i = 1, ..., n) \tag{1}$$

that can be solved for $x_1, ..., x_n$. Geometrically, if one regards $x_1, ..., x_n$ as the coordinates of a point M in the space of n dimensions then the transformation (1) takes an arbitrary point M of the space to another point M' by a well-defined law. The transformations are the ones that are currently used in geometry (homothety, similarity, inversion, or, more simply, rotation, translation, etc.).

The transformation (1) is called the *identity* when the right-hand sides reduce to $x_1, ..., x_n$, respectively; any point is then transformed into itself.

Given a system of differential equations:

$$\frac{dx_1}{X_1} = \frac{dx_2}{X_2} = \dots = \frac{dx_n}{X_n}, \tag{2}$$

this system is said to *admit the transformation* (1) when the application of this transformation to different points of an *arbitrary* integral curve of (2) gives points that all belong to the same new integral curve.

Consider a transformation that depends upon a parameter a that reduces to the identity for a certain numerical value a_0 of this parameter. Set $a - a_0 = \varepsilon$, and suppose that the right-hand side can be developed in powers of ε:

$$x_i' = x_i + \varepsilon\, \xi_i(x_1, ..., x_n) + \dots$$

One will have what one may call an *infinitesimal transformation* by paying attention to only the terms of first order in ε. An infinitesimal transformation is thus completely determined by the n functions ξ_i of $x_1, ..., x_n$. One obtains the same infinitesimal transformation by multiplying all of these functions by the same constant factor. We say that the function ξ_i represents the increment of the variable x_i by the infinitesimal transformation (in reality, the increment is $\varepsilon\xi_i$, but the coefficient ε plays only an auxiliary role).

Given a function $f(x_1, ..., x_n)$, the increment to which the infinitesimal transformation subjects this function is, up to a factor of ε, the first term in the development:

$$f(x_1', ..., x_n') - f(x_1, ..., x_n) = f(x_1 + \varepsilon\, \xi_i, ..., x_n + \varepsilon\, \xi_n) - f(x_1, ..., x_n).$$

It is therefore:

$$\xi_1 \frac{\partial f}{\partial x_1} + \xi_2 \frac{\partial f}{\partial x_2} + \ldots + \xi_n \frac{\partial f}{\partial x_n}.$$

We denote this expression by the symbol *Af*:

$$Af = \xi_1 \frac{\partial f}{\partial x_1} + \xi_2 \frac{\partial f}{\partial x_2} + \ldots + \xi_n \frac{\partial f}{\partial x_n}. \tag{3}$$

We agree to say that *Af* is the *symbol* of the infinitesimal transformation in question.

86. Formula (3) is analogous to the one that gives the total differential of a function *f*:

$$\delta f = \delta x_1 \frac{\partial f}{\partial x_1} + \delta x_2 \frac{\partial f}{\partial x_2} + \ldots + \delta x_n \frac{\partial f}{\partial x_n}.$$

The only difference is that δ is the symbol of an *undetermined* operation, whereas *A* is the symbol of a *determined* operation. The symbol of differentiation becomes the symbol of an infinitesimal transformation when one gives the $\delta x_1, \ldots, \delta x_n$ definite values (i.e., given functions of the variables).

The operation symbolized by *A* is susceptible to being applied not only to finite functions, but also to differential forms. For example, take the principal part (divided by ε) of the increment of dx_i for $A(dx_i)$. Now, one has:

$$dx_i' - dx_i = \varepsilon\, dx_i + \ldots$$

One is thus led to suppose that:

$$A(dx_i) = dx_i = d(Ax_i).$$

One sees by this that *the operation A can be considered to be interchangeable with the operation of (undetermined) differentiation.*

87. Return now to the system of differential equations (2). This system will admit the infinitesimal transformation (3) if the application of that transformation to the different points of an arbitrary integral curve takes the points that are situated along it onto the same new integral curve, *up to second-order infinitesimals.*

It is quite obvious that if the equations (2) admit a transformation that depends upon a parameter *a* for any numerical value of this parameter then it will admit an infinitesimal transformation that corresponds to the values of *a* that are infinitely close to the value a_0 (if it exists) that gives the identity transformation.

If there is a first integral of equations (2), and if these equations (2) admit an infinitesimal transformation *Af* then it is clear that $A(y)$ will also be a first integral. Indeed, *y* has some numerical value *c* at any point *M* of an arbitrary integral curve (*C*). The function *y* is augmented by $\varepsilon A(y)$ at the point M' that is the transform of *M*. This augmentation will be the same for any point M of (C). It is therefore necessary that $A(y)$ have the same numerical value at all of the points of (C). In other words, $A(y)$ is a first integral.

Conversely, if the application of the operation A to an arbitrary first integral gives a first integral again then the system (2) will admit the infinitesimal transformation Af. Indeed, if:

$$c_1, c_2, \ldots, c_{n-1}$$

are constant numerical values that are taken by $n - 1$ independent first integrals:

$$y_1, y_2, \ldots, y_{n-1}$$

at the different points M of an integral curve (C) then the values that these integrals take at the transformed points M' will be the values that the functions:

$$y_1 + \varepsilon A(y_1), \quad y_2 + \varepsilon A(y_2), \ldots, y_{n-1} + \varepsilon A(y_{n-1})$$

take at the points M themselves; they are therefore *constant*. As a result, the points M' will truly generate an integral curve.

II. - *Formation of integral invariants when starting with infinitesimal transformations.*

88. The preceding property show us that *knowing an infinitesimal transformation Af that is admitted by the differential equations* (2) *will permit us to deduce another invariant form from the invariant differential form* Ω, *namely* $A(\Omega)$. If the form Ω is exterior then so is the form $A(\Omega)$, and the new form will have the same degree as the old one did.

There exists a second operation that permits us to deduce another invariant form from an invariant exterior form Ω. Suppose – to fix ideas – that Ω is of third degree, and consider the corresponding trilinear differential form $\Omega(\delta, \delta', \delta'')$. Replace the symbol of undetermined differentiation δ in this form with the symbol of the infinitesimal transformation. We obtain an alternating linear form $\Omega(A, \delta', \delta'')$ with two types of differentials δ', δ'', to which there ultimately corresponds a quadratic exterior form, which we designate by $\Omega(A, \delta)$. This new form is deduced from the first one by an operation that makes sense independently of the choice of variables. If Ω is expressed by means of first integrals y_i of the equations (2) and their differentials then the expression $\Omega(A, \delta)$ will also be expressed in terms of the y_i and the δy_i. As a consequence, *the operation that we just defined will permit us to deduce from any invariant form another invariant form whose degree is diminished by one.*

From this, one has:

$$\Omega(A, \delta) = \xi_1 \frac{\partial \Omega}{\partial(\delta x_1)} + \xi_2 \frac{\partial \Omega}{\partial(\delta x_2)} + \ldots + \xi_n \frac{\partial \Omega}{\partial(\delta x_n)} \quad . \tag{4}$$

89. The two new operations that we just defined are not independent of each other. Suppose – to fix ideas – that Ω is of second degree, and recall the definition of the exterior derivative Ω'. One has (sec. **71**):

$$\Omega'(\delta, \delta', \delta'') = \delta\Omega(\delta', \delta'') - \delta'\Omega(\delta, \delta'') + \delta''\Omega(\delta, \delta')$$

for the condition that the three symbols $\delta, \delta', \delta''$ be interchangeable. Replace the symbol δ with the one for the infinitesimal transformation *Af*. We then have:

$$\Omega'(A, \delta', \delta'') = A(\Omega(\delta', \delta'')) - \delta'\Omega(A, \delta'') + \delta''\Omega(A, \delta');$$

i.e., upon passing to the exterior forms:

$$\Omega'(A, \delta) = A(\Omega(\delta)) - [\Omega(A, \delta)]',$$

or, finally:

(5) $$A(\Omega(\delta)) = \Omega'(A, \delta) + [\Omega(A, \delta)]'.$$

This fundamental formula contains the result of the first operation that was performed on *A* in its left-hand side. As for the two terms in the right-hand side, the first one can be obtained by first applying the operation of exterior derivation to Ω, followed by the second operation, which is associated with *Af*. As for the second term, it is deduced from Ω by the same operations, but in the reverse order.

By definition, *knowing an infinitesimal transformation Af that equations* (2) *admit puts us in possession of an essentially new operation, which is defined by formula* (4) *and permits us to deduce a new invariant form* $\Omega(A, \delta)$ *from an invariant form* $\Omega(\delta)$.

In particular, we may remark that if *y* is a first integral then the first integral $A(y)$ can be obtained, first by differentiation, which gives $\omega(\delta) = \delta y$, and then by the application of operation (4), which gives:

$$\omega(A) = A(y).$$

III. - *Examples.*

90. Consider a continuous material medium in motion whose density is ρ and whose velocity components are *u*, *v*, *w*. As we saw in (sec. **37**), the differential equations of motion for a molecule:

(6) $$\frac{dx}{dt} = u, \qquad \frac{dy}{dt} = v, \qquad \frac{dz}{dt} = w$$

admit the integral invariant:

$$\iiint \rho(\delta x\,\delta y\,\delta z - u\,\delta y\,\delta z\,\delta t - v\,\delta z\,\delta x\,\delta t - w\,\delta x\,\delta y\,\delta t),$$

which corresponds to the invariant form:

$$\Omega = \rho\,[\delta x\ \delta y\ \delta z] - \rho u\,[\delta y\ \delta z\ \delta t] - \rho v\,[\delta z\ \delta x\ \delta t] - \rho w\,[\delta x\ \delta y\ \delta t].$$

Suppose that the motion is *permanent;* i.e., that ρ, u, v, w are independent of t. Equations (6) do not contain time explicitly – i.e., they do not change when one replaces t with $t + \varepsilon$ – so they admit the infinitesimal transformation:

$$\Omega(A, \delta) = \frac{\partial \Omega}{\partial(\delta t)} = -\rho u[\delta y\,\delta z] - \rho v[\delta z\,\delta x] - \rho w[\delta x\,\delta y].$$

The property of this form that it is invariant is physically obvious. Indeed, consider a *tube* of trajectories, and cut this tube by two arbitrary surfaces that determine two areas S and S' in the interior of the tube. The quantity of matter that fills the volume situated between the lateral surface of the tube and the two surfaces S and S' is always the same, so the algebraic flux of the matter traversing the surface that bounds this volume will be zero. Now, the flux that traverses the lateral surface is zero. One will thus have:

$$\iint_S \rho\,(u\ \delta x\ \delta z + v\ \delta z\ \delta x + w\ \delta x\ \delta y) = \iint_{S'} \rho\,(u\ \delta y\ \delta z + v\ \delta z\ \delta x + w\ \delta x\ \delta y).$$

We remark that the invariant form $\Omega(A, \delta)$ is an exact derivative. Indeed, its derivative – if it is not zero – may differ from Ω by only a finite factor. Now, this derivative does not contain δt. One thus has:

$$[\Omega(A, \delta)]' = 0.$$

As a result, the characteristic system of $\Omega(A, \delta)$ reduces to its associated system. It is thus given by the equations:

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dx}{w}.$$

It defines the trajectories of the molecules, but independently of the manner in which these trajectories are described in time.

Formula (5) also shows that the form $\Omega(A, \delta)$ has zero derivative. Indeed, here the form Ω' is identically zero. On the other hand, since Ω does not contain t explicitly, it does not change when one changes t into $t + \varepsilon$, so $A(\Omega)$ is zero. This remark will be applied to the following examples.

91. Now consider a perfect fluid in motion under the action of forces that are derived from a potential. We have seen (sec. **22**) that there exists an absolute invariant form:

$$\omega' = \xi\,[\delta y\ \delta z] + \eta\,[\delta z\ \delta x] + z\,[\delta x\ \delta y] + (\eta\,w - \zeta\,v)[\delta x\ \delta t] + (\zeta\,u - \xi\,w)[\delta y\ \delta t] + (\xi\,v - \eta\,u)[\delta z\ \delta t].$$

It is given by the exterior derivative of a linear form:

$$\omega = u\ \delta x + v\ \delta y + w\ \delta z - E\ \delta t,$$

in which the coefficient E – viz., the energy per unit mass – is expressed by:

$$E = \tfrac{1}{2}(u^2 + v^2 + w^2) - U + \int \frac{dp}{\rho}.$$

Suppose the motion is *permanent;* i.e., that u, v, w, p, ρ are independent of t. As before, one will have a new invariant form:

$$\omega'(A, \delta) = \frac{\partial \omega'}{\partial(\delta t)} = (v\varsigma - w\eta)\delta x + (w\xi - u\varsigma)\delta y + (u\eta - v\xi)\delta z$$

$$= \begin{vmatrix} \delta x & \delta y & \delta z \\ u & v & w \\ \xi & \eta & \zeta \end{vmatrix}.$$

On the other hand, if one starts with the expression:

$$\omega' = [\delta u\ \delta x] + [\delta v\ \delta y] + [\delta w\ \delta z] - [\delta E\ \delta t],$$

then one will find that:

$$\omega'(A, \delta) = \delta E .$$

As a result, E will be a first integral of the equations of motion. We recover *Bernoulli's theorem*, according to which the quantity:

$$\frac{1}{2}(u^2 + v^2 + w^2) - U + \int \frac{dp}{\rho}$$

remains constant along each streamline for a perfect fluid in permanent motion.

However, the form δE is invariant, not only for the differential equations of motion of molecular fluids, but also for the vortex lines, which also admit the invariant form ω'. As a result, the *quantity E remains constant along not only each streamline, but also along each vortex line.*

If the motion is irrotational then the form $\omega'(A, \delta)$, as we originally wrote it, will obviously be identically zero. In this case, energy will be constant over all the fluid mass at any instant.

The equality:

$$\delta E = \begin{vmatrix} \delta x & \delta y & \delta z \\ u & v & w \\ \xi & \eta & \zeta \end{vmatrix}$$

permits us to represent the (spatial) variation of energy at each point M by means of a vector MH that has this point for its origin, and which will be the vector product of the velocity vector (u, v, w) with the vorticity vector (ξ, η, ζ). The derivative of energy in a given direction will be equal to the projection of the vector MH on that direction.

92. Another very general application relates to the problems of dynamics in which the constraints and the forces are independent of time. The infinitesimal transformation $Af = \dfrac{\partial f}{\partial t}$ that the equations of motion admit permits us to deduce from the fundamental integral invariant of dynamics:

$$\iint \omega' = \iint \quad \delta p_i\, \delta q_i - \delta H\, \delta t,$$

the new integral invariant:

$$\int \quad \delta H$$

that is obtained by partial differentiation with respect to δt. One therefore obtains the *generalized energy integral*:

$$H = h$$

under the condition that the function H must be independent of time.

More generally, suppose that the function H does not contain one of the variables p_i and q_i – say q_n , for example. The equations of motion thus admit the infinitesimal transformation $\dfrac{\partial f}{\partial q_n}$, from which one deduces the invariant linear form:

$$\frac{\partial \omega'}{\partial(\delta q_n)} = -\delta p_n .$$

Therefore, if the function H does not contain one of the canonical variables then the conjugate variable will be a first integral of the equations of motion.

IV. - *Applications to the n-body problem.*

93. Consider n mutually attracting material points that are subject to forces that are proportional to their masses and inversely proportional to a given power of their distance. There then exists a function of the forces:

$$U = f \sum_{i,j} \frac{m_i m_j}{r_{ij}^p},$$

in which the exponent p is a given (which is equal to 1 in the case of celestial mechanics), and the quantity r_{ij} denotes the distance between two points M_i and M_j of masses m_i and m_j .

The equations of motion of the system admit a certain number of obvious infinitesimal transformations. First, time is not explicitly contained in these equations. In addition, for any solution of the problem, one may deduce another one from it by displacing the set in space, and also by communicating a supplementary uniform rectilinear motion (which is the same for all points) to each of the n points. One immediately deduces the existence of the following infinitesimal transformations from this:

$$A_0 f = \frac{\partial f}{\partial t},$$

$$A_1 f = \sum_i \frac{\partial f}{\partial x_i} \qquad A_2 f = \sum \frac{\partial f}{\partial y_i}, \qquad A_3 f = \sum \frac{\partial f}{\partial z_i},$$

$$A_4 f = \sum y_i \frac{\partial f}{\partial z_i} - z_i \frac{\partial f}{\partial y_i} + y_i' \frac{\partial f}{\partial z_i'} - z_i' \frac{\partial f}{\partial y_i'}, \qquad A_5 f = \ldots, \qquad A_6 f = \ldots,$$

$$A_7 f = \sum \left(\frac{\partial f}{\partial x_i'} + t \frac{\partial f}{\partial x_i} \right), \quad A_8 f = \sum \left(\frac{\partial f}{\partial y_i'} + t \frac{\partial f}{\partial y_i} \right), \quad A_9 f = \sum \left(\frac{\partial f}{\partial z_i'} + t \frac{\partial f}{\partial z_i} \right).$$

The transformation $A_1 f$ corresponds to a translation parallel to Ox, the transformation $A_i f$, to a rotation around Ox, and the transformation $A_7 f$, to a supplementary motion of constant velocity ε that is parallel to Ox.

Finally, one can point out one last infinitesimal transformation that is based upon considerations of homogeneity. The equations:

$$m_i \frac{d^2 x_i}{dt^2} = \frac{\partial U}{\partial x_i}, \qquad m_i \frac{d^2 y_i}{dt^2} = \frac{\partial U}{\partial y_i}, \qquad m_i \frac{d^2 z_i}{dt^2} = \frac{\partial U}{\partial z_i}$$

remain unaltered if one multiplies all of the coordinates x_i, y_i, z_i by the same constant factor λ, on the condition that t must be multiplied by $\lambda^{1+p/2}$. The components x_i', y_i', z_i' of the velocity are then multiplied by $\lambda^{-p/2}$. Upon taking $\lambda = 1 + \varepsilon$, one will arrive at the new infinitesimal transformation:

$$A_{10} f = \sum x_i \frac{\partial f}{\partial x_i} + y_i \frac{\partial f}{\partial y_i} + z_i \frac{\partial f}{\partial z_i} - \frac{p}{2}\left(x_i' \frac{\partial f}{\partial x_i'} + y_i' \frac{\partial f}{\partial y_i'} \right) + \left(1 + \frac{p}{2}\right) t \frac{\partial f}{\partial t}.$$

If we make the same definition of U then we remark that one will have:

$$A_0 U = A_1 U = \ldots = A_9 U = 0, \quad A_{10} U = -pU.$$

94. Recall the fundamental second-degree integral invariant:

$$\omega' = \sum m_i \,[\delta x_i' \,\delta x_i] + m_i \,[\delta y_i' \,\delta y_i] + m_i \,[\delta z_i' \,\delta z_i] - \left[\sum m_i \,(x_i' \delta x_i' + y_i' \delta y_i' + z' \delta z_i')\,\delta t\right] + [\delta U \,\delta t].$$

Denote the linear form $\omega'(A_i, \delta)$ by ω_i. There then exist eleven invariant linear forms $\omega_0, \omega_1, \ldots, \omega_{10}$. This is easy to see, *a priori*, from formula (5), since the first ten are exact differentials, because ω' does not change under any given one of the first ten infinitesimal transformations. As for ω_{10}, formula (5) gives:

$$(\omega_{10})' = A_{10}(\omega').$$

Now, ω' has a degree of homogeneity (in the sense above) that is equal to $1 - p / 2$. One thus has:

$$(\omega_{10})' = \left(1 - \frac{p}{2}\right)\omega',$$

and ω_{10} will be an exact differential only if $p = 2$; i.e., *if the attraction is proportional to the cube of the distance.*

The calculation of the eleven forms ω_i does not offer any difficulty and gives:

$$
\begin{aligned}
\omega_0 &= \sum m_i\,(x_i'\delta x_i' + y_i'\delta y_i' + z'\delta z_i')\delta t - \delta U = \delta H,\\
\omega_1 &= -\sum m_i\;\delta x_i' = \delta H_1,\\
\omega_2 &= -\sum m_i\;\delta y_i' = \delta H_2,\\
\omega_3 &= -\sum m_i\;\delta z_i' = \delta H_3,\\
\omega_4 &= \sum m_i\,(z_i\,\delta y_i' - y_i\delta z_i' + y_i'\,\delta z_i - z_i'\,\delta y_i = \delta H_4,\\
\omega_5 &= \sum m_i\,(x_i\,\delta z_i' - z_i\delta x_i' + z_i'\,\delta x_i - x_i'\,\delta z_i = \delta H_5,\\
\omega_6 &= \sum m_i\,(y_i\,\delta x_i' - x_i\delta y_i' + x_i'\,\delta y_i - y_i'\,\delta x_i = \delta H_6,\\
\omega_7 &= \sum m_i\,(\delta x_i - t\,x_i' - x_i'\,\delta t) = \delta H_7,\\
\omega_8 &= \sum m_i\,(\delta y_i - t\,y_i' - y_i'\,\delta t) = \delta H_8,\\
\omega_9 &= \sum m_i\,(\delta z_i - t\,z_i' - z_i'\,\delta t) = \delta H_9,\\
\omega_{10} &= -\sum m_i\,(x_i\delta x_i' + y_i\delta y_i' + z_i\delta z_i' + \tfrac{p}{2}x_i'\delta x_i + \tfrac{p}{2}y_i'\delta y_i + \tfrac{p}{2}z_i'\delta z_i)\\
&\qquad + \left(1 + \frac{p}{2}\right)t\,\delta H + pH\,\delta t.
\end{aligned}
$$

We have set:

$$
\begin{aligned}
H &= \tfrac{1}{2}\sum m_i(x_i'^2 + y_i'^2 + z_i'^2) - U,\\
H_1 &= -\sum m_i x_i',\\
H_2 &= -\sum m_i y_i',\\
H_3 &= -\sum m_i z_i',\\
H_4 &= -\sum m_i\,(y_i z_i' - z_i y_i'),\\
H_5 &= -\sum m_i\,(z_i x_i' - x_i z_i'),\\
H_6 &= -\sum m_i\,(x_i y_i' - y_i x_i'),\\
H_7 &= \sum m_i\,(x_i - t\,x_i'),\\
H_8 &= \sum m_i\,(y_i - t\,y_i'),\\
H_9 &= \sum m_i\,(z_i - t\,z_i').
\end{aligned}
$$

One easily verifies that the bilinear covariant of ω_{10} is equal to $\left(1+\frac{p}{2}\right)\omega'$. If $p = 2$ then one will have the new first integral:

$$\sum m_i\,(x_i\,x_i' + y_i\,y_i' + z_i\,z_i') - 2Ht = C,$$

which will give:

$$\sum m_i\,(x_i^2 + y_i^2 + z_i^2) = 2Ht^2 + 2Ct + C'$$

upon integrating; this is the *Jacobi integral.*

The first integrals H_1, H_2, H_3, H_7, H_8, H_9 are the ones that give us the center of gravity theorem. The first integrals H_4, H_5, H_6 are the ones that give us the law of areas.

95. In the preceding section, we *directly* obtained only the *differentials* of the first integrals H and not the integrals themselves. They must be given to us by applying the operation that corresponds to the infinitesimal transformation $A_i f$ to each of the invariant forms ω_i. We have therefore obtained invariant *functions,* i.e., first integrals:

$$\alpha_{ij} = \omega_i(A_j) = \omega'(A_i, A_j) = -\alpha_{ij},$$

which we shall write in the form of a matrix with two indices that will be manifestly antisymmetric. The calculations offer no difficulty. The quantity α_{ij} is found at the intersection of the row i and the column j. The letter M denotes the sum of the masses of the n bodies.

	0	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
0	0	0	0	0	0	0	0	0	0	0	$-pH$
1	0	0	0	0	0	H	$-H_2$	$-M$	0	0	$-\frac{p}{3}H_1$
2	0	0	0	0	$-H_3$	0	H_1	0	$-M$	0	$-\frac{p}{3}H_2$
3	0	0	0	0	H_2	$-H_1$	0	0	0	$-M$	$-\frac{p}{3}H_3$
4	0	0	H_3	$-H_2$	0	H_6	$-H_5$	0	H_3	$-H_8$	$(1-\frac{p}{2})H_4$
5	0	$-H_3$	0	H_1	$-H_6$	0	H_4	$-H_9$	0	H_7	$(1-\frac{p}{3})H_5$
6	0	H_2	$-H_1$	0	H_5	$-H_4$	0	H_8	$-H_7$	0	$(1-\frac{p}{3})H_6$
7	0	M	0	0	0	H_9	$-H_8$	0	0	0	H_7
8	0	0	M	0	$-H_9$	0	H_7	0	0	0	H_8
9	0	0	0	M	H_8	$-H_7$	0	0	0	0	H_9
10	pH	$\frac{p}{3}H_1$	$\frac{p}{3}H_2$	$\frac{p}{3}H_3$	$(\frac{p}{3}-1)H_4$	$(\frac{p}{3}-1)H_5$	$(\frac{p}{3}-1)H_6$	$-H_7$	$-H_8$	$-H_9$	0

We remark that the determinant of the elements of the preceding matrix is zero since it is an antisymmetric determinant of odd degree. There thus exist eleven coefficients λ_i that are not all zero such that the expression $\sum \lambda_i \omega_i$ becomes zero when one applies the operation to it that relates to any of the transformations $A_i f$. One easily sees that λ_{10} is zero. Calculation gives us the expression $\sum \lambda_i \omega_i$, which is defined up to a factor of:

$$\frac{\delta K}{K} + \frac{2-p}{p}\frac{\delta H}{H},$$

upon setting:

$$K = (MH_4 + H_2 H_9 - H_3 H_8)^2 + (MH_5 + H_3 H_7 - H_1 H_9)^2 + (MH_6 + H_1 H_8 - H_2 H_7)^2.$$

In the case of celestial mechanics, $p = 1$. The expression $\frac{\delta K}{K} + \frac{\delta H}{H}$ is the logarithmic differential of HK. This quantity HK is therefore invariant under all of the transformations $A_i f$.

It is then easy to find an interpretation for it by making a convenient choice of coordinate axes. If we take the center of gravity to be the origin, which is permissible, since it is animated with a uniform rectilinear motion, then one will see that H_1, H_2, H_3, H_7, H_8, H_9 are annulled. The aforementioned quantity is then $H(H_4^2 + H_5^2 + H_6^2)$, up to a constant factor; i.e., it is the *product of the square of the kinetic moment of the system in its motion around the center of gravity with the total energy of the system in this motion.* This quantity is evidently independent of the choice of axes and the choice of units.

V. - *Application to the kinematics of rigid bodies.*

96. Consider the motion of a rigid body with respect to three fixed rectangular axes. One knows that at each instant it is defined by a system of vectors with the general resultant (p, q, r) and the moment with respect to the origin (ξ, η, ζ). Suppose that these six quantities are given functions of time. The differential equations of motion of a point of a rigid body are:

$$\frac{dx}{dt} = \xi + qz - ry = X ,$$
$$\frac{dy}{dt} = \eta + rx - pz = Y ,$$
$$\frac{dz}{dt} = \zeta + py - qx = Z .$$

These equations admit an obvious integral invariant. If one considers two infinitely close points:

$$(x, y, z), \qquad\qquad (x + \delta x, y + \delta y, z + \delta z)$$

of the rigid body at the instant t then the distance between these two points will not vary with time. One thus has a differential form:

$$\delta x^2 + \delta y^2 + \delta z^2$$

that is invariant if one considers only points at the same instant, and which becomes invariant in an absolute manner if one completes it by replacing:

$$\delta x,\ \delta y,\ \delta z$$

with:

$$\delta x - X\, \delta t, \quad \delta y - Y\, \delta t, \quad \delta z - Z\, \delta t,$$

respectively.

Let:

$$F = (\delta x - X\, \delta t)^2 + (\delta y - Y\, \delta t)^2 + (\delta z - Z\, \delta t)^2$$

be this invariant form, to which there corresponds the bilinear invariant:

$$F(\delta, \delta') = (\delta x - X\, \delta t)(\delta' x - X\, \delta' t) + (\delta y - Y\, \delta t)(\delta' y - Y \delta' t) + (\delta z - Z\, \delta t)(\delta' z - Z \delta' t).$$

This bilinear form is not alternating, but *symmetric*. Nevertheless, the differential equations of motion admit the infinitesimal transformation:

$$Af = \frac{\partial f}{\partial t}.$$

As a result, one may deduce another invariant form from the form F, namely:

$$\frac{1}{2}\frac{\partial F}{\partial(\delta t)} = -X(\delta x - X\,\delta t) - Y(\delta y - Y\delta t) - Z(\delta z - Z\delta t).$$

The same process may be repeated here, and this time it gives a first integral:

$$\frac{1}{2}\frac{\partial^2 F}{\partial(\delta t)^2} = X^2 + Y^2 + Z^2.$$

This first integral is obviously geometric. The motion of the rigid body is helicoidal, and the preceding integral is equal to the square of the velocity of the point considered – a velocity that remains unchanged throughout the motion.

VI. - *Differential equations that admit an infinitesimal transformation.*

97. In the preceding examples, we supposed that an integral invariant was known. Now, suppose that one knows only an invariant *equation* – for example, the equation:

$$\omega(\delta) \equiv a_1\delta x_1 + a_2\delta x_2 + \ldots + a_n\,\delta x_n = 0.$$

We assume that this equation is invariant; i.e., that it may be written in such a manner that it contains only first integrals $y_1, \ldots, y_{n-1}$ of the given differential equations and their derivatives. In other words, one has:

$$\omega(\delta) \equiv \rho\,[b_1(y)\delta y_1) + b_2(y)\delta y_2 + \ldots + b_{n-1}(y)\delta y_{n-1}],$$

in which the b_i depend upon only the $y_1, \ldots, y_{n-1}$, and ρ is an arbitrary function. If one replaces the symbol of indefinite differentiation δ with the symbol for the infinitesimal transformation Af then one will immediately have:

$$\frac{\omega(\delta)}{\omega(A)} = \frac{b_1(y)\delta y_1 + b_2(y)\delta y_2 + \ldots + b_{n-1}(y)\delta y_{n-1}}{b_1(y)Ay_1 + b_2(y)Ay_2 + \ldots + b_{n-1}(y)Ay_{n-1}},$$

and the right-hand side will obviously be an *invariant linear form.*

The knowledge of an infinitesimal transformation Af that a given system of differential equations admits and the knowledge of a Pfaff equation $\omega(\delta) = 0$ *that is invariant for this system implies the knowledge of a linear integral invariant* $\int \frac{\omega(\delta)}{\omega(A)}$.

For example, suppose that one is dealing with an ordinary differential equation:

$$\frac{dx}{X} = \frac{dy}{Y}.$$

It is invariant by itself. As a result, if it admits an infinitesimal transformation:

$$Af = \xi \frac{\partial f}{\partial x} + \eta \frac{\partial f}{\partial y}$$

then it will also admit the invariant linear form:

$$\frac{X\,\delta y - Y\,\delta x}{X\eta - Y\xi}.$$

Here, since there is only one first integral, that form is necessarily an exact differential. In other words, one knows an integrating factor for the equation; this result is classical.

Most of the differential equations that one knows to be integrable follow from the preceding remark. This is the case for the equations:

$$\frac{dy}{dx} = f(x), \quad \frac{dy}{dx} = f(y), \quad \frac{dy}{dx} = f\left(\frac{y}{x}\right).$$

For example, the last of these equations does not change if one multiplies x and y by the same constant factor $1 + \varepsilon$. It thus admits the infinitesimal transformation:

$$Af = x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y}.$$

As a result, the expression:

$$\frac{dy - f\left(\frac{y}{x}\right)dx}{y - f\left(\frac{y}{x}\right)x}$$

is an exact differential. This property becomes obvious if one sets:

$$y = ax,$$

because then the expression becomes:

$$\frac{du}{u - f(u)} + \frac{dx}{x}.$$

The integration of that exact differential leads to the same calculations as in the classical method.

98. Finally, if one knows nothing *a priori* about a given system of differential equations then the knowledge of an infinitesimal transformation that this system admits allows us to obtain an invariant Pfaff system. Indeed, we look for all of the Pfaff equations $\omega = 0$ that are consequences of the given differential equations and are such that $\omega(A)$ is zero. If one sets:

$$\omega(\delta) = \lambda_1\,\delta x_1 + \lambda_2\,\delta x_2 + \ldots + \lambda_n\,\delta x_n$$

then the coefficients λ_i will be given by the two conditions:

$$\lambda_1 X_1 + \lambda_2 X_2 + \ldots + \lambda_n X_n = 0,$$
$$\lambda_1\,\xi_1 + \lambda_2\,\xi_2 + \ldots + \lambda_n\,\xi_n = 0.$$

The desired set of equations thus forms a Pfaff system that is obtained by annulling all of the determinants with three lines and three columns in the matrix:

$$\left\| \begin{matrix} \delta x_1 & \delta x_2 & \cdots & \delta x_n \\ X_1 & X_2 & \cdots & X_n \\ \xi_1 & \xi_2 & \cdots & \xi_n \end{matrix} \right\|.$$

This system has a significance that is independent of the choice of variables. Now, if one takes the $n - 1$ first integrals $y_1, \ldots, y_{n-1}$, and one n^{th} variable to be the variables then the equations will reduce to:

$$\frac{\delta y_1}{\eta_1} = \frac{\delta y_2}{\eta_2} = \ldots = \frac{\delta y_{n-1}}{\eta_{n-1}} \qquad (\eta_i = Ay_i).$$

The Pfaff system under consideration is therefore invariant, and obviously it is completely integrable, since it reduces to a system of ordinary differential equations in $y_1, \ldots, y_{n-1}$.

For example, if the equations:

$$\frac{dx}{X} = \frac{dy}{Y} = \frac{dz}{Z}$$

admit the infinitesimal transformation:

$$Af = \xi\frac{\partial f}{\partial x} + \eta\frac{\partial f}{\partial y} + \zeta\frac{\partial f}{\partial z}$$

then the total differential equation:

$$\begin{vmatrix} dx & dy & dz \\ X & Y & Z \\ \xi & \eta & \varsigma \end{vmatrix} = 0$$

will be completely integrable. Upon integrating it, one will obtain a first integral of the given equations. After equating this first integral to a constant, one will be left with an ordinary differential equation that admits a known infinitesimal transformation, which can be integrated by quadrature.

VIII. - *Expressing the idea that a given system of differential equations admits a given infinitesimal transformation.*

99. We have not yet indicated the analytical conditions that express whether a given system of differential equations:

(2)
$$\frac{dx_1}{X_1} = \frac{dx_2}{X_2} = \cdots = \frac{dx_n}{X_n}$$

admits a given infinitesimal transformation:

(3)
$$Af = \xi_1 \frac{\partial f}{\partial x_1} + \xi_2 \frac{\partial f}{\partial x_2} + \cdots + \xi_n \frac{\partial f}{\partial x_n}.$$

Set:

(5)
$$Xf = X_1 \frac{\partial f}{\partial x_1} + X_2 \frac{\partial f}{\partial x_2} + \cdots + X_n \frac{\partial f}{\partial x_n}.$$

Our problem is then basically one of expressing the idea that if f is a first integral, ie., if it satisfies the equation:

$$Xf = 0,$$

then Af must also be a first integral. In other words, it amounts to expressing the idea that the equation:

$$X(Af) = 0$$

must be a consequence of the equation:

$$Xf = 0.$$

We may substitute the equation:

$$X(Af) - A(Xf) = 0,$$

for the first equation, *which contains the second-partial derivatives of f*. As an easy calculation shows, this new equation is linear and homogeneous with respect to $\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}$. *The desired condition is therefore,* quite simply, *the existence of an identity of the form:*

$$X(Af) - A(Xf) = \rho\, Xf, \tag{7}$$

in which ρ *denotes a conveniently chosen coefficient.*

This condition is obviously verified if one uses the infinitesimal transformation whose symbol is Xf. This transformation displaces each point M of space along the integral curve that passes through that point; it thus leaves each integral curve invariant. If one directs one's attention to the effect that is produced on the integral curves, which are considered as indivisible, then this particular infinitesimal transformation will play the same role as the *identity* transformation. One will easily see that the effect of applying the infinitesimal transformations that were defined in this chapter *vanishes* in this particular case. The same remark applies to the infinitesimal transformation λXf, in which λ is a given arbitrary factor.

VIII. - *Equations of variation.*

100. The notion of an equation of variation is due to H. Poincaré. One may relate it to the notion of infinitesimal transformation.

Consider a system of differential equations, which we write as:

$$\frac{dx_1}{dt} = X_1, \cdots, \frac{dx_n}{dt} = X_n, \tag{8}$$

in which the right-hand sides are given functions of $x_1, ..., x_n, t$. Let:

$$x_1 = f_1(t), \qquad x_2 = f_2(t), ..., x_n = f_n(t) \tag{9}$$

be a particular solution of this system. Take an infinitely close solution:

$$x_1 = f_1(t) + \varepsilon\, x_1, \quad x_2 = f_2(t) + \varepsilon\, x_2, ..., x_n = f_n(t) + \varepsilon\, x_n,$$

in which ε is an infinitely small constant, and the ξ are unknown functions of t. If we neglect the infinitely small terms of second order then we will obtain the following as a definition of the unknown functions:

$$\frac{d\xi_i}{dt} = \frac{\partial X_i}{\partial x_1}\xi_1 + \frac{\partial X_i}{\partial x_2}\xi_2 + \cdots + \frac{\partial X_i}{\partial x_n}\xi_n \qquad (i = 1, 2, \cdots, n). \tag{10}$$

These are the equations of variation *relative to a particular solution under consideration.*

It may be the case that one knows a particular solution of the equations of variation *independently of the particular solution of the given equations that serve to define the equations of variations.* The quantities $\xi_1, ..., \xi_n$ are then, in reality, definite functions of $x_1, ..., t$ that satisfy the partial differential equations:

$$\frac{\partial \xi_1}{\partial t} + X_1\frac{\partial \xi_i}{\partial x_1} + \cdots + X_n\frac{\partial \xi_i}{\partial x_2} = \xi_1\frac{\partial X_i}{\partial x_1} + \xi_2\frac{\partial X_i}{\partial x_2} + \cdots + \xi_n\frac{\partial X_i}{\partial x_n}. \tag{11}$$

In this case, the given equations obviously admit the infinitesimal transformation:

$$Af = \xi_1 \frac{\partial f}{\partial x_1} + \xi_2 \frac{\partial f}{\partial x_2} + \cdots + \xi_n \frac{\partial f}{\partial x_n}.$$

This transformation is expressed by the equations:

$$\begin{aligned} x_1' &= x_1 + \varepsilon\,\xi_1, \\ &\ldots \\ x_n' &= x_n + \varepsilon\,\xi_n, \\ t' &= t. \end{aligned}$$

The transformed curve of the integral curve (9) has for its equations:

$$x_i + \varepsilon\xi_i = f_i(t)$$

or

$$x_i = f_i(t) - \varepsilon\xi_i .$$

It is also an integral curve since $(-\xi_1, \ldots, -\xi_n)$ constitutes a solution of the equations of variation.

More generally, *any solution* (ξ_i) *of equations* (11) *corresponds to an infinitude of infinitesimal transformations that leave the given system* (8) *invariant,* namely, the transformations:

$$Bf = \xi_1 \frac{\partial f}{\partial x_1} + \cdots + \xi_n \frac{\partial f}{\partial x_n} + \lambda \left(\frac{\partial f}{\partial t} + X_1 \frac{\partial f}{\partial x_1} + \cdots + X_n \frac{\partial f}{\partial x_n} \right), \tag{12}$$

in which λ is an arbitrary function.

Conversely, suppose that one knows an infinitesimal transformation that leaves the given system invariant; it may always put into the form (12). The integral curve (9) is then changed by this transformation into the curve:

$$x_i + \varepsilon\xi_i + \varepsilon\lambda X_i = f_i(t + \varepsilon\lambda),$$

or

$$x_i = f_i(t) - \varepsilon\xi_i + \varepsilon\lambda\,[\, f_i'(t) - X_i] = f_i(t) - \varepsilon\,\xi_i .$$

As a consequence, the equations of variation (11) will admit the solution $(\xi_1, \ldots, \xi_n)$.

All of these properties ultimately result from the fact that equations (11) define an analytical translation of relation (7) only when the coefficient of $\frac{\partial f}{\partial t}$ in Af is zero.

CHAPTER X

COMPLETELY INTEGRABLE PFAFF SYSTEMS

I. - *Frobenius's theorem.*

101. A system of h Pfaff equations:

$$(1) \qquad \begin{cases} \omega_1 \equiv a_{11}dx_1 + a_{12}dx_2 + \cdots + a_{1n}dx_n = 0, \\ \cdots \\ \omega_h \equiv a_{h1}dx_1 + a_{h2}dx_2 + \cdots + a_{hn}dx_n = 0, \end{cases}$$

is completely integrable when it may be put into the form:

$$(2) \qquad dy_1 = dy_2 = \ldots = dy_h = 0.$$

From this, it follows that each form ω_i is linear in $dy_1, \ldots, dy_h$, and, as a result, that ω_i' is zero, on account of (2), i.e., on account of (1).

In order for a Pfaff system to be completely integrable it is necessary and sufficient that the derivatives of its left-hand sides all be annulled on account of the system equations.

In order to prove the converse, we first remark that the property that we just stated does not depend on the choice of variables and does not depend on the choice of the r left-hand sides. In other words, if one writes the equations of the system in the form:

$$\varpi_i \equiv \alpha_{ii}\,\omega_1 + \alpha_{i2}\,\omega_2 + \ldots + \alpha_{ih}\,\omega_h = 0,$$
$$\ldots$$
$$\varpi_h \equiv \alpha_{hi}\,\omega_1 + \alpha_{h2}\,\omega_2 + \ldots + \alpha_{hh}\,\omega_h = 0$$

then the derivatives $\varpi_1', \varpi_2', \cdots, \varpi_h'$ will also be annihilated on account of the system equations.

One has:

$$\varpi_i' = \alpha_{i1}\varpi_1' + \alpha_{i2}\varpi_2' + \ldots + \alpha_{ih}\,\varpi_h' + [d\alpha_{i1}\,\omega_1] + [d\alpha_{i2}\,\omega_2] + \ldots + [d\alpha_{ih}\,\omega_h]\,,$$

and each of the terms on the right-hand side is annihilated, by hypothesis, under the indicated conditions.

Having said this, we suppose that the converse has been proved up to $n - 1$ variables, and prove it for n variables. The ϖ_i' are annulled on account of equations (1), and *a fortiori* if one also makes $dx_n = 0$. As a result, *if one regards x_n as a fixed parameter* then system (1) is reducible to the form:

$$d\,y_1 = 0,$$
$$\dots$$
$$dy_h = 0,$$

in which $y_1, ..., y_n$ are h independent functions of $x_1, ..., x_{n-1}$, but which might also contain the parameter x_n . Now, if one no longer regards x_n as constant then the system will obviously be reducible to the form:

(3)
$$\begin{cases} \varpi_1 \equiv dy_1 + b_1 dx_n = 0, \\ \qquad \cdots \\ \varpi_h \equiv dy_h + b_h dx_n = 0, \end{cases}$$

in which $b_1, ..., b_h$ are functions of $y_1, ..., y_h$, and, for example, of $x_{h+1}, ..., x_n$. One has, moreover:

$$\varpi_1' = [db_1\, dx_n], ..., \varpi_h' = [db_h\, dx_h].$$

On account of equations (3), these formulas reduce to:

$$\varpi_i' = \frac{\partial b_i}{\partial x_{h+1}}[dx_{h+1}\, dx_n] + \cdots + \frac{\partial b_i}{\partial x_{n-1}}[dx_{n-1}\, dx_n]\,.$$

The hypotheses thus imply as a consequence that the coefficients b_i depend upon only the $y_1, ..., y_h, x_n$. But then equations (3) constitute a system of ordinary differential equations that may be reduced to the form:

$$dz_1 = 0, ..., dz_h = 0.$$

The symbols $z_1, ..., z_h$ denote h independent first integrals.

102. The preceding theorem, which is due to Frobenius, permits us (sec. **64**) to express the necessary and sufficient conditions for the complete integrability of the given system by means of the relations:

$$[\omega_1, ..., \omega_h\, \varpi_1'] = 0, ..., [\omega_1, ..., \omega_h\, \varpi_h'] = 0.$$

.

For example, take a Pfaff equation in three variables:

$$\omega \equiv P\,dx + Q\,dy + R\,dz = 0.$$

The condition for complete integrability is:

$$[\omega\omega'] \equiv \left[(Pdx + Qdy + Rdz)\left(\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)dydz + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)dzdx + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dxdy\right)\right]$$

$$\equiv \left\{ P\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) + Q\left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right) + R\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) \right\} [dx\, dy\, dz] = 0 .$$

II. - *Forming the characteristic system of a Pfaff system.*

103. One may give another form to the argument that was presented above by seeking the characteristic Pfaff system of a given arbitrary system (1) in a general manner.

In order for a system such as (1) to be invariant for the differential equations:

$$\frac{dx_1}{X_1} = \cdots = \frac{dx_n}{X_n}, \tag{4}$$

it is necessary and sufficient that the equations of (1) can be expressed by means of first integrals of (4) and their differentials. Therefore, it is first necessary that $\omega_1, ..., \omega_h$ are annihilated on account of (4), and it is therefore necessary that the forms:

$$[\omega_1, ..., \omega_h\, \omega_1'] , ..., [\omega_1, ..., \omega_h\, \omega_h']$$

can be expressed by means of differentials of the first integrals of (4). In other words, *it is necessary that the associated system of the forms:*

$$\omega_1, \omega_2 ..., \omega_h , [\omega_1, ..., \omega_h\, \omega_1'], ..., [\omega_1, ..., \omega_h\, \omega_h']$$

be a consequence of equations (4).

Conversely, if this condition is realized, and if $y_1, ..., y_{n-1}$ are first integrals of (4) then one will be left with equations of the form:

$$\varpi_1 \equiv dy_1 + b_{1,\, h+1}\, dy_{h+1} + ... + b_{1,\, n-1}\, dy_{n-1} = 0,$$
$$...$$
$$\varpi_h \equiv dy_h + b_{h,\, h+1}\, dy_{h+1} + ... + b_{h,\, n-1}\, dy_{n-1} = 0.$$

The form $[\varpi_i , ..., \varpi_h\, \varpi_i']$ does not involve dx_n , so the derivatives $\frac{\partial b_{i,h+1}}{\partial x_n}, \cdots, \frac{\partial b_{i,n-1}}{\partial x_n}$ will all be zero. As a result, equations (1) can be written in such a manner as to involve only the first integrals of system (4) and their differentials. Hence, system (1) is indeed invariant for equations (4).

It results from this that *the characteristic system of* (1) *is nothing but the associated system of the forms:*

$$\omega_1, \omega_2 ..., \omega_h , [\omega_1 ... \omega_h\, \omega_1'], ..., [\omega_1 ... \omega_h\, \omega_h'] .$$

In particular, in order for the system to be completely integrable, it is necessary and sufficient that this system be identical with (1), i.e., that the forms:

$$[\omega_1 \ldots \omega_h\ \omega_1'] , \ldots, [\omega_1 \ldots \omega_h\ \omega_h']$$

be identically zero.

104. One may also obtain the equations of the characteristic system of (1) in the form:

$$\begin{cases} \omega_1 = 0, \omega_2 = 0, \cdots, \omega_h = 0, \\ \begin{Vmatrix} \dfrac{\partial \omega_i'}{\partial (dx_1)} & \dfrac{\partial \omega_i'}{\partial (dx_2)} & \cdots & \dfrac{\partial \omega_i'}{\partial (dx_n)} \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & \cdots & \vdots \\ a_{h1} & a_{h2} & \cdots & a_{hn} \end{Vmatrix} = 0 . \end{cases}$$

In particular, the characteristic system of a Pfaff equation:

$$\omega \equiv a_1\, dx_1 + a_2\, dx_2 + \ldots + a_n\, dx_n = 0$$

is given by the equations:

$$\begin{cases} a_1 dx_1 + a_2 dx_2 + \cdots + a_n dx_n = 0, \\ \dfrac{a_{12} dx_2 + a_{13} dx_3 + \cdots + a_{1n} dx_n}{a_1} = \dfrac{a_{21} dx_1 + a_{23} dx_3 + \cdots + a_{2n} dx_n}{a_n} = \cdots \\ \qquad = \dfrac{a_{n1} dx_1 + \cdots + a_{n,n-1} dx_{n-1}}{a_n}, \end{cases}$$

upon setting:

$$a_{ij} = \frac{\partial a_j}{\partial x_i} - \frac{\partial a_i}{\partial x_j} .$$

We shall return to this system later in Chapter XIV.

III. - *The integration of a completely integrable Pfaff system.*

105. We now return to a completely integrable Pfaff system, which we write:

$$\text{(5)} \qquad \begin{cases} dz_1 = a_{11} dx_1 + \cdots + a_{1q} dx_q , \\ \cdots \\ dz_h = a_{h1} dx_1 + \cdots + a_{hq} dx_q . \end{cases}$$

The integration of this system amounts to the integration of a system of ordinary differential equations in h unknowns $z_1, \ldots, z_h$ of one independent variable x_1.

Indeed, we know that the system admits one and only one solution that corresponds to the given initial values (x_i^0, z_j^0). In order to find the values of the unknown functions $z_1, \ldots, z_h$ that correspond to a given system of numerical values $x_1^1, \cdots, x_q^1$ of the independent variables, we displace ourselves on the integral manifold from the point (x_i^1) to the point (x_i^1), and follow the variation of the z_j. For any such succession of intermediary values of the independent variables, the result will always be the same. For example, set:

$$x_2 - x_2^0 = m_2(x_1 - x_1^0), \cdots, x_q - x_q^0 = m_q(x_1 - x_1^0),$$

in which $m_2, \ldots, m_q$ denote the $q - 1$ quantities:

$$m_i = \frac{x_i^1 - x_i^0}{x_1^1 - x_1^0} \qquad (i = 2, \cdots, q).$$

We have:

$$\text{(6)} \qquad \begin{cases} dz_1 = (a_{11} + a_{12}m_2 + \cdots + a_{1q}m_q)\, dx_1, \\ \qquad\cdots \\ dz_h = (a_{h1} + a_{h2}m_2 + \cdots + a_{hq}m_q)\, dx_1. \end{cases}$$

It will suffice for us to integrate this system of ordinary differential equations and determine the solution that corresponds to the values $z_1^0, \cdots, z_h^0$ of the unknown functions for $x_1 = x_1^0$. Once this solution is determined, we will obtain $z_1^1, \cdots, z_h^1$ by replacing the parametric quantities $m_2, \ldots, m_q$ with the values that were indicated above.

We remark that *the knowledge of one first integral of a system of ordinary differential equations* (6) *in* $q - 1$ *parameters* m_i *does not unavoidably imply the knowledge of a first integral of the Pfaff system* (5).

IV. - *Complete systems.*

106. Return to the completely integrable system (1), in which we denote a system of h independent first integrals by $y_1, \ldots, y_h$. Arbitrarily choose $n - h$ linear differential forms:

$$\omega_{h+1}, \ldots, \omega_h$$

that are mutually independent and independent of the forms $\omega_1, \ldots, \omega_h$. Any linear form in $dx_1, \ldots, dx_n$ may be expressed in only one manner as a linear function of the $\omega_1, \ldots, \omega_h$. Now take an *undetermined* function f and consider its total differential:

$$df = \frac{\partial f}{\partial x_1}dx_1 + \frac{\partial f}{\partial x_2}dx_2 + \cdots + \frac{\partial f}{\partial x_n}dx_n.$$

One may express it linearly by means of $\omega_1, \ldots, \omega_n$, whose coefficients are obviously linear and homogeneous in $\frac{\partial f}{\partial x_1}, \cdots, \frac{\partial f}{\partial x_n}$. Let:

$$df = X_1 f \cdot \omega_1 + X_2 f \cdot \omega_2 + \ldots + X_n f \cdot \omega_n. \tag{7}$$

The n expressions $X_i f$ are linearly independent in $\frac{\partial f}{\partial x_1}, \cdots, \frac{\partial f}{\partial x_n}$.

Having said this, any first integral of the completely integrable system (1) is characterized by the property that its differential, which is considered to be a linear form in $dx_1, \ldots, dx_n$, is annulled under the one condition that equations (1) are verified; in other words, by the property of annulling:

$$X_{h+1} f, \ldots, X_n f.$$

The system of $n - h$ linearly independent partial differential equations:

$$X_{h+1} f = 0, \ldots, X_n f = 0 \tag{8}$$

thus admits h linearly independent solutions $y_1, \ldots, y_h$.

Identity (7) gives us that:

$$dy_i = X_1 y_i \cdot \omega_1 + X_2 y_i \cdot \omega_2 + \ldots + X_n y_i \cdot \omega_n \qquad (i = 1, 2, \ldots, h). \tag{9}$$

Since the y_i are independent functions, the right-hand sides of equations (9) will be linearly independent combinations of the $\omega_1, \omega_2, \ldots, \omega_h$. As a result, *system* (1) *will be equivalent to* (2). *Hence, it will be completely integrable.*

We agree to say that the system of equations (8) forms a *complete system* if it admits the maximum number h of linearly independent solutions. We see that *there is a complete system that corresponds to any completely integrable Pfaff system, and conversely.* The correspondence is such that if the equations of the Pfaff system are:

$$\omega_1 = \omega_2 = \ldots = \omega_h = 0$$

then the equations of the complete system will be:

$$X_{h+1} f = 0, \qquad X_{h+2} f = 0, \ldots, \quad X_n f = 0.$$

107. It is easy to find conditions for a given system of linear first-order partial differential equations to be complete.

Start with identity (7) and exterior differentiate it. We easily get

(9) $$\sum_{h=1}^{h=n} X_h f \cdot \omega'_k + \sum_{i=1}^{i=n} \sum_{j=1}^{j=n} X_i (X_j f)[\omega_i \, \omega_j] = 0.$$

The n covariants ω'_k can be expressed as quadratic exterior forms in the $\omega_1, \ldots, \omega_n$. Let:

(10) $$\omega'_k = \sum_{(ij)}^{1,\ldots,n} c_{ijk} [\omega_i \omega_j].$$

If we equate all of the $[\omega_i \, \omega_j]$ terms in identity (10) to zero then we will find that:

(11) $$X_i (X_j f) - X_j (X_i f) + \sum_{k=1}^{k=n} c_{ijk} X_k f = 0.$$

Note the duality between formulas (10) and (11).

Now suppose that system (1) is completely integrable. From Frobenius's theorem, this signifies that $\omega'_1, \ldots, \omega'_h$ will be annihilated, along with $\omega_1, \ldots, \omega_h$. In other words, one will have:

$$c_{h+1,\, h+j,\, k} = 0 \qquad (i, j = 1, \ldots, n - h;\; k = 1, \ldots, h).$$

As a result, from (12), the combinations:

$$X_{h+i} (X_{h+j} f) - X_{h+j} (X_{h+i} f)$$

will depend linearly upon only the $X_{h+1} f, \ldots, X_n f$. The converse is obvious.

We agree to denote the combination $X(Yf) - Y(Xf)$ by (XY). One sees that *the necessary and sufficient condition for such a system to be complete is that the brackets of all the left-hand sides, when taken two at a time, must be linear combinations of these left-hand sides.*

CHAPTER XI

THE THEORY OF THE LAST MULTIPLIER

I. - *Definition and properties.*

108. Consider a system of differential equations:

$$(1) \qquad \frac{dx_i}{dt} = X_i, \qquad \frac{dy_i}{dt} = Y_i, \qquad \frac{dz_i}{dt} = Z_i$$

that admits an integral invariant of maximum degree n:

$$\Omega = M\,[(\delta x_1 - X_1\,\delta t)(\delta x_2 - X_2\,\delta t)\,\ldots\,(\delta x_n - X_n\,\delta t)].$$

As we have seen (sec. **80**), the condition for this to be the case is that the exterior derivative Ω' must be zero, which gives, by an easy calculation:

$$(2) \qquad \frac{\partial M}{\partial t} + \frac{\partial (MX_1)}{\partial x_1} + \frac{\partial (MX_2)}{\partial x_2} + \ldots + \frac{\partial (MX_3)}{\partial x_3} = 0\,.$$

The coefficient M is known by the name of the *Jacobi multiplier.*

As we have seen, condition (2) expresses the idea that the form Ω can be expressed by means of n independent first integrals:

$$y_1,\, y_2,\, \ldots,\, y_n$$

of the system (1) and their differentials. In other words, that one has an identity:

$$(3) \qquad M\,[(\delta x_1 - X_1\,\delta t)(\delta x_2 - X_2\,\delta t)\,\ldots\,(\delta x_n - X_n\,\delta t) = H(y_1, y_2, \ldots, y_n)[\delta y_1,\, \delta y_2,\, \ldots,\, \delta y_n].$$

It is now possible for us to recover the classical theorems that relate to the Jacobi multiplier.

THEOREM I. – *The quotient of two multipliers M and M' is a first integral.*

Indeed, the two identities (3) that relate to the two multipliers M and M' give:

$$\frac{M}{M'} = \frac{H(y_1, \ldots, y_n)}{H'(y_1, \ldots, y_n)}\,.$$

THEOREM II. - *If one knows p independent first integrals of equations* (1) *then one can determine a multiplier of the system of $n - p$ differential equations that reduces the integration of the given system.*

Suppose that one knows p independent first integrals $y_1, y_2, ..., y_p$, and suppose, as is always permissible, that they are independent functions of p variables $x_1, x_2, ..., x_p$, i.e.:

$$\frac{D(y_1, ..., y_p)}{D(x_1, ..., x_p)} \neq 0 .$$

Equations (1) may then be written:

$$\frac{dy_1}{dt} = 0, \cdots, \frac{dy_p}{dt} = 0 , \tag{4}$$

$$\frac{dx_{p+1}}{dt} = X_{p+1}, \cdots, \frac{dx_n}{dt} = X_n , \tag{5}$$

and if one equates $y_1, ..., y_p$ to arbitrary constants $C_1, ..., C_p$ then the integration of system (1) will reduce to that of system (5), in the right-hand side of which one is supposed to replace $x_1, ..., x_p$ with their values as functions of $x_{p+1}, ..., x_n, t, C_1, ..., C_p$.

This suggests that the form Ω, which is invariant for equations (4) and (5), can obviously be written:

$$\Omega = N [(\delta y_1 \ldots \delta y_p (\delta x_{p+1} - X_{p+1} \delta t) \ldots (\delta x_n - X_n \delta t)].$$

In order to obtain the value of the coefficient N, it is necessary to identify that expression with the original expression. For example, by equating the terms in:

$$[\delta x_1 \delta x_2 \ldots \delta x_n],$$

one will obtain:

$$M = N \frac{D(y_1, ..., y_p)}{D(x_1, ..., x_p)} .$$

Having thus determined the quantity N, one has the identity:

$$N [(\delta y_1 \ldots \delta y_p (\delta x_{p+1} - X_{p+1} \delta t) \ldots (\delta x_n - X_n \delta t)] = H [\delta y_1 \ldots \delta y_p \delta y_{p+1} \ldots \delta y_n],$$

i.e.:

$$[\delta y_1 \ldots \delta y_p \{N(\delta x_{p+1} - X_{p+1} \delta t) \ldots (\delta x_n - X_n \delta t) - H \delta y_{p+1} \ldots \delta y_n\}] = 0.$$

This identity expresses the idea (sec. **64**) that *if one takes into account the linear relations:*

$$\delta y_1 = 0,\ \delta y_2 = 0, ...,\ \delta y_p = 0$$

then one will have that:

$$N [(\delta x_{p+1} - X_{p+1} \delta t) \ldots (\delta x_n - X_n \delta t)] = H (y_1, ..., y_n)[\delta y_{p+1} \ldots \delta y_n]. \tag{6}$$

The left-hand side of this equality is therefore an invariant form for the system of differential equations (5). In other words, *the system* (5) *admits the multiplier*:

$$N = \frac{M}{\dfrac{D(y_1, ..., y_p)}{D(x_1, ..., x_p)}}.$$

THEOREM III. - *If one knows* $n - 1$ *independent first integrals of equations* (1) *then the integration of the equations is achieved by a quadrature.*

It suffices to apply Theorem II in the case of $p = n - 1$. One then sees that the linear differential form:

$$\frac{M}{\dfrac{D(y_1, \cdots, y_{n-1})}{D(x_1, \cdots, x_{n-1})}}(\delta x_n - X_n \delta t)$$

is an exact differential when one supposes that the variables are coupled by relations:

$$y_1 = C_1, \quad y_2 = C_2, \; ..., \; y_{n-1} = C_{n-1}\,.$$

The general solution of equations (1) is thus obtained by equating the integral of a total differential:

$$\int \frac{M}{\dfrac{D(y_1, ..., y_{n-1})}{D(x_1, ..., x_{n-1})}}(dx_n - X_n dt)$$

to a constant.

II. - *Generalizations.*

109. The theorem of the last multiplier can be generalized to the considerably more general case in which one knows an invariant form Ω of *arbitrary degree* $r < n$. Suppose that one knows $n - 1$ independent first integrals $y_1, ..., y_{n-1}$. Choose $n - r$ of these integrals:

$$y_{\alpha_1}, y_{\alpha_2}, \cdots, y_{\alpha_{n-r}}$$

in all possible ways, and consider the *obviously invariant* forms:

$$[\delta y_{\alpha_1} \delta y_{\alpha_2} \cdots \delta y_{\alpha_{n-r}} \Omega].$$

All of these forms are of degree n. *If they are not all zero* then we shall revert to the preceding case that we just studied. *We have a multiplier.* We may even have several, and in certain cases Theorem I can give the last first integral by dividing two of these multipliers.

The exceptional case is the one for which all of the preceding forms are zero. Now, imagine that Ω is expressed by means of $\delta y_1, \ldots, \delta y_{n-1}$ and the differential of an (unknown) n^{th} first integral δy_n. The hypothesis made amounts to saying that Ω *does not contain* δy_n, because if Ω contains, for example, a non-zero term such that:

$$A[\delta y_1 \ldots \delta y_{r-1}\, \delta y_n]$$

then the exterior product of Ω with $\delta y_{r+1} \delta y_{r+2} \ldots \delta y_{n-1}$ would not be zero

If this is the case then Ω will be an exterior form in $\delta y_1, \ldots, \delta y_{n-1}$, which is an exterior form whose coefficients that one may calculate. Each of these coefficients will be a first integral. If at least one of these coefficients is independent of $y_1, \ldots, y_{n-1}$ then one will achieve the integration by equating that coefficient to an arbitrary constant. The only doubtful case is the one for which all of the coefficients are functions of $y_1, \ldots, y_{n-1}$. Now it is clear that in this case the knowledge of the invariant form Ω might not be of any help in achieving this integration. We simply remark that in this case the given equations *do not constitute* the characteristic system of Ω.

We may therefore state the following general theorem:

The knowledge of an invariant differential form Ω *that admits the given system* (1) *of differential equations as its characteristic system permits us, in the most unfavorable case, to achieve the integration of this system by a quadrature when one already knows* $n - 1$ *independent first integrals.*

110. Another generalization of Jacobi's theory of the last multiplier relates to completely integrable Pfaff systems. Let:

$$\omega_1 = 0, \quad \omega_2 = 0, \ldots, \omega_r = 0$$

be a completely integrable system for which one knows an invariant form of maximum degree r:

$$\Omega = M\,[\omega_1\, \omega_2 \ldots \omega_r].$$

The knowledge of $r - 1$ first integrals $y_1, \ldots, y_{r-1}$ of the system permits us to achieve the integration by a quadrature. Indeed, upon equating $y_1, \ldots, y_{r-1}$ to arbitrary constants, the given system will reduce to only one equation – for example, $\omega_r = 0$ – and one will have a formula such as:

$$\Omega = N\,[\delta y_1 \ldots \delta y_{r-1}\, \omega_r],$$

in which the coefficient N can be deduced from M by an easy identification. It results from this that $N\omega_r$ is an invariant form for the single equation that remains to be integrated; viz., $\omega_r = 0$. In other words, $N\omega_r$ is an exact differential. The integration is then achieved by a quadrature.

Finally, the completely general theorem that summarizes all of the cases envisioned is the following one:

The knowledge of a differential form Ω *permits us, in the most unfavorable case, to achieve the integration of the characteristic system of that form by quadrature when one already knows r* – 1 *independent first integrals, where we have denoted the class of that form by r.*

III. - *Case where the independent variable is not specified.*

111. If the given system of differential equations were put into the form:

$$\frac{dx_1}{X_1} = \frac{dx_2}{X_2} = \cdots = \frac{dx_n}{X_n}$$

then any integral invariant of degree n – 1 would be of the form:

$$\Omega = MX_1\,[dx_2\,dx_3 \,\ldots\, dx_n] - MX_2\,[dx_1\,dx_3 \,\ldots\, dx_n] + \ldots -(-1)^n MX_n\,[dx_1\,dx_2 \,\ldots\, dx_{n-1}],$$

and the condition $\Omega' = 0$ would become:

$$\frac{\partial(MX_1)}{\partial x_1} + \frac{\partial(MX_2)}{\partial x_2} + \cdots + \frac{\partial(MX_n)}{\partial x_n} = 0\,.$$

Apart from the difference described, this theory would be identical to the one that was described above.

IV. - *Case in which the given equations admit an infinitesimal transformation.*

112. Take the general case of a completely integrable system:

$$\omega_1 = 0, \qquad \omega_2 = 0, \ldots, \omega_r = 0 \tag{7}$$

that admits an invariant form of degree r, which one may suppose is reduced to the form:

$$\Omega = [\omega_1\,\omega_2 \,\ldots\, \omega_r].$$

Suppose that this system admits a known infinitesimal transformation Af, and form the quantities:

$$\omega_1(A), \quad \omega_2(A), \ldots, \omega_r(A),$$

which we suppose are all zero. One may always suppose that the equations of the system are written in such a manner as to make:

(8) $$\omega_1(A) = 1, \qquad \omega_2(A) = \omega_3(A) = \dots = \omega_r(A) = 0,$$

while Ω keeps the same form.

As we have seen (sec. **88**), the knowledge of the infinitesimal transformation Af permits us to deduce another invariant form $\Omega(A, \delta)$ from the form $\Omega(\delta)$, which, with the hypotheses made here, reduces to:

$$\Omega(A, \delta) = [\omega_2 \dots \omega_r].$$

We denote this new form by the letter Π. We have that:

$$\Omega = [\omega_1\, \Pi].$$

The *associated* (and non-characteristic) system of Π is:

(9) $$\omega_1 = 0, \qquad \omega_2 = 0, \dots, \omega_r = 0.$$

It is completely integrable. This results from an earlier theorem (sec. **98**), but also from the fact that since Ω is expressible by means of r first integrals $y_1, \dots, y_r$ of the given system and their differentials, the associated system to $\Omega(A, \delta)$, like Ω, contains only $y_1, \dots, y_r$, and their differentials. As a result, it will be a system of ordinary differential equations. Hence, it is completely integrable.

Now, form the exterior derivative Π' of the form Π. It is a new invariant form of degree r. One therefore has:

$$\Pi' = m\, \Omega = m\, [\omega_1\, \Pi].$$

The coefficient m is a first integral. However, there are some cases that we must discuss:

1. $m = 0$: Π' is zero, and the system (9) that is associated to Π is its characteristic system. One therefore knows a multiplier of system (9). As a result, when one knows $r - 2$ independent first integrals of this system, the integration will be achieved by a quadrature. A second quadrature will then achieve the integration of the given system (7). This quadrature is obviously $\int \omega_1$.

It is obvious that here Π and Ω are reducible to:

$$\Pi = [\delta y_2\, \delta y_3 \dots \delta y_r], \qquad \Omega = [\delta y_1\, \delta y_2 \dots \delta y_r].$$

When the transformation Af is applied to the first integrals of the given system, it will reduce to:

$$Af = \frac{\partial f}{\partial y_1}.$$

There are an infinitude of ways of choosing the first integrals in such a way that the givens remain the same, i.e., in such a manner that Ω and Af do not change. One may perform an arbitrary transformation on y_2, y_3, ..., y_r of functional determinant 1, and add an arbitrary function of y_2, ..., y_r to y_1. This explains the nature of the simplifications that are presented in the integration.

2. *m is a non-zero constant:* In this case, we suppose that one has integrated system (9), and let y_1, ..., y_r be a system of $r - 1$ independent integrals. One will have:

$$\Pi = H[\delta y_2\, \delta y_3 \dots \delta y_r],$$

in which the coefficient H is *independent* of y_2, ..., y_r (otherwise Π' would be zero), but H is still a first integral of the given system. One thus obtains an r^{th} integral of the given system by simple differentiations.

If we write y_1 in place of H then we have that:

$$\Omega = \frac{1}{m}[\delta y_1 \delta y_2 \cdots \delta y_r], \qquad \Pi = y_1[\delta y_2 \cdots \delta y_r], \qquad Af = my_1 \frac{\partial f}{\partial y_1}.$$

The most general transformation in y_1, ..., y_r that preserves the given data is obtained by performing an *arbitrary* transformation on y_2, ..., y_r and setting:

$$\bar{y}_1 = \frac{y_1}{\dfrac{D(\bar{y}_2, \cdots, \bar{y}_r)}{D(y_2, \cdots, y_r)}}.$$

This explains why the integration of system (9) cannot be simplified and also why, once that integration has been performed, the integration of the system (7) can be deduced from it.

3. *The coefficient m is not constant, but A*(m) *is zero:* The function m is a first integral of system (9). The integration of this system amounts to that of a system of differential equations in $r - 2$ unknown functions. The integration of the given system is deduced from it as in the preceding case.

The form Π is reducible to:

$$\Pi = y_1[\delta m\, \delta y_3 \dots \delta y_r],$$

and one has:

$$\Omega = \frac{1}{m}[\delta y_1\, m\, \delta y_3 \cdots \delta y_r], \qquad Af = my_1 \frac{\partial f}{\partial y_1}.$$

The transformations that preserve the givens are:

$$\begin{cases} \bar{m} = m, \quad \bar{y}_3 = f_3(m, y_3 \cdots y_r), \cdots, \bar{y}_r = f_r(m, y_3, \cdots, y_r), \\ \bar{y}_1 = \dfrac{y_1}{\dfrac{D(f_3, \cdots, f_r)}{D(y_3, \cdots, y_r)}}. \end{cases}$$

This explains the simplifications that are presented in the integration.

4. *The coefficient m is not constant and* $Am = m_1 \neq 0$: In all of what follows, take the general case:

$$Am = m_1, \qquad Am_1 = m_2, \ldots, \quad Am_{i-1} = m_i,$$

and suppose that $m, m_1, \ldots, m_{i-1}$ are i independent first integrals of the given system, and that m_i is a function of $m, \ldots, m_{i-1}$.

The given system then admits i known independent first integrals, and *its integration amounts to the integration of a system of differential equations in* $r - 1$ *unknown functions when one knows a multiplier.*

We look for the reduced form of Ω and Af. One may always set:

$$\Omega = H\,[\delta m\ \delta m_1 \ldots \delta m_{i-1}\ \delta y_{i+1} \ldots \delta y_r],$$

in which $y_{i+1}, \ldots, y_r$ are $r - i$ first integrals of the system (9) and H is a function of $m, \ldots, m_{i-1}, y_{i+1}, \ldots, y_r$. Obviously, one has:

$$Af = m_1 \frac{\partial f}{\partial m} + m_2 \frac{\partial f}{\partial m_1} + \cdots + m_i \frac{\partial f}{\partial m_{i-1}}.$$

We express the fact that the exterior derivative of $\Pi = \Omega(A, \delta)$ is equal to $m\Omega$, or, what amounts to the same thing:

$$A(\Omega) = m\Omega.$$

One has:

$$A(\Omega) = (A(H) + \frac{\partial m_i}{\partial m_{i-1}} H)[\delta m\, \delta m_1 \cdots \delta m_{i-1}\, \delta y_{i+1} \cdots \delta y_r].$$

One will thus have:

$$A(H) + \frac{\partial m_i}{\partial m_{i-1}} H = mH.$$

Let $h(m, m_1, \ldots, m_{i-1})$ be a *particular* solution of that partial differential equation. The latter may be written:

$$A\left(\frac{H}{h}\right) = 0.$$

In other words, H / h is an integral of equations (9). One may then choose $y_{i+1}, \ldots, y_r$ in such a manner as to reduce that function to unity. One will thus have:

$$\Omega = h(m, \ldots, m_{i-1})[\delta m \; \delta m_1 \ldots \delta m_{i-1} \; \delta y_{i+1} \ldots \delta y_r],$$
$$Af = m_1 \frac{\partial f}{\partial m} + m_2 \frac{\partial f}{\partial m_1} + \cdots + m_i \frac{\partial f}{\partial m_{i-1}}.$$

The transformations that preserve the given data are obviously:

$$\bar{m} = m, \quad \bar{m}_1 = m_1, \cdots, \bar{m}_{i-1} = m_{i-1},$$
$$\bar{y}_{i+1} = f_{i+1}(\mu_1, \cdots, \mu_{i-1}, y_{i+1}, \cdots, y_r), \cdots,$$
$$\bar{y}_r = f_r(\mu_1, \cdots, \mu_{i-1}, y_{i+1}, \cdots, y_r),$$

with:

$$\frac{D(\bar{y}_{i+1}, \cdots, \bar{y}_r)}{D(y_{i+1}, \cdots, y_r)} = 1.$$

(We have let $\mu_1, \ldots, \mu_{i-1}$ denote $i - 1$ independent functions of $m, m_1, \ldots, m_{i-1}$ that satisfy $A\mu = 0$.)

The nature of the preceding transformations explains the simplifications that are presented by the integration.

However, it may be the case that $i = r$. In this case, no integration needs to be performed, since one has r independent first integrals by definition.

V. - *Applications.*

113. The theory of the last multiplier can be applied to all of the indicated preceding examples that involve an invariant form whose degree is equal to the number of unknown functions. Recall these cases:

1. The equations that give the motion of the molecules in a continuous medium when one knows the density ρ and the components u, v, w of the velocity as functions of x, y, z, t:

$$\frac{dx}{dt} = u, \qquad \frac{dy}{dt} = v, \quad \frac{dz}{dt} = w.$$

Since the integral invariant is:

$$\Omega = \rho\,[(\delta x - u\,\delta t)(\delta y - v\,\delta t)(\delta z - w\,\delta t)],$$

the multiplier will be ρ. Therefore, if one knows two independent first integrals then the integration is achieved by a quadrature.

If the motion is *steady* then the invariant form:

$$\Pi = \Omega(\square, \delta) = -\rho u\,[\delta y\;\delta z] - \rho v\,[\delta z\;\delta x] - \rho w\,[\delta x\;\delta y]$$

has zero derivative. The equations that give the geometric trajectories:

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w}$$

admit a multiplier r. As a result, if one knows a first integral then the determination of the trajectories will require only a quadrature, and a final quadrature gives t.

2. The equations that give the vortex lines of a given vector field (X, Y, Z) are the characteristic equations of the form:

$$[\delta X\ \delta x] + [\delta Y\ \delta y] + [\delta Z\ \delta z]$$

$$= \left(\frac{\partial Z}{\partial y} - \frac{\partial Y}{\partial z}\right)[\delta y\,\delta z] + \left(\frac{\partial X}{\partial z} - \frac{\partial Z}{\partial x}\right)[\delta z\,\delta x] + \left(\frac{\partial Y}{\partial x} - \frac{\partial X}{\partial y}\right)[\delta x\,\delta y].$$

The equations:

$$\frac{dx}{\frac{\partial Z}{\partial y} - \frac{\partial Y}{\partial z}} = \frac{dy}{\frac{\partial X}{\partial z} - \frac{\partial Z}{\partial x}} = \frac{dz}{\frac{\partial Y}{\partial x} - \frac{\partial X}{\partial y}}$$

thus admit a known multiplier, which is unity.

3. The equations of dynamics, in their canonical form:

$$\frac{dq_i}{dt} = \frac{\partial H}{\partial p_i}, \qquad \frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i},$$

admit the multiplier 1. This results from a direct calculation. It also results from this that the existence of the invariant form:

$$\Omega = \sum_{i=1}^{i=n} [(\delta q_i - \frac{\partial H}{\partial p_i}\delta t)(\delta p_i + \frac{\partial H}{\partial q_i}\delta t)]$$

implies the existence of the invariant form Ω^n:

$$\frac{1}{n!}\Omega^n = \prod_{i=1}^{i=n} [(\delta q_i - \frac{\partial H}{\partial p_i}\delta t)(\delta p_i + \frac{\partial H}{\partial q_i}\delta t)].$$

114. However, the preceding theory of the last multiplier applies not only to material systems for which the canonical equations of Hamilton are valid, but also to any system *with perfect holonomic constraints whose given forces depend upon only the position of the system.*

For such a system, one has Lagrange's equations:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial q_i'}\right)-\frac{\partial T}{\partial q_i}=Q_i(q_1,\cdots,q_n,t)\,.$$

If the Q_i are zero then the introduction of Hamilton's canonical variables will lead to the equations:

$$\frac{dq_i}{dt}=\frac{\partial H}{\partial p_i}\,,\qquad \frac{dp_i}{dt}=-\frac{\partial H}{\partial q_i}\,.$$

It results from this that the complete equations of motion are susceptible to being put into the form:

$$\frac{dq_i}{dt}=\frac{\partial H}{\partial p_i}\,,\qquad \frac{dp_i}{dt}=-\frac{\partial H}{\partial q_i}+Q_i\,.$$

They admit the multiplier 1. In other words, they admit the invariant form:

$$\Omega=[(\delta q_1-\frac{\partial H}{\partial p_1}\delta t)\cdots(\delta q_n-\frac{\partial H}{\partial p_n}\delta t)(\delta p_i+\left(\frac{\partial H}{\partial q_i}-Q_1\right)\delta t)\cdots(\delta p_n+\left(\frac{\partial H}{\partial q_n}-Q_n\right)\delta t)]\,,$$

which can be written in terms of the Lagrange variables:

$$\Omega=\prod_{i=1}^{i=n}[(\delta q_i-q_i'\delta t)(\delta\frac{\partial T}{\partial q_i'}-\left(\frac{\partial T}{\partial q_i}+Q_i\right)\delta t)]\,.$$

If the constraints, as well as the given forces, are independent of time then the equations of motion will admit the infinitesimal transformation:

$$Af=\frac{\partial f}{\partial t}\,,$$

and, as a result, they will also admit the invariant form $\Pi=\Omega(A,\,\delta)$, whose derivative is zero. From the general theorem, the integration of the equations of motion is reduced to the integration of the (geometric) equations of trajectories:

$$\frac{dq_i}{q_i'} = \frac{d\,\dfrac{\partial T}{\partial q_i'}}{\dfrac{\partial T}{\partial q_i} + Q_i} \qquad \text{or} \qquad \frac{dq_i}{\dfrac{\partial H}{\partial p_i}} = \frac{dp_i}{-\dfrac{\partial H}{\partial q_i} + Q_i},$$

to which the theory of the last multiplier may be applied, and a quadrature gives time. Indeed, one obviously has, for example:

$$\Omega = \left[\left(\delta t - \frac{\delta q_i}{q_i'}\right)\Pi\right],$$

since $\Omega(A, \delta)$ is equal to Π.

115. As an example of forces that *depend on time*, but still admit a known infinitesimal transformation, consider the simple case of a point that is moving on a fixed line and is attracted to a fixed point on the line according to a force that is proportional to the distance, with the proportionality factor being a known function of time. The motion is given by the second-order differential equation:

$$\frac{d^2 x}{dt^2} = -k(t)\,x,$$

or by the system:

(10) $$\frac{dx}{dt} = x', \qquad \frac{dx'}{dt} + k(t)\,x = 0.$$

The second-order equation does not change if one changes x into λx, with λ being an arbitrary constant factor. As a result, the system to which it is equivalent admits the infinitesimal transformation whose effect is to change:

$$x,\ x',\ t$$

into

$$(1 + \varepsilon)\,x, \quad (1 + \varepsilon)\,x', \quad t,$$

respectively. The symbol of this transformation is:

$$Af = x\frac{\partial f}{\partial x} + x'\frac{\partial f}{\partial x'}.$$

System (10) admits the invariant form:

$$\Omega = [(\delta x - x'\,\delta t)(\delta x' + kx\,\delta t)],$$

which corresponds to the multiplier 1. Here, the derived form $\Omega(A, \delta)$ is:

$$\varpi = x(\delta x' + kx\,\delta t) - x'(\delta x - x'\,\delta t) = x\,\delta x' - x'\,\delta x + (x'^2 + kx^2)\,\delta t.$$

It is an invariant form. Its exterior derivative is:

$$\varpi' = 2[\delta x\ \delta x'] + 2\,x'\,[\delta x'\ \delta t] + 2kx\,[\delta x\ \delta t] = 2\Omega.$$

Since the coefficient of Ω in the right-hand side is constant, we have (sec. **112**) that it suffices to integrate the *completely integrable* equation $\varpi = 0$ in order to deduce the general solution of the given system by differentiations. The form ϖ is indeed reducible to $y_1\,\delta y_2$. This form may be written, upon changing δ into d:

$$\varpi = x^2\left[d\,\frac{x'}{x} + \left(\frac{x'^2}{x^2} + k\right)dt\right].$$

If one sets:

$$\frac{x'}{x} = u$$

then one will thus be led to integrate the Ricatti equation:

$$\frac{du}{dt} + u^2 + k = 0\,.$$

Suppose we have integrated that equation. We have a first integral in the form:

$$y_2 = \frac{\alpha(t)u + \beta(t)}{\gamma(t)u + \delta(t)} = \frac{\alpha(t)x' + \beta(t)x}{\gamma(t)x' + \delta(t)x}\,.$$

After identifying ϖ with $y_1\,dy_2$, we find, upon taking, for example, the terms in dx':

$$x = y_1 x\,\frac{\alpha\delta - \beta\gamma}{(\gamma x' + \delta x)^2}\,,$$

from which:

$$y_1 = \frac{(\gamma x' + \delta x)^2}{\alpha\delta - \beta\gamma}\,.$$

If one supposes – as is permissible – that the determinant $\alpha\delta - \beta\gamma$ is equal to 1 (or simply constant) then the general solution of the system will be furnished by the equations:

$$\alpha x' + \beta x = C_1,$$
$$\gamma x' + \delta x = C_2,$$

and one has that:

$$x = C_2\,\alpha(t) - C_1\,\gamma(t).$$

In other words, the coefficients $\alpha(t)$ and $\gamma(t)$ that present themselves in the general integral of the Ricatti equation constitute a system of fundamental solutions to the given second-order equation.

Other situations may also present themselves. Suppose we know the general solution of the Ricatti equation, expressed in terms of t and the integration constant y_2. The identity:

$$y_1\, dy_2 = x_2\, [du + (u^2 + k)\, dt]$$

gives:

$$y_1 = x^2 \frac{\partial u}{\partial y_2},$$

from which we obtain x as a function of y_1, y_2, and t. Since we have:

$$u = \frac{\delta y_2 - \beta}{-\gamma y_2 + \alpha},$$

we will obtain:

$$x^2 = y_1\, (\alpha - \gamma y_2)^2,$$

from which, we will get:

$$x = C_1\, \alpha + C_2\, \gamma.$$

116. REMARK. – The theory of Jacobi's last multiplier can be applied to other problems in mechanics than the ones that were pointed out above. For example, take the motion of a material point that is subject to a force that is a function of only its position in space, but the system of reference is associated with a uniform rotational motion around Oz. The equations of motion are of the form:

$$\begin{aligned}
&\frac{d^2x}{dt^2} + 2\alpha \frac{dy}{dt} - X = 0,\\
&\frac{d^2y}{dt^2} - 2\alpha \frac{dx}{dt} - Y = 0,\\
&\frac{d^2z}{dt^2} \qquad\qquad - Z = 0,
\end{aligned}$$

in which X, Y, Z are given functions of x, y, z, t. If we write them as:

$$\begin{aligned}
&\frac{dx}{dt} = x' \quad \frac{dx'}{dt} = -2\alpha y' + X,\\
&\frac{dy}{dt} = y' \quad \frac{dy'}{dt} = \quad 2\alpha x' + Y,\\
&\frac{dz}{dt} = z' \quad \frac{dz'}{dt} = \qquad\qquad + Z
\end{aligned}$$

then we will obtain a system that obviously admits the multiplier 1.

117. The final application that we shall envision is furnished by the integral invariant of hydrodynamics:

$$\Omega = \xi\,[\delta y\ \delta z] + \eta\,[\delta z\ \delta x] + \zeta\,[\delta x\ \delta y] + (\eta\, w - \zeta\, v)[\delta x\ \delta t] + (\zeta\, u - \xi w)[\delta y\ \delta t] + (\xi\, v - \eta u)[\delta z\ \delta t].$$

The characteristic system of this invariant is composed of the two Pfaff equations:

$$\frac{dx - w dt}{\xi} = \frac{dy - v dt}{\eta} = \frac{dz - w dt}{\zeta}.$$

The integral manifolds in spacetime (x, y, z, t) are two-dimensional manifolds that are generated, for example, by a vortex line in its different successive positions.

The integration of this system amounts to that of a system of two differential equations in two unknown functions when one knows a multiplier. The search for molecular trajectories (i.e., the streamlines) requires, in addition, the integration of an ordinary differential equation *that might be arbitrary.*

If the motion is steady then the characteristic manifolds will be given by two quadratures, namely:

$$\int \begin{vmatrix} dx & dy & dz \\ u & v & w \\ \xi & \eta & \varsigma \end{vmatrix} = C,$$

and then if we take the preceding equation into account, which we assume to be solvable in terms of z, then we will get:

$$t + \int \frac{\eta\, dx - \xi\, dy}{\xi v - \eta u} = C'.$$

CHAPTER XII

EQUATIONS THAT ADMIT A RELATIVE LINEAR INTEGRAL INVARIANT

I. - *General method of integration.*

118. Consider a Pfaff form ω and the characteristic system of the relative *integral* invariant $\int \omega$. This is the associated system of the form ω'.

First, suppose that ω has $2n + 1$ variables. Since the form ω' has *even* rank (sec. **59**), its characteristic system will be composed of $2n$ equations, in general. As a result, *there will exist, in general, one and only one system of differential equations that admits a relative integral invariant* $\int \omega$, *where* ω *is an arbitrary Pfaff form in* $2n + 1$ *variables;* this is the case for the integral invariant of dynamics.

In a general manner, let $2n$ be the rank (or the class) of the form ω'. It is easy to indicate a method of integration for the characteristic equations of ω'.

Indeed, let y_1 be a first integral of these equations (it is obtained by an operation of order $2n$). If one couples the variables by the relation $y_1 = C_1$, i.e., if one couples the differentials by the relation $dy_1 = 0$, then the rank of ω' will diminish, and as it always remains even, it will reduce to $2n - 2$. Let y_2 be a first integral of the new characteristic system. If we suppose that:

$$y_1 = C_1, \qquad y_2 = C_2$$

then the rank of ω' reduces to $2n - 4$, and so on. Thus, by successive operations of order:

$$2n,\ 2n - 2,\ \ldots,\ 4,\ 2,$$

one will find n first integrals:

$$y_1, y_2, \ldots, y_n,$$

such that if one supposes that the variables are coupled by the relations:

$$y_1 = C_1, \quad y_2 = C_2, \ldots, y_n = C_n$$

then the rank of ω' will become zero. At that point, since ω' is identically zero, the form ω will be an exact differential. A quadrature will then put it into the form:

$$\omega = dS.$$

The function S depends on n constants $C_1, \ldots, C_n$. If one no longer supposes that the variables are related by the n indicated relations then one will obviously have:

$$\omega = dS + z_1\, dy_1 + z_2\, dy_2 + \ldots + z_n\, dy_n ,$$

and, as a result:

$$\omega' = [dz_1\, dy_1] + [dz_2\, dy_2] + \ldots + [dz_n\, dy_n].$$

Since ω' is of rank $2n$, the $2n$ differentials dy_i and dz_i will be linearly independent. Hence, *the* $2n$ *functions* y_i *and* z_i *will constitute a system of independent first integrals of the given equations,* and the integration is thus accomplished.

By definition, the integration requires $n + 1$ *operations of order:*

$$2n,\ 2n - 2,\ \ldots,\ 4,\ 2,\ 0,$$

according to the differentiations.

REMARK I. – The quantity S serves as only an intermediary here. In general, it is not a first integral of the characteristic equations of the invariant $\int \omega$.

REMARK II. – One sees from the result that was just obtained that *any quadratic exterior form with zero exterior derivative can be put into the form*:

$$[dz_1\, dy_1] + [dz_2\, dy_2] + \ldots + [dz_n\, dy_n] .$$

119. It is important for us to account for the indeterminacy in the choice of functions y_i and z_i that enter into the canonical form. The equality:

$$[dz_1', dy_1'] + [dz_2', dy_2'] + \ldots + [dz_n', dy_n'] = [dz_1\, dy_1] + \ldots + [dz_n\, dy_n]$$

implies the property that the difference:

$$z_1'\, dy_1' + z_2'\, dy_2' + \ldots + z_n'\, dy_n' - (z_1\, dy_1 + z_2\, dy_2 + \ldots + z_n\, dy_n)$$

must be an exact differential dV. Suppose, as is the general case, that $y_1', \ldots, y_n'$ are independent functions of $z_1, \ldots, z_n$. There is then no relation between the y_i and the y_i' . If one expresses V as a function of the y_i and the y_i' then one will deduce that:

$$z_i' = \frac{\partial V}{\partial y_i'}, \qquad z_i = -\frac{\partial V}{\partial y_i} .$$

These equations, which involve an arbitrary function of $2n$ arguments, permit us to express the y' and the z' as functions of the y and z. As a result, the latter n give $y_1', \ldots, y_n'$, and the first n give $z_1', \ldots, z_n'$. This supposes that one does not have:

$$\frac{D\left(\frac{\partial V}{\partial y_1}, \frac{\partial V}{\partial y_2}, \cdots, \frac{\partial V}{\partial y_n}\right)}{D(y'_1, y'_2, \cdots, y'_n)} = 0 .$$

Under the same condition, one can express the y as a function of the y' and the z' by means of the first n equations and thus obtain the z by means of the last equations.

One will similarly treat the case where there exist one or several relations between the y and the y'.

The set of transformations thus defined on the variables y and z – *i.e., on the integral curves of the given equations* – defines an infinite group that plays the same role in this theory that the group of transformations of functional determinant equal to 1 plays in the theory of the Jacobi multiplier.

120. Recall the integration of the characteristic equations of ω'. Suppose that, by an arbitrary procedure, we come to know $N > n$ independent first integrals $y_1, y_2, ..., y_n$, such that upon equating them to arbitrary constants, the rank of ω' becomes zero, i.e., ω becomes an exact differential. A quadrature that corresponds to the differentiations then gives:

$$\omega = dS + z_1\, dy_1 + z_2\, dy_2 + ... + z_n\, dy_n .$$

It is easy to see that $z_1, z_2, ..., z_n$ *are first integrals.*

Indeed, suppose that among the functions y_i and z_i there are $N + r$ independent ones. One may then express the functions z_i as functions of the y_i and r of the others, which we call $t_1, ..., t_r$. Having assumed this, we will then have:

$$\omega' = [dz_1\, dy_1] + [dz_2\, dy_2] + ... + [dz_N\, dy_N].$$

By hypothesis, the characteristic system of ω' is composed of the equations:

$$dy_1 = 0, \quad dy_2 = 0, ..., dy_N = 0.$$

It also consists of the equation:

$$\frac{\partial \omega'}{\partial [dy_i]} \equiv -dz_i + \frac{\partial z_1}{\partial y_i} dy_1 + \frac{\partial z_2}{\partial y_i} dy_2 + \cdots + \frac{\partial z_N}{\partial y_i} dy_N = 0 ;$$

hence, the equation:

$$dz_i = 0.$$

One sees from this that the z_i are first integrals, and, on the other hand, that $N + r$ will be equal to $2n$.

Finally, *the knowledge of N first integrals makes ω an exact differential when one equates them to arbitrary constants and permits us accomplish the integration by one quadrature and differentiations.*

121. In practice, it may happen that one does not seek all of the solutions of the given differential equations, but only the ones for which the N first integrals $y_1, ..., y_N$, have given *numerical* values. One may then proceed in the following manner: Since the form ω' is zero when one annulls $dy_1, ..., dy_N$ it can be put into the form:

$$\omega' = [dy_1 \varpi_1] + [dy_2 \varpi_2] + ... + [dy_N \varpi_N]$$

in an infinitude of ways, with the ϖ_i being conveniently chosen linear forms. Among these N forms ϖ_i, there are $2n - N$ of them that are independent of each other and independent of the dy_i. Suppose that the same is true for $\varpi_1, ..., \varpi_{2n-N}$. The characteristic system of ω' is obviously composed of the equations:

$$dy_1 = dy_2 = ... = dy_N = 0,$$
$$\varpi_1 = \varpi_2 = ... = \varpi_{2n-N} = 0.$$

If we express the idea that the exterior derivative of ω' is zero then we will obtain:

$$[dy_1 \varpi'_1] + [dy_2 \varpi'_1] + ... + [dy_N \varpi'_N] = 0,$$

from which, in particular, upon exterior multiplying by $[dy_2 dy_3 ... dy_\mathrm{N}]$, we will obtain:

$$[dy_1\, dy_2 ... dy_N \varpi'_1] = 0.$$

The form ϖ_1 (as well as the forms $\varpi_2, ..., \varpi_{2n-N}$) is thus an exact differential when one gives fixed numerical values to the y_i. As a result, *the desired solutions are obtained by* $2n - N$ *independent quadratures*:

$$\int \varpi_1 = \gamma_1, ..., \int \varpi_{2n-N} = \gamma_{2n-N}.$$

There is no reason to be surprised at the fact that we have encountered $2n - N$ quadratures here when the search for the general solution requires only one quadrature. Indeed, after performing the $2n - N$ quadratures that were indicated above, we then perform the *single* quadrature:

$$\int \lambda_1 \varpi_1 + \lambda_2 \varpi_2 + ... + \lambda_{2n-N} \varpi_{2n-N} = \text{const.}$$

that has $2n - N$ arbitrary *parameters*, $\lambda_1, ..., \lambda_{2n-N}$.

The preceding integration procedure uses only the invariant form ω', and does not involve the form ω. *Thus, the things that play an essential role here are the knowledge of the absolute integral invariant of the second degree $\int \omega'$ and the property of the form ω' that it is an exact derivative.* The form ω (or the forms ω), whose derivative is ω', plays only an auxiliary role.

II. - *The Poisson brackets and the Jacobi identity.*

122. Let $2n$ be the rank of the exterior derivative ω', and let f and g be two first integrals of its characteristic system. The two differential forms:

$$[\omega'^{n-1} df\, dg] \quad \text{and} \quad [\omega'^{n}]$$

are invariants of maximum degree $2n$. They differ by only a factor, and this factor is a first integral. We set:

$$\frac{1}{(n-1)!}[\omega'^{n-1} df\ dg] = \frac{1}{n!}(f\ g)[\omega'^{n}]$$

or

$$(f\,g)[\omega'^{n}] = n\,[\omega'^{n-1} df\, dg].$$

The quantity $(f\,g)$ thus defined bears the name of *Poisson bracket.* It is an alternating bilinear form in the partial derivatives of f and g.

The bracket of two first integrals is again a first integral.

This theorem, which is due to Poisson in the particular case of canonical equations, had its importance pointed out by Jacobi.

Before going on to the applications of this theorem, we make several remarks.

The condition $(f\,g) = 0$ expresses the idea that the rank of ω' is equal to $2n - 4$. In this case, one says that the integrals f and g are *in involution.*

If this condition is not satisfied then the defining formula of $(f\,g)$ will express the idea that the form:

$$\omega' - \frac{[df\ dg]}{(f\ g)}$$

is of rank $2n - 2$. Indeed, the n^{th} power of this form is:

$$[\omega'^{n}] - \frac{n}{(f\ g)}[\omega'^{n-1} df\ dg] = 0\,.$$

Again, we remark that if one reduces ω' to its normal form:

$$\omega' = [\omega_1\ \omega_2] + [\omega_3\ \omega_4] + \dots + [\omega_{2n-1}\ \omega_{2n}],$$

and if one sets:

$$df = f_1\,\omega_1 + f_2\,\omega_2 + \dots + f_{2n}\,\omega_{2n}\,,$$
$$dg = g_1\,\omega_1 + f_2\,\omega_2 + \dots + g_{2n}\,\omega_{2n}\,,$$

then one will have:

$$(f\,g) = f_1\,g_2 - f_2\,g_1 + f_3\,g_4 - f_4\,g_3 + \dots + f_{2n-1}\,g_{2n} - f_{2n}\,g_{2n-1}\,.$$

We finally remark that from what was said above (sec. **118**), one may always suppose that the ω_i are exact differentials. An easy calculation then gives the following identity, which is due to Jacobi:

$$((f\,g)\,h) + ((g\,h)\,f) + ((h\,f)\,g) = 0,$$

which is applied to three arbitrary first integrals f, g, h.

However, the verification can be carried out without any restrictions on the linear forms ω_1, ω_2, ..., ω_{2n}. It is based upon the identity:

$$(1) \qquad \frac{1}{(n-1)!}[\omega'^{\,n-1}((f\,g)\,dh + (g\,h)\,df + (h\,f)\,dg] = \frac{1}{(n-2)!}[\omega'^{\,n-2} df\;dg\;dh],$$

which is nothing but identity (8) that was proved in sec. **68**. Upon exterior differentiating and remarking that the exterior derivative of the right-hand side is zero, we obtain:

$$[\,\omega'^{\,n-1}\,d(f\,g)\;dh] + [\,\omega'^{\,n-1}\,d(g\,h)\;df] + [\,\omega'^{\,n-1}\,d(h\,f)\;dg] = 0,$$

which is nothing but the Jacobi identity.

123. The method of integration that was pointed out at the beginning of this chapter can be stated in terms of the Poisson brackets. Let:

$$Xf = 0$$

be the equation that expresses the idea that f is a first integral. One first seeks a particular solution y_1 of that equation. One then seeks a particular solution y_2 of the system:

$$Xf = 0, \qquad (y_1\,f) = 0,$$

then a particular solution y_3 of the system:

$$Xf = 0, \quad (y_1\,f) = 0, \quad (y_2\,f) = 0,$$

and so on, up to a particular solution y_n of the system:

$$Xf = 0, \quad (y_1\,f) = 0, \quad (y_2\,f) = 0, \quad ..., (y_{n-1}\,f) = 0.$$

In the case of the canonical equations of dynamics:

$$\frac{dq_i}{dt} = \frac{\partial H}{\partial p_i}, \qquad \frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i},$$

which correspond to the invariant form:

$$\omega' = \sum_{i=1}^{i=n} [\delta p_i \, \delta q_i] - [\delta H \, \delta t],$$

the partial differential equation in the first integrals of the given equation is:

$$\frac{\partial f}{\partial t} + \sum_{i=1}^{i=n} \left(\frac{\partial f}{\partial q_i} \frac{\partial H}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial H}{\partial q_i} \right) = 0.$$

As for the Poisson brackets $(f\ g)$ of the two first integrals, they are defined by the equality:

$$n[\,\omega'^{n-1}\, df\, dg] = (f\ g)[\,\omega'^{n}\,].$$

Equating the terms in:

$$[\delta p_1\, \delta q_1\, \delta p_2 \ldots \delta p_n\, \delta q_n]$$

on both sides, we obtain:

$$(f\ g) = \sum_{i=1}^{i=n} \left(\frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} - \frac{\partial g}{\partial p_i} \frac{\partial f}{\partial q_i} \right).$$

The partial differential equation in the first integrals can also be written by means of the definition of the bracket $(f\ g)$ of two arbitrary functions of q_i, p_i, and t as:

$$\frac{\partial f}{\partial t} - (H\ f) = 0.$$

III. - *Use of known first integrals.*

124. Now, we shall once more take up the problem of integrating the characteristic equations of the differential form ω' by supposing that a certain (arbitrary) number of first integrals y_1, y_2, ..., y_p are known. When one equates these integrals to arbitrary constants C_1, C_2, ..., C_p, the form ω' will have its rank reduced by a certain even number $2p' \leq 2p$ of units. It then suffices to integrate the characteristic equations of that new form – or rather, to find $n - p'$ first integrals in involution. One is then reduced to the problem of sec. **120**.

The preceding method does not generally give us all the possible sets of known integrals. Indeed, from the Poisson-Jacobi theorem, the brackets of p given integrals, when taken two at a time, are themselves first integrals of the equations being integrated. One will then form the brackets $(y_i\, y_j)$, and then, if they provide new integrals, one forms the brackets of these integrals amongst themselves and the given integrals, and so on, until the operation gives us nothing new. This amounts to saying that *one may always, by prior differentiations, suppose that the brackets* $(y_i\, y_j)$ *are functions of the first integrals* y_1, y_2, ..., y_p.

Now, in order to know how many units it takes to reduce the rank of ω' when one supposes that the variables are coupled by relations:

$$y_1 = C_1, \quad y_2 = C_2, \quad ..., y_p = C_p,$$

it will suffice to apply the theorem of sec. **69** to the exterior quadratic form ω' that is constructed from the variables $\delta x_1, ..., \delta x_{2n+1}$, which are coupled by the relations:

$$\delta y_1 = 0, \quad \delta y_2 = 0, ..., \delta y_p = 0.$$

In this case, the coefficients a_{ij} of sec. **69** are the brackets $(y_i\, y_j)$, and the quadratic form $\bar{\Phi}$ is:

$$\bar{\Phi} = \sum (y_i\, y_j)[\xi_i\; \xi_j].$$

The number of units by which the rank of ω' *is reduced is equal to the maximum number* $2p$ *diminished by the rank of the form* Φ.

125. One may account for the fact that all of the possible sets have been obtained from the given first integrals in the following manner.

Perform a linear substitution (with coefficients in $y_1, ..., y_p$) on the p variables $x_1, ..., x_p$ in such a way as to reduce $\bar{\Phi}$ to its normal form:

$$\bar{\Phi} = [\bar{\xi}_1\, \bar{\xi}_2] + \cdots + [\bar{\xi}_{2q-1}\, \bar{\xi}_{2q}] \qquad (2q \leq p).$$

This amounts to replacing the linear forms $\delta y_1, ..., \delta y_p$ with new differential forms:

$$\varpi_1, ..., \varpi_p,$$

which are linear in $\delta y_1, ..., \delta y_p$ with coefficients that are functions of $y_1, ..., y_p$, and are such that one has identically:

$$\xi_1 \delta y_1 + \xi_2 \delta y_2 + \cdots + \xi_p \delta y_p = \bar{\xi}_1 \varpi_1 + \bar{\xi}_2 \varpi_2 + \cdots + \bar{\xi}_p \varpi_p.$$

The exterior quadratic form ω' will then take the form:

$$\omega' = [\varpi_1 \varpi_2] + ... + [\varpi_{2q-1} \varpi_{2q}] + [\varpi_{2q+1} \omega_1] + ... + [\varpi_p \omega_{p-2q}] + [\omega_{p-2q} \omega_{p-2q+2}] + ... + [\omega_{2n-p-1} \omega_{2n-p}],$$

upon introducing $2n - p$ new linear forms $\omega_1, ..., \omega_{2n-p}$.

Let Π denote the form:

$$\Pi = [\varpi_1 \varpi_2] + ... + [\varpi_{2q-1} \varpi_{2q}],$$

and express the idea that the exterior derivative of ω' is zero. If we neglect all of the terms that contain one of the linear forms:

$$\varpi_{2q+1}, ..., \varpi_p; \qquad \omega_{p-2q+1}, ..., \omega_{2n-p}$$

then we will get:

(2) $$\Pi' + [\varpi'_{2j+1} \omega_1] + \cdots + [\varpi'_p \omega_{p-2q}] = 0.$$

As the form Π is constructed from only the functions y_i and their differentials, the same is true for Π'. As a result, no reduction of the similar terms can be made between the different parts of the left-hand side of (2). In particular, it results from this that each of the forms:

$$\varpi'_{2q+1}, \cdots, \varpi'_p$$

will be zero (upon supposing that the forms ϖ_{2q+1}, ..., ϖ_p are zero). As a result, *the Pfaff system*:

$$\varpi_{2q+1} = ... = \varpi_p = 0$$

is completely integrable. We denote a system of first integrals of these equations by:

$$\bar{y}_{2q+1}, \cdots, \bar{y}_p.$$

Furthermore, one always has:

$$\Pi' = 0$$

if one regards the forms ϖ_{2q+1}, ..., ϖ_p as zero. In other words, if one supposes that the $\bar{y}_{2q+1}, \cdots, \bar{y}_p$ are constant then the form Π will be an exact derivative, and, as a result (sec. **118**), it will be reducible to:

$$\Pi = [d\bar{y}_1\, d\bar{y}_2] + \cdots + [d\bar{y}_{2q-1}\, d\bar{y}_{2q}].$$

Finally, one easily sees that one can put ω' into the form:

(3) $$\omega' = [d\bar{y}_1\, d\bar{y}_2] + \cdots + [d\bar{y}_{2q-1}\, d\bar{y}_{2q}] + [d\bar{y}_{2q+1}\, \bar{\omega}_1] + \cdots + [d\bar{y}_p\, \bar{\omega}_{p-2q}] + [\omega_{p-2q+1}\, \omega_{p-2q+2}] + \cdots$$

This ultimately amounts to the following theorem:

One can find p functions:

$$\bar{y}_1, \bar{y}_2, \cdots, \bar{y}_p$$

of the given p first integrals that satisfy the conditions*:*

$$(\bar{y}_1\, \bar{y}_2) = \cdots = (\bar{y}_{2q-1}\, \bar{y}_{2q}) = 1,$$

while all of the other brackets $(\bar{y}_i\, \bar{y}_j)$ *are zero.*

126. Apart from the intrinsic interest of this theorem, its form clarifies the fact stated above that the indicated method of integration has educed all of the possible consequences of the given integrals. The form (3) that was found for ω' permits us to write:

$$\omega = dS + \bar{y}_1\, d\bar{y}_2 + \cdots + \bar{y}_{2q-1}\, d\bar{y}_{2q} + w_1\, d\bar{y}_{2q+1} + \cdots + w_{p-2q} d\bar{y}_p + v_1\, du_1 + \cdots + v_{n-p+q}\, du_{n-p+q},$$

$$\omega' = [d\bar{y}_1 d\bar{y}_2] + \cdots + [d\bar{y}_{2q-1} d\bar{y}_{2q}] + [dw_1 d\bar{y}_{2q+1}] + \cdots + [w_{p-2q} d\bar{y}_p] + [dv_1 du_1] + \cdots + [dv_{n-p+q} du_{n-p+q}].$$

The most general group of transformations on the integral curves that *preserves the givens*, i.e., that leaves ω', $y_1, \dots, y_p$ invariant, is defined by the following equations, in which the primed letters indicate the transformed variables and V denotes an arbitrary function of the arguments u_i, $u_i', \bar{y}_{2q+1}, \cdots, \bar{y}_p$:

$$\begin{gathered}
\bar{y}_i' = \bar{y}_i \qquad (i = 1, 2, \cdots, p), \\
v_1' = \frac{\partial V}{\partial u_1'}, \cdots, v_{n-p+q}' = \frac{\partial V}{\partial u_{n-p+q}'} \\
v_1 = -\frac{\partial V}{\partial u_1}, \cdots, v_{n-p+q} = -\frac{\partial V}{\partial u_{n-p+q}} \\
w_1' = w_1 + \frac{\partial V}{\partial \bar{y}_{2q+1}}, \cdots, w_{p-2q}' = w_{p-2q} + \frac{\partial V}{\partial \bar{y}_{p-2q}}.
\end{gathered}$$

Any unambiguous procedure that permits us, upon starting with ω' and p first integrals $y_1, \dots, y_p$, to deduce another first integral by operations *that have a significance that is independent of the choice of variables* necessarily leads to a first integral that is invariant under the most general group of transformations that preserve ω', $y_1, \dots, y_p$. However, the only functions that are invariant under this group are obviously arbitrary functions of $y_1, \dots, y_p$.

IV. - *Generalization of the Poisson-Jacobi theorem.*

127. The Poisson-Jacobi theorem is immediately generalized if, instead of two first integrals, one knows two invariant linear forms ϖ_1 and ϖ_2. The quantity α that is defined by the equality:

$$n\,[\omega'^{n-1}\varpi_1\varpi_2] = \alpha\,[\omega'^n] \tag{4}$$

is obviously a first integral. It reduces to $(y_1\, y_2)$ if ω_1 and ω_2 are the differentials of the two first integrals y_1 and y_2.

We apply this remark to the case in which, assuming that the characteristic equations of ω' admit two infinitesimal transformations A_1f and A_2f, one has:

$$\varpi_1 = \omega'(A_1, \delta), \qquad \varpi_2 = \omega'(A_2, \delta).$$

In order to calculate the quantity α in this case, we apply the operation that takes an invariant form $\Omega(\delta)$ into the form $\Omega(A_1, \delta)$ to both sides of equality (4). One obtains:

$$n\,(n-1)\,[\omega'^{n-2}\varpi_i\varpi_1\varpi_2] - n\,[\omega'^{n-1}\varpi_1\varpi_2(A_1)] = n\alpha\,[\omega'^{n-1}\omega_1],$$

from which, since the form $[\omega'^{n-1}\omega_1]$ is certainly not zero:

$$\alpha = -\varpi_2(A_1) = \omega'(A_1, A_2) = \varpi_1(A_2).$$

When the generalized Poisson-Jacobi theorem is applied to the two invariant forms $\omega'(A_1, \delta)$ *and* $\omega'(A_2, \delta)$ *it leads to the first integral* $\omega'(A_1, A_2)$ *that is furnished by twice applying the operations that correspond to the infinitesimal transformations* A_1f *and* A_2f.

CHAPTER XIII

EQUATIONS THAT ADMIT AN ABSOLUTE LINEAR INTEGRAL INVARIANT

I. - *General method of integration.*

128. Let ω be a linear differential form. Its bilinear covariant ω' is of even rank, namely $2n$. Two cases may be presented, according to whether the equation $\omega = 0$ does or does not belong to the characteristic system of ω'. We shall first occupy ourselves with the latter case.

I. Obviously, one may set:

$$\omega' = [\omega_1\ \omega_2] + \ldots + [\omega_{2n-1}\ \omega_{2n}].$$

The $2n + 1$ forms $\omega, \omega_1, \ldots, \omega_{2n}$ are independent. In this case, the characteristic equations of ω are (sec. **78**):

$$\omega = \omega_1 = \omega_2 = \ldots = \omega_{2n} = 0.$$

One may easily exhibit a reduced form for ω. Indeed, from operations of order:

$$2n,\ 2n - 2,\ \ldots,\ 2,$$

we successively find n first integrals:

$$y_1, y_2, \ldots, y_n$$

of the characteristic equations of ω', and reduce its rank to zero when we equate them to arbitrary constants. A quadrature puts ω into the form:

$$\omega = du + z_1\, dy_1 + z_2\, dy_2 + \ldots + z_n\, dy_n\,.$$

This is the desired reduced form, which is obtained by operations of order:

$$2n,\ 2n - 2,\ \ldots,\ 2,\ 0,$$

and which, once obtained, gives the general solution of the characteristic equations of ω.

In this case, one sees that the integration of the characteristic equations of ω and the integration of the characteristic equations of ω' are two equivalent problems, and the fact that $\int \omega$ is an *absolute* integral invariant has no more importance in the integration than if $\int \omega$ were a *relative* integral invariant. This is true at least when one follows the method indicated in sec. **118**. This will no longer be the case when one applies the method of sec. **121**.

II. In the former case, one may express:

$$\omega' = [\omega\ \omega_1] + [\omega_2\ \omega_3] + \ldots + [\omega_{2n-2}\ \omega_{2n-1}]$$

in terms of the $2n$ linearly independent forms $\omega, \omega_1, ..., \omega_{2n-1}$. The equations:

$$\omega = 0, \ \omega_2 = \omega_3 = ... = \omega_{2n-1} = 0$$

that are obtained by writing the equations of the associated system of ω', in addition to the equation $\omega = 0$, *in which one supposes that the differentials are coupled by the relation* $\omega = 0$, have an intrinsic significance. This is the associated system of the two forms ω and $[\omega\ \omega']$, and, as a result (sec. **103**), *it is the characteristic system of the Pfaff equation* $\omega = 0$. We call that system (Σ), and denote the characteristic system of ω by (S), which contains the equation $\omega_1 = 0$, as well.

One may obtain a first integral y_1 of the system (Σ) by an operation of order $2n - 1$. Upon equating the system (Σ) of the new form ω – i.e., the characteristic system of the new equation $\omega = 0$ – to an arbitrary constant, the number of its equations will be reduced by two units. One may thus find new integrals:

$$y_2, ..., y_{n-1}$$

by operations of order:

$$2n - 3, ..., 3,$$

such that upon equating them to new arbitrary constants the new system (Σ) that corresponds to ω will contain only one equation, which will obviously be $\omega = 0$. This says that this equation is completely integrable, and a new operation of order 1 will give a new integral y_n, which permits us to write:

$$\omega = z_1\, dy_1 + z_2\, dy_2 + ... + z_n\, dy_n\,.$$

One will thus arrive at the *reduced form* of ω, which effectively involves the minimum number $2n$ of variables, since $2n$ is the number of equations in the characteristic system (S) of ω; i.e., the class of ω.

One finds, with no difficulty, the most general transformation that one can perform on the characteristic variables y_i and z_i that preserves the form ω. The equality:

$$z_1'\, dy_1' + ... + z_n'\, dy_n' = z_1\, dy_1 + ... + z_n\, dy_n$$

gives, upon keeping the most general case:

$$V(y_1', ..., y_n', y_1, ..., y_n) = 0,$$

$$\frac{z_1'}{\dfrac{\partial V}{\partial y_1'}} = \frac{z_2'}{\dfrac{\partial V}{\partial y_2'}} = \cdots = \frac{z_n'}{\dfrac{\partial V}{\partial y_n'}} = \frac{-z_1}{\dfrac{\partial V}{\partial y_1}} = \cdots = \frac{-z_p}{\dfrac{\partial V}{\partial y_n}}\,.$$

These formulas show that the variables $y_1, \ldots, y_n, \frac{z_2}{z_1}, \cdots, \frac{z_n}{z_1}$ are transformed into each other. They are the minimum number of variables by means of which the equation $\omega = 0$ may be written. They are the first integrals of the characteristic system (Σ) of that equation.

If there exist p independent relations:

$$V_1 = 0, \quad V_2 = 0, \ldots, V_p = 0$$

between the y_i and the y_i' then the formulas that define the transformation will be:

$$z_i' = \lambda_1 \frac{\partial V_1}{\partial y_i'} + \lambda_2 \frac{\partial V_2}{\partial y_i'} + \cdots + \lambda_p \frac{\partial V_p}{\partial y_i'},$$
$$-z_i = \lambda_1 \frac{\partial V_1}{\partial y_i} + \lambda_2 \frac{\partial V_2}{\partial y_i} + \cdots + \lambda_p \frac{\partial V_p}{\partial y_i},$$

with p auxiliary unknowns $\lambda_1, \ldots, \lambda_p$.

129. One must remark that, from the point of view of integration, the difference between the two cases where the characteristic system of ω is odd $(2n + 1)$ or even $(2n)$ is that in the first case the integration requires operations of order:

$$2n, 2n - 2, \ldots, 2, 0,$$

whereas, in the second case it requires operations of order:

$$2n - 1, 2n - 3, \ldots, 1.$$

One must also remark that the two cases are practically distinguished from each other in the following manner: Let $2n$ be the rank of ω', i.e., let n be the largest exponent such that the form $[\omega'^n]$ is non-zero. *In the first case,* $[\omega\omega'^n]$ *is non-zero. In the second case,* $[\omega\omega'^n]$ *is zero.*

II. - *Generalization of the Poisson-Jacobi formulas.*

130. I. *Suppose that the form* ω *is of the first type.* – Let f be an arbitrary first integral of its characteristic system. The form $[\omega'^n df]$ is invariant and of maximum order $2n + 1$. One may thus set:

$$[\omega'^n df] = \{f\}[\omega\,\omega'^n],$$

in which $\{f\}$ is a finite quantity that is linear with respect to the first order partial derivatives of the function f. *The quantity* $\{f\}$ *is either a constant or a first integral of the characteristic equations of* ω.

Now let f and g be two first integrals of the characteristic equations of ω. One may define a quantity $(f\,g)$ by the relation:

$$n\,[\omega\,\omega'^n\,df\,dg] = (f\,g)[\omega\,\omega'^n].$$

The quantity $(f\,g)$ *is again a first integral.*

If the form ω has been reduced:

$$\omega = du + z_1\,dy_1 + \ldots + z_n\,dy_n$$

then one has:

$$\{f\} = \frac{\partial f}{\partial u},$$

$$(f\;g) = \sum_{i=1}^{i=n} \frac{\partial f}{\partial z_i}\left(\frac{\partial g}{\partial y_i} - z_i\frac{\partial g}{\partial u}\right) - \frac{\partial g}{\partial y_i}\left(\frac{\partial f}{\partial y_i} - z_i\frac{\partial f}{\partial u}\right).$$

From this, one may deduce the following important identities without difficulty:

$$\{(f\,g)\} = (\{f\}\;g) + (f\;\{g\}),$$
$$((f\,g)\;h) + ((g\;h)\;f) + ((h\;f)\;g) = (f\,g)\{h\} + (g\;h)\{f\} + (h\;f)\{g\}.$$

131. In order to prove these identities directly, we remark that the form $df - \{f\}\,\omega$ is a linear combination of the $2n$ linearly independent forms by means of which ω' may be expressed, since one has:

$$[\omega'^n\,(df - \{f\}\,\omega)] = 0.$$

From this, one immediately derives an identity of the form:

$$n\,[\omega'^{n-1}\,(df - \{f\}\,\omega)(dg - \{g\}\,\omega)] = \lambda[\omega'^n],$$

and exterior multiplication by ω gives $\lambda = (f\,g)$. One then has:

(1) $$n\,[\omega'^{n-1}\,df\,dg] - n\{f\}[\omega\,\omega'^{n-1}\,dg] + n\{g\}[\omega\,\omega'^{n-1}\,df\} = (f\,g)[\omega'^n].$$

When the identity (8) of sec. **68** is applied to the three linear forms $df - \{g\}\,\omega$, $dg - \{g\}\,\omega$, $dh - \{h\}\,\omega$, that gives:

$$(f\,g)[\omega'^{n-1}\,(dh - \{h\}\,\omega)] + (g\;h)[\;\omega'^{n-1}\,(df - \{f\}\,\omega)] + (hf)[\;\omega'^{n-1}\,(dg - \{g\}\,\omega)]$$
$$= (n-1)[\,\omega'^{n-2}\,(df - \{f\}\,\omega)(dg - \{g\}\,\omega)(dh - \{h\}\,\omega)],$$

from which one deduces, upon multiplying by ω, that:

(2) $$[\omega\, \omega'^{n-1}((f\,g)\, dh + (g\,h)\, df + (h\,f)\, dg)] = (n-1)[\omega\, \omega'^{n-2}\, df\, dg\, dh].$$

Having said that, taking the exterior derivative of identity (1) shows that:

$$n[\omega\omega'^{n-1}\, d\{f\}dg] + n[\omega\, \omega'^{n-1}\, df\, d\{g\}] = [\omega'^{n}\, d(f\,g)],$$

i.e., the first identity shows that:

$$(\{f\}\, g) + (f\,\{g\}) = \{(f\,g)\}.$$

Exterior derivation of identity (2) then gives:

$$[\omega'^{n}(f\,g)\, dh + (g\,h)\, df + (h\,f)\, dg)] - [\omega\omega'^{n-1}\, d(f\,g)\, dh] - [\omega\omega'^{n-1}\, d(g\,h)\, df] - [\omega\omega'^{n-1}\, d(h\,f)\, dg]$$
$$= (n-1)[\omega'^{n-1}\, df\, dg\, dh].$$

On the other hand, exterior multiplication of (1) by dh gives:

$$n\,[\omega'^{n-1}\, df\, dg\, dh] - n\{f\}[\omega\omega'^{n-1}\, dg\, dh] + n\{g\}[\omega\omega'^{n-1}\, df\, dh] = (f\,g)[\omega'^{n}\, dh].$$

From this one deduces the final formula:

$$n\,[\omega'^{n-1}\, df\, dg\, dh] = [\{f\}(g\,h) + \{g\}(h\,f) + \{h\}(f\,g)][\omega\, \omega'^{n}],$$

and, from the preceding, the identity has shown that:

$$(f\,g)\{h\} + (g\,h)\{f\} + (h\,f)\{g\} = ((f\,g)\,h) + ((g\,h)\,f + ((h\,f)\,g).$$

132. *Now suppose that the form ω is of the second type.* – Similarly, if one is given two first integrals f and g of the characteristic equations of ω then one will define the quantities $\{f\}$ and $(f\, g)$ by the formulas:

$$n\,[\omega\omega'^{n-1}\, df] = \{f\}[\omega'^{n}]$$
$$n\,[\omega'^{n-1}\, df\, dg] = (f\,g)[\omega'^{n}].$$

If ω is of the reduced form:

$$\omega = z_1\, dy_1 + z_2\, dy_2 + \ldots + z_n\, dy_n$$

then one will have:

$$\{f\} = -\left(z_1\frac{\partial f}{\partial z_1} + z_2\frac{\partial f}{\partial z_2} + \cdots + z_n\frac{\partial f}{\partial z_n}\right),$$

$$(f\;g) = \sum\left(\frac{\partial f}{\partial z_i}\frac{\partial g}{y_i} - \frac{\partial f}{\partial y_i}\frac{\partial g}{\partial z_i}\right).$$

One then verifies the following formulas without difficulty:

(3) $$\{(f\,g)\} = (f\,g) + (\{f\}\;g) + (f\;\{g\})$$
(4) $$((f\,g)\;h) + ((g\,h)\;f) + ((h\,f)\;g) = 0,$$

in which the second one is nothing but the Jacobi identity, since f, g, h are first integrals of the characteristic equations of ω'.

In order to prove the first identity directly we apply the identity (8) of sec. **68** to the three linear forms, ω, df, dg. The relations:

$$n\,[\omega'^{n-1}\,\omega\,df] = \{f\}[\omega'^{n}],$$
$$n\,[\omega'^{n-1}\,df\,dg] = (f\,g)[\omega'^{n}],$$
$$n\,[\omega'^{n-1}\,dg\;\omega] = -\{g\}[\omega'^{n}],$$

lead to the identity:

$$[\omega'^{n-1}(\{f\}\;dg + (f\,g)\;\omega - \{g\}\;df)] = (n-1)[\omega'^{n-2}\,\omega\,df\,dg],$$

which, when exterior differentiated, gives:

$$[\omega'^{n-1}\,d\{f\}\;dg + [\omega'^{n-1}\,df\,d\{g\}] + (f\,g)[\omega'^{n}] - [\omega\omega'^{n-1}\,d(f\,g)]$$
$$= (n-1)[\omega'^{n-1}\,df\,dg].$$

Upon replacing each term by its value and simplifying, one obtains the identity that was to be proved.

III. - *Use of known first integrals.*

133. *Suppose that the form ω is of the first type* and that we know p independent first integrals $y_1, \dots, y_p$ of these characteristic equations. We form the quantities $\{y_i\}$, $(y_i\;y_j)$. If they introduce new integrals then we adjoin them to the given ones and repeat the operation until it gives no new integrals. We may thus suppose that the former result is obtained, i.e., that the quantities $\{y_i\} = a_i$, $(y_i\,y_j) = a_{ij}$ are functions of $y_1, \dots, y_p$.

If we now introduce auxiliary variables $x_1, \dots, x_p$ then we will obtain two forms, one of which is linear:

$$\varphi = a_1\,x_1 + a_2\,x_2 + \dots + a_p\,x_p\,,$$

and the other of which is quadratic:

$$\Phi = \sum a_{ij}\,[\xi_i\,\xi_j]\,.$$

The first form indicates the value of the quantity $\{f\}$ when f is any one of the variables $y_1, \dots, y_p$ that admits $\xi_1, \dots, \xi_p$ for its partial derivatives. The second form, or rather the alternating bilinear form:

$$\sum a_{ij}\,\xi_i\,\eta_j$$

to which it corresponds, indicates the value of the parenthesis $(f\,g)$.

Having said this, we shall reduce the preceding two forms by a convenient linear substitution in the variables ξ_i.

Three cases are possible: If the form Φ has been reduced to:

$$\Phi = [\xi'_1 \xi'_2] + \dots + [\xi'_{2q-1} \xi'_{2q}]$$

then one might have:

$$\varphi = 0,$$
$$\varphi = \xi'_1,$$
$$\varphi = \xi'_{2q+1}.$$

When a linear substitution with coefficients that are functions of the y_i is performed on the δy_i, one will obtain p differential forms $\varpi_1, \dots, \varpi_p$ that satisfy the identity:

$$\xi'_1 \varpi_1 + \xi'_2 \varpi_2 + \dots + \xi'_p \varpi_p = \xi_1 \delta y_1 + \xi_2 \delta y_2 + \dots + \xi_p \delta y_p .$$

Having said this, all of the forms *in case* a):

$$[\omega'^n \varpi_i], \quad [\omega \omega'^{n-1} \varpi_i \varpi_j]$$

are zero, except for:

$$n[\omega \omega'^{n-1} \varpi_1 \varpi_2] = \dots = n\varpi'^{n-1} [\varpi_{2q-1} \varpi_{2q}] = [\omega\, \omega'^n].$$

One easily deduces that:

$$\omega' = [\varpi_1 \varpi_2] + \dots + [\varpi_{2q-1} \varpi_{2q}] + [\varpi_{2q+1} \omega_1] + \dots + [\varpi_p \omega_{p-2q}] + [\omega_{p-2q+1} \omega_{p-2q+2}] + \dots$$

If one equates the y_i to arbitrary constants then the form ω will remain a form of the first type, and the rank of ω' will be reduced by $2p - 2q$ units. This case is identical to the one that was studied in the preceding chapter, in which the given first integrals are the integrals of the characteristic system of ω'.

In case b), one has:

$$[\omega'^n (\varpi_1 - \omega)] = [\omega'^n \varpi_2] = \dots = [\omega'^n \varpi_p] = 0,$$

and ω'^n is reducible to the form:

$$\omega'^n = [(\varpi_1 - \omega)\varpi_2] + [\varpi_3 \varpi_4] + \dots + [\varpi_{2q-1} \omega_{2q}] + [\varpi_{2q+1} \omega_1] + \dots + [\varpi_p \varpi_{p-2q}] + [\omega_{p-2q+1} \omega_{p-2q+2}] + \dots$$

If one equates the y_i to arbitrary constants then the form ω will again remain of the first type, and the rank of ω' will be reduced by $2p - 2q$ units.

In case c), one has:

$$[\omega'^n \varpi_1] = \dots = [\omega'^n \varpi_{2q}] = [\omega'^n (\varpi_{2q+1} - \omega)] = \dots = [\omega'^n \varpi_p] = 0.$$

The form ω' is reducible to:

$$\omega' = [\varpi_1\,\varpi_2] + \ldots + [\varpi_{2q-1}\varpi_{2q}] + [(\varpi_{2q+1} - \omega)\ \omega_1] + \ldots + [\varpi_p\,\omega_{p-2q}] + [\omega_{p-2q+1}\,\omega_{p-2q+2}] + \ldots$$

If one equates the y_i to arbitrary constants then the form ω will become a form of the second type, and the rank of ω' will be reduced by $2p - 2q - 2$ units. The characteristic system of the new equation $\omega = 0$ will be composed of $2n - 2p + 2q + 1$ equations. In this case, the integration will require operations of order:

$$2n - 2p + 2q + 1, \ldots, 3, 1,$$

whereas, in cases a) and b) it will require operations of order:

$$2n - 2p + 2q, \ldots, 2, 0,$$

resp.

In summation, if the exterior product $[\varphi^\Phi]$ is zero then the form ω will remains a form of the first type, and it will become a form of the second type in the contrary case.

134. *Now suppose that the form ω is of the second type.* – We again have two forms:

$$\varphi = a_1\xi_1 + \ldots + a_p\,\xi_p\,,$$
$$\Phi = \sum a_{ij}\,[\xi_i\,\xi_j]\,.$$

Since the coefficients a_{ij} are given by the equations:

$$n\,[\omega'^{n-1} dy_i\,dy_j] = a_{ij}\,[\omega'^n],$$

the rank of ω' will be reduced by $2p - 2q$ units when one equates the integrals y_i to arbitrary constants, if $2q$ is the rank of the form Φ.

Since the form Φ is reduced to its normal form:

$$\Phi = [\xi'_1\,\xi'_2] + \ldots + [\xi'_{2q-1}\,\xi'_{2q}],$$

one can suppose, at the same time, that one has one of the following forms for φ:

a) $\varphi = 0$,

b) $\varphi = \xi'_1$,

c) $\varphi = \xi'_{2q+1}$.

From identity (3), *case* a) requires that all of the brackets $(y_i\,y_j)$ must be zero, i.e., that the form Φ must be identically zero. One thus has $q = 0$. In this case, one obviously has:

$$\omega' = [\omega\,\omega_1] + [\varpi_1\,\omega_2] + \ldots + [\varpi_p\,\omega_{p+1}] + [\omega_{p+2}\,\omega_{p+3}] + \ldots$$

The form ω remains a form of the second type, since the number n is diminished by p units.

In case b), one has:

$$n\,[\omega'^{n-1}\,\omega\,\varpi_1] = n\,[\omega'^{n-1}\,\varpi_1\,\varpi_2] = \ldots = n\,[\omega'^{n-1}\,\varpi_{2q-1}\varpi_{2q}] = [\omega'^{n}],$$

and ω' is reducible to the form:

$$\omega' = [\varpi_1\,\varpi_2] + \ldots + [\varpi_{2q-1}\varpi_{2q}] + [(\omega + \varpi_2)\,\omega_1] \\ + [\varpi_{2q+1}\,\omega_2] + \ldots + [\varpi_p\,\omega_{p-2q+1}] + [\omega_{p-2q+2}\,\omega_{p-2q+3}] + \ldots$$

If one equates the y_i to arbitrary constants then the form ω will remain a form of the second type since the rank of ω' will be diminished by $2p - 2\,q$ units.

In case c), ω' is reducible to the form:

$$\omega' = [\varpi_1\,\varpi_2] + \ldots + [\varpi_{2q-1}\,\varpi_{2q}] + [\omega\,\varpi_{2q+1}] + [\varpi_{2q+2}\,\omega_1] + \ldots + [\varpi_p\,\omega_{p-2q-1}] + [\omega_{p-2q}\,\omega_{p-2q+1}] + \ldots$$

If one equates the y_i to arbitrary constants then the form ω will become a form of the first type since the rank of ω' will be diminished by $2p - 2q$ units.

In summation, if the product $[\varphi^{\Phi}]$ is zero then the form ω' will remain a form of the second type, and in the contrary case it will become a form of the first type.

135. In summation, we have obtained four essentially distinct reduced problems by abstracting from two cases of *a*), one of which was treated in the preceding chapter, and the other of which corresponds to knowing p first integrals in involution for the characteristic system of the equation $\omega = 0$.

One may examine the four reduced problems a little more carefully and ask whether all of the possible results of the known first integrals have been obtained. The method of answering this question is the same as the one that was used in the preceding chapter. It is based upon the reduction of ω to a canonical form that involves p conveniently chosen functions of $y_1, \ldots, y_p$, and other independent first integrals. Once this canonical form is obtained, one deduces from it the equations of the largest group of transformations that preserve the given data when they act on the integral curves

We shall rapidly indicate the canonical forms of ω and ω' in each of the four cases, with the calculations that bring us to that form being made in the same manner as in the preceding chapter (sec. **125**).

1. *The form ω is of the first type, and the forms φ and Φ are reducible to:*

$$\varphi = \xi_1', \quad \Phi = [\xi_1'\,\xi_2'] + \ldots + [\xi_{2q-1}'\,\xi_{2q}'].$$

In this case, one has:

$$\omega' = [(\varpi_1 - \omega)\,\varpi_2] + [\varpi_3\varpi_4] + \ldots + [\varpi_{2q-1}\varpi_{2q}] + [\varpi_{2q+1}\,\omega_1] + \ldots + [\varpi_p\,\omega_{p-2q}] + [\omega_{p-2q+1}\,\omega_{p-2q+2}] + \ldots$$

If one sets:

$$\Pi = [\varpi_1\, \varpi_2] + \ldots + [\varpi_{2q-1}\, \varpi_{2q}]$$

then the exterior derivative of ω' will give, if one neglects terms in:

$$\varpi_{2q+1}\,,\ \ldots,\ \varpi_p\,,\ \omega_{p-2q+1}\,,\ \ldots,\ \omega_{2n-p}\,,$$

the identity:

$$\Pi' - [\Pi\, \varpi_2] + [\omega\, \varpi_2'] + [\varpi_{2q+1}'\ \ \omega_1] + \ldots + [\varpi_p'\ \ \omega_{p-2q}] = 0.$$

One may then set:

$$\varpi_2 = \frac{d\bar{y}_2}{\bar{y}_2}, \quad \varpi_{2q+1} = d\bar{y}_{2q+1}, \cdots, \quad \varpi_p = d\bar{y}_p .$$

Since the exterior derivative of the form $\dfrac{1}{y_2}\Pi$ is zero, one may then set:

$$\Pi = \bar{y}_2([d\bar{y}_1\, d\bar{y}_2] + \cdots + [d\bar{y}_{2q-1}\, d\bar{y}_{2q}]) .$$

Finally, one has:

$$\omega' = \left[\frac{d\bar{y}_2}{\bar{y}_2}\,\omega\right] + \bar{y}_2[d\bar{y}_1\ d\bar{y}_2] + \cdots + \bar{y}_2[d\bar{y}_{2q-1}\, d\bar{y}_{2q}]\ \ + [d\bar{y}_{2q+1}\, \varpi_1] + \cdots + [d\bar{y}_p\, \varpi_{p-2q}] + \cdots$$

This result may be put into a more intuitive form by setting:

$$\bar{\varpi} = \frac{1}{\bar{y}_2}\,\omega - \bar{y}_1\, d\bar{y}_2 - \cdots - \bar{y}_{2q-1}\, d\bar{y}_{2q} .$$

Indeed, one then gets:

$$\bar{\varpi}' = [d\bar{y}_{2q+1}\bar{\varpi}_1] + \cdots + [d\bar{y}_p \bar{\varpi}_{p-2q}] + [\bar{\varpi}_{p-2q+1}\bar{\varpi}_{p-2q+2}] + \cdots + [\bar{\varpi}_{2n-p-1}\bar{\varpi}_{2n-p}] .$$

In this form, one sees quite clearly that all of the possible results have been deduced from the known integrals.

In addition, one has the canonical relations:

$$\{\bar{y}_1\} = 1, \qquad \{\bar{y}_i\} = 0, \qquad (i = 2, \cdots, p);$$
$$(\bar{y}_1\, \bar{y}_2) = (\bar{y}_3\, \bar{y}_4) = \cdots = (\bar{y}_{2q-1}\, \bar{y}_{2q}) = \bar{y}_2,$$

and all of the other brackets are zero.

2. *The form* ω *is of the first type, and the forms* φ *and* Φ *are reducible to*:

$$\varphi = \xi_{2q+1}', \qquad \Phi = [\xi_1'\, \xi_2'] + \ldots + [\xi_{2q-1}'\, \xi_{2q}'].$$

In this case, one has:

$$\omega' = [\varpi_1\, \varpi_2] + \ldots + [\varpi_{2q-1}\, \varpi_{2q}] + [(\varpi_{2q+1} - \omega)\, \omega_1] + \ldots + [\varpi_p\, \varpi_{p-2q}] + [\omega_{p-2q+1}\, \omega_{p-2q+2}] + \ldots$$

Exterior derivation of the right-hand side easily shows that one may set:

$$\begin{aligned}
&\varpi_{2q+2} = d\bar{y}_{2q+2}, \cdots, \varpi_p = d\bar{y}_p, \\
&\varpi_{2q+1} = d\bar{y}_{2q+1} + \bar{y}_1\, d\bar{y}_2 + \cdots + \bar{y}_{2q-1}\, d\bar{y}_{2q}, \\
&\Pi = [d\bar{y}_1\, d\bar{y}_3] + [d\bar{y}_3\, d\bar{y}_4] + \cdots + [d\bar{y}_{2q-1}\, d\bar{y}_{2q}].
\end{aligned}$$

Upon setting:

$$\bar{\omega} = -\omega + dy_{2q+1} + \bar{y}_1\, d\bar{y}_2 + \cdots + \bar{y}_{2q-1}\, d\bar{y}_{2q},$$

one obtains:

$$\bar{\omega} = [\bar{\omega}\,\bar{\omega}_1] + [d\bar{y}_{2q+2}\, \bar{\omega}_2] + \cdots + [d\bar{y}_p\, \bar{\omega}_{p-2q}] + [\bar{\omega}_{p-2q+1}\, \bar{\omega}_{p-2q+2}] + \cdots,$$

a formula that makes it obvious that all of the possible results have been derived from the given integrals.

In addition, one has obtained the canonical relations:

$$\begin{aligned}
&\{\bar{y}_{2q+1}\} = 1, \\
&(\bar{y}_1\, \bar{y}_2) = \cdots = (\bar{y}_{2q-1}\, \bar{y}_{2q}) = 1, \\
&(\bar{y}_1\, \bar{y}_{2q+1}) = -\bar{y}_1,\ (\bar{y}_3\, \bar{y}_{2q+1}) = -\bar{y}_3, \ldots,
\end{aligned}$$

in which all of the other quantities $\{\bar{y}_i\}$, $(\bar{y}_i\, \bar{y}_j)$ are zero.

3. *The form* ω *is of the second type, and the forms* φ and Φ *are reducible to*:

$$\varphi = \xi_1', \quad \Phi = [\xi_1'\, \xi_2'] + \ldots + [\xi_{2q-1}'\, \xi_{2q}'].$$

In this case, one has:

$$\omega' = [\varpi_1\, \varpi_2] + \ldots + [\varpi_{2q-1}\, \varpi_{2q}] + [(\omega + \varpi_2)\, \omega_1] + [\omega_{2q+1}\, \omega_2] + \ldots + [\varpi_p \omega_{p-2q+1}] + \ldots$$

Again, set:

$$\Pi = [\varpi_1 \varpi_2] + \ldots + [\varpi_{2q-1} \varpi_{2q}],$$

and exterior differentiate ω', while neglecting the terms in:

$$\omega + \varpi_2, \quad \varpi_{2q+1}, \ldots, \varpi_p, \quad \omega_{p-2q+2}, \ldots$$

We get:

$$\Pi' = [\Pi\ \omega_1] + [\varpi_2'\, \omega_1] + [\varpi_{2q+1}'\, \omega_2] + \ldots + [\omega_p'\, \omega_{p-2q+1}] = 0.$$

This identity permits us to set:

$$\varpi_{2q+1} = d\bar{y}_{2q+1}, \cdots, \varpi_p = d\bar{y}_p.$$

One then sees that if we regard $\bar{y}_{2q+1}, \cdots, \bar{y}_p$ as constants then ϖ_2' will be equal to $-\Pi$, which is of rank $2q$, since the equation $\varpi_2 = 0$ is a part of the associated system to ϖ_2'. One may then suppose that:

$$\varpi_2 = -(\bar{y}_1\, d\bar{y}_2 + \cdots + \bar{y}_{2q-1}\, d\bar{y}_{2q}).$$

Finally, if one sets:

$$\bar{\varpi} = \omega + \varpi_2 = \omega - \bar{y}_1\, d\bar{y}_2 - \cdots - \bar{y}_{2q-1}\, d\bar{y}_{2q}$$

then one will get:

$$\bar{\varpi}' = [\bar{\varpi}\ \bar{\omega}_1] + [d\bar{y}_{2q+1}\ \bar{\omega}_2] + \cdots + [d\bar{y}_p\ \bar{\varpi}_{p-2q+1}] + \cdots$$

One sees that all of the possible results of the known integrals have been deduced, and one arrives, in addition, at the canonical relations:

$$\{\bar{y}_1\} = -\bar{y}_1, \quad \{\bar{y}_3\} = -\bar{y}_3, \ldots, \{\bar{y}_{2q-1}\} = -\bar{y}_{2q-1},$$
$$(\bar{y}_1\ \bar{y}_2) = (\bar{y}_3\ \bar{y}_4) = \cdots = (\bar{y}_{2q-1}\ \bar{y}_{2q}) = 1,$$

with all of the other quantities $\{\bar{y}_i\}$, $(\bar{y}_i\ \bar{y}_j)$ being zero.

4. *The form ω is of the second type, and the forms φ and Φ are reducible to:*

$$\varphi = \xi'_{2q+1}, \quad \Phi = [\xi'_1 \xi'_2] + \ldots + [\xi'_{2q-1}\ \xi'_{2q}].$$

In this case, one has:

$$\omega' = [\varpi_1\ \varpi_2] + \ldots + [\varpi_{2q-1}\ \varpi_{2q}] + [\omega\ \varpi_{2q+1}] + [\varpi_{2q+2}\ \omega_1] + \ldots + [\varpi_p\ \omega_{p-2q-1}] + \ldots$$

If the notation retains its same significance that it had in part II then one will have, by neglecting terms in:

$$\varpi_{2q+3}, \ldots, \varpi_p, \omega_{p-2q}, \ldots, \omega_{2n-1-p},$$

the identity:

$$\Pi' + [\Pi\ \varpi_{2q+1}] - [\omega\ \varpi'_{2q+1}] + [\varpi'_{2q+2}\ \omega_1] + \ldots + [\varpi'_p\ \omega_{p-2q-1}] = 0.$$

The exterior derivatives $\varpi'_{2q+1}, \varpi'_{2q+2}, \ldots, \varpi'_p$ are zero, along with $\varpi_{2q+1}, \ldots, \varpi_p$. One may therefore set:

$$\varpi_{2q+1} = \frac{d\bar{y}_{2q+1}}{\bar{y}_{2q+1}}, \varpi_{2q+2} = d\bar{y}_{2q+2}, \cdots, \varpi_p = d\bar{y}_p.$$

The exterior derivative of the form $\bar{y}_{2p+1}\Pi$ will then be zero when one regards $\bar{y}_{2q+2}, \ldots, \bar{y}_p$ collectively as constants. One may therefore set:

$$\bar{y}_{2q+1}\Pi = [d\bar{y}_1\ d\bar{y}_2] + \cdots + [d\bar{y}_{2q-1}\ d\bar{y}_{2q}].$$

Finally, upon setting:

$$\bar{\omega} = \bar{y}_{2q+1}\,\omega - \bar{y}_1\,d\bar{y}_2 - \cdots - \bar{y}_{2q-1}\,d\bar{y}_{2q}\,,$$

one gets:

$$\bar{\omega}' = [d\bar{y}_{2q+2}\,\bar{\omega}_1] + \cdots + [d\bar{y}_p\,\bar{\omega}_{p-2q-1}] + \cdots$$

One sees clearly that all of the possible results have been deduced from the known integrals. In addition, one obtains the canonical relations:

$$\{\bar{y}_{2q+1}\} = \bar{y}_{2q+1}\,,$$
$$(\bar{y}_1\,\bar{y}_2) = (\bar{y}_3\,\bar{y}_4) = \cdots = (\bar{y}_{2q-1}\,\bar{y}_{2q}) = \bar{y}_{2q+1}\,,$$

with all of the other quantities $\{y_i\}$, $(y_i\,y_j)$ being zero.

CHAPTER XIV

DIFFERENTIAL EQUATIONS THAT ADMIT AN INVARIANT PFAFF EQUATION

I. - *General method of integration.*

136. We have already (sec. **104**) encountered the characteristic system of a Pfaff equation:

(7) $$\omega \equiv a_1 dx_1 + a_2 dx_2 + \ldots + a_r dx_r = 0.$$

It is composed of the equations:

(8) $$\omega = 0, \qquad \frac{\dfrac{\partial \omega'}{\partial (dx_1)}}{a_1} = \frac{\dfrac{\partial \omega'}{\partial (dx_2)}}{a_2} = \cdots = \frac{\dfrac{\partial \omega'}{\partial (dx_r)}}{a_r},$$

of which the last $r - 1$ provide the associated system of the quadratic exterior form ω' when one supposes that the variables are coupled by the relation $\omega = 0$.

This characteristic system has likewise been encountered in the preceding chapter (sec. **128**) in the context of a Pfaff expression ω of the second type.

The number of independent equations of the characteristic system (2) is always odd. Indeed, on account of the relation $\omega = 0$, one may put ω' into the form:

$$\omega' = [\omega_1\ \omega_2] + \ldots + [\omega_{2n-1}\ \omega_{2n}] \qquad (\text{mod } \omega),$$

if we denote the linear differential forms, which are mutually independent and independent of ω, by $\omega_1, \omega_2, \ldots, \omega_{2n}$. The characteristic system of equation (1) is then defined by the equations:

$$\omega = \omega_1 = \omega_2 = \ldots = \omega_{2n} = 0.$$

The integer n is, as one knows, the greatest integer such that the form $[\omega\ \omega'^n]$ is non-zero. The class of the equation $\omega = 0$ is equal to the degree of that form.

137. It is easy to recover a canonical form for equation (1). Indeed, let y_1 be an arbitrary first integral of the characteristic system (2). If one equates y_1 to an arbitrary constant C_1 and dy_1 to zero then the rank of the characteristic system of the new equation (1) will be reduced by at least one unit, and, *since its rank is odd,* it will be reduced by at least two units. Let y_2 be a first integral of the new characteristic system. If one sets:

$$y_1 = C_1, \quad y_2 = C_2, \quad dy_1 = 0, \quad dy_2 = 0$$

then the rank of the characteristic system of the given equation will be reduced by at least four units, and so on. Finally, after at least $n + 1$ equations the equation $\omega = 0$ will be verified identically. In other words, that equation will be of the form:

$$z_1\,dy_1 + z_2\,dy_2 + \ldots + z_q\,dy_q + dy_{q+1} = 0 \qquad (q \le n).$$

The integer q is, moreover, equal to n; otherwise equation (1) could be written by means of less than $2n + 1$ variables.

Hence, *if the characteristic system of equation* (1) *is of rank* $2n + 1$ *then this equation will be reducible to the form*:

$$dy_{n+1} + z_1\,dy_1 + z_2\,dy_2 + \ldots + z_n\,dy_n = 0,$$

and the quantities:

$$y_1, \ldots, y_{n+1}; \qquad z_1, \ldots, z_n$$

will constitute a system of independent first integrals of the characteristic equations.

One sees that by this method *the reduction of system* (1) *to its canonical form – and, as a result, the integration of its characteristic system – requires* n+1 *successive operations of order:*

$$2n + 1, \quad 2n - 1, \ldots, 3, 1$$

and differentiations.

138. One may remark, as in chapter XII (sec. **120**), that knowing $N \ge n+1$ first integrals:

$$y_1, y_2, \ldots, y_N$$

such that equation (1) is verified identically by equating these integrals to arbitrary constants will permit us to accomplish the integration of the characteristic equation by differentiation. Indeed, equation (1) may – in one and only one manner – be put into the form:

$$dy_N + z_1\,dy_1 - z_2\,dy_2 + \ldots + z_{N-1}\,dy_{N-1} = 0,$$

and one proves that the coefficients $z_1, \ldots, z_{N-1}$ are again first integrals of the characteristic equations.

More generally, one may propose to see to what the integration of the characteristic system reduces when one knows a certain number r of independent first integrals of this system.

139. *First integrals in involution.* We say that two first integrals f and g of the characteristic system of equation (1) are *in involution* if one has:

$$[\omega\,\omega'^{\,n-1}\,df\,dg] = 0. \tag{9}$$

This definition is obviously independent of the choice of variables, and is also independent of the arbitrary factor by which one may multiply the left-hand side of equation (1).

The property of two first integrals being in involution implies the important consequence that the *rank of the characteristic system will be reduced by four units when one supposes that the variables are coupled by two relations:*

$$f = C, \quad g = C',$$

in which C *and* C' *are two arbitrary constants.* Indeed, if one supposes that $df = dg = 0$ as well as $\omega = 0$, then condition (3) will express the idea that the rank of ω' is less than $2n - 2$, and as a result, is equal to $2n - 4$.

II. - *Use of known integrals.*

140. *Case where one knows p first integrals* $y_1, \ldots, y_p$ *that are pair-wise independent and in involution.* - In this case, from the developments of chapter VI (sec. **67**), it results that when one supposes the differentials are coupled by the relations:

$$\omega = 0, \quad dy_1 = 0, \ldots, dy_p = 0,$$

the rank of ω' will be reduced by $2n - 2p$. If one supposes that the variables are coupled by the relations:

$$y_1 = C_1, \quad y_2 = C_2 = , \ldots, y_p = C_p,$$

then the characteristic system of equation (1) will be of rank $2n - 2p + 1$, and its integration will require operations of order:

$$2n - 2p + 1, \quad 2n - 2p - 1, \ldots, 3, 1,$$

along with its differentiations.

The case that we shall now examine is the one in which the rank of the characteristic system is instantly reduced by the maximum number of units $2p$.

141. *Case where the given first integrals are not all in pair-wise involution.* – In this case, when one equates the given first integrals to arbitrary constants, the reduction of rank of the characteristic system will not attain its upper limit $2p$. On the other hand, *one may determine an absolute linear integral invariant for the characteristic equations,* which, in certain cases, may produce a reduction of the integration problem that is at least as large as in the first case, which is apparently the most favorable case.

Indeed, suppose that y_1 and y_2 are two first integrals of the characteristic equations that are not in involution. One will have:

$$n\,[\omega\omega'^{n-1} dy_1\, dy_2] = A[\omega\, \omega'^{n}],$$

in which the coefficient A is non-zero. There exist an infinitude of (unknown) functions m such that:

$$\varpi = m\,\omega$$

is an invariant form; i.e., it may be expressed by means of first integrals of the characteristic equations and their differentials. For such a form, one has:

$$\varpi' = m\,\omega' + [dm\,\omega],$$

and, as a result:

$$[\varpi\,\varpi'^{n-1}\,dy_1\,dy_2] = m^n\,[\omega\,\omega'^{n-1}\,dy_1\,dy_2],$$
$$[\varpi\,\varpi'^{n}\,] = m^{n+1}[\omega\,\omega'^{n}\,].$$

By comparison, one thus has:

$$n[\varpi\,\varpi'^{n-1}dy_1\,dy_2] = \frac{A}{m}[\varpi\,\varpi'^{n}\,]\,.$$

The two forms between brackets are obviously invariant. As a result, A / m is a first integral. Hence

$$\frac{A}{m}\varpi = A\omega$$

is an invariant form. This is the conclusion that we hoped to reach:

If one knows two first integrals y_1, y_2 *such that the function* A*, which is defined by the equality*:

$$n\,[\omega\,\omega'^{n-1}\,dy_1\,dy_2] = A\,[\omega\,\omega'^{n}\,],$$

is not zero then the linear form $A\omega$ *will be an absolute invariant form.*

In addition, we remark that the minimum number of variables by means of which the form $A\omega$ can be expressed are obviously the $2n + 1$ first integrals of the given characteristic equations. *The characteristic system of the form* $A\omega$ *thus agrees with that of equation* (1) and, as a result, it is of odd rank. *The form* $A\omega$ *is then of the first type.*

142. One may attach the preceding theorem to a method of integration that is susceptible to a vast generalization and consists of integrating the characteristics of the *form* $u\,\omega$, where u is an *auxiliary* variable. Indeed, it is obvious that to any solution of these equations there corresponds a solution of the characteristic equations of the *equation* $\omega = 0$, namely, the one that is obtained by eliminating the auxiliary variable u from the relations that define the solution.

The form $u\,\omega$ is obviously of the second type, and the general method of integration of its characteristic equations that was discussed in sec. **128** is identical to the one that was recalled in sec. **137** for the characteristic equations of the equation $\omega = 0$. However, the advantage becomes obvious if one knows, *a priori,* first integrals of the characteristic equations, because one may apply the method that was discussed in sec. **134** to the integration of the characteristic equations of the form $u\omega$. In particular, if one knows two first integrals y_1 and y_2 of the characteristic equations of the equation $\omega = 0$ then one will have:

$$\{y_1\} = 0, \qquad \{y_2\} = 0,$$

and if one then calculates the bracket $(y_1\, y_2)$ that was defined by (sec. **132**):

$$(n + 1)[\,(u\,\omega)'^n\, dy_1\, dy_2] = (y_1\, y_2)\ (u\,\omega)'^{n+1}$$

then this will give, upon developing and equating the terms which contain du:

$$(y_1\, y_2) = \frac{A}{u},$$

in which A is the quantity that was defined in the preceding section. One may continue to apply the general method while preserving the variable u by forming the quantity $\left\{\frac{A}{u}\right\}$, which is the bracket of that quantity with y_1 and y_2, and so on. One may thus remark that the form $A\,\omega$ is itself invariant since A / u is a first integral of the characteristic equations of the form $u\omega$.

III. - *Application to first-order partial differential equations.*

143. The problem of integrating the characteristic equations of a Pfaff equation finds an immediate application in the theory of first-order partial differential equations. Indeed, to integrate an equation:

$$F\left(z, x_1, x_2, \cdots, x_n; \cdots, \frac{\partial z}{\partial x_1}, \frac{\partial z}{\partial x_2}, \cdots, \frac{\partial z}{\partial x_n}\right) = 0,$$

or, if we employ the classical notation:

$$F(z, x_1, x_2, \ldots, x_n\,; p_1, p_2, \ldots, p_n) = 0, \tag{10}$$

is to determine $(n + 1)$ functions $z, p_1, p_2, \ldots, p_n$ of $x_1, x_2, \ldots, x_n$ that satisfy equation (4) and the Pfaff equation:

$$\omega \equiv dz - p_1\, dx_1 - p_2\, dx_2 - \ldots - p_n\, dx_n = 0. \tag{11}$$

Now, if one imagines that one of the $2n + 1$ arguments z, x_i, p_i has been expressed in equation (4) as a function of the $2n$ others then the Pfaff equation (5) will contain only $2n$ variables, and its characteristic system will necessarily be of *odd* rank $2n - 1$. As a result, on account of equation (4) the Pfaff equation (5) will be reducible to the canonical form:

(12) $$dZ - P_1\, dX_1 - \dots - P_{n-1}\, dX_{n-1} = 0,$$

in which $Z, X_1, \dots, X_{n-1}, P_1, \dots, P_{n-1}$, are $2n - 1$ independent functions. They are first integrals of the characteristic equations of equation (5).

Having said this, suppose that one wants to bring equation (5) into the canonical form (6). As the integration of equation (4) amounts, in essence, to the determination of a number, which is equal to $n + 1$ (with the given relation (4)), of independent relations between $z, x_1, \dots, x_n, p_1, \dots, p_n$ such that equation (5) results, it will suffice to establish a number, which is equal to n and implies equation (6) as well, of independent relations between $Z, X_1, \dots, P_{n-1}$, in order to arrive at this result. But this is possible in a general manner by taking:

$$Z = f(X_1, \dots, X_{n-1}),$$

$$P_1 = \frac{\partial f}{\partial X_1}, \cdots, P_{n-1} = \frac{\partial f}{\partial X_{n-1}},$$

in which f denotes an arbitrary function of its arguments. More generally, one will establish an arbitrary number $h \leq n$ of independent relations:

$$\Phi_1(Z, X_1, \dots, X_{n-1}) = 0, \quad \Phi_2 = 0, \dots, \Phi_h = 0,$$

between $Z, X_1, \dots, X_{n-1}$, and combine them with the relations that are obtained by eliminating the homogenous parameters $\lambda_1, \dots, \lambda_h$ from the equations:

$$\lambda_1\left(\frac{\partial \Phi_1}{\partial X_1} + P_1 \frac{\partial \Phi_1}{\partial Z}\right) + \lambda_2\left(\frac{\partial \Phi_2}{\partial X_1} + P_1 \frac{\partial \Phi_2}{\partial Z}\right) + \cdots + \lambda_h\left(\frac{\partial \Phi_h}{\partial X_1} + P_1 \frac{\partial \Phi_h}{\partial Z}\right) = 0,$$
$$\dots$$
$$\lambda_1\left(\frac{\partial \Phi_1}{\partial X_{n-1}} + P_{n-1} \frac{\partial \Phi_1}{\partial Z}\right) + \lambda_2\left(\frac{\partial \Phi_2}{\partial X_{n-2}} + P_{n-1} \frac{\partial \Phi_2}{\partial Z}\right) + \cdots + \lambda_h\left(\frac{\partial \Phi_h}{\partial X_{n-1}} + P_{n-1} \frac{\partial \Phi_h}{\partial Z}\right) = 0.$$

144. The equations:

$$X_1 = a_1, \dots, X_{n-1} = a_{n-1},$$
$$P_1 = b_1, \dots, P_{n-1} = b_{n-1},$$
$$Z = c,$$

define one-dimensional multiplicities that are characteristic multiplicities of the Pfaff equation (5) (when one supposes that the variables are coupled by relation (4)). *These are what one calls the characteristics of the partial differential equation* (4). One sees immediately that any surface integral is generated by characteristics.

It is easy to form the differential equations of the characteristics. Indeed, they are the equations of the associated system of ω' if one supposes that the differentials of the variables are coupled by the relation $\omega = 0$ and *also by the relation* $dF = 0$. One thus obtains the equations (sec. **104**) by adding the equations:

$$\left\| \begin{matrix} \dfrac{\partial \omega'}{\partial(dz)} & \dfrac{\partial \omega'}{\partial(dx_1)} & \dfrac{\partial \omega'}{\partial(dx_2)} & \cdots & \dfrac{\partial \omega'}{\partial(dp_1)} & \cdots & \dfrac{\partial \omega'}{\partial(dp_n)} \\ 1 & -p_1 & -p_2 & \cdots & 0 & \cdots & 0 \\ \dfrac{\partial F}{\partial z} & \dfrac{\partial F}{\partial x_1} & \dfrac{\partial F}{\partial x_2} & \cdots & \dfrac{\partial F}{\partial p_1} & \cdots & \dfrac{\partial F}{\partial p_n} \end{matrix} \right\| = 0,$$

which may be written:

$$\frac{dx_1}{\dfrac{\partial F}{\partial p_1}} = \cdots = \frac{dx_n}{\dfrac{\partial F}{\partial p_n}} = \frac{-dp_1}{\dfrac{\partial F}{\partial x_1} + p_1 \dfrac{\partial F}{\partial z}} = \cdots = \frac{-dp_n}{\dfrac{\partial F}{\partial x_n} + p_n \dfrac{\partial F}{\partial z}}, \tag{13}$$

to equation (5). One then recovers the classical equations.

IV. - *Cauchy's method.*

145. The method that we just presented amounts, in essence, to the integration of the characteristic equations and the reduction of equation (5) to its canonical form (6). Moreover, this reduction will result in the integration if it is itself directed in a convenient manner (sec. **137**). Whatever procedure is used to integrate the characteristic equations, it is easy to see that the reduction of equation (5) to its normal form is always possible once the integration of the characteristic equations has been performed. It suffices to determine the first integrals that, for a given numerical value x_n^0 of x_n reduce them to:

$$z, x_1, ..., x_{n-1}, p_1, ..., p_n,$$

respectively.

If one denotes these first integrals by:

$$Z, X_1, ..., X_{n-1}, P_1, ..., P_n,$$

which are necessarily coupled by the relation:

$$F(Z, X_1, ..., X_{n-1}, x_n^0; P_1, ..., P_n) = 0,$$

then relation (5), which may, as one knows, be expressed by means of first integrals, will obviously reduce to:

$$dZ - P_1\,dX_1 - ... - P_{n-1}\,dX_{n-1} = 0.$$

This is the principle of *Cauchy's method.*

V. - *Lagrange's method.*

146. Lagrange's method of the *complete integral* also adheres easily to the preceding viewpoint. The equation:

$$V(z, x_1, ..., x_n ; a_1, ..., a_n) = 0 \tag{14}$$

will define a complete integral if it defines a function of z that satisfies equation (4) for any n arbitrary *constants* $a_1, ..., a_n$. Equation (4) is, moreover, the only one that satisfies all of the functions of z that are defined by (8), because the elimination of the $a_1, ..., a_n$, between equation (8) and the equations:

$$\left\{\begin{array}{l} \dfrac{\partial V}{\partial x_1} + p_1 \dfrac{\partial V}{\partial z} = 0, \\ \qquad \cdots \\ \dfrac{\partial V}{\partial x_n} + p_n \dfrac{\partial V}{\partial z} = 0, \end{array}\right. \tag{15}$$

which are derived from them, leads in general (and this is what we have supposed) to only one relation, which is naturally equation (4).

Since equation (4) is the result of the elimination of the $a_1, ..., a_n$, from the $(n + 1)$ equations (8) and (9) the integration of equation (4) will amount to satisfying the Pfaff equation (5) while supposing that the $3n + 1$ variables z, x_i, p_i, a_i are coupled by the $(n + 1)$ relations (8) and (9). Now, on account of these relations, one has:

$$\begin{aligned} 0 &= \frac{\partial V}{\partial z} dz + \frac{\partial V}{\partial x_1} dx_1 + \cdots + \frac{\partial V}{\partial a_1} da_1 + \cdots \\ &= \frac{\partial V}{\partial z}(dz - p_1 dx_1 - \cdots - p_n dx_n) + \frac{\partial V}{\partial a_1} da_1 + \cdots + \frac{\partial V}{\partial a_n} da_n. \end{aligned}$$

The Pfaff equation (5) is itself reduced to its normal form by setting:

$$X_1 = a_1, ..., X_{n-1} = a_{n-1}, Z = a_n,$$

$$P_1 = -\frac{\dfrac{\partial V}{\partial a_1}}{\dfrac{\partial V}{\partial a_n}}, \cdots, P_{n-1} = -\frac{\dfrac{\partial V}{\partial a_{n-1}}}{\dfrac{\partial V}{\partial a_n}}.$$

One sees that the characteristics are defined by the equations:

$$V = 0,$$

$$\frac{\partial V}{\partial a_1} + b_1 \frac{\partial V}{\partial a_n} = 0, \cdots, \frac{\partial V}{\partial a_{n-1}} + b_{n-1} \frac{\partial V}{\partial a_n} = 0 .$$

This is a classical result.

147. We now apply the theorem of sec. **141** to the particular case of an equation in two independent variables:

(16) $$F(x, y, z, p, q) = 0.$$

The knowledge of two independent first integrals u and v of the characteristic equations leads, when they are not in involution, to the determination of a linear integral invariant for the characteristic equations. This invariant is $A\omega$, where A is defined the equality:

$$[\omega\, du\, dv] = A[\omega\, \omega'],$$

or rather, since we have supposed that the variables are coupled by relation (10) here:

$$[dF\, \omega\, du\, dv] = A[dF\; \omega\, \omega'].$$

If we take the terms in $[dx\, dz\, dp\, dq]$ in both sides of this equation in particular then we will find that:

$$A = \frac{1}{\dfrac{\partial F}{\partial q}} \begin{vmatrix} \dfrac{\partial F}{\partial x} + p\dfrac{\partial F}{\partial z} & \dfrac{\partial F}{\partial p} & \dfrac{\partial F}{\partial q} \\ \dfrac{\partial u}{\partial x} + p\dfrac{\partial u}{\partial z} & \dfrac{\partial u}{\partial p} & \dfrac{\partial u}{\partial q} \\ \dfrac{\partial v}{\partial x} + p\dfrac{\partial v}{\partial z} & \dfrac{\partial v}{\partial p} & \dfrac{\partial v}{\partial q} \end{vmatrix}.$$

Hence, if the determinant of the right-hand side is non-zero then the expression $A(dz - p\, dx - q\, dy)$ *will be an invariant form for the characteristic equations.*

VI. – *First-order partial differential equations that admit an infinitesimal transformation.*

148. If the first-order partial differential equation:

$$F(z, x_1, ..., x_n, p_1, ..., p_n) = 0$$

admits an infinitesimal transformation Af that acts on the variables z, x_1, ..., p_n then this would signify that any system of $n + 1$ relations between these $2n + 1$ variables that defines an integral multiplicity is changed by the transformation into another system of $n + 1$ relations that also defines an integral multiplicity. As a result, on account of equation (4), the Pfaff equation:

$$\omega \equiv dz - p_1\, dx_1 - \ldots - p_n\, dx_n = 0$$

will admits the infinitesimal transformation Af. It then immediately results from this (sec. **97**) that the linear form:

$$\frac{\omega(\delta)}{\omega(A)} = \frac{\delta z - p_1 \delta x_1 - p_2 \delta x_2 - \cdots - p_n \delta x_n}{A(z) - p_1 A(x_1) - p_2 A(x_2) - \cdots - p_n A(x_n)},$$

will be invariant for the system of differential equations of the characteristics.

The knowledge of one infinitesimal transformation thus implies the knowledge of a linear integral invariant for the characteristic equations and, as a result, the integration of the given equation, which, since it is a problem of the second type that requires operations of order:

$$2n + 1, \quad 2n - 1, \ldots, 3, 1,$$

is converted into a problem of the first type that requires operations of order:

$$2n, \quad 2n - 2, \ldots, 2, 0.$$

149. A classic example of this is the case in which the given equation (1) does not depend explicitly on z. It is obvious then that from any solution of the equation one can deduce another solution by adding an arbitrary constant to z. In other words, the given equation admits the infinitesimal transformation:

$$Af = \frac{\partial f}{\partial z}.$$

The absolute integral invariant that admits the characteristic equations is then:

$$\int \omega_\delta = \int \delta z - p_1\, \delta x_1 - \ldots - p_n\, \delta x_n.$$

The method of integration of these equations is, of its nature, a result of the theory of chapter XII. Here, the characteristic equations of ω' are:

$$\frac{dx_1}{\frac{\partial F}{\partial p_1}} = \cdots = \frac{dx_n}{\frac{\partial F}{\partial p_n}} = \frac{-dp_1}{\frac{\partial F}{\partial x_1}} = \cdots = \frac{-dp_n}{\frac{\partial F}{\partial x_n}} .$$

Once we have determined $n - 1$ first integrals that are pair-wise in involution the integration of the characteristic equations of ω will be converted into a quadrature, since the expression ω will become an exact differential when on equates the $n - 1$ first integrals to arbitrary constants.

V11. - *Jacobi's first method.*

150. Jacobi's first method for integrating first-order partial differential equations is related to the preceding considerations. Jacobi converted equation (4), which was supposed arbitrary, into an equation in which the unknown function no longer figured, namely:

$$F\left(z, x_1, \cdots, x_n, -\frac{\frac{\partial V}{\partial x_1}}{\frac{\partial V}{\partial z}}, \cdots, -\frac{\frac{\partial V}{\partial x_n}}{\frac{\partial V}{\partial z}}\right) = 0 .$$

To abbreviate, we set:

$$\frac{\partial V}{\partial z} = u ,$$

and the characteristic equations to be integrated are those of the absolute invariant form:

$$\delta V - u\,(\delta z - p_1\,\delta x_1 - \ldots - p_n\,\delta x_n),$$

in which the $2n + 3$ variables are related by the relation (4). They admit the *relative* integral invariant:

$$\int u(\delta z - p_1\,\delta x_1 - \ldots - p_n\,\delta x_n), \tag{17}$$

and they are the characteristic equations of the integral invariant that one integrates with the methods of chapter XII.

Jacobi's method approaches the one that we indicated in sec. **142**, with the difference that the latter method used the integral (11) as *absolute* integral invariant, but Jacobi's method uses it as a *relative* integral invariant. Moreover, Jacobi's method leads to operations of order:

$$2n + 2,\ 2n,\ \ldots,\ 2,\ 0,$$

instead of:

$$2n + 1,\ 2n - 1,\ \ldots,\ 1.$$

Its advantage is that it permits us to use the knowledge of the given first integrals by applying the Poisson-Jacobi theorem. However, this advantage is preserved by the method of sec. **142**, which deduced all of the possible consequences of given first integrals.

VIII. - *Reducing certain differential equations to a first-order partial differential equation.*

151. One may now place oneself at a viewpoint that is the opposite of the one that was taken in the preceding sections.

First, consider a Pfaff equation with an even number $2s$ of variables, but suppose that only $s + 1$ of these coefficients are non-zero:

$$\omega \equiv a_1\, dx_1 + a_2\, dx_2 + \ldots + a_{s+1}\, dx_{s+1} = 0.$$

The characteristic equations of this Pfaff equation are obviously the same as those of the first-order partial differential equation in s independent variables $x_1, x_2, \ldots, x_s$ that is obtained by setting:

$$x_{s+1} = a, \quad a_1 + p_1\, a_{s+1} = 0, \ldots, a_{s+1}\, p_s\, a_{s+1} = 0,$$

and eliminating $x_{s+1}, x_{s+2}, \ldots, x_{2s}$, between these $s + 1$ equations. Of course, it is necessary to suppose that the elimination is possible and that it gives only one relation.

152. Now consider a system of differential equations that admit a relative linear integral invariant $\int \omega$, such that the form ω is in $2s + 1$ variables and $[\omega'^s]$ is non-zero. The differential equations considered are the characteristic equations of ω'. Their integration can be converted into the integration of a first-order partial differential equation that does not contain the unknown function explicitly if the s coefficients of the differentials are zero in ω:

$$\omega \equiv a_1\, dx_1 + a_2\, dx_2 + \ldots + a_{s+1}\, dx_{s+1}\,.$$

Indeed, consider the Pfaff equation:

$$dV - \omega \equiv dV - a_1\, dx_1 - a_2\, dx_2 - \ldots - a_{s+1}\, dx_{s+1} = 0,$$

and set:

$$p_1 = a_1, \quad p_2 = a_2, \ldots, p_{s+1} = a_{s+1}\,.$$

The elimination of the $x_{s+2}, \ldots, x_{2s+1}$ between these $s + 1$ equations leads to one relation:

$$F(x_1, \ldots, x_{s+1}\,;\, p_1, \ldots, p_{s+1}) = 0, \tag{18}$$

which is nothing but the partial differential equation that was alluded to above. The differential equations of the characteristics of that equation are formed from the characteristic equations of ω', to which one adds the equation:

$$\mathrm{d}V - \omega = 0.$$

One easily accounts for the fact that the method of integration that was indicated in chapter XII of the characteristic equations of ω' leads to the same operations as the search for the characteristics of the partial differential equation (12).

If the invariant $\int \omega$ is that of the equations of dynamics:

$$\omega = p_1\,\delta q_1 + \ldots + p_n\,\delta q_n - H\,\delta t$$

then equation (12) will be nothing but the Jacobi equation:

$$\frac{\partial V}{\partial t} + H\left(t, q_1, \cdots q_n;\quad \frac{\partial V}{\partial q_1}, \cdots, \frac{\partial V}{\partial q_n}\right) = 0\,.$$

153. Jacobi's method for integrating the equations of dynamics is therefore based fundamentally upon the equivalence of two integration problems, the problem of integrating the characteristic system of a relative linear integral invariant $\int \omega$, and the problem of integrating the characteristic equations of a first-order partial differential equation that admits an infinitesimal transformation (*for example*, one that does not explicitly contain the unknown function). The nature of the problem is determined in the two cases by the existence of an integral invariant $\int \omega$.

This method of reducing to one partial differential equation is useful only if the form ω has $2s + 1$ variables with s zero coefficients, but it is hard to believe that in the case where that peculiarity is not present the integration of the characteristic equations of ω' would be a problem that is more complicated than the search for the characteristics of a first-order differential equation that does not explicitly contain the unknown functions, or, what amounts to the same thing, the integration of a system of *canonical* differential equations. Basically, *the importance of the canonical equations stems uniquely from their property of admitting an integral invariant* $\int \omega$ *, and not on the simplicity of their form.* The existence of the integral invariant is the fundamental property from which everything else is derived.

IX. - *Remarks on the nature of the principal practical applications of Jacobi's method.*

154. In fact, most of the rich variety of applications of Jacobi's method in dynamics have their origin in the simplifications that the search for a complete integral of Jacobi's partial differential equation presents, an equation that is obtained as a sum of functions in each of which only one of the variables $q_1, \ldots, q_n$ other than t appears. However, these simplifications may be exhibited independently of any recourse to the theory of first-order partial differential equations and complete integrals.

Indeed, let ω be a linear differential form in $2s + 1$ variables that we denote by:

$$x_1, \ldots, x_{2s}, t.$$

Suppose that ω may be decomposed into a sum of p terms:

$$\omega = \omega_1 + \omega_2 + \ldots + \omega_p ,$$

such that the form ω is constructed from a certain number $2h$ of variables x and the variable t in such a manner that the variables x that enter into the formation of any two of the forms $\omega_1, \ldots, \omega_p$ are different. As a result, one will have:

$$s = h_1 + h_2 + \ldots + h_p .$$

If one supposes that quadratic exterior form ω' is of rank $2s$ then it will be necessary that the forms ω_1', ω_2', ..., ω_p' must be of rank $2h_1$, $2h_2$, ..., $2h_p$, resp. The reduction of each of these p forms to its canonical form will thus imply the same reduction for ω'. As a result, *integrating the characteristic equations of* ω' *amounts to integrating the characteristic equations of* ω_2', ..., ω_p', and the p corresponding problems may be solved independently of each other.

An even greater simplification is produced if the numbers k_i of the x variables (that are different for different forms ω_i) that enter into the composition of these forms at the same time as t are not all even. In this case, the variable t will be a first integral of the characteristic equations of ω'. Indeed, if one gives an arbitrary constant value to t then the rank of the quadratic form ω_i' will be reduced to at most:

$$\begin{array}{ll} k_i & \text{for even } k_i , \\ k_{i-1} & \text{for odd } k_i . \end{array}$$

Now, $2s$ is equal to the sum of all the k_i. The rank of ω' will therefore be less than $2s$ for constant t, which was to be proved. Furthermore, one sees that there cannot be *two* numbers k_i that are both odd, and when one makes t constant, the reduction of ω' to its normal form will be furnished by the reductions of ω_1', ..., ω_p' to their normal forms when one likewise makes t constant.

CHAPTER XV

DIFFERENTIAL EQUATIONS THAT ADMIT SEVERAL LINEAR INTEGRAL INVARIANTS

I. - *Case in which one knows as many integral invariants as there are unknowns.*

155. We will not go into the general problem of integrating differential equations that admit an arbitrary number of integral invariants in these lessons. We will restrict ourselves to the particularly simple case in which a system of n first-order ordinary differential equations in n unknown functions admits n (independent) linear invariant forms:

$$\omega_1,\ \omega_2,\ \ldots,\ \omega_n\ ;$$

i.e., n absolute linear integral invariants:

$$\int \omega_1,\ \int \omega_2,\ \ldots,\ \int \omega_n\ .$$

In this particular case, the given differential equations can be written:

$$\omega_1 = \omega_2 = \ldots = \omega_n = 0. \tag{1}$$

Since the quadratic exterior forms $\omega_1', \omega_2', \ldots, \omega_n'$ are invariant, they may be expressed in terms of $\omega_1, \omega_2, \ldots, \omega_n$ by formulas such as:

$$\omega_s' = \sum_{(ik)}^{1,\ldots,n} c_{iks}[\omega_i\ \omega_k] \qquad (s = 1, 2, \cdots, n). \tag{2}$$

The coefficients c_{iks} are obviously first integrals of the given differential equations. We shall see that one may always convert them into a form where they are constant.

Indeed, suppose that among the integrals c_{iks} there are a certain number r of independent ones, which we denote by:

$$y_1,\ y_2,\ \ldots,\ y_r\ .$$

The c_{iks} are thus well-defined functions of these r integrals. Each differential dy_i is, in turn, an invariant form that can be expressed linearly by means of $\omega_1, \omega_2, \ldots, \omega_n$:

$$dy_i = b_{i1}\,\omega_1 + b_{i2}\,\omega_2 + \ldots + b_{in}\,\omega_n \qquad (i = 1, 2, \ldots, r).$$

The coefficients b_{ik} are, in turn, first integrals. If there are r' of them that are mutually independent and independent of the y_i then their differentials $dy_{r+1}, \ldots, dy_{r+r'}$ can also be expressed linearly as functions of the ω_i, and the coefficients might provide new first integrals, and so on. We will arrive at a point where these operations cease, and one will arrive at a certain number $\rho \leq n$ of first integrals $y_1, ..., y_\rho$ such that the coefficients c_{iks} in formula (2) and the coefficients b_{ik} in the formula:

(3) $$dy_i = b_{i1}\omega_1 + b_{i2}\omega_2 + ... + b_{in}\omega_n \qquad (i = 1, 2, ..., \rho)$$

are well-defined functions of the $y_1, y_2, ..., y_\rho$.

Having said this, suppose, for the sake of specificity, that the determinant that is obtained by taking the first ρ columns of the matrix of b_{ik} is non-zero. One may then substitute the invariant forms $dy_1, ..., dy_\rho$ for the ρ invariant forms $\omega_1, ..., \omega_\rho$. If one attributes arbitrary constant values to the $y_1, ..., y_\rho$ then the system of equations to be integrated will admit $n - \rho$ invariant forms $\omega_{\rho+1}, ..., \omega_n$, and one will have:

$$\omega_s' = \sum_{(ik)}^{\rho+1,\dots,n} c_{iks}[\omega_i\,\omega_k] \qquad (s = \rho+1, \cdots, n),$$

in which the coefficients c_{iks} are now constant.

156. Therefore, take the case in which the coefficients c_{iks} in formula (2) are all constant. Conversely, it is easy to see that the existence of relations such as (2) implies, as a consequence, the property of the forms $\omega_1, ..., \omega_n$ that they must be invariant with respect to the differential equations (1). The characteristic system of the set of forms $\omega_1, \omega_2, ..., \omega_n$ is obtained (sec. **78**) by adding the equations of the associated system to $\omega_1', \omega_2', ..., \omega_n'$ – equations that are all a consequence of equations (1) – to equations (1).

If one substitutes linear combinations *with constant coefficients:*

(4) $$\begin{cases} \varpi_1 = a_{11}\omega_1 + a_{12}\omega_2 + \cdots + a_{1n}\omega_n, \\ \cdots \\ \varpi_n = a_{n1}\omega_1 + a_{n2}\omega_2 + \cdots + a_{n1n}\omega_n, \end{cases}$$

for the forms $\omega_1, ..., \omega_n$ then these n new forms will again be invariant, and one will again have relations:

$$\varpi_s' = \sum_{(ik)}^{1,\dots,n} \bar{c}_{iks}[\varpi_i\,\varpi_k],$$

with new constants $\bar{c}_{iks}$. We say that the matrix of $\bar{c}_{iks}$ has the same *structure* as the matrix of c_{iks}. It is possible that one can choose the constant coefficients a_{ij} in the substitution (4) in such a manner that only the $\varpi_1, \cdots, \varpi_\nu$ appear in the expression of the first $\nu < n$ derivatives $\varpi_1', \cdots, \varpi_\nu'$; i.e., in such a manner that one will have:

$$\bar{c}_{\nu+i,k,s} = \bar{c}_{k,\nu+i,s} = \bar{c}_{\nu+i,\nu+j,s} = 0 \qquad (k, s = 1, 2, ..., \nu;\ i, j = 1, ..., n-\nu).$$

In this case, the forms $\varpi_1, ..., \varpi_\nu$ are invariant for the *completely integrable* system of Pfaff equations:

$$\varpi_1 = \varpi_2 = ... = \varpi_\nu = 0.$$

If one knows how to integrate this system, and if one equates these first integrals to arbitrary constants then the given system will be converted into a system that is analogous to the first one, except that n will be replaced by $n - \nu$.

We say that the matrix of c_{iks} is *simple* if it is impossible to find a linear substitution with constant coefficients (4) that brings about the preceding reduction. We then say that *the given system of differential equations can be converted into successive systems such that the matrix of* c_{iks} *is simple for each of them.* A particular integration problem corresponds to each matrix.

157. Leaving aside this method of reduction for the moment, we imagine a second system of differential equations:

$$(1') \qquad \varpi_1 = \varpi_2 = ... = \varpi_n = 0$$

that admit the n invariant forms ϖ_i with the relations:

$$(2') \qquad \varpi'_s = \sum_{(ik)}^{1,...,n} c_{iks}[\varpi_i\,\varpi_k],$$

where the coefficients c_{iks} have the same numerical values as in formula (2). Let

$$y_1, y_2, ..., y_n,$$
$$z_1, z_2, ..., z_n,$$

respectively, be two systems of independent first integrals, the first one being associated with equations (1), and the second one, with equations (1′). $\varpi_1, \varpi_2, ..., \varpi_n$ can then be expressed by means of the dy_i and their differentials. *It is possible to choose the first integrals z_i in such a fashion that the ϖ_i can be expressed by means of the z_i and the dz_i in the same way that the ω_i are expressed by means of the y_i and the dy_i.* This amounts to saying that if this condition is not realized then one can, at least, find functions:

$$f_1(y_1, ..., y_n), ..., f_n(y_1, ..., y_n)$$

such that upon setting:

$$z_1 = f_1(y_1, ..., y_n), ..., z_n = f_n(y_1, ..., y_n),$$

the ϖ_i will become equal to the ω_i, respectively. In order to this, it suffices to integrate the total differential equations:

$$\text{(5)}\qquad \begin{aligned} \varpi_1 - \omega_1 &= 0, \\ &\cdots \\ \varpi_n - \omega_n &= 0, \end{aligned}$$

in which the z_i are unknown functions of the independent variables y_i. This Pfaff system (5) is *completely integrable* (sec. **101**), because, if one takes equations (5) into account then the exterior derivatives:

$$\varpi'_{si} - \omega'_s = \sum c_{iks}[\varpi_i\, \varpi_k] - \sum c_{iks}[\omega_i\, \omega_k],$$

of the left-hand sides will all be zero. It is therefore possible to satisfy the stated conditions, and in an infinitude of ways (that depend on n arbitrary constants).

In particular, this proves that the integrations of the two systems (1) and (1′) are two problems of *essentially* the same nature, in the sense that any method *that uses only the given property of the* $\omega_1, \ldots, \omega_n$ *that they are invariant forms for the integration* can be applied to systems (1) and (1′) in a parallel fashion, such that any progress in the integration of (1) implies an equivalent progress in the integration of (1′).

II. - *The group that preserves the given invariants.*

158. Return to system (1), and imagine that a choice of n independent first integrals:

$$y_1, \ldots, y_n$$

has been made.

It is possible to find an infinitude of other systems of n first integrals:

$$\bar{y}_1, \bar{y}_2, \cdots, \bar{y}_n$$

such that the forms ω_i may be expressed by means of the $\bar{y}_i$ and their differentials in the same manner as they are by means of the y_i and their differentials. In order to do this, it suffices to integrate the Pfaff system:

$$\text{(6)}\qquad \begin{aligned} \bar{\omega}_1 - \omega_1 &= 0, \\ &\cdots \\ \bar{\omega}_n - \omega_n &= 0, \end{aligned}$$

in which $\bar{\omega}_s$ denotes the same function of the $\bar{y}_i$ and $d\bar{y}_i$ that ω_s is of y_i and dy_i. In this Pfaff system, we regard the arguments $\bar{y}_1, \ldots, \bar{y}_n$ as unknown functions of the independent variables $y_1, \ldots, y_n$. Such a system is completely integrable for the same reason that was indicated in relation to system (5). Therefore, there exist functions:

$$\text{(7)}\qquad \bar{y}_s = f_s(y_1, \cdots, y_n; C_1, \cdots, C_n) \qquad (s = 1, \cdots, n),$$

that depend on n arbitrary constants and satisfy the conditions stated above.

Equations (7) define an infinitude of transformations that act on the first integrals $y_1, ..., y_n$ and *preserve the givens of the problem;* i.e., they leave the forms $\omega_1, ..., \omega_n$ invariant. These transformations form a group G because, since they are characterized by the property of preserving $\omega_1, ..., \omega_n$, it will be obvious that if one performs a transformation of the form (7), followed by another one, then one will again obtain a transformation of the same form. The group G is a finite group of n parameters; *it is the largest group that preserves the given invariant forms when it is applied to the first integrals of the given system.* As one easily concedes, just what the knowledge of these n forms gives us depends on the nature of that group. Moreover, this is a general fact that applies to all the cases where one knows, *a priori*, integral invariants, systems of invariant equations, infinitesimal transformations, etc. The nature of the largest group of transformations that preserves the known information when applied to the first integrals of given differential equations (or, what amounts to the same thing, to their integral curves, which are regarded as being indivisible) has an overshadowing importance in the integration of that system.

In the case that occupies us, one sees, in particular, that it is impossible to obtain any first integral without integration ([7]) by starting only with the fact that $\omega_1, ..., \omega_n$ are invariant forms. Nevertheless, the fact that the forms $\omega_1, ..., \omega_n$ are invariant will, by itself, permit us to *individualize* a first integral – y_1, for example – which, as a result, becomes equal to one of the integrals $\bar{y}_i$ that are defined by formula (7). However, this is obviously impossible, because equations (6) always admit a solution such that given numerical values of $y_1, ..., y_n$ will correspond to *arbitrary* numerical values of $\bar{y}_1, \cdots, \bar{y}_n$.

159. The constants c_{iks} play an important role in relation to the group G. They are what one calls the *structure constants* of that group in group theory. The method of reduction that was indicated above (sec. **156**) is based precisely upon the decomposition of G into a normal series of subgroups. The case where the matrix of c_{iks} is *simple* corresponds to the *simple* groups.

One knows that the structure constants of a group are not arbitrary. One may verify this here by saying that the exterior derivatives of the $\omega_1', ..., \omega_n'$ are zero. If we use the expressions (2) for $\omega_1', ..., \omega_n'$ then the exterior derivative of ω_s' will be (sec. **73**):

$$\sum_{(ik)}^{1,\ldots,n} c_{iks}([\omega_i'\,\omega_k] - [\omega_i\,\omega_k']) = \sum_{(\alpha\beta\gamma)}^{1,\ldots,n}\left(\sum_{i=1}^{i=n} c_{\alpha\beta i}c_{i\gamma s} + c_{\beta\gamma i}c_{i\alpha s} + c_{\gamma\alpha i}c_{i\alpha s}\right)[\omega_\alpha\omega_\beta\omega_\gamma].$$

One thus has the necessary relations:

$$\sum_{i=1}^{i=n} c_{\alpha\beta i}\,c_{i\gamma s} + c_{\beta\gamma i}\,c_{i\alpha s} + c_{\lambda\alpha i}\,c_{i\beta s} = 0 \qquad (\alpha, \beta, \gamma, s = 1, 2, ..., n).$$

([7]) This is also true for an arbitrary sequence of *invertible* operations that are applied to $\omega_1, ..., \omega_n$ and which are capable of being performed no matter what the nature of the coefficients of these forms.

In group theory, one proves that they are sufficient for the existence of a group that admits the c_{iks} for its structure constants.

III. - *Examples.*

160. Suppose that all of the constants c_{iks} are zero. It is obvious then that since the forms ω_1, ..., ω_n are exact differentials, the integration will require only n independent quadratures. Since the forms ω_1, ..., ω_n are reducible to:

$$\omega_1 = dy_1, \qquad \omega_2 = dy_2, \ldots, \omega_n = dy_n,$$

the group G will have:

$$y'_s = y_s + C_s \qquad (s = 1, 2, \ldots, n)$$

for its equations.

The preceding case presents itself when $n = 1$.

We look for all possible cases when $n = 2$. Along with the case that we just examined, one may have:

$$\omega'_1 = a\,[\omega_1\,\omega_2],$$
$$\omega'_2 = b\,[\omega_1\,\omega_2],$$

in which the coefficients a and b are not both zero. Suppose, for example, that $b \neq 0$. If one takes $a\omega_2 - b\omega_1$ to be a new form $\bar{\omega}_1$ then one will immediately see that one has:

$$\bar{\omega}'_1 = 0,$$
$$\omega'_2 = [\bar{\omega}_1\,\omega_2].$$

One quadrature gives:

$$\bar{\omega}_1 = dy_1.$$

If one equates y_1 to an arbitrary constant then ω_2 will become an exact differential, and a second quadrature will complete the integration. By changing the notations slightly, one may suppose that:

$$\omega_1 = \frac{dy_1}{y_1}$$
$$\omega_2 = \frac{dy_2}{y_1}.$$

The group G will then have the equations:

$$y'_1 = C_1 y_1,$$
$$y'_2 = C_1 y_2 + C_2.$$

161. We shall not discuss the general case for $n = 3$. We only point out the most interesting case, in which one reduces formulas (2) to:

$$\omega_1' = [\omega_1 \; \omega_2],$$
$$\omega_2' = [\omega_1 \; \omega_3],$$
$$\omega_3' = [\omega_2 \; \omega_3].$$

In this case, *the integration of equations* (1) *amounts to the integration of a Ricatti equation.*

Indeed, consider the Pfaff equation:

$$dt + \omega_1 + t\omega_2 + \tfrac{1}{2}t^2\omega_3 = 0, \tag{8}$$

in which t is regarded as an unknown function of the variables – both dependent and independent – that appear in the given differential equations. *This equation is completely integrable.* Indeed, one verifies without difficulty that the exterior derivative of its left-hand side is zero if one takes the equation itself into account (and if one uses the expressions for $\omega_1', \omega_2', \omega_3'$). As a result, as one knows, one may convert this integration into the integration of an ordinary differential equation that is obviously a Ricatti equation. Now, if one denotes a system of independent first integrals of the given equations (1) by y_1, y_2, y_3 then the expressions $\omega_1, \omega_2, \omega_3$ may be expressed by means of three quantities y_1, y_2, y_3 and their differentials. The solution of equation (1) for general t is thus a function of y_1, y_2, y_3 (and an arbitrary constant C). As a result, if one has integrated the Ricatti equation (8) in its classical form:

$$t = \frac{\alpha + C\beta}{\gamma + C\delta}$$

then the mutual relations between the four functions $\alpha, \beta, \gamma, \delta$ will furnish three first integrals of the given equations, and one easily shows that they are independent.

IV. - *Generalizations.*

162. We nevertheless do not insist on the use of the foregoing theory, which, in order to be appropriately developed, requires a very extended knowledge of the theory of groups. One sees how the latter is necessarily introduced if one pushes the method of integrating differential equations that admit given integral invariants to its limit. We point out only that the method that was indicated in sec. **142** can be generalized to an arbitrary system of differential equations that admit an invariant form, invariant Pfaff equations, etc. It consists of forming any linear integral invariants that the given system of equations implies as independent first integrals *by the introduction of auxiliary variables.* An example will suffice to make the spirit of this method comprehensible.

Suppose that one has integrated a system of differential equations (Σ) in 4 variables:

$$\omega_1 = \omega_2 = \omega_3 = 0,$$

and each of these equations $\omega_1 = 0$, $\omega_2 = 0$, $\omega_3 = 0$ are invariant for the given system. One introduces three new variables u_1, u_2, u_3, and one will consider the three forms:

$$\bar{\omega}_1 = u_1\omega_1, \quad \bar{\omega}_2 = u_2\omega_2, \quad \bar{\omega}_3 = u_3\omega_3.$$

The integration of the characteristic equations (Σ) of these three forms implies that of the given differential equations by the elimination of u_1, u_2, u_3 between the relations that define an arbitrary solution of (Σ). We then form the exterior derivatives $\bar{\omega}_1', \bar{\omega}_2', \bar{\omega}_3'$. If one supposes that one has:

$$\begin{aligned}
\bar{\omega}_1' &\equiv a_1[\omega_2\omega_3] \qquad (\bmod\ \omega_1),\\
\bar{\omega}_2' &\equiv a_2[\omega_2\omega_1] \qquad (\bmod\ \omega_2),\\
\bar{\omega}_3' &\equiv a_3[\omega_1\omega_2] \qquad (\bmod\ \omega_3),
\end{aligned}$$

with coefficients a_1, a_2, a_3 that are functions of the original variables, then one will have:

$$\begin{aligned}
\bar{\omega}_1' &\equiv \frac{a_1u_1}{u_2u_3}[\bar{\omega}_2\bar{\omega}_3] \qquad (\bmod\ \bar{\omega}_1),\\
\bar{\omega}_2' &\equiv \frac{a_2u_2}{u_3u_1}[\bar{\omega}_3\bar{\omega}_1] \qquad (\bmod\ \bar{\omega}_2),\\
\bar{\omega}_3' &\equiv \frac{a_3u_3}{u_1u_2}[\bar{\omega}_1\bar{\omega}_2] \qquad (\bmod\ \bar{\omega}_3).
\end{aligned}$$

The coefficients:

$$\bar{a}_1 = \frac{a_3u_1}{u_2u_3}, \quad \bar{a}_2 = \frac{a_2u_2}{u_3u_1}, \quad \bar{a}_3 = \frac{a_3u_3}{u_1u_2}$$

are thus first integrals of the system (Σ), which is characteristic for the *forms* $\bar{\omega}_1, \bar{\omega}_2, \bar{\omega}_3$. As a result, the same is true for:

$$\sqrt{\bar{a}_2\bar{a}_3} = \frac{\sqrt{a_2a_3}}{u_1},$$

and the form:

$$\sqrt{\bar{a}_2\bar{a}_3}\,\bar{\omega}_1 = \sqrt{a_2a_3}\,\omega_1$$

is again an invariant form. However, it does not contain the auxiliary variables u_1, u_2, u_3. It is thus an invariant form for the given equations (Σ), and the same is true for $\sqrt{a_3a_1}\,\omega_2$, $\sqrt{a_1a_2}\,\omega_3$. As a result, *if any of the coefficients a_1, a_2, a_3 are zero then the given differential system will admit three invariant linear forms,* and one is confronted with the problem that was treated in this chapter.

Naturally, this will not always be the case, but *in any case one will have the means to deduce all of the possible consequences of the known information about the given equations.*

CHAPTER XVI

DIFFERENTIAL EQUATIONS THAT ADMIT GIVEN INFINITESIMAL TRANSFORMATIONS

I. - *Reduction of the problem.*

163. We have already considered differential equations that admit infinitesimal transformations, but these equations were assumed to admit an integral invariant or a Pfaff invariant. We shall now take the most general viewpoint, which will furnish us, moreover, with an illustration of the theories that were sketched out in the preceding chapter.

Consider a system of n ordinary differential equations (or a completely integrable system of n Pfaff equations):

(1) $$\omega_1 = \omega_2 = \ldots = \omega_n = 0,$$

and suppose that this system admits a certain number $r \leq n$ of infinitesimal transformations:

$$A_1 f, A_2 f, \ldots, A_r f .$$

We then look for the consequences that one might deduce from the knowledge of these r infinitesimal transformations by integrating. This is a problem that has been solved by S. Lie. We confine ourselves to its essential generalities.

Consider the matrix of quantities $\omega_i(A_k)$ that is obtained by replacing the differentiation symbol in the form ω_i with the symbol of the infinitesimal transformation $A_k f$. Suppose that the determinant that is formed from the first r rows and the first r columns of the matrix:

(2) $$\left\| \begin{matrix} \omega_1(A_1) & \omega_1(A_2) & \cdots & \omega_1(A_r) \\ \omega_2(A_1) & \omega_2(A_2) & \cdots & \omega_2(A_r) \\ \cdots & \cdots & \cdots & \cdots \\ \omega_n(A_1) & \omega_n(A_2) & \cdots & \omega_n(A_r) \end{matrix} \right\|$$

is zero. One may then substitute linear combinations of the left-hand sides of equations (1) for those left-hand sides in such a way that the matrix becomes:

$$(3) \qquad \left\| \begin{matrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ . & . & . & . \\ 0 & 0 & \cdots & 1 \\ 0 & 0 & \cdots & 0 \\ . & . & . & . \\ 0 & 0 & \cdots & 0 \end{matrix} \right\|$$

i.e., in such a manner that all of the $\omega_i(A_j)$ are zero, except for:

$$\omega_1(A_1) = \omega_2(A_2) = ... = \omega_r(A_r) = 1.$$

It is obvious that if n is greater than r then the new forms $\omega_1, ..., \omega_n$ will not be perfectly determined; one may once more perform an arbitrary linear substitution on:

$$\omega_{r+1}, ..., \omega_n ,$$

and one may add an arbitrary linear combination of the $\omega_{r+1}, ..., \omega_n$ to each of the forms $\omega_1, ..., \omega_r$.

If equations (1) have been put into the form:

$$dy_1 = dy_2 = ... = dy_n = 0$$

then it will be obvious that, since the quantities $\omega_i(A_j) = A_j(y_i)$ are presumably first integrals, the new forms $\omega_1, ..., \omega_n$ that are obtained by reducing the matrix of $\omega_i(A_j)$ to its canonical form can always be presumed to have been constructed from the y_i and their differentials. The following two consequences result from this, along with what was said above:

1. *Whenever the matrix of $\omega_i(A_k)$ is reduced to its normal form* (3), *the Pfaff system:*

$$(4) \qquad \omega_{r+1} = ... = \omega_n = 0$$

will be an invariant system.

2. *Each of the linear forms $\omega_1, ..., \omega_n$ is an invariant form, up to a linear combination of the left-hand sides of the preceding invariant Pfaff system.*

164. Before we proceed, we first remark that if the system (1) admits two infinitesimal transformations Af and Bf then it will admit the infinitesimal transformation Cf whose symbol is defined by:

$$Cf = A(Bf) - B(Af).$$

Assume, with no loss of generality, that the symbols of the infinitesimal transformations that one may deduce from the r given transformations, when taken pair-wise, are linear combinations of the $A_1 f, ..., A_r f$. In other words, suppose that one has:

(5) $$A_i(A_k f) - A_k(A_i f) = \sum_{s=1}^{s=r} \gamma_{iks} A_s f \qquad (i, k = 1, 2, \cdots, r).$$

With this hypothesis, we shall prove that *the Pfaff system* (4) *is completely integrable.*

In order to prove this, it is necessary that we return to the definition of the bilinear covariant $\omega'(\delta, \delta')$ of a linear form ω in the case that we have not considered up till now in which the two differentiation symbols δ, δ' *are not interchangeable.* If one sets:

$$\omega(\delta) = a_1 \delta x_1 + a_2 \delta x_2 + ... + a_n \delta x_n$$

then one will have:

$$\delta\omega(\delta') - \delta'\omega(\delta) = a_1(\delta\delta' x_1 - \delta'\delta x_1) + ... + a_n(\delta\delta' x_n - \delta'\delta x_n)$$
$$+ \sum \left(\frac{\partial a_k}{\partial x_i} - \frac{\partial a_i}{\partial x_k} \right)(\delta x_i \delta' x_k - \delta x_k \delta' x_i),$$

or rather, if we agree to set:

$$\delta'' = \delta\delta' - \delta'\delta$$

then we will have:

(6) $$\delta\omega(\delta') - \delta'\omega(\delta) = \omega(\delta'') + \omega'(\delta, \delta').$$

We apply this formula to the case in which the symbols δ and δ' are replaced by the symbols $A_i f$ and $A_k f$. It will then be convenient to replace δ'' with the symbol:

$$A_i(A_k f) - A_k(A_i f) = \sum \gamma_{iks} A_s f.$$

Finally, suppose that one takes any one of the forms $\omega_{r+1}, ..., \omega_n$ to be ω, which we may, as we have seen, presume to be expressed in terms of $y_1, ..., y_n$ and their differentials. One will have:

$$\omega'_{r+\alpha} = \sum c_{\lambda,u,r+\alpha} [\omega_\lambda \, \omega_\mu].$$

However:

$$\omega_{r+\alpha}(A_i) = \omega_{r+\alpha}(A_k) = \omega_{r+\alpha}(A_s) = 0.$$

Therefore, one has the relation:

$$\sum c_{\lambda,\mu,r+\alpha} [\omega_\lambda(A_i) \, \omega_\mu(A_k) - \omega_\mu(A_i) \, \omega_\lambda(A_k)] = 0.$$

It results from this that the coefficients $c_{\lambda,\mu,r+\alpha}$, will be zero when the indices λ, μ are both less than or equal to r, since the preceding relation then obviously reduces to:

$$c_{i,k,r+\alpha} = 0 \qquad (i, k = 1, 2, ..., r).$$

As a consequence, since the exterior derivatives ω'_{r+1}, ..., ω'_n are all zero when one takes equations (4) into account, system (4) will be completely integrable (sec. **101**

II. - *Case in which there are as many infinitesimal transformations as unknown functions.*

165. Now suppose that the system (4), which is an absolutely arbitrary completely integrable Pfaff system, has been integrated. Similarly, simply suppose that a solution of this system (4) is known. An infinitude of solutions in the given system will correspond to that solution that are obtained by integrating the equations:

$$\omega_1 = \omega_2 = \ldots = \omega_r = 0. \tag{7}$$

This is a system for which one knows r invariant forms ω_1, ..., ω_r. One is then reduced to the problem that was treated in the preceding chapter.

Here, it is easy to determine, *a priori*, the coefficients c_{iks} that enter into the expressions ω'_1, ..., ω'_r :

$$\omega'_s = \sum_{(ik)}^{1,\cdots,n} c_{iks} [\omega_i \, \omega_k].$$

Indeed, we apply formula (6) after replacing the symbol δ with the symbol A_α , the symbol δ' with the symbol $A_{\beta'}$, and the symbol δ'' with $\sum_{\rho=1}^{\rho=r} \gamma_{\alpha\beta\rho} A_\rho$ in it. As all of the ω_i (A_k) are equal to 0 or to 1, i.e., to constants, formula (6) will reduce to:

$$0 = \gamma_{\alpha\beta s} + c_{\alpha\beta s} .$$

Therefore, one will have:

$$c_{\alpha\beta s} = - \gamma_{\alpha\beta s} .$$

166. We now restrict ourselves to the case in which the coefficients γ_{iks} are constants. In this case, one proves that *the given infinitesimal transformations* $A_1 f$, ..., $A_r f$ *will generate an r-parameter group* Γ whose structure coefficients are the γ_{iks} . One sees that the system (7) falls into the category of the two systems that we studied in the preceding chapter (sec. **156**), and the group G that corresponds to it has the same structure as the group Γ that admits the given differential system (7). *This group G is the largest group that, when applied to the first integrals* y_1, ..., y_r , *preserves the following rule: These integrals are interchangeable with the given infinitesimal transformations.*

Indeed, denote an arbitrary function of y_1, ..., y_r by f. It is obvious that one may determine – in one and only one manner – r Pfaff expressions ϖ_1, ..., ϖ_r such that one has identically – i.e., for any differentials dy_1, ..., dy_r and any arguments $\frac{\partial f}{\partial y_1}, \cdots, \frac{\partial f}{\partial y_r}$:

$$df \equiv \frac{\partial f}{\partial y_1} dy_1 + \cdots + \frac{\partial f}{\partial y_r} dy_r = \varpi_1 A_1 f + \cdots + \varpi_r A_r f .$$

In this identity, if we replace the symbol of indeterminate differentiation d with the symbol $A_k f$ then we will have:

$$A_k f = \varpi_1(A_k)\, A_1 f + \ldots + \varpi_k(A_k)\, A_r f.$$

As a result, all of the $\varpi_i(A_k)$ will be zero except for:

$$\varpi_1(A_1) = \varpi_2(A_2) = \ldots = \varpi_r(A_r) = 1.$$

Finally, it results that *the forms* ϖ_i *are identical with the forms* ω_i . We then perform a transformation of the group G on the $y_1, \ldots, y_r$, and these quantities become $\bar{y}_1, \cdots, \bar{y}_r$. The function f of the $y_1, \ldots, y_r$ becomes a function $\bar{f}$ of the $\bar{y}_1, \cdots, \bar{y}_r$, the symbols $A_1 f, \ldots, A_r f$ become $\bar{A}_1 \bar{f}, \cdots, \bar{A}_r \bar{f}$, and one will have:

$$d\bar{f} = \bar{\omega}_1 \bar{A}_1 \bar{f} + \cdots + \bar{\omega}_r \bar{A}_r \bar{f} .$$

However, since the $\bar{\omega}_i$ are forms in the $\bar{y}_i$ and their differentials, just as the ω_i are forms in the y_i and their differentials, the coefficient of $\frac{\partial \bar{f}}{\partial \bar{y}_k}$ in $\bar{A}_i \bar{f}$ will be the same function of the $\bar{y}_1, \cdots, \bar{y}_r$ that the coefficient of $\frac{\partial f}{\partial y_k}$ in $A_i f$ is as a function of the $y_1, \ldots, y_r$. In other words, the given infinitesimal transformations transform the $\bar{y}_i$ in the same way that they transform the y_i .

Here, one sees once more that the group G is the largest group of transformations that preserves the given data when applied to the first integrals.

III. - *Application to second-order differential equations*.

167. We have already treated the case $n = r = 1$ directly; let us take several other examples. A second-order differential equation of the form:

$$\frac{d^2 y}{dx^2} = F\left(\frac{dy}{dx}\right)$$

is equivalent to the system:

$$dy - y'\, dx = 0,$$
$$dy' - F(y')\, dx = 0,$$

which admits the two infinitesimal transformations:

$$Af = \frac{\partial f}{\partial x}, \qquad Bf = \frac{\partial f}{\partial y}.$$

In order to reduce this matrix of quantities $\omega_i(A_k)$ to its normal form, it is necessary to take:

$$\omega_1 = dx - \frac{dy'}{F(y')},$$

$$\omega_2 = dy - \frac{y'dy'}{F(y')}.$$

These two invariant forms are exact differentials, and one gets the desired general solution by two independent quadratures:

$$x - \int \frac{dy'}{F(y')} = C_1, \qquad y - \int \frac{y'dy'}{F(y')} = C_2.$$

Now, take a second-order differential equation of the form:

$$y\frac{d^2y}{dx^2} = F\left(\frac{dy}{dx}\right).$$

It admits a translation that is parallel to the x-axis and a homothety with a center O that correspond to the two infinitesimal transformations:

$$Af = \frac{\partial f}{\partial x}, \qquad Bf = x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y}.$$

The given equation is equivalent to the system:

$$dy - y'\,dx = 0,$$
$$y\,dy' - F(y')\,dx = 0.$$

In order to make the matrix of $\omega_i(A_k)$ normal, it is necessary to take:

$$\omega_1 = dx - \frac{x}{y}dy - \frac{y - xy'}{F(y')}dy',$$
$$\omega_2 = \frac{dy}{y} - \frac{y'}{F(y')}dy'.$$

Since one has:

$$A(Bf) - B(Af) = Af$$

here, one will have, as is easily verified:

$$\omega_1' = -[\omega_1\ \omega_2],$$
$$\omega_2' = 0.$$

As a result, the integration is carried out by means of two quadratures:

$$y = C_1 e^{\int \frac{y' dy'}{F(y')}},$$
$$x = C_1 \int \frac{1}{F(y')} e^{\int \frac{y' dy'}{F(y')}} dy' + C_2.$$

IV. - *Generalizations. Examples.*

168. In the case of system (1), one may arrive at n Pfaff equations that admit r infinitesimal transformations:

$$A_1 f, \ldots, A_r f,$$

such that the rank of the matrix of $\omega_i(A_k)$ becomes less than r (this is certainly the case if $r > n$). Therefore let ρ be the rank of this matrix, and suppose, as is permissible, that the determinant that is constructed from the first ρ rows and the first ρ columns is not zero. For any index s, one will then have $r - \rho$ relations of the form:

$$\omega_s(A_{r+1}) = \lambda_{11}\,\omega_s(A_1) + \ldots + \lambda_{1\rho}\,\omega_s(A_\rho),$$
$$\ldots$$
$$\omega_s(A_r) = \lambda_{r-\rho,1}\,\omega_s(A_1) + \ldots + = \lambda_{r-\rho,\rho}\,\omega_s(A_\rho).$$

The coefficients λ_{ij} that were introduced into these relations are first integrals. Indeed, for any linear combination ω of $\omega_1, \ldots, \omega_n$ – in particular, for the differentials $dy_1, \ldots, dy_n$ of n independent first integrals – one will have the same relations:

$$A_{\rho+1}(y_s) = \lambda_{11} A_1(y_s) + \ldots + \lambda_{1\rho} A_\rho(y_s),$$

which implies that the values of $\lambda_{11}, \ldots, \lambda_{1\rho}$ depend upon only the $A_i(y_k)$; i.e., upon the $y_1, \ldots, y_n$.

We will not pursue the general case further; it is based upon the same principles as the preceding one.

169. EXAMPLE I. – Consider the differential equation:

$$(1 + y'^2)^{3/2} = R y''$$

of plane curves that have a ray of given curvature. It is equivalent to system:

$$\omega_1 \equiv dy - y'\,dx = 0,$$
$$\omega_2 \equiv R\,dy' - (1 + y'^2)^{3/2}\,dx = 0.$$

This system admits the three infinitesimal transformations that correspond to a translation parallel to Ox, a translation parallel to Oy, and a rotation around O. One may calculate the effects of these transformations on not only x and y, but also on y'. One finds without difficulty that:

$$A_1 f = \frac{\partial f}{\partial x},$$
$$A_2 f = \frac{\partial f}{\partial y},$$
$$A_3 f = -y\frac{\partial f}{\partial x} + x\frac{\partial f}{\partial y}(1 + y'^2)\frac{\partial f}{\partial y'}.$$

The matrix of quantities $\omega_i(A_k)$ is the following one:

$$\left\| \begin{matrix} -y' & 1 & x + yy' \\ -(1+y'^2)^{3/2} & 0 & y(1-y'^2)^{3/2} + R(1+y'^2) \end{matrix} \right\|.$$

One therefore has:

$$\omega_s(A_3) = -\left(y + \frac{R}{\sqrt{1+y'^2}} \right)\omega_s(A_1) + \left(x - \frac{Ry'}{\sqrt{1+y'^2}} \right)\omega_s(A_2).$$

As a result, one obtains two first integrals of the given system by simple differentiations, and the general solution is furnished by the formulas:

$$x = \frac{Ry'}{\sqrt{1+y'^2}} + C_1,$$
$$y = \frac{R}{\sqrt{1+y'^2}} + C_2,$$

or

$$(x_1 - C_1)^2 + (y - C_2)^2 = R^2.$$

170. EXAMPLE II. – Consider the third-order differential equation:

$$y''' = \frac{3y'y''^2}{1+y'^2}$$

that defines the plane curves of constant curvature. It is equivalent to the system:

$$\omega_1' \equiv dy - y' dx = 0,$$
$$\omega_2' \equiv dy' - y'' dx = 0,$$
$$\omega_3 \equiv dy'' - \frac{3y'y''}{1+y'^2} dy' = 0\,.$$

This admits four infinitesimal transformations, which correspond to a translation parallel to *Ox*, a translation parallel to *Oy*, a rotation around *O*, and a homothety with center *O*. The symbols of these transformations, which are regarded as operating on x, y, x', y', and y'', are:

$$A_1 f = \frac{\partial f}{\partial x},$$
$$A_2 f = \frac{\partial f}{\partial y},$$
$$A_3 f = -y\frac{\partial f}{\partial x} + x\frac{\partial f}{\partial y} + (1+y'^2)\frac{\partial f}{\partial y'} + 3y'y''\frac{\partial f}{\partial y''},$$
$$A_4 f = x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} - y''\frac{\partial f}{\partial y''}.$$

Here, the matrix of quantities $\omega_i(A_k)$ is:

$$\begin{Vmatrix} -y' & 1 & x+yy' & y-xy' \\ -y'' & 0 & 1+y'^2+yy' & -xy'' \\ 0 & 0 & 0 & -y'' \end{Vmatrix}.$$

It is of rank 3, and the determinant that is obtained by taking the first, the second, and the fourth column, for example, is non- zero. From this, one deduces the relations:

$$\omega_s(A_3) = -\left(y + \frac{1+y'^2}{y''}\right)\omega_s(A_1) + \left(x - y'\frac{1+y'^2}{y''}\right)\omega_s(A_2),$$

which lead to the two first integrals:

$$u = x - y'\frac{1+y'^2}{y''},$$
$$v = y + \frac{1+y'^2}{y''}.$$

In order to continue the integration, choose linear combinations $\varpi_1, \varpi_2, \varpi_3$ such that the principal determinant of the matrix of the $\varpi_i(A_k)$ is reduced to its normal form. In order to do this, one may take:

$$\bar{\omega}_1 = dx - \left(\frac{1}{y''} + \frac{3xy'}{1+y'^2}\right)dy' + x\frac{dy''}{y''},$$

$$\bar{\omega}_2 = dy - \left(\frac{y'}{y''} + \frac{3yy'}{1+y'^2}\right)dy' + y\frac{dy''}{y''},$$

$$\bar{\omega}_3 = \frac{3y'dy'}{1+y'^2} - \frac{dy''}{y''}.$$

On the other hand:

$$du = \bar{\omega}_1 + u\bar{\omega}_3,$$
$$dv = \bar{\omega}_2 + v\bar{\omega}_3,$$

and:

$$\bar{\omega}_1' = -[\bar{\omega}_1\,\bar{\omega}_3],$$
$$\bar{\omega}_2' = -[\bar{\omega}_2\,\bar{\omega}_3],$$
$$\bar{\omega}_3' = 0.$$

As a result, $\bar{\omega}_3$ is an exact differential, and one gets the missing first integral *by a quadrature.* The general solution of the given equation is furnished by the formulas:

$$\frac{(1+y'^2)^{3/2}}{y''} = C_3,$$
$$x = \frac{C_3 y'}{\sqrt{1+y'^2}} + C_1$$
$$y = -\frac{C_3}{\sqrt{1+y'^2}} + C_2.$$

Here, one sees that the group G that preserves the given data is:

$$\bar{C}_1 = C_1, \qquad \bar{C}_2 = C_2, \qquad \bar{C}_3 = aC_3, \qquad \left(\text{because} \quad \bar{\omega}_3 = \frac{dC_3}{C_3}\right),$$

with an arbitrary constant a. It is because of the fact that there is only one parameter that the integration reduces to a quadrature. In the preceding exercise, the group G reduced to the identity transformation, and the solution was obtained without integration

171. REMARK. – For all of the examples in which one arrives at n invariant linear forms, one obtains integral invariants of all degrees by constructing an arbitrary exterior form with constant coefficients from the $\omega_1, \ldots, \omega_n$. This is why one has the invariant integral $\iiint \bar{\omega}_1\bar{\omega}_2\bar{\omega}_3$

in the latter exercise, which, if one is limited to sets of states that correspond to the same value of x, reduces to:

$$\iiint \frac{dy\,dy'\,dy''}{y''^2}.$$

As a result, if one considers an arbitrary family of circumferences that depend on three parameters, and if one cuts the circles of that family by an arbitrary parallel to the y-axis then the integral $\iiint \frac{dy\,dy'\,dy''}{y''^2}$, when taken over the family of circles under consideration, will be independent of x. It will be, moreover, equal to $\iiint \frac{dC_1\,dC_2\,dC_3}{C_3}$ if one denotes the coordinates of the center by C_1 and C_2 and those of the ray by C_3.

CHAPTER XVII

APPLICATION OF THE PRECEDING THEORIES TO THE *n*-BODY PROBLEM

I. - *Reduction of the number of degrees of freedom.*

172. We have already seen (sec. **123**) how the method of integration that was discussed in Chapter XII is applied to the canonical equations of dynamics:

$$\frac{dq_i}{dt} = \frac{\partial H}{\partial p_i}, \qquad \frac{dp_i}{dt} = -\frac{\partial H}{\partial p_i}.$$

We suppose the function H is arbitrary. If that function is independent of time then the function H will be a first integral (sec. **92**), and one is left with the integration of the equations:

$$\frac{dq_i}{\dfrac{\partial H}{\partial p_i}} = -\frac{dp_i}{\dfrac{\partial H}{\partial q_i}},$$

whose first integrals are solutions of the equation:

$$(H\,f) = 0$$

and a quadrature.

173. A little later on, we shall study the reduction in the integration of the n-body problem that is produced by accounting for the previously determined (sec. **93**) infinitesimal transformations that the equations of motion admit. We suppose, as is permissible, that the system of n bodies is referred to its center of gravity, i.e., that the $3n$ coordinates x_i, y_i, z_i, and the $3n$ velocity components x_i', y_i', z_i' are coupled by the relations:

$$\sum m_i x_i = 0, \qquad \sum m_i y_i = 0, \quad \sum m_i z_i = 0,$$
$$\sum m_i x_i' = 0, \qquad \sum m_i y_i' = 0, \quad \sum m_i z_i' = 0.$$

Let U be a function of forces that is assumed to be homogeneous and of degree $-p$ with respect to the coordinates. The equations of motion then admit the five infinitesimal transformations:

$$A_0 f = \frac{\partial f}{\partial t}$$

$$A_1 f = \sum\left(y_i \frac{\partial f}{\partial z_i} - z_i \frac{\partial f}{\partial y_i} + y_i' \frac{\partial f}{\partial z_i'} - z_i' \frac{\partial f}{\partial y_i'} \right),$$

$$A_2 f = \sum\left(z_i \frac{\partial f}{\partial x_i} - x_i \frac{\partial f}{\partial z_i} + z_i' \frac{\partial f}{\partial x_i'} - x_i' \frac{\partial f}{\partial z_i'} \right),$$

$$A_3 f = \sum\left(x_i \frac{\partial f}{\partial y_i} - y_i \frac{\partial f}{\partial x_i} + x_i' \frac{\partial f}{\partial y_i'} - y_i' \frac{\partial f}{\partial x_i'} \right),$$

$$A_4 f = \sum\left[x_i \frac{\partial f}{\partial x_i} + y_i \frac{\partial f}{\partial y_i} + z_i \frac{\partial f}{\partial z_i} - \frac{p}{2}\left(x_i' \frac{\partial f}{\partial x_i'} + y_i' \frac{\partial f}{\partial z_i} + z_i' \frac{\partial f}{\partial z_i'} \right)\right] + \left(1 + \frac{p}{2}\right) t \frac{\partial f}{\partial t}.$$

On the other hand:

$$\omega' = \sum \{m_i [\delta x_i' \, \delta x_i] + m_i [\delta y_i' \, \delta y_i] + m_i [\delta z_i' \, \delta z_i]\} - [\delta H \, \delta t],$$

if one sets:

$$H = \frac{1}{2}\sum m_i (x_i'^2 + y_i'^2 + z_i'^2) - U.$$

174. The five invariant linear forms:

$$\omega_i = \omega'(A_i, \delta)$$

are:

$$(1) \qquad \begin{cases} \omega_0 = \delta H, \\ \omega_1 = \delta H_1, \\ \omega_2 = \delta H_2, \\ \omega_3 = \delta H_3, \\ \omega_4 = -\sum m_i \left(x_i \, \delta x_i' + y_i \, \delta y_i' + z_i \, \delta z_i' + \frac{p}{2} x_i' \, \delta x_i + \frac{p}{2} y_i' \, \delta y_i + \frac{p}{2} z_i' \, \delta z_i\right) \\ \qquad\qquad + \left(1 + \frac{p}{2}\right) t \, \delta H + pH \, \delta t, \end{cases}$$

in which one sets:

$$(2) \qquad \begin{cases} H_1 = \sum m_i (y_i' z_i - z_i' y_i), \\ H_2 = \sum m_i (z_i' x_i - x_i' z_i), \\ H_3 = \sum m_i (x_i' y_i - y_i' x_i). \end{cases}$$

Finally, one has:

$$\omega_4' = A_4(\omega') = \left(1 - \frac{p}{2}\right)\omega'.$$

The matrix of quantities $a_{ij} = \omega'(A_i, A_j)$ has already been addressed in the most general case (sec. **95**). We reproduce it below.

	0	**1**	**2**	**3**	**4**
0	0	0	0	0	$-pH$
1	0	0	H_3	$-H_2$	$\left(1-\frac{p}{2}\right)H_1$
2	0	$-H_3$	0	H_1	$\left(1-\frac{p}{2}\right)H_2$
3	0	H_2	$-H_1$	0	$\left(1-\frac{p}{2}\right)H_3$
4	pH	$\left(\frac{p}{2}-1\right)H_1$	$\left(\frac{p}{2}-1\right)H_2$	$\left(\frac{p}{2}-1\right)H_3$	0

175. We now recognize five invariant linear forms, and the matrix of coefficients a_{ij} is defined by the operation of the generalized Poisson brackets:

$$N\,[\omega'^{N-1}\,\omega_i\,\omega_j] = a_{ij}\,[\omega'^{N}].$$

We apply the theory of Chapter XII (sec. **125**). Construct the auxiliary form:

$$\Phi(\xi) = \sum_{(ij)}^{0,1\ldots,4} a_{ij}[\xi_i\,\xi_j].$$

It may be written:

$$\Phi(\xi) = pH\,[\xi_4\,\xi_0] + \frac{p-2}{2}\,[\xi_4\,(H_1\,\xi_1 + H_2\,\xi_2 + H_3\,\xi_3)] + H_1\,[\xi_2\,\xi_3] + H_2\,[\xi_3\,\xi_1] + H_3\,[\xi_1\,\xi_2].$$

It is of rank 4, and its reduction to normal form:

$$\Phi = [\,\xi_4'\,\xi_0'\,] + [\,\xi_1'\,\xi_2'\,]$$

can be accomplished by setting:

$$\xi_0' = pH\xi_0 + \frac{p-2}{2}\,(H_1\,\xi_1 + H_2\,\xi_2 + H_3\,\xi_3),$$

$$\begin{aligned}
\xi_4' &= \xi_4\,, \\
\xi_1' &= \alpha_1\,\xi_1 + \alpha_2\,\xi_2 + \alpha_3\,\xi_3\,, \\
\xi_2' &= \beta_1\,\xi_1 + \beta_2\,\xi_2 + \beta_3\,\xi_3\,, \\
\xi_3' &= \xi_0\,,
\end{aligned}$$

in which the α_i and β_i are chosen – as is always possible – in such a manner as to make:

$$\alpha_2\,\beta_3 - \beta_2\,\alpha_3 = H_1\,, \quad \alpha_3\,\beta_1 - \beta_3\,\alpha_1 = H_2 \quad \alpha_1\,\beta_2 - \beta_1\,\alpha_2 = H_3\,.$$

One may add the supplementary conditions:

$$\alpha_1\beta_1 + \alpha_2\beta_2 + \alpha_3\beta_3 = 0,$$
$$\alpha_1^2 + \alpha_2^2 + \alpha_3^2 = \beta_1^2 + \beta_2^2 + \beta_3^2 = \sqrt{H_1^2 + H_2^2 + H_3^2}\,.$$

However, if one defines five linear forms ϖ_0, ϖ_1, ϖ_2, ϖ_3, ϖ_4 by the identity:

$$\xi_0\,\omega_0 + \xi_1\,\omega_1 + \xi_2\,\omega_2 + \xi_3\,\omega_3 + \xi_4\,\omega_4 = \xi_0'\,\varpi_0 + \xi_1'\,\varpi_1 + \xi_2'\,\varpi_2 + \xi_3'\,\varpi_3 + \xi_4'\,\varpi_4$$

then one will obtain, without difficulty:

$$\varpi_4 = \omega_4\,,$$

$$\varpi_0 = \frac{2}{p-2}\,\frac{H_1dH_1 + H_2dH_2 + H_3dH_3}{H_1^2 + H_2^2 + H_3^2}\,,$$

$$\varpi_3 = dH - \frac{2pH}{p-2}\,\frac{H_1dH_1 + H_2dH_2 + H_3dH_3}{H_1^2 + H_2^2 + H_3^2}\,,$$

$$\varpi_1 = \frac{\alpha_1dH_1 + \alpha_2dH_2 + \alpha_3dH_3}{\sqrt{H_1^2 + H_2^2 + H_3^2}}\,,$$

$$\varpi_2 = \frac{\beta_1dH_1 + \beta_2dH_2 + \beta_3dH_3}{\sqrt{H_1^2 + H_2^2 + H_3^2}}\,.$$

When the auxiliary form Φ has been reduced to $[\xi_4'\,\xi_0'] + [\xi_1'\,\xi_2']$, we will have:

$$\omega' = [\varpi_4\varpi_0] + [\varpi_1\varpi_2] + [\omega_5\varpi_3] + [\omega_6\,\omega_7] + \ldots;$$

i.e., when we perform the calculations:

$$(3)\qquad \begin{cases} \omega' = \dfrac{2}{p-2}\left[\omega_4 \dfrac{H_1 dH_1 + H_2 dH_2 + H_3 dH_3}{H_1^2 + H_2^2 + H_3^2}\right] \\ \qquad + \dfrac{H_1[dH_2 dH_3 + H_2[dH_3 dH_1] + H_3[dH_1 dH_2]}{H_1^2 + H_2^2 + H_3^2} + \Omega, \end{cases}$$

in which we have set:

$$(4)\qquad \Omega = \left[\omega_5\left(dH - \frac{2pH}{p-2}\frac{H_1 dH_1 + H_2 dH_2 + H_3 dH_3}{H_1^2 + H_2^2 + H_3^2}\right)\right] + [\omega_6 \omega_7] + \cdots$$

176. If one equates the four first integrals H, H_1, H_2, H_3 to arbitrary constants then the rank of ω' will be reduced by six units. It thus passes from $6n - 6$ to $6n - 12$, which corresponds to a problem with $3n - 6$ degrees of freedom (which will be 3 in the case of the three-body problem) as a consequence. However, the corresponding characteristic system contains arbitrary parameters.

There is a (theoretical) procedure for reducing the number of degrees of freedom while completely avoiding the introduction of arbitrary parameters. After annulling the exterior derivative in the right-hand side of equation (3) and taking into account the relation:

$$\omega_4' = \frac{2-p}{2}\omega',$$

one will obtain:

$$\Omega' = \left[\frac{H_1\, dH_1 + H_2\, dH_2 + H_3\, dH_3}{H_1^2 + H_2^2 + H_3^2}\,\Omega\right].$$

This relation expresses the idea that the exterior derivative of the quadratic form:

$$\frac{1}{\sqrt{H_1^2 + H_2^2 + H_3^2}}\Omega = \frac{1}{\sqrt{H_1^2 + H_2^2 + H_3^2}}\omega' - \frac{2}{p-2}\left[\omega_4 \frac{H_1\, dH_1 + H_2\, dH_2 + H_3\, dH_3}{(H_1^2 + H_2^2 + H_3^2)^{3/2}}\right]$$

$$- \frac{H_1[dH_2\, dH_3] + H_2[dH_3\, dH_1] + H_3[dH_1\, dH_2]}{(H_1^2 + H_2^2 + H_3^2)^{3/2}}$$

$$= \frac{2}{2-p}\left(\frac{1}{\sqrt{H_1^2 + H_2^2 + H_3^2}}\omega_4\right)' - \frac{H_1[dH_2\, dH_3] + H_2[dH_3\, dH_1] + H_3[dH_1\, dH_3]}{(H_1^2 + H_2^2 + H_3^2)^{3/2}}$$

is zero. This property is, moreover, evident in the right-hand side of the preceding equality, whose first term has a zero exterior derivative since it is an exact exterior derivative. We shall see that the same is true for the second term.

In order to interpret the second term, consider the vector (OS) of length $\gamma = \sqrt{H_1^2 + H_2^2 + H_3^2}$, which represents the kinetic moment of the system with respect to the origin, and which has H_1, H_2, H_3 for its projections. If one imagines a surface element $d\sigma$ that is described by the point S, and if one calls the direction cosines of the normal to that surface element α_1, α_2, α_3 then one will have:

$$[dH_2\, dH_3] = \alpha_1\, d\sigma, \quad [dH_3\, dH_1] = \alpha_2\, d\sigma, \quad [dH_1\, dH_2] = \alpha_3\, d\sigma,$$

and, as a result:

$$\frac{H_1[dH_2\, dH_3] + H_2[dH_3\, dH_1] + H_3[dH_1\, dH_2]}{(H_1^2 + H_2^2 + H_3^2)^{3/2}} = \frac{\alpha_1 H_1 + \alpha_2 H_2 + \alpha_3 H_3}{\gamma^2}\, d\sigma = \frac{\cos\psi\, d\sigma}{\gamma^2} = d\omega,$$

in which ψ represents the angle that OS makes with the normal to the element, and $d\omega$ represents the solid angle that the surface element subtends at the origin. If one denotes the co-latitude (viz., the angle with Oz) and the longitude (viz., the angle between the plane zOS and the plane xOz) of the point S by θ and φ, respectively, then one will have, on the one hand:

$$d\omega = \sin\theta\, [d\theta\, d\varphi].$$

As a result, the form under consideration can be regarded as the exterior derivative of the linear form $-\cos\theta\, d\varphi$.

We then set:

$$\varpi = \frac{2}{2-p}\frac{\omega_4}{\gamma} + \cos\theta\, d\varphi. \tag{5}$$

We see that *the characteristic equations of the relative integral invariant* $\int \varpi$ *are:*

$$\begin{cases} dH - \dfrac{2p}{p-2} H \dfrac{d\gamma}{\gamma} = 0, \\ \omega_5 = 0, \omega_6 = 0, \omega_7 = 0, \cdots \end{cases} \tag{6}$$

II. - *Equations of motion with respect to a moving reference frame.*

177. It is easy to interpret this system. In order to fix ideas, we place ourselves in the case of the three-body problem of celestial mechanics ($p = 1$). First, we calculate the quantities $\omega_5(A_i)$,

$\omega_6(A_i)$, etc. In order to do this, we shall apply the operation that corresponds to $A_i f$ to both sides of formula (3), while writing it in terms of dH, ω_6, ω_7, ..., ω_{11}. We immediately have:

$$\begin{aligned} dH &= \omega_5(A_0)\, dH - \omega_7(A_0)\, \omega_6 + \omega_6(A_0)\, \omega_7 + \ldots + \omega_{10}(A_0)\, \omega_{11}, \\ 0 &= \omega_5(A_1)\, dH - \omega_7(A_1)\, \omega_6 + \omega_6(A_1)\, \omega_7 + \ldots + \omega_{10}(A_1)\, \omega_{11}, \\ &\ldots \\ 0 &= \omega_5(A_4)\, dH - \omega_7(A_4)\, \omega_6 + \omega_6(A_4)\, \omega_7 + \ldots + \omega_{10}(A_4)\, \omega_{11}. \end{aligned}$$

We therefore find that all of the $\omega_\alpha(A_i)$ are zero for $\alpha \geq 5$, except for $\omega_5(A_0)$, which is equal to 1.

On the other hand, we remark that the quantities $A_i H + 2H \dfrac{A_i\gamma}{\gamma}$ are all zero, since the function $K = H\gamma^2$ will be invariant for each of the infinitesimal transformations under consideration. We see that *the characteristic system* (6) *of* $\dfrac{1}{\gamma}\Omega$ *can be defined by all linear combinations of the equations of motion that enjoy the property of being verified identically when one replaces the symbol of undetermined differentiation in them with the symbol of any one of the infinitesimal transformations* $A_1 f$, $A_2 f$, $A_3 f$, $A_4 f$.

178. This result is what permits us to geometrically interpret the system (6).

In order to do that, imagine different possible reference systems, each of which is defined by three rectangular coordinate axes, a time origin, and units of length, time, and mass. We fix the unit of mass once and for all, and impose the condition on the other units that the constant of universal attraction must have a fixed numerical value. The unit of length is also arbitrary. Finally, we fix the origin of the axes, which will be the center of gravity of the three-body system, as well as the time origin.

The remaining reference systems depend on four arbitrary parameters. Three of them fix the orientation of the axes, and the fourth one fixes the units.

One may make a reference system correspond to each *state* of the three bodies (as defined by their positions, velocities, and time, and depending on 13 variables) according to a law that is determined in advance, in such a manner as to reduce the number of quantities that fix the state of the three bodies with respect to that reference system by 4 units. *For example*, one may choose the line that joins the center of gravity to the body A_1 to be the x-axis, the plane of the three bodies to be the xy-plane, and the distance OA_1 to be the unit of length. The state of the three bodies is then defined by the two coordinates of A_2, the six projections of the velocities of A_1 and A_2 onto the three axes, and finally, the time t. One may fix the choice of moving reference system that corresponds to a given state by another law. While always taking Oz to be perpendicular to the plane of the three bodies, one may take Ox to be parallel to $A_1 A_2$, and take the length of the side $A_1 A_2$ to be the unit of length. One may also choose the axes according to either of the preceding laws, but choose the units in such a manner as to make the measure of the kinetic moment OS equal to 1. One may also take Oz to be perpendicular to the plane of the three bodies, take the plane zOS to be the xz-plane, and choose the units in such a way as to make the measure of OS equal to 1. Under this latter hypothesis, the nine quantities that determine the state of the three bodies with respect to the moving reference system will be the two coordinates of A_1, the two coordinates of A_2, time, and finally, the six components of the velocities of A_1 and A_2, which makes 11 quantities, although they will be coupled by the two relations $\gamma = 1$, $\varphi = 0$.

Now suppose that we have made a choice of correspondence between each three-body state and a moving reference frame by one of the preceding laws or any other law imaginable, and let:

$$q_1, q_2, ..., q_9$$

be the nine quantities that determine the state of the three bodies with respect to the corresponding moving reference system. The state of the three bodies will be determined with respect to a fixed reference system if one knows, along with $q_1, ..., q_9$, the four parameters u_1, u_2, u_3, u_4 that define the position of the moving reference system with respect to the fixed reference system. These four parameters will be, for example, the three parameters that depend on the nine direction cosines and the ratio of the moving unit of length to fixed unit of length. Finally, the quantities (19 in number, but reducing to 13):

$$x_i\,, y_i\,, z_i\,, x_i'\,, y_i'\,, z_i'\,, t$$

that fix the state of the three bodies with respect to the fixed reference system are definite functions of the 13 quantities:

$$q_1, q_2, ..., q_9, \quad u_1, u_2, u_3, u_4.$$

Conversely, the latter are definite functions of the former. However, *when the* 9 *quantities* $q_1, ..., q_9$ *are considered to be functions of the* $x_i\,, y_i\,, z_i\,, x_i', y_i', z_i', t$ *they will obviously be invariant under each of the infinitesimal transformations* $A_1 f, ..., A_4 f$, because performing one of these transformations amounts to changing the fixed system of reference and then altering the quantities u_1, u_2, u_3, u_4 that define the relation between the moving reference system and the fixed reference system, but without altering the quantities q_i that define the state of the three bodies with respect to the moving reference system.

One may also say that if one looks for all of the linear differential forms in $dx_i\,, dy_i\,, ..., dy_i'$ that enjoy the property of being annulled when one replaces the symbol d in them with the symbols $A_1 f, ..., A_4 f$ then one will find all of the linear combinations in $dq_1, ..., dq_9$, and uniquely at that.

In particular, equations (8) for the characteristic system of Ω have left-hand sides that are linear in $dq_1, ..., dq_9$. Since they are 8 in number, these equations (6) may be put into the form:

$$dq_i - C_i\,dq_9 = 0 \qquad (i = 1, 2, ..., 8),$$

and, since they are completely integrable the C_i depend only on the q_i. In other words, we say that system (6) is a system of ordinary differential equations in $q_1, ..., q_8$. *It therefore defines the motion of the three bodies with respect to the moving reference frame.*

179. It is now easy to effectively form the equations of system (6). In order to do this, we start with the relative integral invariant $\int \varpi$, in which we have set:

$$\varpi = \frac{2}{\gamma}\omega_4 + \cos\theta\, d\varphi,$$

and imagine that we have expressed all of the quantities x_i , y_i , ..., z'_i , t in terms of the q_1, ..., q_9, u_1, ..., u_4. First of all, we know that the differentials du_1, du_2, du_3, du_4 cannot appear in the defining expression for ϖ' , which can be constructed in terms of the linear forms $dq_i - C_i\, dq_9$, uniquely. As a result, in order to calculate ϖ' we can regard u_1, ..., u_4 as fixed parameters. Moreover, when the coefficients of the form ϖ' are expressed in terms of the dq_1, ..., dq_9 they cannot contain the variables u_1, ..., u_4, since then the exterior derivative ϖ' would not be zero then. In order to carry out the calculation, one can therefore not only regard u_1, ..., u_4 as fixed parameters, but *one can also give them arbitrary numerical values.* In particular, one may thus give them numerical values that correspond to the case in which the fixed reference system agrees with the moving reference system. In other words, *in order to form* ϖ', *one may give the values* X_i, Y_i, ..., Z'_i , T, which define the state of the three bodies with respect to the moving reference system, to the quantities x_i , y_i , ..., z'_i , t in ϖ. As we have seen, these thirteen quantities reduce down to nine.

180. In particular, we examine the case in which the moving unit of length is chosen in such a manner as to reduce γ by one unit (the moving unit of length is then found to be fixed). In this case, one has:

$$\varpi = 2\,\omega_4 + \cos\,\theta\, d\varphi.$$

If one adds an exact differential then one will get the relative integral invariant $\int \omega + \cos\,\theta\, d\varphi$ for the desired differential equations.

Since the z-axis is assumed to be normal to the plane of the three bodies, and the x-axis is assumed to be parallel to A_1A_2 , for example, the position of the triangle will depend on three quantities ξ_1, ξ_2, ξ_3. However, one will have:

$$\omega + \cos\,\theta\, d\varphi \;= \sum m_i\,(X'_i\, dX_i + Y'_i\; dY_i) - H\, dT + \cos\theta\, d\varphi$$

$$= \eta_1\, d\xi_1 + \eta_2\, d\xi_2 + \eta_3\, d\xi_3 + \eta_4\, d\xi_4 - H\, dt,$$

in which we have set:

$$\xi_4 = \varphi, \qquad \eta_4 = \cos\theta.$$

The desired equations of relative motion are then:

$$\frac{d\xi_i}{dt} = \frac{\partial H}{\partial \eta_i}, \qquad \frac{d\eta_i}{dt} = -\frac{\partial H}{\partial \xi_i} \quad (i = 1, 2, 3, 4).$$

They are canonical and admit the first integral H = const.

For example, one may take ξ_1, ξ_2, ξ_3 *to be the lengths of the sides of the triangle* $A_1 A_2 A_3$.

If one supposes that the motion is planar then θ will be zero, and there will be no more than six unknown functions ξ_1, ξ_2, ξ_3, η_1, η_2, η_3.

181. Once the motion of the three bodies with respect to the moving reference system is known, the absolute motion will be determined by a quadrature. Indeed, if we first know the projections of the kinetic moment *OS* onto the moving axes then we will know the ratio of the moving units to the fixed ones by giving it the constant number *C* that measures *OS* with respect to the fixed reference. One may then take *OS* to be the fixed *z*-axis, where the position of the fixed axes depends on an unknown angle. This angle will be given by a quadrature. Indeed, it suffices to remark that the invariant form ω_4 (when expressed by means of the *fixed* coordinates) is an exact differential when one takes into account the relations that are assumed to be obtained, and which define the relative motion. Indeed, the formula:

$$2\omega_4' = \omega' = 2\left[\frac{d\gamma}{\gamma}\omega_4\right] + \frac{H_1[dH_2\, dH_3] + [H_2\, dH_3\, dH_1] + H_3[dH_1\, dH_2]}{\gamma^2} + \Omega$$

shows that under these conditions ω_4' is zero (since H_1 and H_2 are zero). The integration is thus accomplished by means of the formula:

$$\int \omega_4 = \text{const.}$$

One may remark that this quadrature may be performed when one has determined the motion from only the geometric viewpoint without having found the time (by a quadrature, as one knows). In other words, the two quadratures that give time (in the relative motion) and the orientation that defines the fixed axes with respect to the moving axes *can be performed independently of each other.*

III. - *Case in which the area constants are all zero.*

182. The preceding theory essentially assumed that $H_1^2 + H_2^2 + H_3^2 \neq 0$. We study the motions for which all three area constants will be zero. In this case, it is necessary to suppose that the 18 quantities:

$$x_i,\ y_i,\ z_i,\ x_i',\ y_i',\ z_i'$$

are not only coupled by the relations:

$$\sum m_i x_i = 0, \qquad \sum m_i y_i = 0, \qquad \sum m_i z_i = 0,$$
$$\sum m_i x_i' = 0, \qquad \sum m_i y_i' = 0, \qquad \sum m_i z_i' = 0,$$

but also by the relations:

$$\sum m_i (y_i' z_i - z_i' y_i) = 0, \qquad \sum m_i (z_i' x_i - x_i' z_i) = 0, \qquad \sum m_i (x_i' y_i - y_i' x_i) = 0.$$

It is easy to show that the plane of the triangle of the three bodies remains fixed, because the components of the three velocities that are normal to the plane are all zero, at least if the three bodies are not all in a straight line.

We may thus suppose that the z_i and the z_i' are zero, and there then remain five relations between the 12 quantities x_i, y_i, x_i', y_i', namely:

$$\sum m_i x_i = 0, \qquad \sum m_i y_i = 0,$$
$$\sum m_i x_i' = 0, \qquad \sum m_i y_i' = 0,$$
$$\sum m_i (x_i' y_i - y_i' x_i) = 0.$$

All totalled, there are then seven dependent variables and one independent variable (time). However, ω', which is of even rank, cannot have a rank that is equal to the number of differential equations of motion. *Therefore, the characteristic system of ω' does not agree with that of the equations of motion.*

Here we have three infinitesimal transformations $A_0 f$, $A_3 f$, $A_4 f$ with:

$$\omega'(A_0, \delta) = \delta H, \quad \omega'(A_3, \delta) = 0, \quad \omega'(A_4, \delta) = \omega_4.$$

The seven differential equations of motion can then be put into the form of Pfaff equations:

$$\varpi_1 = 0, \quad \varpi_2 = 0, \ldots, \varpi_7 = 0,$$

and one may suppose that:

$$\varpi_1(A_3) = \ldots = \varpi_6(A_3) = 0, \qquad \varpi_7(A_3) = 1.$$

The form ω', which can be expressed by means of the $\varpi_1, \ldots, \varpi_7$, *certainly does not contain* ϖ_7, since otherwise the form $\omega'(A_3, \delta)$ would not be identically zero. As a result, *the characteristic system of ω' will be the completely integrable system:*

$$\varpi_1 = \varpi_2 = \ldots = \varpi_6 = 0.$$

It gives the motion of the three bodies *independently of the orientation of the triangle of the three bodies around the center of gravity.* Once this system is integrated, *the integration will be given by a quadrature.* Indeed, the relation $\varpi_7(A_3) = 1$ assures us that ϖ_7 will have the property of being an invariant form for the differential equation $\varpi_7 = 0$.

Now, return to the form ω' of rank 6. Its characteristic system admits the two infinitesimal transformations $A_0 f$ and $A_4 f$, which give rise to two invariant linear forms $\omega_0 = \delta H$, which we denote by ϖ_1, and ω_4, which we denote by ϖ_2. We assume, as is permissible, that one has $\varpi(A_0) = \varpi(A_4) = 0$ for each of the forms $\varpi_3, \ldots, \varpi_6$. A calculation that is analogous to the one that was made in the general case gives:

$$\omega' = -\frac{1}{H}[\varpi_1 \varpi_2] + \Omega = -\frac{1}{H}[\delta H \varpi_2] + \Omega,$$

in which Ω has rank 4 and is formed from ϖ_2, ϖ_4, ϖ_5, ϖ_0.

As we saw above, one has $\omega' = 2\,\omega_4' = 2\,\omega_2'$, and, as a result:

$$\Omega = 2\varpi_2' + \left[\frac{\delta H}{H}\varpi_2\right] = \frac{2}{\sqrt{H}}(\sqrt{H}\varpi_2)'.$$

The form $\frac{1}{2}\sqrt{H}\cdot\Omega$ is therefore an exact derivative; as a result it admits the following equations:

$$\varpi_3 = \varpi_4 = \varpi_5 = \varpi_6 = 0 \tag{7}$$

for its characteristic system.

This system may be integrated by some equations of order:

$$4, \quad 2, \quad 0.$$

By definition, the equations of motion will be given by operations of order 4 and 2 after two quadratures.

We remark that from the expression (1) for $\omega_4 = \varpi_2$, the form $\sqrt{H}\varpi_2$, which plays the role of relative invariant for system (7), is equal to:

$$\sqrt{H}\varpi_2 = -\sqrt{H}\sum m_i(x_i\,\delta x_i' + y_i\,\delta y_i' + \frac{1}{2}x_i'\,\delta x_i + \frac{1}{2}y_i'\,\delta y_i) + \delta(H^{3/2}t).$$

System (7) is easy to interpret: It gives the motion of the three bodies with respect to a moving reference system, which one may make to correspond with each state of the three bodies according to a determined law *for which the origin of time no longer is necessarily fixed.* For example, one may choose the actual instant for the moving time origin, and fix the unit of length by the condition that the energy H will have a given fixed numerical value. The equations of the system are obtained by starting with the form $\sqrt{H}\varpi_2$ in which one includes the *moving* coordinates. Obviously, under the hypotheses envisioned, one may substitute the form:

$$\sum m_i(x_i'\,\delta x_i + y_i'\,\delta y_i).$$

Here, the quantities of motion of the three bodies form a system of vectors that is equivalent to zero. One may thus regard the quantity of motion of the bodies A_i as the resultant of two vectors u_i and u_k that are directed along the sides $A_i A_k$ and $A_i A_j$ and counted positively when they are projections of $A_k A_i$ and $A_j A_i$. One then has, upon denoting the three sides of the triangle by r_1, r_2, r_3:

$$\omega = u_1\,\delta r_1 + u_2\,\delta r_2 + u_3\,\delta r_3.$$

One has, moreover:

$$H = \frac{1}{2m_1}\left(u_2^2 + u_3^2 + u_2 u_3 \frac{r_2^2 + r_3^2 - r_1^2}{r_2 r_3}\right) + \cdots - f\left(\frac{m_2 m_3}{r_1} + \frac{m_3 m_1}{r_2} + \frac{m_1 m_2}{r_3}\right) = h.$$

The equations of relative motion are then:

$$\frac{dr_1}{\frac{\partial H}{\partial u_1}} = \frac{dr_2}{\frac{\partial H}{\partial u_2}} = \frac{dr_3}{\frac{\partial H}{\partial u_3}} = \frac{-du_1}{\frac{\partial H}{\partial r_1}} = \frac{-du_2}{\frac{\partial H}{\partial r_2}} = \frac{-dr_3}{\frac{\partial H}{\partial r_3}}.$$

IV. - *Case in which the vis viva constant is zero.*

183. The preceding theory implicitly supposed that the *vis viva* constant was non-zero. If we suppose that it is zero then the variables will be subject to a new relation:

$$\frac{1}{2}\sum m_i(x_i'^2 + y_i'^2) - U = 0.$$

There are now only six dependent variables and one independent variable. The invariant form $\omega'(A_0, \delta)$, is identically zero here, just as the form $\omega'(A_3, \delta)$ is.

The system of equations of motion may be put into the form:

$$\varpi_1 = \varpi_2 = \ldots = \varpi_6 = 0,$$

and one may suppose (sec. **163**) that:

$$\varpi_1(A_0) = \varpi_2(A_0) = \ldots = \varpi_4(A_0) = 0, \qquad \varpi_5(A_0) = 1, \qquad \varpi_6(A_0) = 0,$$
$$\varpi_1(A_3) = \varpi_2(A_3) = \ldots = \varpi_4(A_3) = 0, \qquad \varpi_5(A_3) = 1, \qquad \varpi_6(A_3) = 0.$$

When the form ω' is expressed in terms of the ϖ_i, it will obviously contain neither ϖ_5 nor ϖ_6. Finally, suppose that:

$$\varpi_2(A_4) = \varpi_3(A_4) = \varpi_4(A_4) = 0, \qquad \varpi_1(A_4) = 1,$$

and $\omega'(A_4, \delta) = \varpi_2$. One will have:

$$\omega' = 2\,\omega_2' = [\varpi_1\varpi_2] + [\varpi_3\varpi_4].$$

The form ϖ_2 is of the second type, and the equations:

$$\varpi_2 = \varpi_3 = \varpi_4 = 0$$

form a completely integrable system that is characteristic for the equation $\varpi_2 = 0$. They define the motion of the three bodies with respect to a moving reference system whose time origin is variable. Here, for example, one may choose the side r_3 of the triangle to be a unit of length. The equations to be integrated then constitute the characteristic system of the Pfaff *equation:*

$$2\,d(u_1 r_1 + u_2 r_2 + u_3) - u_1\,dr_1 - u_2\,dr_2 = 0,$$

in which the quantities u_1, u_2, u_3, r_1, r_2 are coupled by the relation:

$$\frac{1}{2m}\left(u_2^2 + u_3^2 + 2u_2 u_3 \cos A_1\right) + \cdots - f\left(\frac{m_2 m_3}{r_1} + \frac{m_3 m_1}{r_2} + m_1 m_2\right) = 0\,.$$

If one sets:

$$r_1 = x, \quad r_2 = y, \quad u_1 r_1 + u_2 r_2 + u_3 = z, \quad u_1 = 2p, \quad u_2 = 2q$$

then one will be reduced to the integration of the first order partial differential equation:

$$\begin{aligned}
2\left(\frac{1}{m_2} + \frac{1}{m_3}\right)p^2 + 2\left(\frac{1}{m_1} + \frac{1}{m_3}\right)q^2 + \frac{2}{m_3}\frac{x^2 + y^2 - 1}{xy}\,pq \\
+(z - 2px - 2qy)\left(\frac{p}{m_2}\frac{x^2 + 1 - y^2}{x} + \frac{q}{m_1}\frac{y^2 + 1 - x^2}{y}\right) \\
+\frac{1}{2}\left(\frac{1}{m_1} + \frac{1}{m_2}\right)(z - 2px - 2qy)^2 - f\left(\frac{m_2 m_3}{x} + \frac{m_1 m_3}{y} + m_1 m_2\right) = 0\,.
\end{aligned}$$

Once this equation has been integrated, one will obtain the general solution of the characteristic system of ω' by differentiations because once ϖ_2 has been put into the form $Z_1\,dY_1 + Z_2\,dY_2$, one will deduce the first integrals Y_1, Y_2, Z_1, Z_2 of this system from it by differentiation.

However, the equations of motion are not, moreover, completely integrable. It is necessary to integrate the equations:

$$\varpi_5 = \varpi_6 = 0.$$

They constitute a system of differential equations that admit the two infinitesimal transformations $A_0 f$, $A_1 f$, and the matrix:

$$\begin{Vmatrix} \varpi_5(\mathrm{A}_0) & \varpi_5(\mathrm{A}_3) \\ \varpi_6(\mathrm{A}_0) & \varpi_6(\mathrm{A}_3) \end{Vmatrix}$$

is reduced to its normal form:

$$\begin{Vmatrix} 1 & 0 \\ 0 & 1 \end{Vmatrix},$$

precisely. On the other hand, because one has:

$$A_0(A_3 f) - A_3(A_0 f) = 0,$$

the two transformations $A_0 f$, $A_2 f$ will be interchangeable, and one will then have:

$$\varpi_5' = 0,$$
$$\varpi_6' = 0.$$

As a result, the integration is accomplished by means of two independent quadratures: One gives the orientation of the triangle $A_1 A_2 A_3$, and the other one gives the time.

CHAPTER XVIII

INTEGRAL INVARIANTS AND THE CALCULUS OF VARIATIONS

I. - *Extremals attached to a relative integral invariant.*

184. We have already seen in chapter 1 (sec. **9**) that the differential equations for the extremals of the integral:

$$I = \int F(q_1, ..., q_n ; q_1', ..., q_n' ; t)\, dt,$$

agree with the characteristic equations of the relative integral invariant $\int \omega$, when one sets:

$$\omega = \sum_{i=1}^{i=n} \frac{\partial F}{\partial q_i'} \delta q_i - \left(\sum_{i=1}^{i=n} q_i' \frac{\partial F}{\partial q_i'} - F \right) \delta t ,$$

and regards $q_1, ..., q_n, q_1', ..., q_n', t$, as $2n + 1$ independent variables.

In the calculus of variations, one regards $q_1, ..., q_n$ as arbitrary functions of t and $q_1', ..., q_n'$, as their derivatives. In the $n + 1$-dimensional space of $(q_1, ..., q_n, t)$, any extremal will enjoy the property that when the integral I is taken over a given arc of that curve it will be stationary when compared to the arcs of infinitely close curves that admit the same origin and the same extremity. However, one may also place oneself in the $2n + 1$-dimensional space of $(q_1, ..., q_n, q_1', ..., q_n', t)$. An extremal curve will then enjoy the property that when I is taken over a given arc of that curve it will be stationary when compared to infinitely close curves for which the initial and final values *of only the coordinates* $q_1, ..., q_n, t$ are the same as for the given extremal. If one takes the second point of view then $q_1', ..., q_n'$ will be functions of t that have no relationship to the derivatives of $q_1, ..., q_n$ with respect to t, *a priori*.

185. More generally, start with a linear differential form ω in $2n + 1$ variables. Suppose that the form ω' is of rank $2n$ and finally suppose that n of the coefficients of the differentials in ω are identically zero. We may therefore set:

$$\omega = a_1\, \delta t + a_2\, \delta x_2 + ... + a_n\, \delta t_n - b\, \delta t,$$

since the quantities $a_1, ..., a_n, b$ are functions of the $2n + 1$ independent variables $x_1, ..., x_n, y_1, ..., y_n, t$.

The characteristics of the relative integral invariant ω – or, what amounts to the same thing, the quadratic exterior form ω' – are given by a system of ordinary differential equations:

(1) $$\frac{dx_i}{dt} = X_i, \qquad \frac{dy_i}{dt} = Y_i,$$

if one supposes, as we do, that t is not a first integral of the characteristic equations.

Having said this, consider an arc of a curve in $2n + 1$-dimensional space that connects the point $M(x_i^0, y_i^0, t^0)$ to the point $M(x_i^1, y_i^1, t^1)$, and form the integral:

$$I = \int_{M_0}^{M_1} a_1\,dx_1 + a_2\,dx_2 + \cdots - b\,dt\,.$$

We calculate the variation of that integral when one passes from the arc of the curve under consideration to an arc of a curve infinitely close to the given one that connects the point $(x_i^0 + \delta x_i^0, y_i^0 + \delta y_i^0, t^0 + \delta t^0)$ to the point $(x_1^i + \delta x_i^1, y_i^1 + \delta y_i^1, t^1 + \delta t^1)$. We will have:

$$\delta I = [\omega_\delta]_0^1 + \int_{M_0}^{M_1} (\delta\omega_d - d\,\omega_\delta) = [\omega_\delta]_0^1 + \int_{M_0}^{M_1} \omega'(d, \delta)\,.$$

If we would like the integral to be stationary relative to all of the curves that are infinitely neighboring close to the given curve then it will be necessary to have:

$$\omega'(d, \delta) = 0$$

when one displaces along the arc of the given curve for any δx_i, δy_i, δt. In other words, *it is necessary that the arc of the curve must belong to a characteristic of the form* ω'. The value of the integral will be stationary for all the arcs of the infinitely close curves on which ω_δ is zero at the origin and the curve extremity; i.e., the arcs on which $x_1, ..., x_n$, t has the same initial and final values as the arc of the given curve.

186. Now suppose that we restrict the field of curves that are infinitely close to the given curve to curves for which the functions x_i, y_i, and t satisfy the first n characteristic equations:

(2) $$\frac{dx_i}{dt} = X_i.$$

We assume that the n functions $X_1, ..., X_n$ are independent of the $y_1, ..., y_n$, which permits us to take *arbitrary* functions of t for the x_i. Finally, we suppose that the initial and final values of $y_1, ..., y_n$, t are the same for the varied curves as for the primitive curve. With these conditions, one has:

$$\delta I = -\int \omega'(d, \delta).$$

It is easy to see that there is no term in $dy_1, ..., dy_n$ in $\omega'(d, \delta)$. Indeed, the coefficient of δy_1 will be:

$$\frac{\partial a_1}{\partial y_1}dx_1 + \frac{\partial a_2}{\partial y_1}dx_2 + \cdots + \frac{\partial a_n}{\partial y_1}dx_n - \frac{\partial b}{\partial y_1}dt\,.$$

If one takes the characteristic equations (2) into account then this coefficient will necessarily be annulled. It is therefore zero when one displaces along the extremal. Since only $\delta x_1, \ldots, \delta x_n$ enter under the $\int$ sign in the expression for δI, the coefficients of $\delta x_1, \ldots, \delta x_n$, δt will become zero. As a result, *the extremals will be given by the characteristic equations of* ω'.

II. - *The least-action principle of Maupertuis.*

187. Suppose that the Hamiltonian function H is independent of time. Consider the set of motions for which the function H has a given constant value h. The corresponding trajectories are the characteristics of the linear integral invariant $\int \omega$, with:

$$\omega = \sum_{i=1}^{i=n} p_i\,\delta q_i - h\,\delta t,$$

or, what amounts to the same thing, of the integral invariant $\int \varpi$, with:

$$\varpi = \sum_{i=1}^{i=n} p_i\,\delta x_i\,.$$

Indeed, the form ϖ differs from ω only by an exact differential. This form ϖ is constructed from $2n$ variables, which are coupled by the relation:

$$H = h,$$

and only n of these coefficients are zero. The characteristic equations are:

$$\frac{dq_1}{\dfrac{\partial H}{\partial p_1}} = \cdots = \frac{dq_n}{\dfrac{\partial H}{\partial p_n}} = \frac{-dp_1}{\dfrac{\partial H}{\partial q_1}} = \cdots = \frac{-dp_n}{\dfrac{\partial H}{\partial q_n}}\,.$$

As a result, the trajectories are extremals of the integral:

$$\int p_1\,dq_1 + \ldots + p_n\,dq_n$$

in the $2n - 1$ dimensional space (q_i, p_i), whether one considers all of the curves for which the initial and final values of $q_1, \ldots, q_n$ are given or only those curves that satisfy the equations:

$$\frac{dq_1}{\frac{\partial H}{\partial p_1}} = \cdots = \frac{dq_n}{\frac{\partial H}{\partial p_n}},$$

and, of course, the condition that $H = h$.

188. Take the second viewpoint, for example. Suppose that the q_i are the position parameters of the system, and that the p_i are the components of the quantity of motion. If one denotes the *vis viva* by $2T$ and decomposes it into terms of second degree, first degree, and degree zero, in q_1', ..., q_n' then one will have:

$$H = \sum q_i' \frac{\partial T}{\partial q_i'} - T - U = T_2 - T_0 - U.$$

Substitute the variables q_i' for the p_i. By hypothesis, one has:

$$T_2(q') = T_0 + U + h,$$

$$\varpi = \sum_{i=1}^{i=n} \frac{\partial T}{\partial q_i'} \delta q_i = \sum_{i=1}^{i=n} \frac{\partial T_2}{\partial q_i'} \delta q_i + T_1(\delta q_i).$$

Finally, suppose that one has:

$$\frac{dq_1}{q_1'} = \frac{dq_2}{q_2'} = \cdots = \frac{dq_n}{q_n'} = \frac{\sqrt{T_2(dq)}}{\sqrt{T_2(q')}} = \frac{\sqrt{T_2(dq)}}{\sqrt{T_0 + U + h}}.$$

The quantity under the $\int$ sign in the integral I is:

$$\varpi_d = \sum q_i' \frac{\partial T_2}{\partial q_i'} \cdot \frac{\sqrt{T_2(dq)}}{\sqrt{T_0 + U + h}} + T_1(dq) = \sqrt{2(T_0 + U + h) \cdot 2T_2(dq)} + T_1(dq).$$

Therefore, if one sets:

$$2T = \sum a_{ij}\, q_i'\, q_j' + 2 \sum b_i\, q_i' + 2T_0$$

then one will arrive at the following theorem, which constitutes the *principle of least action in the sense of Maupertuis:*

Trajectories are extremals of the integral:

$$\int \left(\sqrt{2(T_0 + U + h) \cdot \sum a_{ij} dq_i\, dq_j} + \sum b_i\, dq_i \right)$$

relative to all of the infinitely close trajectories, subject to the constraints that they have the same initial and final position of the system and satisfy the vis viva theorem H = h, with a given vis viva constant.

One recovers the principle in its classical form when $T_1 = T_0 = 0$

189. EXAMPLE. – In the case of a free moving point that is referred to fixed axes, the trajectories will be extremals of the integral:

$$\int \sqrt{2(U+h)}\, ds.$$

If the point is referred to axes that are rotating around Oz with a constant angular velocity α, and if, moreover, the time-independent force field is directed along the axes then one will have:

$$\begin{aligned} 2T &= m\,[(x' + \alpha y)^2 + (y' - \alpha x)^2 + z'^2] \\ &= m\,(x'^2 + y'^2 + z'^2) - 2m\alpha\,(x\,y' - y\,x') + m\alpha^2\,(x^2 + y^2). \end{aligned}$$

For a point of mass 1, the trajectories will be extremals of the integral:

$$\int \sqrt{\alpha^2(x^2+y^2)+2U+2h}\, ds - \alpha(x\,dy - y\,dx).$$

III. - *Generalizations.*

190. Everything that was done in the case where time does not explicitly enter into H can also be done in the case where H does not contain one of the other variables q_i and p_i . To fix ideas, take the case of a free material point of mass 1 that is subject to a central force that is a function of the distance. Consider all of the motions that it makes around the given plane, which we take to be the xy-plane, and which obey the law of areas with a given constant C. The real motions are given by a system of differential equations that admit the relative integral invariant:

$$\int \omega = \int \left[r'\delta r + r^2\theta'\,\delta\theta - \left(\tfrac{1}{2}r'^2 + \tfrac{1}{2}r^2\theta'^2 - U\right)\right]\delta t,$$

which obviously reduces to:

$$\int \omega = \int r'\delta r - \left(\tfrac{1}{2}r'^2 + \tfrac{1}{2}\frac{C^2}{r^2} - U\right)\delta dt.$$

The form ϖ depends upon only the variables r, r', and t, and one of its characteristic equations is:

$$\frac{dr}{dt} = r'.$$

Iresults from this that if one is given the values r_0 and t_0 for initial conditions and r_1 and t_2 for final conditions then the actual motion that satisfies these conditions will be the one that makes the integral:

$$L = \int_{t_0}^{t_1} r' dr - \left(\tfrac{1}{2} r'^2 + \tfrac{1}{2} \frac{C^2}{r^2} - U \right) dt = \int_{t_0}^{t_1} \left[\tfrac{1}{2} \left(\frac{dr}{dt} \right)^2 - \tfrac{1}{2} \frac{C^2}{r^2} - U \right] dt$$

stationary relative to all of the infinitely close motions that satisfy the same conditions at the limits and verify the law of areas with the area constant C

IV. - *Application to the propagation of light in an isotropic medium.*

191. Consider an isotropic medium whose index of refraction n is known at each point. Fermat's principle then leads us to define light rays as the extremals of the integrals:

$$\int n\, ds = \int n \sqrt{dx^2 + dy^2 + dz^2}.$$

If one introduces an auxiliary variable t then one will be dealing with the case of an integral:

$$\int F(x, y, z; x', y', z', t),$$

with:

$$F = n \sqrt{x'^2 + y'^2 + z'^2}.$$

The linear relative integral invariant $\int \omega$, whose light rays are then the characteristics, is defined by the formula:

$$\omega = \frac{\partial F}{\partial x'} \delta x + \frac{\partial F}{\partial y'} \delta y + \frac{\partial F}{\partial z'} \delta z - \left(x' \frac{\partial F}{\partial x'} + y' \frac{\partial F}{\partial y'} + z' \frac{\partial F}{\partial z'} - F \right) \delta t,$$

which becomes:

$$\omega = n \left(\frac{x'}{\sqrt{x'^2 + y'^2 + z'^2}} \delta x + \frac{y'}{\sqrt{x'^2 + y'^2 + z'^2}} \delta y + \frac{z'}{\sqrt{x'^2 + y'^2 + z'^2}} z' \right),$$

or furthermore:

$$\omega = n\, (\alpha\, \delta x + \beta\, \delta y + \gamma\, \delta z),$$

in which α, β, γ denote the directions cosines of an arbitrary direction. The form ω therefore depends upon 5 variables, in reality. It is easy to form its characteristic equations and to show that they contain, in particular, the equations:

$$\frac{dx}{\alpha} = \frac{dy}{\beta} = \frac{dz}{\gamma}.$$

The direction (α, β, γ) is obviously nothing but the tangent to the light ray under consideration.

192. The property of $\int \omega$ that it is a relative integral invariant leads to the property of a pencil of light rays that says that if one describes a closed curve (C) that encircles the pencil then when the integral $\int n \cos\theta\, \delta s$ is taken over that curve it will be independent of the chosen curve, in which we have let θ denote the angle between the tangent to (C) at a point M and the tangent to the pencil of light that passes through M. One may easily prove that the necessary and sufficient condition for the rays of a congruence to all be normal to the same surface is that this integral must be zero for any pencil of rays that is taken from the congruence. This corresponds to the theorem of Malus that the rays of a congruence that are normal to a surface will be normal to an infinitude of surfaces. The condition for this to be the case is that the quadratic exterior form ω' must be zero, or, more precisely, that the alternating bilinear form $\omega'(\delta, \delta')$ must be zero, in which one considers δ to be the symbol of differentiation with respect to one of the parameters of the congruence and δ' to be the symbol of differentiation with respect to the other parameter.

The light rays that propagate in the medium under consideration depend on the four parameters u_1, u_2, u_3, u_4. One calls a transformation that is performed on these parameters that changes any congruence of rays that are normal to one surface into another congruence of rays that are normal to another surface a *Malus transformation.* As we have seen, the form ω' is expressible in terms of the u_i and their differentials. The most general Malus transformation is obviously defined by the equation:

$$\omega'(u', du') = k\,\omega'(u, du),$$

in which k is an unknown function. Taking the exterior derivative of both sides of this equation immediately gives:

$$[dk\ \omega'] = 0.$$

Since ω' is of rank 4, this is possible only if $dk = 0$; i.e., if k is a *constant.* As a result, the desired transformation can be obtained by expressing the idea that the linear form:

$$\omega(u', du') - k\,\omega(u, du)$$

must be an exact differential:

$$n(x', y', z')(\alpha' dx' + \beta' dy' + \gamma' dz') = k\, n(x, y, z)(\alpha\, dx + \beta\, dy + \gamma\, dz) + dV.$$

For example, define a light ray by the coordinates (x_0, y_0) of a point where it intersects the xy-plane, and the direction cosines of its tangent at this point by α_0, β_0, γ_0. One will have:

$$n(x_0', y_0', z_0')(\alpha_0'\, dx_0' + \beta_0'\, dy_0') - kn(x_0, y_0, 0)(\alpha_0\, dx_0 + \beta_0\, dy_0) = dV.$$

1st *Case. – There is no relation between* x'_0, y'_0, z'_0. In this case, V is a known function of x_0, y_0, x'_0, y'_0, and one has:

$$-kn(x_0, y_0, z_0)\alpha_0 = \frac{\partial V}{\partial x_0}, \qquad -kn(x_0, y_0, 0)\beta_0 = \frac{\partial V}{\partial y_0},$$

$$n(x'_0, y'_0, 0)\alpha'_0 = \frac{\partial V}{\partial x'_0}, \qquad n(x'_0, y'_0, 0)\beta'_0 = \frac{\partial V}{\partial y'_0}.$$

The first two equations give x'_0 and y'_0; the last two then give α'_0 and β'_0.

2nd *Case. – There is one and only one relation between* x_0, y_0, x'_0, *and* y'_0. Let:

$$F(x_0, y_0;\ x'_0, y'_0) = 0$$

be that relation. If one denotes an arbitrary function V of x_0, y_0, x'_0, y'_0 and introduces an auxiliary parameter λ then one will have:

$$-kn(x_0, y_0, 0)\,\alpha_0 = \frac{\partial V}{\partial x_0} + \lambda\frac{\partial F}{\partial x_0}, \qquad -kn(x_0, y_0, 0)\beta_0 = \frac{\partial V}{\partial y_0} + \lambda\frac{\partial F}{\partial y_0},$$

$$n(x'_0, y'_0, 0)\alpha'_0 = \frac{\partial V}{\partial x'_0} + \lambda\frac{\partial F}{\partial x'_0}, \qquad n(x'_0, y'_0, 0)\beta'_0 = \frac{\partial V}{\partial y'_0} + \lambda\frac{\partial F}{\partial y'_0}.$$

When the first two of these four equations are combined with the equation $F = 0$ they will give x'_0, y'_0, and λ; the last two then give α'_0 and β'_0.

3rd *Case. –* x'_0 *and* y'_0 *are given functions of* x_0 *and* y_0:

$$x'_0 = f(x_0, y_0), \qquad y'_0 = g(x_0, y_0).$$

V is then a function of x_0, y_0, and one has:

$$n(x'_0, y'_0, 0)\left(\alpha'_0\frac{\partial f}{\partial x_0} + \beta'_0\frac{\partial g}{\partial x_0}\right) = kn(x_0, y_0, 0)\alpha_0 + \frac{\partial V}{\partial x_0},$$

$$n(x'_0, y'_0, 0)\left(\alpha'_0\frac{\partial f}{\partial y_0} + \beta'_0\frac{\partial g}{\partial y_0}\right) = kn(x_0, y_0, 0)\beta_0 + \frac{\partial V}{\partial y_0};$$

these equations then determine α'_0 and β'_0.

193. The form ω' is invariant, and we have seen above the characteristic property of congruences of rays for which that form is identically zero. The invariant form $\frac{1}{2}\omega'^2$ has applications in optics. Its developed expression is:

$$\begin{aligned} &= n\,[dn\,(\alpha\,\delta x + \beta\,\delta y + \gamma\,\delta z)(\delta\alpha\,\delta x + \delta\beta\,\delta y + \delta\gamma\,\delta z)] \\ &\quad - n^2\,([\delta\beta\,\delta\gamma\,\delta y\;\delta z] + [\delta\gamma\,\delta\alpha\;\delta z\;\delta x] + [\delta\alpha\,\delta\beta\;\delta x\;\delta y]). \end{aligned}$$

For example, take all of the light rays that traverse a given surface element $d\sigma$ and whose tangents at the points of intersection are parallel to the lines that are interior to an infinitely small nappe of a cone $d\omega$. The rays under consideration will depend on four parameters u_1, u_2, u_3, u_4, where the first two, for example, define the position of the point of intersection of the element $d\sigma$, and the last two define the orientation of the tangent of that point. Take the *state* each light ray to be characterized by the corresponding point of intersection (x, y, z) and the direction cosines (α, β, γ) of the tangent to that point. The first three quantities x, y, z depend upon only u_1 and u_2, so any cubic exterior form in δx, δy, δz is zero. As a result, up to a sign, the invariant $\frac{1}{2}\omega'^2$ reduces to:

$$\tfrac{1}{2}\omega'^2 = n^2\,([\delta\beta\,\delta\gamma\,\delta y\;\delta z] + [\delta\gamma\,\delta\alpha\;\delta z\;\delta x] + [\delta\alpha\,\delta\beta\;\delta x\;\delta y]).$$

However, if one denotes the direction cosines of the normal to the element $d\sigma$ by λ, μ, ν then one will have:

$$\begin{aligned} &[\delta y\;\delta z] = \lambda\,d\sigma, && [\delta z\;\delta x] = \mu\,d\sigma, && [\delta x\;\delta y] = \nu\,d\sigma, \\ &[\delta\beta\;\delta\gamma] = \alpha\,d\omega, && [\delta\gamma\;\delta\alpha] = \beta\,d\omega, && [\delta\alpha\;\delta\beta] = \gamma\,d\omega. \end{aligned}$$

As a result:

$$\tfrac{1}{2}\omega'^2 = n^2(\lambda\,\alpha + \mu\,\beta + \nu\,\gamma)\,d\sigma\,d\omega = n^2\cos\theta\;d\sigma\,d\omega,$$

in which θ denotes the angle between the normal to the surface and the (mean) direction of the light rays that traverse the surface.

Having said this, if one considers an arbitrary set of light rays that depend upon four parameters then one can take the point on each ray where that ray pierces a given surface (S). All of the rays that pass through this same point form a solid cone, and the integral invariant $\int \frac{1}{2}\omega'^2$ that relates to the given set may be given by the formula:

$$I = \int n^2\cos\theta\;d\sigma\,d\omega,$$

in which $d\sigma$ denotes the surface element of S, and $d\omega$ denotes the nappe of an elementary cone of rays that starts from the same point of S and makes an angle θ with the normal to S.

For example, take the set of light rays that traverse a volume bounded by a closed surface (S), and take each ray at the point where it *leaves* the volume. For that set, one will have:

$$I = \int n^2 d\sigma \int \cos\theta \; d\omega.$$

However, if one takes the longitude φ and co-latitude θ on a sphere of radius 1 to be the coordinates then the integral $\int \cos\theta \; d\omega$ will be equal to:

$$\iint \sin\theta \cos\theta \; d\theta \; d\varphi,$$

which is taken over the hemisphere $0 \le \theta \le \frac{\pi}{2}$; it is thus equal to π. As a result, one has:

$$I = \pi \iint n^2 \, d\sigma.$$

If the medium has index 1 then the rays will be rectilinear, and the integral I will be equal to the product of the area of the surface with π

194. As an application of the general methods of integration that were discussed in chapter XVI, we propose to determine the path of light rays in an isotropic medium in which the index of refraction n depends upon only one of the rectangular coordinates z. Here, one knows the invariant ω', as well as three infinitesimal transformations that correspond to a translation parallel to Ox, a translation parallel to Oy, and a rotation around Oz:

$$A_1 f = \frac{\partial f}{\partial x}, \quad A_2 f = \frac{\partial f}{\partial y}, \quad A_3 f = x\frac{\partial f}{\partial y} - y\frac{\partial f}{\partial x} + \alpha\frac{\partial f}{\partial \beta} - \beta\frac{\partial f}{\partial \alpha}.$$

Since these three transformations leave the form:

$$\omega_\delta = n\,(\alpha\,\delta x + \beta\,\delta y + \gamma\,\delta z)$$

invariant, the linear invariant forms $\omega'(\delta, \mathrm{A}_i)$ will reduce to:

$$\begin{aligned} \delta\omega(A_1) &= \delta(n\,\alpha), \\ \delta\omega(A_2) &= \delta(n\,\beta), \\ \delta\omega(A_3) &= \delta[n(\beta\,x - \alpha\,y)]. \end{aligned}$$

One thus has three first integrals:

$$n\alpha, \qquad n\beta, \qquad n(\beta x - \alpha y).$$

Set:

$$n\alpha = a, \quad n\beta = b, \quad \beta x - \alpha y = c.$$

The last relation shows that any light ray is in a plane parallel to Oz. As a result, one has that:

$$\omega_\delta = \delta\left(ax + by + \int \sqrt{n^2 - a^2 - b^2}\,dz\right)$$
$$-\left(x - a\int \frac{dz}{\sqrt{n^2 - a^2 - b^2}}\right)\delta a - \left(y - b\int \frac{dz}{\sqrt{n^2 - a^2 - b^2}}\right)\delta b\,.$$

As a result, one has:

$$x = a\int \frac{dz}{\sqrt{n^2 - a^2 - b^2}} + a', \qquad y = b\int \frac{dz}{\sqrt{n^2 - a^2 - b^2}} + b'$$

for the trajectories of the light rays.

CHAPTER XIX

FERMAT'S PRINCIPLE
AND THE INVARIANT PFAFF EQUATION OF OPTICS

I. - *Fermat's principle.*

195. In the preceding chapter, we considered an integral invariant of the optics of isotropic media, while supposing that the index of refraction was *independent of time.*

Now take an arbitrary medium, in which we suppose that the propagation of light waves is defined by a Monge equation:

$$(1) \qquad F(x, y, z, t; dx, dy, dz, dt) = 0$$

that is homogeneous in dx, dy, dz, dt. This signifies that the wave emanating from a light signal that is emitted at the instant t at the point (x, y, z) will have:

$$F(x, y, z, t; \mathrm{X} - x, \mathrm{Y} - y, \mathrm{Z} - z, dt) = 0$$

for its equation at the instant $t + dt$.

As one knows, relative to the point (x, y, z) and at the instant t the wave surface has:

$$F(x, y, z, t; X - x, Y - y, Z - z, 1) = 0$$

for its equation.

In such a medium, a light ray is defined by taking x, y, z to be three functions of t that satisfy equation (1) and, moreover, a supplementary condition that constitutes what one calls *Fermat's principle.* Among all of the curves that satisfy equation (1) – or, as on says – among all of the *integral curves* of the Monge equation (1), the light ray that emanates from a given point (x_1, y_1, z_1) is the one that minimizes the time $t_1 - t_0$ that is necessary for the light to go from the first point to the second. In other words, *light rays are extremals of the Mayer problem that is defined by the Monge equation* (1).

196. We briefly recall how Fermat's principle leads to the formation of differential equations that define the light rays. Imagine a light ray that starts from a point (x_0, y_0, z_0) at the instant t_0 and reaches the point (x_1, y_1, z_1) at the instant t_1. If one is given an arbitrary integral curve of equation (1) that is infinitely close to the light ray then one can suppose that x, y, z, t are expressed as functions of a parameter u for the light ray, as well as the integral curve, such that the values 0 and 1 of that parameter correspond to the instants t_0 and t_1 for the light ray, respectively. Let:

$$x + \delta x, \quad y + \delta y, \quad z + \delta z, \quad t + \delta t$$

be functions of u that relate to the varied curve. Denote the derivatives of x, y, z, t with respect to

u by x', y', z', t'. If one writes equation (1) in the form:

(2) $$F(x, y, z, t; x', y', z', t') = 0$$

and varies that equation then one will obtain:

$$\frac{\partial F}{\partial x}\delta x+\frac{\partial F}{\partial y}\delta y+\frac{\partial F}{\partial z}\delta z+\frac{\partial F}{\partial t}\delta t+\frac{\partial F}{\partial x'}\delta x'+\frac{\partial F}{\partial y'}\delta y'+\frac{\partial F}{\partial z'}\delta z'+\frac{\partial F}{\partial t'}\delta t'=0\,.$$

We multiply the left-hand side of the latter equation by $\lambda\,du$, with λ being a given function of u, and integrate from 0 to 1. We have:

$$\int_0^1\left[\lambda\frac{\partial F}{\partial x}\delta x+\lambda\frac{\partial F}{\partial y}\delta y+\lambda\frac{\partial F}{\partial z}\delta z+\lambda\frac{\partial F}{\partial t}\delta t+\lambda\frac{\partial F}{\partial x'}\delta\frac{dx}{du}+\lambda\frac{\partial F}{\partial y'}\delta\frac{dy}{du}\right.$$
$$\left.+\lambda\frac{\partial F}{\partial z'}\delta\frac{dz}{du}+\lambda\frac{\partial F}{\partial t'}\delta\frac{dt}{du}\right]du=0,$$

or, after integrating by parts:

(3) $$\begin{cases}\left[\lambda\left(\dfrac{\partial F}{\partial x'}\delta x'+\dfrac{\partial F}{\partial y'}\delta y'+\dfrac{\partial F}{\partial z'}\delta z'+\dfrac{\partial F}{\partial t'}\delta t'\right)\right]_0^1\\ +\displaystyle\int_0^1\left\{\left[\lambda\frac{\partial F}{\partial x}-\frac{d}{du}\left(\lambda\frac{\partial F}{\partial x'}\right)\right]\delta x+\cdots+\left[\lambda\frac{\partial F}{\partial t}-\frac{d}{du}\left(\lambda\frac{\partial F}{\partial t'}\right)\right]\delta t\right\}du=0.\end{cases}$$

If the integral curve in a neighborhood of the light ray satisfies the initial and final conditions that were imposed then one will have:

$$(\delta x)_0 = (\delta y)_0 = (\delta z)_0 = (\delta x)_1 = (\delta y)_1 = (\delta z)_1 = 0,$$

and, as a result:

$$\lambda_1\left(\frac{\partial F}{\partial t'}\right)_1(\delta t)_1+\int_0^1\left\{\left[\lambda\frac{\partial F}{\partial x}-\frac{d}{du}\left(\lambda\frac{\partial F}{\partial x'}\right)\right]\delta x\;+\cdots+\left[\lambda\frac{\partial F}{\partial t}-\frac{d}{du}\left(\lambda\frac{\partial F}{\partial t'}\right)\right]\delta t\right\}du=0.$$

One may specify the functions δx, δy, δz arbitrarily, provided that they are annihilated at the limits of the interval. We then determine the function λ by the condition that the coefficient of δt in the quantity under the $\int$ sign must be zero. In order for $(\delta t)_1$ to be zero for any varied integral curve, it is necessary and sufficient that the coefficients of δx, δy, δz in the quantity under the $\int$ sign must also be zero.

In other words, *if one introduces an auxiliary quantity* λ *then the light rays will be given by equation* (2), *when combined with the equations:*

$$(4)\qquad \begin{cases} \lambda\dfrac{\partial F}{\partial x}-\dfrac{d}{du}\left(\lambda\dfrac{\partial F}{\partial x'}\right)=0, \\ \lambda\dfrac{\partial F}{\partial y}-\dfrac{d}{du}\left(\lambda\dfrac{\partial F}{\partial y'}\right)=0, \\ \lambda\dfrac{\partial F}{\partial z}-\dfrac{d}{du}\left(\lambda\dfrac{\partial F}{\partial z'}\right)=0, \\ \lambda\dfrac{\partial F}{\partial t}-\dfrac{d}{du}\left(\lambda\dfrac{\partial F}{\partial t'}\right)=0. \end{cases}$$

Moreover, besides equation (2), the elimination of λ gives the three equations:

$$(4')\qquad \frac{\dfrac{\partial F}{\partial x}-\dfrac{d}{du}\left(\dfrac{\partial F}{\partial x'}\right)}{\dfrac{\partial F}{\partial x'}}=\frac{\dfrac{\partial F}{\partial y}-\dfrac{d}{du}\left(\dfrac{\partial F}{\partial y'}\right)}{\dfrac{\partial F}{\partial z'}}=\frac{\dfrac{\partial F}{\partial z}-\dfrac{d}{du}\left(\dfrac{\partial F}{\partial z'}\right)}{\dfrac{\partial F}{\partial z'}}=\frac{\dfrac{\partial F}{\partial t}-\dfrac{d}{du}\left(\dfrac{\partial F}{\partial t'}\right)}{\dfrac{\partial F}{\partial t'}},$$

to which, we agree to append:

$$\frac{dx}{du}=x',\quad \frac{dy}{du}=y',\quad \frac{dz}{du}=z',\ \frac{dt}{du}=t'.$$

One immediately sees that when equation (2) is differentiated with respect to u then along with equations ($4'$), this will give $\dfrac{dx'}{dt}, \dfrac{dy'}{dt}, \dfrac{dz'}{dt}, \dfrac{dt'}{du}$ as four equations of the first degree, and the values that one derives will not depend on u. As is natural, the parameter u appears only in the final equations, which are of the form:

$$\frac{dx}{x'}=\frac{dy}{y'}=\frac{dz}{z'}=\frac{dt}{t'}=\frac{dx'}{X}=\frac{dy'}{Y}=\frac{dz'}{Z}=\frac{dt'}{T},$$

in which X, Y, Z, T are given functions of $x, y, z, t, x', y', z', t'$ that are homogeneous of second degree in x', y', z', t' and satisfy:

$$x'\frac{\partial F}{\partial x}+y'\frac{\partial F}{\partial y}+z'\frac{\partial F}{\partial z}+t'\frac{\partial F}{\partial t}+X\frac{\partial F}{\partial x'}+Y\frac{\partial F}{\partial y'}+Z\frac{\partial F}{\partial z'}+T\frac{\partial F}{\partial t'}=0.$$

In reality, *the differential equations of light rays are first-order ordinary differential equations in x, y, z, t,* $\dfrac{x'}{t'}, \dfrac{y'}{t'}, \dfrac{z'}{t'}$, *and these seven quantities are assumed to be coupled by relation* (2).

II. - *The invariant Pfaff equation of optics.*

197. Now consider a family of light rays that depend upon a parameter α, and take each of the light rays in an interval of time (t_0, t_1) that depends on α and correspond to a point of departure (x_0, y_0, z_0) that varies with α and a point of arrival (x_1, y_1, z_1) that likewise varies with α. If, for each light ray, one denotes the arbitrary auxiliary function that figures in equations (4) by λ, and applies formula (3) to it then one will obtain:

$$\lambda_1\left[\left(\frac{\partial F}{\partial x'}\right)_1 \delta x_1 + \left(\frac{\partial F}{\partial y'}\right)_1 \delta y_1 + \left(\frac{\partial F}{\partial z'}\right)_1 \delta z_1 + \left(\frac{\partial F}{\partial t'}\right)_1 \delta t_1\right]$$
$$= \lambda_0\left[\left(\frac{\partial F}{\partial x'}\right)_0 \delta x_0 + \left(\frac{\partial F}{\partial y'}\right)_0 \delta y_0 + \left(\frac{\partial F}{\partial z'}\right)_0 \delta z_0 + \left(\frac{\partial F}{\partial t'}\right)_0 \delta t_0\right].$$

From this, it results that when *the differential equation of the light rays* is considered to be a first-order system of differential equations in x, y, z, t, $\frac{x'}{t'}, \frac{y'}{t'}, \frac{z'}{t'}$ that are coupled by (2), *it will admit the invariant Pfaff equation:*

$$\omega_\delta \equiv \frac{\partial F}{\partial x'}\delta x + \frac{\partial F}{\partial y'}\delta y + \frac{\partial F}{\partial z'}\delta z + \frac{\partial F}{\partial t'}\delta t = 0 .$$

This Pfaff equation, which also depends upon only the relations between x', y', z', t', is a function of six variables. Its characteristic system is a system of ordinary differential equations, which, as a result, can only be identical to the equations of the light rays.

We thus arrive at the conclusion that *the light rays are characteristics of the Pfaff equation:*

$$\frac{\partial F}{\partial x'}\delta x + \frac{\partial F}{\partial y'}\delta y + \frac{\partial F}{\partial z'}\delta z + \frac{\partial F}{\partial t'}\delta t = 0 . \tag{5}$$

This is the invariant Pfaff equation of optics.

198. In practice, the Monge equation (1) is written in the form:

$$\Omega\left(x, y, z, t; \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right) \equiv F\left(x, y, z, t; \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}, 1\right) = 0 .$$

By setting:

$$\frac{dx}{dt} = \dot{x}, \qquad \frac{dy}{dt} = \dot{y}, \qquad \frac{dz}{dt} = \dot{z},$$

it is easy to form the invariant Pfaff equation. Indeed, one has:

$$x'\frac{\partial F}{\partial x'} + y'\frac{\partial F}{\partial y'} + z'\frac{\partial F}{\partial z'} + t'\frac{\partial F}{\partial t'} = 0.$$

As a result, equation (5) can be written:

$$t'\frac{\partial F}{\partial x'}\delta x + t'\frac{\partial F}{\partial y'}\delta y + t'\frac{\partial F}{\partial z'}\delta z - \left(x'\frac{\partial F}{\partial x'} + y'\frac{\partial F}{\partial y'} + z'\frac{\partial F}{\partial z'} \right)\delta t = 0.$$

Since the left-hand side is homogeneous in x', y', z', t', one can replace its arguments by $\dot{x}$, $\dot{y}$, $\dot{z}$, 1, respectively. One thus has the form:

(6) $$\frac{\partial \Omega}{\partial \dot{x}}\delta x + \frac{\partial \Omega}{\partial \dot{y}}\delta y + \frac{\partial \Omega}{\partial \dot{z}}\delta z - \left(\dot{x}\frac{\partial \Omega}{\partial \dot{x}} + \dot{y}\frac{\partial \Omega}{\partial \dot{y}} + \dot{z}\frac{\partial \Omega}{\partial \dot{z}} \right)\delta t = 0$$

for the invariant Pfaff equation.

For example, take a medium in which the wave surface is a sphere, and let c / n be the velocity of propagation of light, where c is the velocity in a vacuum and n is the index of refraction (which is a function of x, y, z, t). Here, the Monge equation is:

$$n^2\left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2\right] - c^2 = 0,$$

and the invariant Pfaff equation is:

$$n^2(\dot{x}\,\delta x + \dot{y}\,\delta y + \dot{z}\,\delta z) - c^2\delta t = 0 .$$

Upon setting:

$$\alpha = \frac{n\dot{x}}{c}, \quad \beta = \frac{n\dot{y}}{c}, \quad \gamma = \frac{n\dot{z}}{c}$$

it becomes:

$$n(\alpha\,\delta x + \beta\,\delta y + \gamma\,\delta z) - c\,\delta t = 0.$$

α, β, γ are then the direction cosines of the tangent to the light rays.

If n does not depend on time then the law of propagation of light will admit the infinitesimal transformation $\frac{\partial f}{\partial t}$, and, as a result, the differential equations that give the light rays will admit the invariant *form*:

$$\delta t - \frac{n}{c}(\alpha\,\delta x + \beta\,\delta y + \gamma\,\delta z) .$$

The differential equations that give the (geometric) curves that are described by the light rays admit, in turn, the *relative* integral invariant:

$$\int n(\alpha\,\delta x + \beta\,\delta y + \gamma\,\delta z).$$

We thus recover the viewpoint of the preceding chapter (sec. **191**).

199. As one knows (sec. **152**), the characteristic equations of the Pfaff invariant of optics can be converted into the characteristic equations of a first-order partial differential equation. (The converse is also true, but we shall leave it at that.)

The existence of an integral invariant will be assured any time the law of propagation of light admits an infinitesimal transformation. In all of these cases, one may convert the search for light rays into a problem in the *ordinary* calculus of variations.

For example, take the case in which the law of propagation of light is given by the Monge equation:

$$n^2\,(dx^2 + dy^2 + dz^2) - c^2\,dt^2 = 0,$$

where the index of refraction can depend on x, y, t, *but not on* z. One then has the infinitesimal transformation:

$$Af = \frac{\partial f}{\partial z}.$$

The form:

$$\frac{\omega(\delta)}{\omega(A)} = \frac{n(\alpha\,\delta y + \beta\,\delta y + \gamma\,\delta z)}{n\gamma} = \delta z + \frac{\alpha}{\gamma}\delta x + \frac{\beta}{\gamma}\delta y - \frac{c}{n\gamma}\delta t$$

is an invariant form. Once the coordinates x and y are known as a function of t, one will get z by a quadrature. As for the differential equations that give x and y as functions of t, they will admit the relative integral invariant:

$$\int \frac{\alpha}{\gamma}\delta x + \frac{\beta}{\gamma}\delta y - \frac{c}{n\gamma}\delta t,$$

or, what amounts to the same thing, the integral invariant:

$$\int \xi\,\delta x + \eta\,\delta y - \zeta\,\delta t,$$

in which ξ, η, ζ are three quantities that are coupled by the relation:

$$1 + \xi^2 + \eta^2 = \frac{n^2}{c^2}\zeta^2.$$

In particular, the equations of the characteristics consist of the equations:

$$\frac{dx}{\xi} = \frac{dy}{\eta} = \frac{-dt}{\frac{n^2}{c^2}\zeta} = \frac{\sqrt{dt^2 - \frac{n^2}{c^2}(dx^2 + dy^2)}}{\frac{n}{c}}.$$

The light rays render the integral:

$$\int -\xi\, dx - \eta\, dy + \zeta\, dt = \int \sqrt{\frac{c^2}{n^2}dt^2 - dx^2 - dy^2}$$

stationary.

III. - *Fermat's principle is independent of the spacetime framing.*

200. It is important to remark that the invariant Pfaff equation of optics is coupled to the Monge equation that defines the law of propagation of light in a manner that is *independent of the framing chosen for space and time.* In other words, the equation:

$$\frac{\partial F}{\partial x'}\delta x + \frac{\partial F}{\partial y'}\delta y + \frac{\partial F}{\partial z'}\delta z + \frac{\partial F}{\partial t'}\delta t = 0$$

is a *covariant* of the equation:

$$F(x, y, z, t;\ x', y', z', t') = 0$$

relative to any change of variables performed on x, y, z, t. This result is basically Fermat's principle itself. However, one also recover this equation in the following manner, in which nothing distinguishes one of the variables x, y, z, t from the others.

Consider the Pfaff system:

$$\frac{dx}{x'} = \frac{dy}{y'} = \frac{dz}{z'} = \frac{dt}{t'}, \tag{19}$$

such that $x, y, z, t, \frac{x'}{t'}, \frac{y'}{t'}, \frac{z'}{t'}$ are assumed to be coupled by relation (2), and look for the *derived* system of (7). It is the system that is composed of Pfaff equations that are linear combinations of equations (7), and which enjoy the property that the exterior derivative of their left-hand side is zero on account of equations (7). If one sets:

$$\frac{x'}{\dot{x}} = \frac{y'}{\dot{y}} = \frac{z'}{\dot{z}} = \frac{t'}{1},$$

as above, then any linear combination of equations (7) will be of the form:

$$u\,(dx - \dot{x}\,dt) + v\,(dy - \dot{y}\,dt) + w\,(dz - \dot{z}\,dt) = 0\,.$$

If one takes equations (7) into account then the exterior derivative of the left-hand side will reduce to:

$$[dt\,(u\,d\dot{x} + v\,d\dot{y} + w\,d\dot{z})].$$

On account of equations (7) and the derivative of equation (2), the condition for this to be zero is:

$$[dt\ (u\ d\dot{x} + v\ d\dot{y} + w\ d\dot{z})(dx - \dot{x}\ dt)(dy - \dot{y}\ dt)(dz - \dot{z}\ dt)\ d\mathrm{F}] = 0,$$

or, upon simplifying:

$$\left[dx\,dy\,dz\,dt\,(u\,d\dot{x} + v\,d\dot{y} + w\,d\dot{z})\left(\frac{\partial F}{\partial \dot{x}}d\dot{x} + \frac{\partial F}{\partial \dot{y}}d\dot{y} + \frac{\partial F}{\partial \dot{z}}d\dot{z}\right)\right] = 0.$$

This gives:

$$\frac{u}{\dfrac{\partial F}{\partial \dot{x}}} = \frac{v}{\dfrac{\partial F}{\partial \dot{y}}} = \frac{w}{\dfrac{\partial F}{\partial \dot{z}}}.$$

The system derived from system (7) is therefore simply the Pfaff equation:

$$\frac{\partial F}{\partial \dot{x}}(dx - \dot{x}\,dt) + \frac{\partial F}{\partial \dot{y}}(dy - \dot{y}\,dt) + \frac{\partial F}{\partial \dot{z}}(dz - \dot{z}\,dt) = 0,$$

whose characteristics are the light rays.

It results from this that, even in optics, the time coordinate does not play a role that is essentially different from the one that is played by the spatial coordinates. The fundamental laws of optics are not necessarily related to the classical notions of space and time, and they behave just as they do in the theory of relativity.

201. For example, if one chooses a convenient framing for the universe (i.e., spacetime) then the laws of propagation of light in a gravitational field that is produced by a unique mass (which is reduced to a point) will be furnished by the Schwarzschild equation:

$$\frac{dr^2}{1 - \dfrac{2m}{r}} + r^2(d\theta^2 + \sin^2\theta\,d\varphi^2) - \left(1 - \frac{2m}{r}\right)dt^2 = 0.$$

These laws admit the infinitesimal transformation $\dfrac{\partial f}{\partial t}$. When the light rays are considered from only the space point of view, they will thus be defined as realizing an extremum of the integral:

$$\int \sqrt{\frac{dr^2}{\left(1-\frac{2m}{r}\right)^2}+\frac{r^2(d\theta^2+\sin^2\theta\,d\varphi^2)}{1-\frac{2m}{r}}}.$$

Propagation takes place in a plane that passes through the center of attraction, and if one supposes that this plane is defined by $\varphi = 0$ then one will have to realize an extremum for the integral:

$$\int \sqrt{\frac{dr^2}{\left(1-\frac{2m}{r}\right)^2}+\frac{r^2 d\theta^2}{1-\frac{2m}{r}}}\,.$$

By taking advantage of the existence of the infinitesimal transformation $\frac{\partial f}{\partial \theta}$, the integration offers no difficulty, and gives:

$$\theta = \int \frac{C\frac{dr}{r^2}}{\sqrt{1-\frac{C^2\left(1-\frac{2m}{r}\right)}{r^2}}}\ .$$

BIBLIOGRAPHY

H. POINCARÉ. - *Les méthodes nouvelles de la Mécanique céleste,* t. III, Paris, Gauthier-Villars, 1899.

P. APPELL. - *Traité de Mécanique rationelle,* t. II, chap. XXV, Paris, Gauthier-Villars.

H. POINCARÉ. – "Sur les résidus des intégrales doubles," Acta Mathem., t. IX, 1887, pp. 321-380.

- "Analysus situs," Journal Ec. Polyt., 1895.

TH. DE DONDER. – "Étude sur les invariants intégraux," Rend. Circ. Mat. Palermo, t. XV (1901), pp 66-131; t. XVI (1902), pp 155-179.

- "Sur les invariants intégraux," Atti del IV Congresso internazionale dei Matematici, t. II, Roma, 1909, pp. 129-137.
- "Sur le multiplicateur de Jacobi généralisé," Bulletin de l'Acad. Royale de Belgique (classes des sciences), 1908, pp. 129-137.
- "Sur les invariants intégraux relatifs," *ibid*., 1909, pp. 66-83.
- "Application du multiplicateur généralisé," *ibid.*, 1909, pp. 610-621.
- "Sur le multiplicateur généralisé," *ibid*., pp. 268-286.
- "Sur les invariants intégraux relatifs et leurs applications à la Physique mathématique," *ibid.*, 1911, pp. 50-70.
- "Quelques remarques sur le multiplicateur de Jacobi et le multiplicateur généralisé," *ibid*., 1911, pp. 740-749.
- "Introduction à la théorie des invariants intégraux," *ibid.*, 1913, pp. 1043-1073.
- "Applications nouvelles des invariants intégraux," Mémoires de l'Acad. Royale de Belgique (classes des sciences) (2), I, (1904).
- "Sur les équations canonique de Hamilton-Volterra," ibid., (2), III.

G. KOENIGS. – "Application des invariants intégraux à la réduction au type canonique d'un systéme quelconque d'équations differentielles," C.R. Acad. des Sc. Paris, t. CXXI, (1895), pp. 875-878.

- "Sur les invariants intégraux," *ibid.*, t. CXXII, (1896), pp. 25-27.

S. LIE. – "Ueber Integralinvarianten und Differentialgleichungen," Videnskabsselskabets Skrifter, Christiania, 1902, no. 1 (73 pp.).

K. ZORAWSKI. – "Ueber gewisse Transformationseigenschaften der vielfachen Integrale," Bull. Acad. des sciences de Cracovie (sc. math. et natur.), 1909, pp. 483-542.

R. HARGREAVES. – "Integral forms and their connexion with physical equations," Trans. Cambridge Philosoph. Society; t. XXI (1912), pp. 107-122.

R. DONTOT. – "Sur les invariants intégraux de la propagation par ondes," Bull. Soc. Math. de France, t. XLII (1914), pp. 53-91.

E. VESSIOT. – "Sur les invariants intégraux et quelques points d'optique géométrique," Bull. Soc. Math. de France, t. XLII (1914), pp. 142-167.

- "Sur un invariant intégral de l'Hydrodynamique et son application à la relativité," C.R. Acad. des Sc. De Paris, 30 Dec. 1918.

E. GOURSAT. – "Sur les invariants intégraux," Journal Math. pures et appliq*é*es (5), t. IV, (1908), pp. 331-365.

- "Sur quelques points de la théorie des invariants intégraux," *ibid.*, (7), t. I, (1915), pp. 241-250.

- "Sur certains systémes d'equations aux différentiels totales et sur une généralisation du probléme de Pfaff," Ann. Fac. Sc. Toulouse, t. VII, (1915).

E. CARTAN. – "Sur l'intégration des systémes différentiels complétement intégrables," C.R. Acad. des Sc. Paris, t.. CXXXV, (1902), pp. 1415-1417; 1564-1566.

T. CHELLA. – "Vantaggiche si possono trarre da noti invarianti e differenziali in alcuni problemi d'integrazione," Annali R. Scuola norm. sup. Pisa (sc. fis.-mat.), t. XI, (1910), pp. 1-137.

On the symbolic calculus that is applied to exterior differential forms, and on certain related symbolic calculi, one may consult, in addition to the preceding:

E. CARTAN. – "Sur certains expressions différentiels et le probléme de Pfaff," Ann. Ec. Norm., (3), t. XVI, (1899), pp. 239-332.

A. BOREL. – "Sur les transformations et extensions de la formule de Stokes," Ann. Fac. Sc. Toulouse, t. IV, (1912); t. VI, (1914), t. VII, (1915)

Finally, on the subject of integral invariants of continuous groups of transformations, where the viewpoint is a little different from that of H. Poincaré, one may consult:

S. Lie. – "Die Theorie der Integralinvarianten ist ein corollar der Theorie der Differentialinvarianten," Ber. Sächs. Gesellsch., Leipzig, 1897, pp. 342-357.

– "Ueber die Integralinvarianten und ihre Verwertung für die Theorie der Differentialgleichungen," *ibid.*, 1897, pp. 369-410.

K. ZORAWSKI. – "Ueber Integralinvarianten der continuerlichen Transformationsgruppen," Bull. Acad. Sc. Cracovie (sc. math. et nat.), 1895, pp. 127-130.

E. CARTAN. – "Le principe de dualité et certains intégrales multiples de l'espace tangentiel et de l'espace réglé," Bull. Soc. Math. France, t. XXIV (1896), pp. 140-177.

TH. DE DONDER. – "Sur un probléme relatif aux invariants intégraux," Bull. Acad. royale de Belgique (classe des sc.), 1912, pp. 583-590.

R. DELTHEIL. – "Sur la théorie des probabilités géométriques," Thesis, Toulouse, Private Ed., 1920.