"Sur certaines expressions différentielles et le problème de Pfaff," Ann. sci. de l'E. N. S. (3) **16** (1899), 239-332.

ON

CERTAIN DIFFERENTIAL EXPRESSIONS

AND THE

PFAFF PROBLEM

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The Pfaff problem has been the object of numerous papers. I have no intention of passing all of them in review ([1]). The most prominent are those of Pfaff himself, and then those of Grassmann, Natani, Clebsch, Lie, Frobenius, and Darboux. The problem in question is, in summary, the solution of a total differential equation, and later on it is joined with that of the reduction of a linear *expression* in total differentials – or *Pfaff expression* – to a canonical form by means of a convenient change of variables.

Pfaff ([2]) was the first to give the result that a total differential equation can always be verified by a system of integral equations whose number does not exceed $n / 2$ if n is even and $(n + 1) / 2$ if n is odd. His method is based upon the gradual reduction of the number of differential elements in the equation, each reduction by one unit being provided by the complete integration of a system of ordinary differential equations and a change of variables.

Grassmann applied the principles of the calculus of extensions to the same problem in the second edition of his *Ausdehnungslehre* ([3]). At its basis, his method is the same as that of Pfaff, but only applies to equations that can be converted into a *general* equation in an even number of variables by a gradual reduction. It gives the necessary and sufficient condition for the equation to be verified by a system of m integral equations. His results have an extremely concise form.

Natani ([4]) and Clebsch ([5]) successively reduced the number of differential elements in the equation, but – and this represents a great advance – each reduction required only the

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([1]) For the bibliography, consult, for example, FORSYTH, *Theory of differential equations*, Part I, Chapter III. In this work, the Pfaff problem is presented in a very interesting manner from the historical standpoint (Chap. IV and XII).

([2]) *Methodus generalis aequationes differentiarum partialium necnon aequationes differentiales vulgares, utrasque primi ordinis, inter quotcumque variabiles complete integrandi* (Abh. d. K.-P. Akad. d. Wiss. zu Berlin (1814-1815), 76-136.

([3]) *Die Ausdehnungslehre, vollständig und in strenger Form bearbeite*. Berlin, 1862.

([4]) *Journal de Crelle*, **58**, January, 1860, 301-328.

([5]) *Ibid.*, **60**, September, 1860, 193-251.

search for one integral of a system of differential equations. However, as in the Pfaff method, one must make a change of variable each time. Nevertheless, Natani sought, without arriving at very simple results, to form the successive auxiliary systems directly by only knowing the integrals that had already been found for the preceding systems.

In a second paper ([1]), Clebsch solved the problem in a very elegant manner in the case of *general* equations in an even number of variables. One must seek an integral of a certain number of successive complete systems, and each equation of a system of that series will depend linearly upon the partial derivatives of just one of these previously-found integrals, except for an equation that is common to all of these systems that depends upon only the coefficients of the given equation. His method does not extend to the other case, and furthermore, Clebsch has never complete solved the case of a general system of an odd number of variables. In the same paper, Clebsch indicated the manner of deducing the most general integral system from a particular integral system.

Lie ([2]) was, in short, the first one to occupy himself with the reduction of a Pfaff *expression*. He exhibited the invariant character of a certain integer number (viz., the *class* of the Pfaff expression, following Frobenius) that completely determines the canonical form to which it can reduce. His method is based upon the theory of contact transformations. The reduction is obtained as in the first method of Clebsch, except that it is combined with the integration method of Mayer for first-order, partial differential equations.

Frobenius, in his beautiful paper in the *Journal de Crelle* ([3]), employed a completely new method. It is based upon the consideration of that which one calls the *bilinear covariant* that is associated with the Pfaff expression. The equivalence conditions – i.e., the possible reduction to the same form – of two Pfaff equations are then the algebraic equivalence conditions for two forms that are linear and bilinear with respect to the differential elements. He thus arrives at the notion of *class*. His method of reduction is analogous to that of Natani and Clebsch, except that the successive complete systems are formed without changing the variables, and their equations depend upon partial derivatives of all the preceding integrals that were found.

Finally, in a paper that was contemporaneous to that of Frobenius, but published five years later ([4]), Darboux began with the same bilinear covariant, whose invariance properties permitted him to deduce the first auxiliary system that is common to all of the methods for reducing the class of the Pfaff expression. One also deduces the fundamental formulas of the theory of contact transformations from it in a very elegant manner.

The present paper constitutes an exposition of the Pfaff problem that is based upon the consideration of certain symbolic differential expressions that are integer and homogeneous with respect to the differentials in n variables, the coefficients being arbitrary functions of these variables. These expressions can be subject to the ordinary rules of calculation, on the condition that one does not change the order of the differentials of a product. The calculation of these quantities is, in short, that of

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([1]) *Ibid.*, **61**, September, 1860, 146-179.

([2]) Most especially, see: "Theorie des Pfaffschen Problems," Arch. for Math. og Nat., **II** (1877), 338-379.

([3]) "Ueber das Pfaffsche Problem," Journal de Crelle **82** (1877), 230-315.

([4]) "Sur le problème de Pfaff," Bulletin des Sciences mathématiques (3) **VI** (1882), 14-36, 49-68.

differential expressions that are placed under a multiple integral sign ([1]). This calculation also presents numerous analogies with the Grassmann calculus. It is, moreover, identical to the geometric calculations that Burali-Forti did in a recent book ([2]).

It is clear that if one makes a change of variables then any differential expression of degree p is changed into a differential expression of degree p in the new differentials. In the case of a Pfaff expression, which is of degree one, one can associate it with another differential expression of second degree that is a covariant with respect to the changes of variables, and which is nothing but the bilinear covariant of Frobenius and Darboux. I call it the *derivative* of the Pfaff expression. However, thanks to the notion of symbolic differential expressions, *this covariant is the first term in a sequence of symbolic covariants of third, fourth, … degree that are deduced intuitively from the Pfaff expression and its derivative by multiplications.* They constitute the second, third, … derivatives of the Pfaff expressions, the p^{th} one being of degree $p + 1$.

One understands how much one can deduce from the consideration of these derivatives, thanks to their invariant character. *They are the only quantities that intervene in the statement of all the results of the theory*, and their form is very simple.

The consideration of these derivatives permits one to find all of the results that are already known in a manner that is, so to speak, intuitive; however, it has allowed me to discover some others. Among others, I will point out the *extension of the second Clebsch method to the reduction of arbitrary Pfaff expressions* ([3]), of either even or odd class, and in an arbitrary number of variables. It has also allowed me to completely present the theory of *singular integrals* of a Pfaff equation ([4]).

This memoir is divided into five parts. In the *first* one, I present the principles of the calculus of differential expressions that intervene in what follows. In the *second* one, I introduce the derivatives of a Pfaff expression and the notion of class, and I prove the necessary and sufficient condition for a Pfaff expression to be of class p. The result is extremely simple, viz., *that the p^{th} derivative has all of its coefficients zero.* I then introduce what I call the "adjoint complete system" and then discuss the reduction of an expression to its canonical form, either by successive changes of variables (i.e., the method of Natani and Clebsch) or without changes of variables (i.e., the Frobenius method).

The *third part* is dedicated to the solution of a Pfaff equation, a problem that admits general solutions that depend upon the reduction of the left-hand side to its canonical form, and singular solutions that are obtained by annulling all of the coefficients of a certain derivative.

The *fourth part* is dedicated to the following two problems:

([1]) Cf., CARTAN, "Le principe de dualité et certaines intégrales multiples de l'espace tangentiel et de l'espace réglé," Bulletin de la Société mathématique de France, **XXV**, 1-39.

([2]) *Introduction à la Géométrie différentielle, suivant la méthode de Grassmann* (Gauthier-Villars, 1898).

([3]) See below, Chap. IV, §§ 69, 70, 75.

([4]) Apart from the classical case of singular integrals of the equation in three variables, I know of only a paper of Frisiani, which I have not consulted and which is entitled: "Sull' integrazione delle equazioni differenziali ordinarie di primo ordine e lineari fra un numero qualunque di variabili (Effer. astr. di Milano, 1848). Following Forsyth, he has discussed the possibility of satisfying a Pfaff equation by equations that are fewer in number than the canonical number.

Solve a Pfaff equation by means of a given number r of unknown relations.

Solve a Pfaff equation by means of a given number r of relations, among which, h of them are given in advance.

These two problems admit general solutions and singular solutions. The former are given by the search for an integral of several successive complete integrals, the equations of these systems containing the derivatives of all the integrals that were already found linearly. As for the singular solutions, they are the solutions of an analogous problem, but in which the given relations between the variables are greater in number and can be formed by differentiations.

In the very general case where the desired solutions are not *singular* solutions of the Pfaff equation – or, more precisely – do not annul all of the coefficients of the $(2r - 2)^{\text{th}}$ derivative of the expression, the form of the complete systems can be simplified for the calculations in such a manner that each equation depends upon no more than the derivatives of just one of the preceding integrals that were found. In particular, this method gives the generalization of the second Clebsch method.

Finally, the fifth part is dedicated to the applications of the theory to the integration of first-order, partial differential equations, whether ordinary or homogeneous. I have also indicated how the consideration of derivatives lends itself to the establishment of the fundamental formulas of the theory of contact transformations.

I. – Differential expressions.

1. Being given n variables $x_1, x_2, \ldots, x_n$, consider some purely symbolic expressions ω that are deduced by means of a finite number of addition or multiplication *signs* from n differentials $dx_1, dx_2, \ldots, dx_n$ and certain coefficients that are functions of $x_1, x_2, \ldots, x_n$; these expressions are *homogeneous* in $dx_1, dx_2, \ldots, dx_n$, in the usual sense of the word. Since they are purely symbolic, we restrict ourselves to not changing the order of the terms whenever one has an addition or multiplication sign or of the factors that are united by that sign.

Subject to the usual rules of calculation, these expressions can be put into the form of homogeneous integer polynomials in $dx_1, dx_2, \ldots, dx_n$. The degree of these polynomials will be, by definition, the degree of the corresponding expression ω. The differentials of the first degree are further called *Pfaff expressions;* they are of a form analogous to the following one:

(1) $$A_1\,dx_1 + A_2\,dx_2 + \ldots$$

As examples of differential expressions of higher order, one might give the following ones:

(2) $$A_1\,dx_2\,dx_1 + A_2\,dx_3\,dx_2\,,$$

(3) $$(A_1\,dx_1 + A_2\,dx_2)\,(B_1\,dx_1\,dx_2 + B_2\,dx_1\,dx_1) + A_1\,dx_1\,dx_2\,dx_1\,,$$

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2. *Monomial differential expressions.* – These are the ones that are deduced by multiplication signs from a certain coefficient and certain differentials $dx_1, dx_2, \ldots, dx_n$, repeated or not; for example, one might have the following:

(4) $$A\,dx_1\,dx_2\,dx_1\,dx_4\,dx_3\,dx_2\,.$$

Beyond these differential expressions, the simplest are the ones that one deduces by addition signs from a certain number of monomial differential expressions of the same degree; they have the form of polynomials in $dx_1, dx_2, \ldots, dx_n$, such as the expression (2).

Apart from these particular expressions, we also consider the ones that one deduces by multiplication signs from a certain number of the preceding differential expressions, such as the expression:

(5) $$(A_1\,dx_1 + A_2\,dx_2)\,(B_1\,dx_1\,dx_2 + B_2\,dx_1\,dx_3) + (C_1\,dx_1\,dx_2 + C_2\,dx_2\,dx_4).$$

3. *Rank of a differential in a differential expression.* – Consider a differential that enters in a certain place in a differential expression. If that differential expression is a monomial expression then the rank distinguishes the place that the differential occupies in the monomial; therefore, the differential dx_4 in the expression (4) occupies the fourth rank.

If one is dealing with a polynomial differential expression then the rank of a differential is the one that it occupies in the monomial that it enters into.

Finally, in the general case, if one subjects an arbitrary differential expression to the usual rules of calculation in such a manner as to transform it into a polynomial expression, but taking care to respect the order of the differentials in each product, then the rank of a given differential is the one that it will have in the polynomial expression thus obtained. For example, in the expression (3), the differential dx_1, which enters into the second term in the second parenthesis, is of third rank. In a differential expression that is the product of several polynomial differential expressions, the differentials of the first factor, which are assumed to be of degree h, have ranks 1, 2, …, h; those of the second factor, which are assumed to be of degree k, have ranks h + 1, h + 2, …, h + k, and so on.

4. *Value of a differential expression.* – By convention, in order to define the value of a differential expression ω – of degree h, for example – we consider $x_1, x_2, \ldots, x_n$ to be functions of *h indeterminate* parameters $\alpha_1, \alpha_2, \ldots, \alpha_h$ that are assumed to have ranks that are in a certain order that we call the *natural order*.

This being the case, one considers all the h! permutations of the letters $\alpha_1, \alpha_2, \ldots, \alpha_h$. Let $(\beta_1, \beta_2, \ldots, \beta_h)$ be one of these permutations. One makes that permutation correspond to the value that is taken by the expression ω according to the usual rules of calculation when one replace the differentials that occupy the 1st, 2nd, …, hth rank with the corresponding derivatives that are taken with respect to $\beta_1, \beta_2, \ldots, \beta_h$, respectively. One precedes the quantity thus determined with a + or – sign according to whether the permutation $(\beta_1, \beta_2, \ldots, \beta_h)$ presents an even or odd number of inversions. The algebraic sum of the h! quantities thus obtained is, by definition, the value of the given differential expression.

Therefore, the value of the expression (2) is:

$$\left(A_1\frac{\partial x_1}{\partial\alpha_1}\frac{\partial x_1}{\partial\alpha_2}+A_2\frac{\partial x_3}{\partial\alpha_1}\frac{\partial x_2}{\partial\alpha_2}\right)-\left(A_1\frac{\partial x_1}{\partial\alpha_2}\frac{\partial x_1}{\partial\alpha_1}+A_2\frac{\partial x_3}{\partial\alpha_2}\frac{\partial x_2}{\partial\alpha_1}\right).$$

5. *Equivalent differential expressions.* – Two differential expressions are called *equivalent* when, being of the same degree, they have the same value for any parameters that one chooses in order to define that value.

It results from the definition given above that one can, without changing the value of a differential expression, apply all of the usual rules of calculation to it, on the condition that one leave the *rank* of the differential unaltered – i.e., on the condition that one does not invert the order of the differentials in the products that one forms. Indeed, these modifications change none of the h! quantities that serve to define the value of the differential expression.

It results from this that an arbitrary differential expression is equivalent to a polynomial differential expression and that, moreover, one can invert the order of the monomials in that polynomial expression in an arbitrary manner, and likewise reduce two monomials that differ only by the coefficients into just one monomial.

That is why the expression (3) is equivalent to the polynomial expression:

$$(3') \qquad A_1\,B_1\,dx_1\,dx_1\,dx_2 + (A_1\,B_2 + C)\,dx_1\,dx_2\,dx_1 + A_2\,B_1\,dx_2\,dx_1\,dx_2 + A_2\,B_2\,dx_2\,dx_2\,dx_1\,.$$

6. *Value of a monomial differential expression.* – If one seeks to find the value of a monomial differential expression such as:

$$A\,dx_{m_1} dx_{m_2} \cdots dx_{m_h}$$

following the rules that were given above, $m_1, m_2, \ldots, m_h$ being h of the indices 1, 2, …, n (distinct or not) then one finds the product of A with the functional determinant of x_{m_1}, x_{m_2}, …, x_{m_h} with respect to $\alpha_1, \alpha_2, \ldots, \alpha_h$ quite simply. It immediately results from this that if a monomial differential expression contains two identical differentials then it must have the value zero; one says that it is *identically zero*. It likewise results from the theory of determinants that one can invert the order of the differentials in a monomial expression in an arbitrary manner, on the condition that one change the sign of the coefficient if that substitution amounts to an odd number of transpositions, or furthermore if the two permutations of the indices of the differentials are of opposite parity. For example, one has:

$$A\,dx_1\,dx_2\,dx_3 = A\,dx_2\,dx_3\,dx_1 = A\,dx_3\,dx_1\,dx_2$$
$$= -A\,dx_2\,dx_1\,dx_3 = -A\,dx_3\,dx_2\,dx_1 = -A\,dx_1\,dx_3\,dx_2\,.$$

7. *Reduction of a differential expression to its simplest form.* – It results from the preceding that one can always put an arbitrary differential expression into the form of a polynomial expression such that each monomial of the latter expression does not contain identical differentials, and the differentials that it does contain are arranged by order of increasing indices. We say that under these conditions the expression is *reduced to its simplest form*. That is why the simplest form for the expression:

$$(A_1\,dx_1 + A_2\,dx_2 + A_3\,dx_3 + A_4\,dx_4)\,(B_1\,dx_2\,dx_3 + B_2\,dx_1\,dx_4)$$

is

$$A_1\,B_1\,dx_1\,dx_2\,dx_3 - A_2\,B_2\,dx_1\,dx_2\,dx_4 - A_3\,B_2\,dx_1\,dx_3\,dx_4 + A_4\,B_1\,dx_2\,dx_3\,dx_4\,.$$

8. *Identically zero differential expressions.* – These are the ones whose value is zero no matter what the parameters are that one makes $x_1, x_2, \ldots, x_n$ depend upon.

A differential expression in n variables and of degree greater than n is necessarily zero, because if one puts it into the form of a polynomial expression then all of the monomials must have at least two identical differentials.

A differential expression of degree $h \leq n$ will be identically zero if, upon reducing it to its simplest form, the coefficients of all of the monomials are zero. One accounts for this by taking $\alpha_1, \alpha_2, \ldots, \alpha_h$ to be h arbitrary ones of the variables $x_1, x_2, \ldots, x_n$.

9. *Inversion of the factors in a product of differential expressions.* – Consider a (symbolic) product ω of differential expressions $\omega_1, \omega_2, \ldots, \omega_m$.

$$\omega = \omega_1\ \omega_2 \ldots \omega_m\,.$$

Imagine that we invert two of the factors ω_μ , ω_ν of this product, which are assumed to be of order h and k, resp., and suppose that these two factors are separated by one or more other factors ω_p of total degree p. It is clear that such an operation amounts to making a certain substitution of the ranks of the differentials of any of these monomials of ω when it is reduced to a polynomial expression.

If this substitution is even then the value of ω does not change, while if it is odd then the sign does change.

Now, in order to perform this operation, one can first make ω_ν come before ω_μ, which demands k $(h + p)$ transpositions, and then make ω_μ appear after the group of factors ω_p, which requires hp transpositions. Therefore, in all, one has $hk + (h + k)\,p$ transpositions. The differential expression ω is thus multiplied by $(-1)^{hk+(h+k)p}$.

In particular, suppose that the two factors considered have degrees of the same parity. $p(h + k)$ is then even, and ω is multiplied by $(-1)^{hk}$. Therefore, the transposition of the two factors into a product of differential expressions does not change this product if the factors are pair-wise even and changes the sign of this product if they are pair-wise odd.

It results from this that *if a differential expression ω is the product of several other differential expressions, among which one finds two that are identical and of odd degree, then the expression ω is identically zero.*

10. *Powers of a differential expression.* – One calls the symbolic product of p expressions that are identical to ω the p^{th} *power* of a differential expression ω.

The p^{th} power of a monomial is identically zero, because it is a monomial expression that contains some identical differentials.

The p^{th} power of a differential expression of odd degree is also identically zero, because it is a product that contains two identical factors of odd degree.

It therefore suffices to consider differential expressions ω of even degree. Reduced to its simplest form, ω is a sum of m monomials of the same degree:

$$\omega = \omega_1 + \omega_2 + \ldots + \omega_m \,.$$

One immediately sees that the square of ω is:

$$\omega^2 = 2\,(\omega_1\,\omega_2 + \omega_1\,\omega_3 + \ldots + \omega_1\,\omega_m + \omega_2\,\omega_3 + \ldots + \omega_{m-1}\,\omega_m),$$

because the squares of $\omega_1, \omega_2, \ldots, \omega_m$ are zero and the product of two monomials of even degree is independent of the order of the factors. One likewise verifies that:

$$\omega^3 = 2 \cdot 3\,(\omega_1\,\omega_2\;\omega_3 + \omega_1\;\omega_2\;\omega_4 + \ldots + \omega_{m-2}\,\omega_{m-1}\,\omega_m),$$

and, in a general manner, that ω^p is obtained by multiplying the sum of all the products of p of the m monomials $\omega_1, \omega_2, \ldots, \omega_m$ by $p!$.

11. *Change of variables in a differential expression.* – Imagine that one performs a change of variables on $x_1, x_2, \ldots, x_n$ by taking the new variables to be n independent functions $y_1, y_2, \ldots, y_n$ of $x_1, x_2, \ldots, x_n$. Conversely, $x_1, x_2, \ldots, x_n$ are then independent functions of $y_1, y_2, \ldots, y_n$.

This being the case, replace the old variables in a differential expression ω in $x_1, x_2, \ldots, x_n$ with the new variables and the differentials $dx_1, dx_2, \ldots, dx_n$ with:

$$\frac{\partial x_1}{\partial y_1}dy_1 + \frac{\partial x_1}{\partial y_2}dy_2 + \cdots + \frac{\partial x_1}{\partial y_n}dy_n,$$
$$\cdots\cdots\cdots\cdots\cdots\cdots,$$
$$\frac{\partial x_n}{\partial y_1}dy_1 + \frac{\partial x_n}{\partial y_2}dy_2 + \cdots + \frac{\partial x_n}{\partial y_n}dy_n.$$

We thus obtain a certain differential expression ϖ of the same degree in $y_1, y_2, \ldots, y_n$, and in which each differential dy will have the same rank as the differential dx in ω that provided it.

It results from this that the value of ϖ is equal to the value of ω if one expresses the variables as functions of the same parameters α, because in the $h!$ quantities that define the value of ω, one replaces, in short, derivatives such as $\partial x_i / \partial \beta_r$ by the expressions:

$$\frac{\partial x_i}{\partial y_1}\frac{\partial y_1}{\partial \beta_r} + \frac{\partial x_i}{\partial y_2}\frac{\partial y_2}{\partial \beta_r} + \cdots + \frac{\partial x_i}{\partial y_n}\frac{\partial y_n}{\partial \beta_r},$$

which are obviously equal to them.

It immediately results from this that if the expressions $\omega_1, \omega_2, \ldots, \omega_m$ transform into $\varpi_1, \varpi_2, \ldots, \varpi_m$ by a change of variables then the expression:

$$\omega = \omega_1 \omega_2 \ldots \omega_m$$

transforms into:

$$\varpi = \varpi_1 \varpi_2 \ldots \varpi_m.$$

That property is paramount in the applications that we will make of this theory.

For example, one has:

$$dx_1\, dx_2 = \left(\frac{\partial x_1}{\partial y_1}dy_1 + \frac{\partial x_1}{\partial y_2}dy_2\right)\left(\frac{\partial x_2}{\partial y_1}dy_1 + \frac{\partial x_2}{\partial y_2}dy_2\right) = \left(\frac{\partial x_1}{\partial y_1}\frac{\partial x_2}{\partial y_2} - \frac{\partial x_1}{\partial y_2}\frac{\partial x_2}{\partial y_1}\right)dy_1 dy_2,$$

which agrees with the well-known property of functional determinants that is expressed by the equality:

$$\frac{D(x_1, x_2)}{D(\alpha_1, \alpha_2)} = \frac{D(x_1, x_2)}{D(y_1, y_2)}\frac{D(y_1, y_2)}{D(\alpha_1, \alpha_2)}.$$

II. – Application of the preceding theorems to Pfaff expressions.

12. *Derived expression of a Pfaff expression.* – Being given a Pfaff expression in n variables:

$$\omega = A_1\, dx_1 + A_2\, dx_2 + \dots + A_n\, dx_n\,,$$

one calls the second-degree differential expression that is defined by the equality:

$$\omega' = dA_1\, dx_1 + dA_2\, dx_2 + \dots + dA_n\, dx_n\,,$$

the *derived expression.*

The fundamental property of that derivative is the following:

Theorem. – *If a change of variables transforms the Pfaff expression ω into an expression ϖ then that same change of variables transforms the derived expression ω' into the derived expression ϖ'.*

Indeed, suppose that with the new variables $y_1, y_2, \dots, y_n$, ω becomes:

$$\varpi = B_1\, dy_1 + B_2\, dy_2 + \dots + B_n\, dy_n\,.$$

If one lets α, β denote two arbitrary parameters then one has:

$$\text{(6)} \qquad A_1 \frac{\partial x_1}{\partial \alpha} + A_2 \frac{\partial x_2}{\partial \alpha} + \cdots + A_n \frac{\partial x_n}{\partial \alpha} = B_1 \frac{\partial y_1}{\partial \alpha} + B_2 \frac{\partial y_2}{\partial \alpha} + \cdots + B_n \frac{\partial y_n}{\partial \alpha}\,,$$

$$\text{(6)} \qquad A_1 \frac{\partial x_1}{\partial \beta} + A_2 \frac{\partial x_2}{\partial \beta} + \cdots + A_n \frac{\partial x_n}{\partial \beta} = B_1 \frac{\partial y_1}{\partial \beta} + B_2 \frac{\partial y_2}{\partial \beta} + \cdots + B_n \frac{\partial y_n}{\partial \beta}\,.$$

Differentiate the first of these equations with respect to β and the second one with respect to α, and subtract the two equations thus obtained. We will have:

$$\text{(8)} \qquad \begin{aligned} &\left(\frac{\partial A_1}{\partial \alpha}\frac{\partial x_1}{\partial \beta} - \frac{\partial A_1}{\partial \beta}\frac{\partial x_1}{\partial \alpha}\right) + \dots + \left(\frac{\partial A_n}{\partial \alpha}\frac{\partial x_n}{\partial \beta} - \frac{\partial A_n}{\partial \beta}\frac{\partial x_n}{\partial \alpha}\right) \\ &= \left(\frac{\partial B_1}{\partial \alpha}\frac{\partial y_1}{\partial \beta} - \frac{\partial B_1}{\partial \beta}\frac{\partial y_1}{\partial \alpha}\right) + \dots + \left(\frac{\partial B_n}{\partial \alpha}\frac{\partial y_n}{\partial \beta} - \frac{\partial B_n}{\partial \beta}\frac{\partial y_n}{\partial \alpha}\right). \end{aligned}$$

The left-hand side of (8) is nothing but the value of ω' relative to the two parameters α, β; the right-hand side is the value of ϖ' with the same two parameters.

Since these two values are equal for any α and β, the change of variables transforms ω' into a differential expression that is equivalent to ϖ', and which, in turn, after the making the reductions, is nothing but ϖ'. The theorem is thus proved ([1]).

([1]) The consideration of the derivative ω', or, what amounts to the same thing, of the bilinear covariant of ω, forms the basis for the beautiful research of Frobenius and Darboux on the Pfaff problem (*loc. cit.*).

13. *Derivatives of higher order.* – Along with the derivative of a Pfaff expression ω, we also consider other differential expressions of higher order ω'', ω''', …, which we define in the following manner:

$$\omega'' = \omega\omega' = (A_1\, dx_1 + \ldots + A_n\, dx_n)\,(dA_1\, dx_1 + \ldots + dA_n\, dx_n), \tag{9}$$

$$\omega''' = \tfrac{1}{2}\omega'^2 = \tfrac{1}{2}(dA_1\, dx_1 + dA_2\, dx_2 + \ldots + dA_n\, dx_n)^2 = \sum_{i,j} dA_i\, dx_i\, dA_j\, dx_j\,, \tag{10}$$

$$\omega^{\text{IV}} = \omega\omega''' = (A_1\, dx_1 + \ldots + A_n\, dx_n)\left(\sum_{i,j} dA_i\, dx_i\, dA_j\, dx_j\right). \tag{11}$$

In a general manner, the derivative of order $2m - 1$, $\omega^{(2m-1)}$, of a Pfaff expression ω will be the m^{th} power of ω', divided by $m!$, or the sum of all the m products of n monomials $dA_1\, dx_1$, $dA_2\, dx_2$, …, $dA_n\, dx_n$. The derivative of order $2m$, $\omega^{(2m)}$, will be the product of ω with $\omega^{(2m-1)}$. The p^{th} derivative is of degree $p + 1$.

These derivatives enjoy the same property as the derivative ω'. It is obvious that *if a change of variables transforms ω into ϖ then the same change of variables will transform the p^{th} derivative of ω into the p^{th} derivative of ϖ*, because that derivative is deduced by multiplying the two differential expressions ω and ω', which are transformed into ϖ and ϖ'.

14. *Exact differential Pfaff expressions.* – Suppose that the Pfaff expression ω is an exact differential form. It is then clear that under a change of variables it can be put into the form:

$$\varpi = dy_1\,.$$

Conversely, suppose that the derivative ω' of a Pfaff expression:

$$\omega = A_1\, dx_1 + A_2\, dx_2 + \ldots + A_n\, dx_n$$

is identically zero. I say that ω is an exact differential. The theorem is true for $n = 1$. Suppose that it is true up to $n - 1$, and prove that it is true for n. If one sets $dx_1 = 0$ in ω and regards x_1 as a constant then one obtains a Pfaff expression ω_1 in $n - 1$ variables whose derivative ω_1' is deduced from ω' by the same operations. It then results that this derivative ω_1' is identically zero, and that ω_1 is, in turn, an exact differential du. Now, if one no longer regards x_1 as a constant then one sees that one has:

$$\omega = du + \left(A_1 - \frac{\partial u}{\partial x_1}\right) dx_1\,,$$

and, by a change of variables, one can assume that:

$$\omega = A_1\, dx_1 + dx_1 .$$

Upon calculating ω', which must remain zero, one finds that:

$$\omega' = dA_1\, dx_1 = \frac{\partial A_1}{\partial x_2} dx_2 dx_1 + \frac{\partial A_1}{\partial x_3} dx_3 dx_1 + \cdots + \frac{\partial A_1}{\partial x_n} dx_n dx_1 = 0.$$

One then sees that the derivatives of A_1 with respect to x_2, x_3, …, x_n are zero, and consequently depend only upon x_1, so:

$$\omega = d\left(x_2 + \int A_1 dx_1\right)$$

is an exact differential.

The conditions for a Pfaff expression to be an exact differential are thus given by the equation:

$$\omega' = dA_1\, dx_1 + dA_2\, dx_2 + \ldots + dA_n\, dx_n = 0,$$

or, in finite terms:

$$\frac{\partial A_i}{\partial x_j} - \frac{\partial A_j}{\partial x_i} = 0 \qquad (i, j = 1, 2, \ldots, n). \tag{12}$$

15. *Class of a Pfaff expression.* – In the case that we just examined, one can, by a change of variables, put ω into a form that contains just the one variable y_1 explicitly. In the general case, it can happen that by a change of variables ω takes a form ϖ that contains just the p variables $y_1, y_2, \ldots, y_p$ explicitly:

$$\varpi = B_1\, dy_1 + B_2\, dy_2 + \ldots + B_p\, dy_p ,$$

in which the $B_1, B_2, \ldots, B_p$ depend upon only $y_1, y_2, \ldots, y_p$.

One calls the minimum number of variables, by means of which, one can express a Pfaff expression by a convenient change of variables the *class* ([1]) of that expression. A Pfaff expression of the first class is an exact differential.

16. *Necessary condition for a Pfaff expression to be of class p.* – If a Pfaff expression ω is of class p then one can, by a change of variables, put it into the form of a Pfaff expression ϖ in p variables. Thus, consider the p^{th} derivative of ϖ, which is of degree p + 1. Since that differential expression is in p variables and of degree p + 1, it is identically zero. It then results that the p^{th} derivative of ω, which is equal to it, is also identically zero.

Therefore, *in order for a Pfaff expression to be of class p, it is necessary that its p^{th} derivative be identically zero.* ([2]).

([1]) That expression was introduced by Frobenius, *loc. cit.*
([2]) Cf., GRASSMAN, *loc. cit.*

17. *Converse of the preceding theorem.* – We shall now prove that, conversely, if the p^{th} derivative of a Pfaff expression is identically zero then that expression is of class at most p. Since the theorem is true for $p = 1$, we shall suppose that it has been proved for p 1, 2, …, $p - 1$ and prove it for p.

Thus, consider a Pfaff expression:

$$\omega = A_1\, dx_1 + A_2\, dx_2 + \ldots + A_n\, dx_n$$

whose p^{th} derivative is identically zero, so:

$$\omega^{(p)} = \frac{1}{\left(\frac{p+1}{2}\right)!}\,\omega'^{\frac{p+1}{2}}$$

if p is odd, or:

$$\omega^{(p)} = \frac{1}{\left(\frac{p}{2}\right)!}\,\omega\omega'^{\frac{p}{2}}$$

if p is even.

It is clear that if n is less than or equal to p then that Pfaff expression has class at most p. Therefore, suppose that it has been proved that a Pfaff expression in 1, 2, …, $n - 1$ variables whose p^{th} derivative is zero has class at most p, and prove it for an expression in n variables.

If we regard x_1 in ω as a constant and make $dx_1 = 0$ then we obtain an expression ω_1 in $n - 1$ whose p^{th} derivative $\omega_1^{(p)}$ is therefore identically zero, and in turn, from the hypothesis that was made, ω_1 has class at most p. One can thus make a change of variables such that ω_1 is transformed into:

$$\varpi_1 = B_2\, dy_2 + B_3\, dy_3 + \ldots + B_{p+1}\, dy_{p+1}\,,$$

where $y_2, y_3, \ldots, y_{p+1}$ are p functions of $x_1, x_2, \ldots, x_n$, and where the B's are functions of $y_2, y_3, \ldots, y_{p+1}$, as well as the constant x_1. Now, if one no longer regards x_1 as a constant in ω then one will obviously obtain:

$$\omega = A_1\, dx_1 + B_2\left(dy_2 - \frac{\partial y_2}{\partial x_1}dx_1\right) + \ldots + B_{p+1}\left(dy_{p+1} - \frac{\partial y_{p+1}}{\partial x_1}dx_1\right).$$

Finally, after changing the notations, one has:

$$\omega = A_1\, dx_1 + A_2\, dx_2 + \ldots + A_{p+1}\, dx_{p+1}\,,$$

where $A_2, A_3, \ldots, A_{p+1}$ depend upon only $x_1, x_2, \ldots, x_{p+1}$.

This being the case, two cases can be present themselves: Either A_1 is independent of $x_1, x_2, \ldots, x_{p+1}$ or A_1 depends upon only these $p + 1$ variables.

18. In the first case, one can always suppose that one has taken $A_1 = x_{p+2}$. If one then groups the terms in $\omega^{(p)}$ that contain dx_{p+2} then one easily verifies that one obtains:

$$dx_{p+2}\ dx_1\,\omega_1^{(p-2)}\,,$$

where ω_1 has the same significance as it did above. Since the derivative $\omega^{(p)}$ is identically zero, the same must be true for the group of terms in that derivative that contain dx_{p+2}, and consequently, for $\omega_1^{(p-2)}$. Since the Pfaff expression ω_1 has its $(p-2)^{\text{th}}$ derivative equal to zero, it is of order at most $p - 2$. In other words, one can suppose that A_p and A_{p+1} are zero and that $A_2, A_3, \ldots, A_{p-1}$ depend upon only $x_1, x_2, \ldots, x_{p-1}$. The expression ω then becomes an expression in only p variables $x_1, x_2, \ldots, x_{p-1}, x_{p-2}$, and the theorem is proved.

19. In the second case, one comes down to an expression ω in $p + 1$ variables $x_1, x_2, \ldots, x_{p+1}$. Then consider the differential expression of $(p + 1)^{\text{th}}$ degree:

$$\omega^{(p-1)}\,df,$$

where f denotes an arbitrary function of $x_1, x_2, \ldots, x_{p+1}$; it is of the form:

$$H\,dx_1\,dx_2\,\ldots\,dx_{p+1} = \left(\alpha_1\frac{\partial f}{\partial x_1}+\alpha_2\frac{\partial f}{\partial x_2}+\cdots+\alpha_{p+1}\frac{\partial f}{\partial x_{p+1}}\right)dx_1\,dx_2\,\ldots\,dx_{p+1},$$

in which the α's are functions of x that depend upon only the coefficients A. If a change of variables transforms ω into ϖ and the function f of $x_1, x_2, \ldots, x_{p+1}$ into the function φ of $y_1, y_2, \ldots, y_{p+1}$ then that change of variables will transform $\omega^{(p-1)}\,df$ into $\varpi^{(p-1)}\,d\varphi$, and in turn, any function f that annuls the first of these two expressions will be transformed into a function φ that annuls the second one, and conversely. Now, the equation:

$$\omega^{(p-1)}\,df = 0,$$

or

$$\alpha_1\frac{\partial f}{\partial x_1}+\alpha_2\frac{\partial f}{\partial x_2}+\cdots+\alpha_{p+1}\frac{\partial f}{\partial x_{p+1}} = 0$$

is a partial differential equation that is linear in f and admits p independent integrals. One can make a change of variables by taking y_1 to be one of these integrals, or furthermore, one can, by changing the notations, suppose that x_1 is one of these integrals; i.e., that one has:

$$\omega^{(p-1)}\,dx_1 = 0.$$

The coefficient of dx_1 in the left-hand side of this equality is nothing but $\omega^{(p-1)}$, where one has set $dx_1 = 0$. If one then regards x_1 as a constant then it is the $(p-1)^{\text{th}}$ derivative of ω_1, where ω_1 has the same significance as it did above. Since the $(p-1)^{\text{th}}$ derivative of the expression ω_1 is zero, it therefore has class at most $p - 1$. In other words, one can suppose that:

$$\omega = A_1\, dx_1 + A_2\, dx_2 + \ldots + A_p\, dx_p ,$$

where $A_2, A_3, \ldots, A_p$ depend upon only $x_1, x_2, \ldots, x_p$. If A_1 is independent of $x_1, x_2, \ldots, x_p$ then one comes down to the first case, and the theorem is proved. If A_1 depends upon only $x_1, x_2, \ldots, x_p$ then ω takes the form of an expression in p variables, and the theorem is likewise proved.

20. *Introduction of a remarkable complete system.* – Consider a Pfaff expression of class p in n variables:

$$\omega = A_1\, dx_1 + A_2\, dx_2 + \ldots + A_n\, dx_n$$

and the equation that one obtains by equating the differential expression $\omega^{(p-2)}\, df$ to zero, where f denotes an arbitrary function of $x_1, x_2, \ldots, x_n$. Upon writing down that this expression is identically zero, one obtains a certain number of partial differential equations for f that are linear and of first order.

Consider a transformation of variables that makes ω now depend upon only p variables:

$$\varpi = B_1\, dy_1 + B_2\, dy_2 + \ldots + B_p\, dy_p ,$$

and let φ be the function of $y_1, y_2, \ldots, y_p$ that f is transformed into. It is clear that the two equations:

(13) $$\omega^{(p-2)}\, df = 0,$$

(14) $$\varpi^{(p-2)}\, d\varphi = 0$$

transform into each other under the change of variables, or that the system of partial differential equations for f that is equivalent to equation (13) is transformed into the system of partial differential equations for φ that is equivalent to equation (14). Now, this latter system is, first of all, comprised of the equations:

(15) $$\frac{\partial \varphi}{\partial y_{p+1}} = \frac{\partial \varphi}{\partial y_{p+2}} = \ldots = \frac{\partial \varphi}{\partial y_n} = 0.$$

Because $\varpi^{(p-2)}$ is not identically zero (since otherwise ϖ, and in turn, ω, would not be of class p), the coefficient of $dy_1\, dy_2 \ldots dy_{p-1}$, for example, in $\varpi^{(p-2)}$ is not zero, and consequently equations (15) are obtained by annulling the coefficients of:

$$dy_1\, dy_2 \ldots dy_{p-1}\, dy_{p+1}, \ldots, \qquad dy_1\, dy_2 \ldots dy_{p-1}\, dy_n$$

in the right-hand side of (14).

Other than equations (15), equations (14) provides one and only one equation for φ that one obtains by taking the coefficient of $dy_1\, dy_2 \ldots dy_p$, namely:

(16) $$\beta_1 \frac{\partial \varphi}{\partial y_1} + \beta_2 \frac{\partial \varphi}{\partial y_2} + \cdots + \beta_p \frac{\partial \varphi}{\partial y_p} = 0$$

in (14).

Equation (14) is therefore equivalent to the system of equations (15) and (16). Since the β's are functions of $y_1, y_2, \ldots, y_p$, that system is obviously a complete system that admits $p - 1$ independent integrals that are functions of $y_1, y_2, \ldots, y_p$.

Upon returning to equation (13), we see that it is equivalent to a complete system that admits p independent integrals. The integration of that system – by the Mayer method, for example – amounts to the integration of a system of ordinary differential equations in p variables.

We call this system *the adjoint complete system to the Pfaff expression* ([1]).

21. *Example.* – Consider, for example, the Pfaff expression in five variables:

(17) $$\omega = x_1 x_3\, dx_2 + x_1 x_2\, dx_3 + (x_1 + x_3 x_5)\, dx_4 + x_3 x_4\, dx_5 .$$

Here, one has, upon performing the calculations:

$$\omega' = x_3\, dx_1\, dx_2 + x_2\, dx_1\, dx_3 + dx_1\, dx_4 + x_5\, dx_3\, dx_4 + x_4\, dx_3\, dx_5 ,$$

$$\omega''' = \tfrac{1}{2}\omega'^2 = x_3 x_5\, dx_1\, dx_2\, dx_3\, dx_4 - x_4\, dx_1\, dx_3\, dx_4\, dx_5 + x_3 x_4\, dx_1\, dx_2\, dx_3\, dx_5 ,$$

so

$$\omega^{\mathrm{IV}} = \omega\, \omega''' = 0.$$

The expression ω is therefore of fourth class. The adjoint complete system is then given by the equation:

$$\omega''\, df = \omega\omega'\, df = 0,$$

and must therefore admit three independent integrals. Upon performing the calculations, one finds for the preceding equation:

$$\begin{aligned}\omega''\, df = (\, & x_3^2 x_5\, dx_1\, dx_2\, dx_4 + x_3^2 x_4\, dx_1\, dx_2\, dx_5 + x_2 x_3\, dx_1\, dx_3\, dx_4 \\ & + x_2 x_3 x_4\, dx_1\, dx_3\, dx_5 + x_3 x_4\, dx_1\, dx_4\, dx_5 \\ & + x_1 x_3 x_5\, dx_2\, dx_3\, dx_4 + x_1 x_3 x_4\, dx_2\, dx_3\, dx_5 - x_1 x_4 x_3\, dx_3\, dx_4\, dx_5) \\ & \times \left(\frac{\partial f}{\partial x_1} dx_1 + \frac{\partial f}{\partial x_2} dx_2 + \cdots + \frac{\partial f}{\partial x_5} dx_5 \right) = 0,\end{aligned}$$

which gives the system for f:

$$x_4 \frac{\partial f}{\partial x_2} - x_3 x_4 \frac{\partial f}{\partial x_4} + x_3 x_5 \frac{\partial f}{\partial x_3} = 0,$$

([1]) In the case where p is even and n is equal to p, it is the first auxiliary system that one finds in all of the methods of reduction.

$$x_1 \frac{\partial f}{\partial x_1} - x_2 \frac{\partial f}{\partial x_2} + x_3 \frac{\partial f}{\partial x_3} = 0.$$

This is indeed a complete system that admits the three independent integrals x_1 / x_3, $x_2x_3 + x_4$, $x_4 x_5$.

22. *Properties of the integrals of the adjoint complete system.* – Consider a Pfaff expression ω of class p and one of the p independent integrals of the adjoint complete system. Make a change of variables by taking that particular integral to be one of the new variables y_1 . The expression ω then becomes a certain expression ϖ in $y_1, y_2, \ldots, y_n$, and one has:

$$\varpi^{(p-2)}\, dy_1 = 0.$$

That equality expresses the idea that if one regards y_1 as a constant in ϖ and one sets $dy_1 = 0$ then the expression ϖ_1 thus obtained has its $(p - 2)^{\text{th}}$ derivative equal to zero. In other words, the expression ϖ has class at most $p - 2$; moreover, it certainly does not have a lower class, or else the introduction of a term in dy_1 could not make ϖ have class p.

Conversely, if ϖ_1 has class $p - 2$ then its $(p - 2)^{\text{th}}$ derivative is zero, or furthermore, the expression $\varpi^{(p-2)}\, dy_1$ is zero.

An integral of the adjoint complete system is therefore a function f that reduces the class of the Pfaff expression considered by two units when one equates it to an arbitrary constant.

Naturally, this statement implicitly assumes that at the same time that one couples $x_1, x_2, \ldots, x_n$ by the relation:

$$f(x_1, x_2, \ldots, x_n) = a$$

one couples the differentials by the relation:

$$df = \frac{\partial f}{\partial x_1} dx_1 + \frac{\partial f}{\partial x_2} dx_2 + \cdots + \frac{\partial f}{\partial x_n} dx_n = 0.$$

Therefore, in the example that was previously treated, if one takes the integral x_1 / x_3 of the adjoint complete system then the substitution of ax_3 for x_1 and adx_3 for dx_1 must reduce the class of ω by two units. Indeed, ω becomes:

$$(18) \qquad \begin{cases} \omega = ax_3^2 dx_2 + ax_2x_3dx_2 + x_3(a + x_5)dx_4 + x_3x_4dx_5 \\ \quad = x_3 d[ax_2x_3 + (a + x_5)x_4], \end{cases}$$

and is no longer of class two.

23. *Reduction of a Pfaff expression of class p to a canonical form.* – Given a Pfaff expression ω of class p, let f_1 be an integral of the adjoint complete system. Consider the equations:

(19) $$\omega^{(p-4)}\, df_1\, df = 0,$$

where f denotes an arbitrary function of $x_1, x_2, \ldots, x_n$. If one makes a change of variables by taking f_1 to be one of the variables y_1 then if that change of variables transforms ω into ϖ and f into φ then the preceding equation becomes:

(20) $$\varpi^{(p-4)}\, dy_1\, d\varphi = 0.$$

If one regards y_1 as a constant in ϖ and φ, and one makes $dy_1 = 0$ everywhere then this equation can be further written:

(21) $$\omega_1^{(p-4)} d\varphi = 0.$$

Since ϖ_1 has class $p - 2$, one sees that it is equivalent to the adjoint complete system to ϖ_1 . This system admits $p - 3$ independent integrals that are functions of $y_2, y_3, \ldots, y_n$, and also the constant y_1 . Upon going back to equation (20) and no longer regarding y_1 as a constant, one sees that the equation is equivalent to a complete system that admits $p - 2$ independent integrals, among which is y_1 .

Finally, equation (19) is equivalent to a complete system that admits $p - 2$ independent integrals, among which one finds the function f_1 itself.

Those of the integrals f of the complete system that is equivalent to (19) *that are independent of f_1 are functions such that the relations:*

(22) $$\begin{cases} f = a, & f_1 = a_1, \\ df = 0, & df_1 = 0, \end{cases}$$

reduce the class of ω by four units. The proof is absolutely the same as in the preceding case.

From the Mayer method, these functions are given by the integration of a system of ordinary differential equations in $p - 2$ variables.

It is indeed clear that when it is practical to infer one of the variables as a function of the $n - 1$ other ones from $f_1 = a$, it will suffice to integrate the adjoint complete system to the Pfaff expression that results from ω by that substitution.

One can then continue step-by-step. Letting f_2 denote an independent integral of f_1 in equation (19), one considers the equation:

(23) $$\omega^{(p-6)}\, df_1\, df_2\, df = 0.$$

This equation is equivalent to a complete system that admits $p - 5$ independent integrals of f_1 and f, and these integrals are functions f such that the relations:

(24) $$\begin{cases} f = a, & f_1 = a_1, \quad f_2 = a_2, \\ df = 0, & df_1 = 0, \quad df_2 = 0 \end{cases}$$

reduce the class of ω by six units, and so on.

Having done that, two cases can present themselves, according to whether p is even or odd.

24. *Canonical form for an expression of even class.* – If p is even and equal to $2m$, for example, then the $(m-1)^{\text{th}}$ complete system will be given by the equation:

$$\omega' \, df_1 \, df_2 \dots df_{m-2} \, df = 0, \tag{25}$$

and the m^{th} one will be, in turn:

$$\omega \, df_1 \, df_2 \dots df_{m-2} \, df_{m-1} \, df = 0. \tag{26}$$

It is clear that this will give functions f_m such that the relations:

$$\begin{cases} f_1 = a_1, \quad f_2 = a_2, \quad \cdots, \quad f_{m-1} = a_{m-1}, \quad f_m = a_m, \\ df_1 = 0, \quad df_2 = 0, \quad \cdots, \quad df_{m-1} = 0, \qquad df_m = 0 \end{cases} \tag{27}$$

render ω identically zero. If one then takes the new variables to be $y_1 = f_1$, $y_2 = f_2$, …, $y_m = f_m$ and m other arbitrary functions that are independent of the latter then ω will take the form:

$$\varpi = B_1 \, dy_1 + B_2 \, dy_2 + \dots + B_m \, dy_m .$$

It is clear that the m coefficients B are mutually independent functions that are independent of $y_1, y_2, \dots, y_m$, since other wise ϖ would have a class that was lower than $2m$. One can thus take m independent variables other than $y_1, y_2, \dots, y_m$. Upon changing notations, we have the following theorem:

Theorem. – *Given an arbitrary Pfaff expression of class* $2m$*, one can always put it into the form:*

$$\omega = p_1 \, dx_1 + p_2 \, dx_2 + \dots + p_m \, dx_m \tag{28}$$

by a change of variables, where $x_1, x_2, \dots, x_m$; $p_1, p_2, \dots, p_m$ *are* $2m$ *independent variables.*

This reduction can be accomplished by the search for *one* integral of m systems of ordinary differential equations in $2m$, $2m - 2$, …, 4, 2 variables, respectively ([1]).

25. In the example that was treated above, one had $m = 2$; we then found an integral x_1 / x_3 of the first complete system. The second one is provided by the equation:

$$[a \, x_3^2 \, dx_2 + a \, x_2 \, x_3 \, dx_3 + x_3 \, (a + x_5) \, dx_4 + x_3 \, x_4 \, dx_5] \tag{29}$$

([1]) Equations (25) differ only in form from the equations that present themselves when one uses the Frobenius method of reduction.

$$\times\left(\frac{\partial f}{\partial x_2}dx_2+\frac{\partial f}{\partial x_3}dx_3+\cdots+\frac{\partial f}{\partial x_5}dx_5\right)=0,$$

and can be put into the form:

(30)
$$\frac{\dfrac{\partial f}{\partial x_2}}{a\,x_3^2}=\frac{\dfrac{\partial f}{\partial x_3}}{a\,x_2x_3}=\frac{\dfrac{\partial f}{\partial x_4}}{x_3(a+x_5)}=\frac{\dfrac{\partial f}{\partial x_5}}{x_3x_4}.$$

One easily finds an integral, namely:

$$a\,x_2\,x_3+(a+x_5)\,x_4=b=x_1\,x_2+x_4\,x_5+\frac{x_1x_4}{x_3}.$$

By setting, for example:

$$x_1 = a\,x_3, \qquad x_5=-a+\frac{b-a\,x_2x_3}{x_4},$$

and substituting this in (17), while regarding a and b are variables, one finds, after making all reductions:

$$\omega=-x_3\,(x_4+x_2\,x_3)\,da+x_3\,db.$$

Here, the variables x_1, x_2, p_1, p_2 in the canonical form are:

$$\frac{x_1}{x_3},\; x_1\,x_2+x_4\,x_5+\frac{x_1x_4}{x_3},\; -x_3\,(x_4+x_2\,x_3),\; x_3\,.$$

26. *Canonical form for an expression of odd class.* – If p is odd and equal to $2m + 1$, for example, then the m^{th} complete system is:

(31)
$$\omega'\,df_1\,df_2\,\ldots\,df_{m-1}\,df=0.$$

Thus, if f_m is an independent integral of $f_1, f_2, \ldots, f_{m-1}$ then the relations:

(32)
$$\begin{cases} f_1=a_1, & f_2=a_2, \quad \cdots, \quad f_m=a_m, \\ df_1=0, & df_2=0, \quad \cdots, \quad df_m=0 \end{cases}$$

makes ω have first class; i.e., a *exact differential dz.* The following theorem results:

Theorem. – *Given an arbitrary Pfaff expression of class* $2m + 1$, *one can always put it into the form:*

(33)
$$\omega=dz-p_1\,dx_1-p_2\,dx_2-\ldots-p_m\,dx_m$$

by a change of variables, where $x_1, x_2, \ldots, x_m, z, p_1, p_2, \ldots, p_m$ *are* $2m + 1$ *independent variables.*

This reduction can be accomplished by the search for *one* integral of the m systems of ordinary differential equations in $2m + 1$, $2m - 1$, …, 5, 3 variables, respectively, and by a quadrature.

27. For example, take:

$$\omega = x_3\, dx_1 + x_1\, dx_2 - x_3\, x_5\, dx_4 - x_3\, x_4\, dx_5 + x_2\, dx_6 .$$

One finds that:

$$\begin{aligned} \omega' &= dx_2\, dx_6 - x_5\, dx_3\, dx_4 - x_4\, dx_3\, dx_5 , \\ \omega''' &= - x_5\, dx_2\, dx_3\, dx_4\, dx_6 - x_4\, dx_2\, dx_3\, dx_4\, dx_6 , \\ \omega^{\text{V}} &= 0. \end{aligned}$$

The expression ω therefore has class five at most; one easily confirms that since ω^{IV} is identically zero, ω is effectively of class five.

Here, the adjoint complete system is:

$$\omega''' \, df = 0,$$

which decomposes into:

$$\frac{\partial f}{\partial x_1} = 0,$$

$$x_4 \frac{\partial f}{\partial x_4} - x_5 \frac{\partial f}{\partial x_5} = 0.$$

One can take x_2 to be one of the integrals of that complete system. Then, take $x_2 = a_1$, and form the complete system:

$$\omega' \, df = 0,$$

which is:

$$\frac{\partial f}{\partial x_1} = 0,$$

$$\frac{\partial f}{\partial x_4} = 0,$$

$$x_4 \frac{\partial f}{\partial x_4} - x_5 \frac{\partial f}{\partial x_5} = 0,$$

here. The function x_2 is an integral of this system. Thus, upon setting $x_2 = a_1$, $x_3 = a_2$, ω must become an exact differential. Indeed, one finds that:

$$\omega = d(a_1\, x_1 - a_2\, x_4\, x_5 + a_1\, x_6),$$

and if one no longer regards a_1 and a_2 as constants then one gets ([1]):

$$\omega = d(a_1\,x_1 - a_2\,x_4\,x_5 + a_1\,x_6) - x_6\,da_1 + x_4\,da_2\,,$$
$$\omega = d(x_1\,x_2 - x_3\,x_4\,x_5 + x_2\,x_6) - x_4\,dx_2 + x_4\,x_5\,dx_3\,.$$

28. *Remark.* – The adjoint complete system for a Pfaff expression, when put into its canonical form, reduces to the equation:

$$p_1\frac{\partial f}{\partial p_1} + p_2\frac{\partial f}{\partial p_2} + \cdots + p_m\frac{\partial f}{\partial p_m} = 0$$

in the case where the class is even, and to the equation:

$$\frac{\partial f}{\partial z} = 0$$

in the case where the class is odd. It thus admits the independent integrals:

$$x_1,\ x_2,\ \ldots,\ x_m,\quad \frac{p_1}{p_m},\ \frac{p_2}{p_m},\ \ldots,\ \frac{p_{m-1}}{p_m}$$

in the former case, and the independent integrals:

$$x_1,\ x_2,\ \ldots,\ x_m,\quad p_1,\ p_2,\ \ldots,\ p_m$$

in the latter case.

One sees that all of the integrals of the complete system that one encounter in the reduction satisfy the adjoint complete system, since the m integrals that are used here are $x_1, x_2, \ldots, x_m$.

Later on (IV, 69, 70, 75), one will give a new form into which one can put the equations of the successive complete systems that facilitate the reduction.

([1]) The Frobenius method of reduction for the expressions of odd class differs from the one that we just presented in that it begins by determining a function f such that $\omega - df$ is only of class $2m$.

III. – Total differential equations.

29. The total differential equations that we shall occupy ourselves with are the ones that one obtains by equating a Pfaff expression to zero.

To solve an equation of that nature is to find a system of finite relations between x_1, x_2, …, x_m such that these relations between the variables and the ones that one deduces between their differentials annul the Pfaff expression.

We shall first propose to find, in a general manner, all of the systems of equations that annul a Pfaff expression. In that regard, any Pfaff expression can be assumed to be of odd class, since one can always reduce the class of a Pfaff expression of even class by one unit because by dividing by a convenient factor. Thus, the equation:

$$p_1\, dx_1 + p_2\, dx_2 + \ldots + p_m\, dx_m = 0$$

can be written

$$dx_m + \frac{p_1}{p_m} dx_1 + \frac{p_2}{p_m} dx_2 + \cdots + \frac{p_{m-1}}{p_m} dx_{m-1} = 0,$$

where the left-hand side has class $2m - 1$.

30. *Minimum number of equations that annul a Pfaff expression*. – Consider a Pfaff expression ω of class $2m + 1$ (or $2m + 2$) that has been put into canonical form, and seek to solve the equation:

(1) $$\omega = dz - p_1\, dx_1 - p_2\, dx_2 - \ldots - p_m\, dx_m = 0$$

by means of a minimum number of relations between $x_1, x_2, \ldots, x_m, z, p_1, p_2, \ldots, p_m$, and the other variables that do not enter into ω explicitly. A first solution is provided by equating $x_1, x_2, \ldots, x_m, z$ to $m + 1$ arbitrary constants, which gives $m + 1$ relations. I say that it is not possible to satisfy equation (1) with a smaller number of relations.

Indeed, equation (1) expresses the idea that there is at least one relation between z, x_1, x_2, …, x_m. Suppose that there are exactly $h + 1$ of them, and, more precisely, these relations can be put into the form:

(2) $$\begin{cases} x_1 = \varphi_1(x_{h+1}, x_{h+2}, \ldots, x_m), \\ x_2 = \varphi_2(x_{h+1}, x_{h+2}, \ldots, x_m), \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ x_h = \varphi_h(x_{h+1}, x_{h+2}, \ldots, x_m), \\ z = \psi(x_{h+1}, x_{h+2}, \ldots, x_m). \end{cases}$$

Since the variables x_{h+1}, x_{h+2}, …, x_m are not coupled by any relation, equation (1) gives:

(3)
$$\begin{cases} \dfrac{\partial\psi}{\partial x_{h+1}} - p_1\dfrac{\partial\varphi_1}{\partial x_{h+1}} - p_1\dfrac{\partial\varphi_1}{\partial x_{h+1}} - \cdots - p_h\dfrac{\partial\varphi_h}{\partial x_{h+1}} - p_{h+1} = 0, \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots, \\ \dfrac{\partial\psi}{\partial x_m} - p_1\dfrac{\partial\varphi_1}{\partial x_m} - p_1\dfrac{\partial\varphi_1}{\partial x_m} - \cdots - p_h\dfrac{\partial\varphi_h}{\partial x_m} - p_m = 0. \end{cases}$$

These $m - h$ new relations (3) are mutually independent and independent of equations (2). Along with (2), they form a system of $m + 1$ equations that solve the problem.

At the same time, one sees that in order to get the most general solution, it suffices to append to these equations an arbitrary number of other equations that form an algebraically compatible system with the first ones. One can take $\varphi_1, \varphi_2, \ldots, \varphi_h$ to be absolutely arbitrary functions, the number h being equal to 0, 1, 2, …, m. The particular solution that was obtained above is obtained by taking $h = m$, so the functions φ and ψ are then constants.

31. In order to make what we just said about the general solution to the equation:

(1)
$$\omega = dz - p_1\, dx_1 - p_2\, dx_2 - \ldots - p_m\, dx_m = 0$$

precise and complete, one directly looks for all of the systems of relations that satisfy that equation, and for which a given system of values $x_1^0, \ldots, x_m^0, z_0, p_1^0, \ldots, p_m^0$ constitutes a simple example; i.e., such that in the neighborhood of this system of values, a certain number $2m - h + 1$ of the variables can be expressed as holomorphic functions of h other ones, or, what amounts to the same thing, one looks for all systems of $2m - h + 1$ relations such that the left-hand sides of these relations are holomorphic in the neighborhood of this system of values and the functional determinants of these left-hand sides with respect to the $2m - h + 1$ variables are not zero for the same system of values, moreover. The h different variables of the $2m - h + 1$ variables with respect to which one can solve the system will be called the *h independent variables.*

This being the case, one can always suppose that z is not one of the independent variables; indeed, otherwise, one would have:

$$1 - p_1\frac{\partial x_1}{\partial z} - p_2\frac{\partial x_2}{\partial z} - \cdots - p_m\frac{\partial x_m}{\partial z} = 0.$$

This equality shows that $x_1, x_2, \ldots, x_m$ cannot all be independent variables, since then the left-hand side would reduce to 1 and one of the derivatives $\partial x_1 / \partial z, \ldots, \partial x_m / \partial z$ – the first one, for example – would be non-zero for $(x_1^0, \ldots, p_m^0)$. This shows that one can deduce z as a holomorphic function of x_1 and $h - 1$ other independent variables, and that, in turn, one can replace z with x_1 as an independent variable.

One can likewise suppose that among the h independent variables that are taken from the x and p, there are not two of them such as x_1 and p_1, in other words, that these h variables have h distinct indices. Indeed, otherwise one would have:

$$\frac{D(x_1, p_1)}{D(x_1, p_1)} + \frac{D(x_2, p_2)}{D(x_1, p_1)} + \cdots + \frac{D(x_m, p_m)}{D(x_1, p_1)} = 0$$

upon considering the derivative ω', which must be zero for the system of relations considered.

The first term in that equality is equal to 1. It then results that at least one of the indices is not represented in any of the h independent variables, because otherwise all of the terms that follow the first one would be zero. If the unrepresented indices were, for example, the last $m - \alpha$ indices then only the last $m - \alpha$ terms in the equality could be non-zero, and, in turn, at least one of the quantities:

$$\frac{\partial x_{\alpha+1}}{\partial x_1}, \frac{\partial x_{\alpha+2}}{\partial x_1}, \ldots, \frac{\partial x_m}{\partial x_1}; \frac{\partial p_{\alpha+1}}{\partial x_1}, \ldots, \frac{\partial p_m}{\partial x_1}$$

would be non-zero for $(x_1^0, \ldots, p_m^0)$, namely, $\partial x_{\alpha+1} / \partial x_1$. One could then deduce z as a holomorphic function of $x_{\alpha+1}$ and substitute $x_{\alpha+1}$ for x_1 as an independent variable. The indices 1 and $a + 1$ would then be represented just one time amongst the h independent variables. If there were another pair of variables, such as (x_2, p_2), then one would repeat the same operation, in such a manner that one would finally arrive at h independent variables with all of their indices distinct. This proves, in particular, that h cannot exceed m.

32. This being the case, suppose that the h independent variables are:

$$x_1, x_2, \ldots, x_\alpha; \; p_{\alpha+1}, p_{\alpha+2}, \ldots, p_h .$$

One will then have relations of the form:

$$(4) \qquad \begin{cases} z - x_{\alpha+1} p_{\alpha+1} - \cdots - x_h p_h = w(x_1, \ldots, x_\alpha, p_{\alpha+1}, \ldots, p_h), \\ x_{h+1} = u_{h+1}(x_1, \ldots, x_\alpha, p_{\alpha+1}, \ldots, p_h), \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \\ x_m = u_m(x_1, \ldots, x_\alpha, p_{\alpha+1}, \ldots, p_h), \\ p_{h+1} = v_{h+1}(x_1, \ldots, x_\alpha, p_{\alpha+1}, \ldots, p_h), \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \\ p_m = v_m(x_1, \ldots, x_\alpha, p_{\alpha+1}, \ldots, p_h), \end{cases}$$

the functions $u_{h+1}, \ldots, u_m, v_{h+1}, \ldots, v_m, w$ being holomorphic in the neighborhood of $(x_1^0, \ldots, x_\alpha^0, p_{\alpha+1}^0, \ldots, p_h^0)$ and subject to the sole condition that for this system of values they must take the values:

$$x_{h+1}^0, \ldots, x_m^0, \; p_{h+1}^0, \ldots, p_m^0, \; z^0 - x_{\alpha+1}^0 p_{\alpha+1}^0 - \ldots - x_h^0 p_h^0,$$

respectively.

Upon substituting this in the total differential equation (1), it takes the form:

$$dw - p_1\, dx_1 - \ldots - p_\alpha\, dx_\alpha + x_{\alpha+1}\, dp_{\alpha+1} + \ldots + x_h\, dp_h - v_{h+1}\, du_{h+1} - v_m\, du_m = 0,$$

which immediately gives the values for $p_1, p_2, \ldots, p_\alpha, x_{\alpha+1}, \ldots, x_h$, namely:

$$(5)\qquad \left\{\begin{aligned} p_1 &= \frac{\partial w}{\partial x_1} - v_{h+1}\frac{\partial u_{h+1}}{\partial x_1} - \cdots - v_m \frac{\partial u_m}{\partial x_1}, \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \\ p_\alpha &= \frac{\partial w}{\partial x_\alpha} - v_{h+1}\frac{\partial u_{h+1}}{\partial x_\alpha} - \cdots - v_m \frac{\partial u_m}{\partial x_\alpha}, \\ x_{\alpha+1} &= -\frac{\partial w}{\partial p_{\alpha+1}} + v_{h+1}\frac{\partial u_{h+1}}{\partial p_{\alpha+1}} + \cdots + v_m \frac{\partial u_m}{\partial x_{\alpha+1}}, \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \\ x_h &= -\frac{\partial w}{\partial p_h} + v_{h+1}\frac{\partial u_{h+1}}{\partial p_h} + \cdots + v_m \frac{\partial u_m}{\partial x_h}. \end{aligned}\right.$$

Formulas (4) and (5) resolve the question. The solution thus depends upon $2m - 2h + 1$ arbitrary functions of h arguments and h of them can take the values 0, 1, 2, …, m. If we combine these two groups of formulas then we get the general solution of equation (1) that admits the system of values $(x_1^0, \ldots, p_m^0)$ as a simple element in the form of the following relations:

$$
(6)\qquad \left\{
\begin{aligned}
z &= w - p_{\alpha+1}\frac{\partial w}{\partial p_{\alpha+1}} - \cdots - p_h \frac{\partial w}{\partial p_h} + v_{h+1}\left(p_{\alpha+1}\frac{\partial u_{h+1}}{\partial p_{\alpha+1}} + \cdots + p_h \frac{\partial u_{h+1}}{\partial p_h}\right) + \cdots \\
&\qquad + v_m \left(p_{\alpha+1}\frac{\partial u_m}{\partial p_{\alpha+1}} + \cdots + p_h \frac{\partial u_m}{\partial p_h}\right), \\
p_1 &= \frac{\partial w}{\partial x_1} - v_{h+1}\frac{\partial u_{h+1}}{\partial x_1} - \cdots - v_m \frac{\partial u_m}{\partial x_1}, \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \\
p_\alpha &= \frac{\partial w}{\partial x_\alpha} - v_{h+1}\frac{\partial u_{h+1}}{\partial x_\alpha} - \cdots - v_m \frac{\partial u_m}{\partial x_\alpha}, \\
x_{\alpha+1} &= -\frac{\partial w}{\partial p_{\alpha+1}} + v_{h+1}\frac{\partial u_{h+1}}{\partial p_{\alpha+1}} + \cdots + v_m \frac{\partial u_m}{\partial p_{\alpha+1}}, \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \\
x_h &= -\frac{\partial w}{\partial p_h} + v_{h+1}\frac{\partial u_{h+1}}{\partial p_h} + \cdots + v_m \frac{\partial u_m}{\partial p_h}, \\
x_{h+1} &= u_{h+1}, \\
&\ldots\ldots, \\
x_m &= u_m, \\
p_{h+1} &= v_{h+1}, \\
&\ldots\ldots, \\
p_m &= v_m,
\end{aligned}
\right.
$$

where $u_{h+1}, \ldots u_m, v_{h+1}, \ldots, v_m, w$ are holomorphic functions of $x_1, \ldots, x_\alpha, p_{\alpha+1}, \ldots, p_h$ that are holomorphic in the neighborhood of $(x_1^0, \ldots, p_h^0)$ and take given values for that system of values, while the first-order, partial derivatives of the first $2m - 2h$ of them take given values that are easy to calculate (always for the same system of values).

In particular, for $h = m$, there is only one arbitrary function w of m arguments – namely, of $x_1, \ldots, x_\alpha, p_{\alpha+1}, \ldots, p_m$ – and one has:

$$
(7)\qquad \begin{cases}
z = w - p_{\alpha+1}\dfrac{\partial w}{\partial p_{\alpha+1}} - \cdots - p_m \dfrac{\partial w}{\partial p_m}, \\
p_1 = \dfrac{\partial w}{\partial x_1}, \\
\quad \ldots\ldots\ldots, \\
p_\alpha = \dfrac{\partial w}{\partial x_\alpha}, \\
x_{\alpha+1} = -\dfrac{\partial w}{\partial p_{\alpha+1}}, \\
\quad \ldots\ldots\ldots\ldots\ldots, \\
x_m = -\dfrac{\partial w}{\partial p_m}.
\end{cases}
$$

In particular, if one takes w to be a linear function of $p_{\alpha+1}, \ldots, p_m$ then one recovers formulas (2) and (3), with a simple change of notations.

33. *General solution of an arbitrary Pfaff equation.* – We just solved the particular Pfaff equations (1). From that, if one is given an arbitrary Pfaff expression of class $2m + 1$ or $2m + 2$ then one will only have to reduce it to its canonical form. The equation to be solved will then be of the form (1), and equations (6) will provide the general solution of the problem. One sees that if $\omega^{(2m+2)}$ is the first derivative of even order that is annulled identically then in order to annul ω one must have a system of at least $m + 1$ equations between the variables, and then one will have an infinitude of them that depend upon an arbitrary function of m arguments.

34. *Singular solutions.* – The preceding conclusion can nevertheless be incorrect in certain particular cases. It can happen that the first derivative of even order of a Pfaff expression ω that is identically zero is $\omega^{(2m)}$, so one can either annul that expression by means of a system of less than m relations between $x_1, x_2, \ldots, x_n$ or by means of a system of at most m relations, but those relations do not enter into formula (7). This case can present itself when the change of variables that reduces ω to its canonical form is illusory for the system of values of the variables that satisfy these relations. That is why the third-order expression:

$$dx_1 - x_1\, x_2\, dx_3$$

can be annulled by means of the single equation:

$$x_1 = 0,$$

which indeed translates into the system of two equations:

$$x_1 = x_1\, x_2 = 0$$

with the canonical variables (x_1, x_2, x_3, x_4). Moreover, that is why one can satisfy the equation:

$$p_1\,dx_1 + \ldots + p_m\,dx_m = 0$$

with the system of m relations:

$$p_1 = p_2 = \ldots = p_m = 0,$$

which does not fall into the general type.

It is therefore important to find all of the solutions that do not fall into the general type. In order to do this, we shall give a very simple criterion.

35. *Conditions for a solution to be singular*. – We shall prove that *if the Pfaff expression* ω *is of class* $2m - 1$ *or* $2m$ *then in order for a solution to be singular, it is necessary that this solution should annul all of the coefficients of the* $(2m - 2)^{\text{th}}$ *derivative of* ω*, which is assumed to be put into its simplest form.*

We suppose that the coefficients of ω are holomorphic functions of the variables, and we consider only solutions, which are general or singular, that are defined by a certain number h of equations whose left-hand sides are holomorphic in the neighborhood of an arbitrary system of values that satisfies these equations, and the functional determinants of these h left-hand sides with respect to h arbitrary ones of the variables are not all zero for this same system of values.

This being the case, we shall prove that if the coefficients of $\omega^{(2m-2)}$ are not all zero for an arbitrary system of values of the variables that corresponds to a given solution then that solution is *general* – i.e., one can obtain it by the procedure presented above.

Indeed, first consider the equation:

$$\omega^{(2m-2)}\,df = 0. \tag{8}$$

If ω has class $2m$ then this equation is equivalent to the adjoint complete system to ω and admits $2m - 1$ independent integrals. This complete system is thus formed from $n - 2m + 1$ independent equations. If ω has class $2m - 1$, and if one takes the variables y_1, y_2, …, y_n such that ω depends upon only y_1, y_2, …, y_{2m-1} explicitly then equation (8) is obviously equivalent to the system:

$$\frac{\partial f}{\partial y_{2m}} = \frac{\partial f}{\partial y_{2m+1}} = \ldots = \frac{\partial f}{\partial y_n} = 0,$$

and, in turn, to a complete system that admits $2m - 1$ independent integrals and forms $n - 2m + 1$ independent integrals. In any case, equation (8) *furnishes a complete system that shall call the* COMPLETE SYSTEM *that is adjoint to the total differential equation* $\omega = 0$ *and which admits* $2m - 1$ *independent integrals.*

36. This being the case, we return to our particular solution and let:

$$(x_1^0, x_2^0, \ldots, x_n^0)$$

be an arbitrary system of values that correspond to that solution. By hypothesis, the coefficients of $\omega^{(2m-2)}$, which has degree $2m - 1$, are not all zero for this system of values. For example, suppose that the coefficient of:

$$dx_1\, dx_2 \,\dots\, dx_{2m-1}$$

is not zero. Since the expression $\omega^{(2m-2)}$ results from the product of $\omega^{(2m-1)}$ with ω', it then results that the coefficients of the various monomials in $\omega^{(2m-1)}$ that are formed from the differentials $dx_1, dx_2, \dots, dx_{2m-1}$ are not all zero, always for the same system of values. For example, suppose that this is true for the coefficient of:

$$dx_1\, dx_2 \,\dots\, dx_{2m-2} .$$

We can continue in that way and assume that:

The coefficient of $dx_1\, dx_2 \dots dx_{2m-1}$ in $\omega^{(2m-2)}$ is not zero,
" $dx_2\, dx_3 \dots dx_{2m-2}$ in $\omega^{(2m-4)}$ ",
…………………………………………………………..,
" $dx_r\, dx_{r+1} \dots dx_{2m-r}$ in $\omega^{(2m-2r)}$ ",
…………………………………………………………..,
" dx_m in ω ".

Under these conditions, consider the adjoint complete system to the equation $\omega = 0$. In order to form it, take the total coefficients of the monomials:

$$\begin{aligned} &dx_1\, dx_2 \dots dx_{2m-1}\, dx_{2m} , \\ &dx_1\, dx_2 \dots dx_{2m-1}\, dx_{2m+1}, \\ &\dots\dots\dots\dots, \\ &dx_1\, dx_2 \dots dx_{2m-1}\, dx_n \end{aligned}$$

in the left-hand side of (8). From the hypotheses that were made, we will thus have $n - 2m + 1$ equations solved for $\frac{\partial f}{\partial x_{2m}}, \frac{\partial f}{\partial x_{2m+1}}, \dots, \frac{\partial f}{\partial x_n}$, while the coefficients of the other derivatives are holomorphic in the neighborhood of $x_1^0, x_2^0, \dots, x_n^0$. Since the complete system contains exactly $n - 2m + 1$ equations, it is determined completely. We see, moreover, that from the theory of complete systems *this system admits* $2m - 1$ *independent integrals that are holomorphic in the neighborhood of* $x_1^0, x_2^0, \dots, x_n^0$ *and reduce to* $x_1 - x_1^0, x_2 - x_2^0, \dots, x_{2m-1} - x_{2m-1}^0$ *for* $x_{2m} = x_{2m}^0, x_{2m+1} = x_{2m+1}^0, \dots, x_n = x_n^0$. We take u_1 to be the one of these integrals that reduces to $x_1 - x_1^0$. Upon neglecting terms of degree two and higher, that integral is therefore of the form:

$$u_1 = x_1 - x_1^0 + \alpha_{2m}(x_{2m} - x_{2m}^0) + \alpha_{2m+1}(x_{2m+1} - x_{2m+1}^0) + \dots + \alpha_n(x_n - x_n^0).$$

If one equates u_1 to a constant, and one takes into account the fact that $du_1 = 0$ then the expression ω no longer has class $2m - 2$ or $2m - 3$, since its $(2m - 2)^{\text{th}}$ derivative is annulled, and its class cannot be reduced by more than two units.

37. We now consider the equation:

$$\omega^{(2m-4)}\, du_1\, df = 0, \tag{9}$$

which, from the preceding, is equivalent to the adjoint complete system to the equation $\omega = 0$, where one makes $u_1 = \text{const}$.

This complete system admits $2m - 3$ independent integrals, and if one does not regard u_1 as a constant then it admits $2m - 2$ independent integrals. It is composed of $n - 2m + 2$ independent equations. In order to find them, here, it suffices to take the coefficients of $dx_1\, dx_2 \,\ldots\, dx_{2m-2}\, dx_{2m-1}$, $dx_1\, dx_2 \,\ldots\, dx_{2m-2}\, dx_{2m}$, …, $dx_1\, dx_2 \,\ldots\, dx_{2m-2}\, dx_n$. These coefficients contain the derivatives $\frac{\partial f}{\partial x_{2m-1}}$, $\frac{\partial f}{\partial x_{2m}}$, …, $\frac{\partial f}{\partial x_n}$, respectively, multiplied by a coefficient that is non-zero, by hypothesis, as well as some terms in $\frac{\partial f}{\partial x_1}$, $\frac{\partial f}{\partial x_2}$, …, $\frac{\partial f}{\partial x_{2m-2}}$.

Upon equating these coefficients to zero, one has $n - 2m + 2$ independent equations that one can consider as being solved for $\frac{\partial f}{\partial x_{2m-1}}$, $\frac{\partial f}{\partial x_{2m}}$, …, $\frac{\partial f}{\partial x_n}$, while the coefficients of the other derivatives are holomorphic in a neighborhood of x_1^0, x_2^0, …, x_n^0. These $n - 2m + 2$ are the equations of the desired complete system. One sees, moreover, that this system admits $2m - 2$ holomorphic independent integrals that reduce to $x_1 - x_1^0$, $x_2 - x_2^0$, …, $x_{2m-2} - x_{2m-2}^0$ for:

$$x_1 = x_1^0,\ x_2 = x_2^0,\ \ldots,\ x_{2m-2} = x_{2m-2}^0,$$

respectively. The first one is nothing but u_1 . We denote the second one by u_2 . Up to terms of higher degree, u_2 has the form:

$$u_2 = x_2 - x_2^0 + \beta_{2m-1}(x_{2m-1} - x_{2m-1}^0) + \beta_{2m}(x_{2m} - x_{2m}^0) + \ldots + \beta_n(x_n - x_n^0).$$

We then continue by forming:

$$\omega^{(2m-6)}\, du_1\, du_2\, df = 0, \tag{10}$$

an equation that is equivalent to a complete system that admits a holomorphic integral u_3 that reduces to $x_3 - x_3^0$ for:

$$x_{2m-2} = x_{2m-2}^0,\quad x_{2m-1} = x_{2m-1}^0,\ \ldots,\qquad x_n = x_n^0,$$

and so on, up to the complete system:

$$\omega\, du_1\, du_2 \ldots du_{m-1}\, df = 0,$$

which will admit the holomorphic integral u_m that reduces to $x_m - x_m^0$ for:

$$x_{m+1} = x_{m+1}^0, \qquad x_{m+2} = x_{m+2}^0, \quad \ldots, \qquad x_n = x_n^0 .$$

38. We thus finally arrive at m holomorphic functions, in all:

$$u_1, u_2, \ldots, u_m,$$

reduce to:

$$x_1 - x_1^0, x_2 - x_2^0, \ldots, x_m - x_m^0,$$

respectively, when one sets:

$$x_{m+1} - x_{m+1}^0 = x_{m+2} - x_{m+2}^0 = \ldots = x_n - x_n^0 = 0.$$

Moreover, ω becomes zero when one gives constant values to these m functions, in such a way that one has an equality of the form:

$$\omega = C_1\, du_1 + C_2\, du_2 + \ldots + C_m\, du_m . \tag{11}$$

The C coefficients are holomorphic functions in a neighborhood of $x_1^0, x_2^0, \ldots, x_n^0$. Indeed, if one makes a change of variables by taking:

$$\begin{array}{llll} y_1 = u_1, & y_2 = u_2, & \ldots, & y_m = u_m, \\ y_{m+1} = x_{m+1} - x_{m+1}^0, & & \ldots, & y_n = x_n - x_n^0 \end{array}$$

then any holomorphic function of the old variable in the neighborhood of $x_1^0, x_2^0, \ldots, x_n^0$ is a holomorphic function of the new ones in the neighborhood of $y_1 = y_2 = \ldots = y_n = 0$, and conversely. In particular, ω remains holomorphic in the neighborhood of zero y's, and since it must contain only $dy_1, dy_2, \ldots, dy_m$, it then results that $C_1, C_2, \ldots, C_n$ are holomorphic.

One sees, moreover, that C_m^0 is non-zero, because the developed expression (11) gives a quantity for the coefficient of dx_m whose values for $x_1^0, x_2^0, \ldots, x_n^0$, which is, by hypothesis, non-zero, is nothing but C_m^0. It then results that in a neighborhood of $x_1^0, x_2^0, \ldots, x_n^0$ the total differential equation is equivalent to the equation:

$$\begin{aligned} du_m + \frac{C_1}{C_m} du_1 + \frac{C_2}{C_m} du_2 + \ldots + \frac{C_{m-1}}{C_m} du_{m-1} \\ = du_m - v_1\, du_1 - \ldots - v_{m-1}\, du_{m-1} = 0, \end{aligned} \tag{12}$$

where the v are again holomorphic. Finally, if one reduces $u_1, u_2, \ldots, u_m, v_1, v_1, \ldots, v_{m-1}$ to their terms of first degree then one obtains $2m - 1$ expressions of first degree in $x_1, x_2, \ldots, x_n$ *that must be independent.*

Indeed, it is only these terms of first degree that are involved with expressing the value of the coefficients of $\omega^{(2m-2)}$ when one sets $x_1 = x_1^0, x_2 = x_2^0, \ldots, x_n = x_n^0$. If these $2m - 1$ quantities are not independent then they furnish a Pfaff expression of class at most $2m - 2$, and consequently $\omega^{(2m-2)}$ will be zero for $x_1 = x_1^0, x_2 = x_2^0, \ldots, x_n = x_n^0$, which is contrary to hypothesis.

39. It ultimately results that if $v_1, v_1, \ldots, v_{m-1}$ take the values $v_1^0, v_2^0, \ldots, v_{m-1}^0$ for $x_1 = x_1^0, x_2 = x_2^0, \ldots, x_n = x_n^0$ then one can make a change of variables by taking the new variables to be $u_1, u_2, \ldots, u_m, v_1 - v_1^0, v_2 - v_2^0, \ldots, v_{m-1} - v_{m-1}^0$, and $n - 2m + 1$ quantities $x_i - x_i^0$. Any holomorphic function of the old variables in the neighborhood of x_i^0 will be holomorphic in the new ones in a neighborhood of their zero values. The solution considered will then transform into a solution that contains the system of zero values of the variables and annuls the expression:

$$du_m - v_1\, du_1 - v_2\, du_2 - \ldots - v_{m-1}\, du_{m-1}\,.$$

It can be provided only by the general process of solution of that total differential equation.

One will have an infinitude of systems of $m, m + 1, \ldots, 2m - 1$ dependent functions of arbitrary variables, to which one adds, if necessary, arbitrary equations in an arbitrary number if n is greater than $2m - 1$.

The problem that we just solved is, in short, the following one:

Find all solutions of the equation $\omega = 0$ *that admit a point (or system of values)* $(x_1^0, x_2^0, \ldots, x_n^0)$ *that does not annul all of the coefficients of the* $(2m - 2)^{\text{th}}$ *derivative of* ω *for a simple point.*

One sees that all of these solutions are given by formulas that all fall into a finite number of types that depend upon arbitrary functions.

40. *Search for singular solutions.* – From the foregoing, *we call a solution whose points all annul the coefficients of* $\omega^{(2m-2)}$, *which is assumed to be reduced to its simplest form, a* SINGULAR SOLUTION.

If one equates all of these coefficients to zero then one has a system of equations that can be algebraically incompatible, and then there is no singular solution; they can also decompose into several other incompatible ones. Each of them can be put into a form such that the left-hand sides of the h equations that comprise them are holomorphic with respect to an arbitrary system of values of variables that satisfy the system, and such that the functional determinants of these h left-hand sides with respect to h arbitrary ones of these variables are not zero for the same system of values, moreover.

This being the case, consider a well-defined singular solution and an arbitrary simple point $(x_1^0, x_2^0, \ldots, x_n^0)$ of that solution. If the two conditions that were enumerated above are verified for this point then one can deduce h of these variables as holomorphic functions of the $n - h$ other ones and substitute them into ω. One then has a new Pfaff expression that is holomorphic in a neighborhood of $x_1^0, x_2^0, \ldots, x_n^0$, *and which will be of class at most* $2m - 2$. One will come down to looking for solutions to the equation thus obtained by equating this expression to zero.

If the second condition, which relates to the functional determinants, is not realized for all the points $(x_1^0, x_2^0, \ldots, x_n^0)$ of the solution considered then one has a higher-order solution of the singularity. One will get all the solutions by adding to the h solutions that were found above, the ones that one obtains by equating all of the functional determinants of their left-hand sides with respect to h of the variables to zero. One thus obtains a new system of $k > h$ equations that one can put into a form that satisfies the two conditions stated above. One proceeds for the second system as one did for the first one, and so on.

These operations necessarily terminate, because one is necessarily dealing with a *finite* number of total differential equations of lower order than the given equation. Each of them can lead to other total differential equations whose order will be less than their own. It is indeed clear that this will conclude after a finite sequence of these operations.

41. *Examples.* – Take the Pfaff expression:

$$\omega = x_5\, dx_1 + x_3\, dx_2 + x_1\, dx_4 + x_1\, dx_5 . \tag{13}$$

Here, one has:

$$\begin{cases} \omega' = dx_1 dx_4 + dx_3 dx_2, \\ \omega'' = -x_5\, dx_1 dx_2 dx_3 - x_3\, dx_1 dx_2 dx_4 - x_1\, dx_2 dx_3 dx_4 \\ \qquad\qquad + x_1\, dx_1 dx_4 dx_5 - x_1\, dx_2 dx_3 dx_5, \\ \omega^{\mathrm{IV}} = -x_1\, dx_1 dx_2 dx_3 dx_4 dx_5, \\ \omega^{\mathrm{VI}} = 0. \end{cases} \tag{14}$$

Thus, $m = 3$, here. The singular solutions are the ones for which one has:

$$x_4 = 0,$$

since the only coefficient of ω^{IV} is x_1. If one replaces the variable x_1 with zero in ω then one gets the equation:

$$\varpi = x_3\, dx_2 = 0.$$

The general solution to that equation is:

$$x_2 = a,$$

and the singular solution is:

$$x_3 = 0.$$

As a result, the singular solutions of the equation $\omega = 0$ are:

(15) $$x_1 = 0, \qquad x_2 = a$$
and
(16) $$x_1 = 0, \qquad x_2 = 0,$$

and the ones that one gets by adding arbitrary equations to each of these solutions.

Here, the general solution is given by at least three equations, whereas one has singular solutions that are comprised of just two equations.

42. Once again, consider the Pfaff expression that has already served as an example:

$$\omega = x_1\, x_3\, dx_2 + x_1\, x_2\, dx_3 + (x_1 + x_3\, x_5)\, dx_4 + x_3\, x_4\, dx_5\,.$$

Here, one finds:

(17) $$\begin{cases} \omega'' = x_3^2 x_5 dx_1 dx_2 dx_4 + x_3^2 dx_4 x_1 dx_2 dx_5 + x_2 x_3 x_5 dx_1 dx_3 dx_4 \\ \qquad + x_2^2 x_4 dx_1 dx_3 dx_5 + x_3 x_4 dx_1 dx_4 dx_5 + x_1 x_3 x_5 dx_2 dx_3 dx_4 \\ \qquad + x_1 x_3 x_4 dx_2 dx_3 dx_5 - x_1 x_4 dx_3 dx_4 dx_5 , \\ \omega^{\text{IV}} = 0. \end{cases}$$

One thus has $m = 2$. The singular solutions are obtained annulling the coefficients of ω''. One thus finds:
(18) $$x_1\, x_4 = x_3\, x_4 = x_3\, x_5 = 0.$$

This system decomposes into three others:

$$(18)_a \quad \begin{cases} x_1 = 0, \\ x_3 = 0, \end{cases} \qquad (18)_b \quad \begin{cases} x_3 = 0, \\ x_4 = 0, \end{cases} \qquad (18)_c \quad \begin{cases} x_4 = 0, \\ x_5 = 0. \end{cases}$$

The first system, as well as the second one, annuls ω identically; they thus constitute two singular solutions. The third one gives:

(19) $$\varpi = x_1\, x_3\, dx_2 + x_1\, x_2\, dx_3 = 0,$$

and $m = 1$ for ϖ. The general solution of equation (19) is immediate; it is:

(20) $$x_2\, x_3 = a.$$

As for the singular solutions, they are obtained by annulling the $x_1\, x_3$ and $x_1\, x_2$ coefficients of ϖ. One thus has two cases: Either:

$$x_1 = 0$$

or

$$x_2 = x_3 = 0$$

The system $(18)_c$ thus gives the singular solutions of the original equation:

(21)	$x_4 = 0,$	$x_5 = 0,$	$x_2 x_3 = a,$	
(22)	$x_4 = 0,$	$x_5 = 0,$	$x_1 = 0,$	
(23)	$x_4 = 0,$	$x_5 = 0,$	$x_2 = 0,$	$x_3 = 0.$

The last one enters into the singular solution $(18)_b$, moreover.

43. Finally, consider the equation:

(24) $$\omega = A_1\, dx_1 + A_2\, dx_2 + A_3\, dx_3 = 0,$$

where the A are functions of x_1, x_2, x_3 . In the general case, ω will be of third class, and in turn, $m = 2$. The singular solutions will be furnished by annulling the coefficients of $\omega^{(2m-2)} = \omega'$. Now:

$$\omega'' = \left[A_1\left(\frac{\partial A_2}{\partial x_3} - \frac{\partial A_3}{\partial x_2}\right) + A_3\left(\frac{\partial A_3}{\partial x_1} - \frac{\partial A_1}{\partial x_3}\right) + A_2\left(\frac{\partial A_1}{\partial x_2} - \frac{\partial A_2}{\partial x_1}\right)\right] dx_1\, dx_2\, dx_3 ,$$

here.

If the quantity inside the brackets is identically zero then one can satisfy equation (24) by just one equation that depends upon one arbitrary parameter. Otherwise, in certain cases, upon annulling that quantity one might get a singular solution that might be formed from the only relation thus obtained, but which will generally need to be completed with another relation. That is why upon taking:

(25) $$\omega = x_1(1 - x_1^2 - x_2^2)\, dx_1 + x_2 x_3^2 dx_2 + x_3^3 dx_2 ,$$

the equation that is obtained by annulling the coefficient of ω'' is:

$$2x_1 x_2 x_3 (x_1^2 + x_2^2 + x_3^2 - 1) = 0.$$

This equation decomposes into four other ones, and when each of them is treated separately, one is finally led to the following singular solutions:

$(25)_a$	$x_1 = 0,$	$x_2^2 + x_3^2 = a,$
$(25)_b$	$x_2 = 0,$	$2x_1^2 - x_1^4 + x_3^4 = a,$
$(25)_c$	$x_1 = 0,$	$x_3 = a,$
$(25)_d$	$x_1^2 + x_2^2 + x_3^2 = 1.$	

Of course, from each of these solutions one deduces an infinitude of other ones by adding arbitrary equations to the equations that determine them.

IV. – Systems formed from several finite equations and one total differential equation.

44. Given a total differential equation:

(26) $$\omega = A_1\,dx_1 + A_2\,dx_2 + \ldots + A_n\,dx_n = 0$$

and a system of h finite equations in the variables:

(27) $$\begin{cases} f_1(x_1, x_2, \ldots, x_n) = 0, \\ f_2(x_1, x_2, \ldots, x_n) = 0, \\ \qquad\qquad\qquad\qquad ....., \\ f_h(x_1, x_2, \ldots, x_n) = 0, \end{cases}$$

one addresses the problem of satisfying equation (26) by means of a system of equations that consist of equations (27).

We suppose, as always, that the left-hand sides of equations (27) satisfy the two fundamental conditions that were stated above that relate to all the systems of equations that we are concerned with.

Before solving the problem, we shall prove a theorem that is important in itself, and which has already helped us implicitly.

45. *Class of a Pfaff expression, when one supposes that the variables are coupled by given relations.* – Consider the Pfaff expression ω. Suppose that one deduces h of the variables as functions of the $n - h$ other ones from equations (27), and one substitutes them into ω. That expression contains more than $n - h$ variables. I say that *the class of that new expression is the smallest whole number p such that the coefficients of the expression:*

(28) $$\omega^{(p)}\,df_1\,df_2 \ldots df_h\,,$$

which is assumed to be reduced to its simplest form, are all zero by virtue of relations (27).

Indeed, suppose that the functional determinant of $f_1, f_2, \ldots, f_h$, with respect to $x_1, x_2, \ldots, x_h$ is not identically zero. One can then take the new variable to be:

$$y_1 = f_1, \qquad y_2 = f_2, \ \ldots, \qquad y_h = f_h, \qquad y_{h+1} = f_{h+1}, \qquad \ldots, \qquad y_n = f_n,$$

and any holomorphic function of the old variables will be a holomorphic function of the new ones, and conversely. If one lets ϖ denote what ω becomes under this change of variables then expression (28) transforms into:

(29) $$\varpi^{(p)}\,dy_1\,dy_2 \ldots dy_h\,.$$

It is indeed clear, moreover, that each coefficient of (28) is a linear combination with holomorphic coefficients of the coefficients of (29), and conversely. (The coefficients of these expressions are taken once the reduction to the simplest form has been performed.) Now, the coefficients of the expression (29) are the same as those of $\varpi^{(p)}$, where one has removed the terms that contain the differentials $dy_1, dy_2, \ldots, dy_h$. Therefore, if one takes the relations (27) into account in the coefficients of (28) then this amounts to setting, on the one hand:

$$dy_1 = dy_2 = \ldots = dy_h = 0,$$

and on the other:

$$y_1 = y_2 = \ldots = y_h = 0$$

in $\varpi^{(p)}$.

Let ϖ_0 be what ϖ becomes when one makes these substitutions. It is easy to see that ϖ' changes into ϖ'_0 under these substitutions. Indeed, if:

$$\varpi = B_1\, dy_1 + \ldots + B_h\, dy_h + B_{h+1}\, dy_{h+1} + \ldots + B_n\, dy_n$$

then one has:

$$\varpi_0 = B_{h+1}^0 dy_{h+1} + \cdots + B_n^0 dy_n,$$

from which, one infers that:

$$\varpi'_0 = \sum_{i,j} \frac{\partial B_{h+i}^0}{\partial y_{h+j}} dy_{h+i} dy_{h+j} = \sum_{i,j} \left(\frac{\partial B_{h+i}}{\partial y_{h+j}}\right)_0 dy_{h+i} dy_{h+j},$$

where the index 0 expresses the idea that one sets $y_1 = y_2 = \ldots = y_h = 0$. One indeed sees that ϖ'_0 is deduced from ϖ' by setting:

$$y_1 = y_2 = \ldots = y_h = dy_1 = dy_2 = \ldots = dy_h = 0.$$

Under the latter substitution, ϖ changes into ϖ_0 and ϖ' into ϖ'_0, so it is clear that ϖ'^q changes into ϖ'^q_0 and that $\varpi\varpi'^q$ changes into $\varpi_0\varpi'^q_0$, and in other words, that $\varpi^{(p)}$ changes into $\varpi_0^{(p)}$.

One sees from this that the necessary and sufficient condition for the coefficients of (28) to be zero by virtue of (27) is that $\varpi_0^{(p)}$ must be identically zero, or, since ϖ_0 is what ω becomes when one derives $x_1, x_2, \ldots, x_h$ from (27), that the class of ω is at most p, after the variables in it are linked by the relations (27). This conclusion proves the theorem.

46. *General solutions to the proposed problem.* – After that, we return to our problem and suppose that m is the smallest whole number such that the coefficients of:

(30) $$\omega^{(2m)}\, df_1\, df_2 \ldots df_h,$$

are all zero by virtue of (27). The general solutions will be the ones, by virtue of which, the coefficients of:

(31) $$\omega^{(2m-2)}\, df_1\, df_2 \ldots df_h,$$

will not all be zero. In particular, the functional determinants of $f_1, f_2, \ldots, f_h$ with respect to any h of the variables will not all be zero for these solutions, since otherwise the expression $df_1\, df_2 \ldots df_h$ would have all of its coefficients zero (those coefficients are these functional determinants themselves) and the same would be true for the expression (31).

From this, if $(x_1^0, x_2^0, \ldots, x_n^0)$ denotes an arbitrary system of values that correspond to a well-defined general solution then one can derive h of the variables as holomorphic functions of the other ones from (27) in a neighborhood of $(x_1^0, x_2^0, \ldots, x_n^0)$, and if one substitutes them into ω then one will come down to solving a total differential equation whose left-hand side will be of order $2m$ or $2m - 1$, while the derivative $\omega^{(2m-2)}$ does not have all of its coefficients zero for the system of values (x_i^0) considered. One will arrive at this stage by considering m successive complete systems and determining a holomorphic integral for each of them.

47. Here, one can give the following form to these complete systems. The first one will be equivalent to the equation:

$$\omega^{(2m-2)}\, df_1\, df_2 \ldots df_h\, df = 0, \tag{32}$$

where the variables are assumed to be coupled by the relations (27). If f_{h+1} is a holomorphic integral that does not reduce to a constant by virtue of (27) then the complete second system will be:

$$\omega^{(2m-2)}\, df_1\, df_2 \ldots df_h\, df_{h+1}\, df = 0, \tag{33}$$

and so on, up to:

$$\omega\, df_1\, df_2 \ldots df_{h+m-1}\, df = 0,$$

which will give a holomorphic integral f_{h+m}. We will thus have m independent holomorphic functions $f_{h+1}, f_{h+2}, \ldots, f_{h+m}$, while likewise taking (27) into account. One can, moreover, arrange that the equation to be solved is put into the form:

$$df_{h+m} = \varphi_{h+1}\, df_{h+1} - \ldots - \varphi_{h+m-1}\, df_{h+m-1} = 0, \tag{34}$$

where the φ are also holomorphic in a neighborhood of $x_1^0, \ldots, x_n^0$. One will thus be in a position to find all of the solutions that admit the point $(x_1^0, \ldots, x_m^0)$ as a simple point.

48. *Singular solutions.* – In order to have singular solutions, one must append the relations that one obtains by annulling all of the coefficients of the differential expression:

$$\omega^{(2m-2)}\, df_1\, df_2 \ldots df_h \tag{31}$$

to equations (27). One will thus have a new system of relations, and one will be, in short, reduced to a problem that is analogous to the first one, except that the integer h will become larger. For this new problem, we will have a new value m' for m that is equal to at most m, and it will admit general solutions and singular solutions that will be given by the solutions of a third problem, where h will again be augmented. It is clear that h cannot exceed the number $n - 1$, so these operations will terminate.

49. *Solution of a total differential equation by means of a given number of equations*. – The new problem that we propose is the following one: Given the total differential equation (26), solve that equation by means of r finite relations between the variables, among which $h < r$ are given relations (27).

To that effect, we shall first prove that *if a system of r relations annuls the Pfaff expression ω then all of the coefficients of $\omega^{(2r)}$ are zero by virtue of these relations.*

50. Indeed, let:

$$(35)\qquad \begin{cases} f_1(x_1, x_2, \ldots, x_n) = 0, \\ f_2(x_1, x_2, \ldots, x_n) = 0, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \\ f_r(x_1, x_2, \ldots, x_n) = 0 \end{cases}$$

be the system of r relations that annul ω. One supposes that the left-hand sides always verify the same two fundamental hypotheses. If the functional determinant of $f_1, f_2, \ldots, f_r$ with respect to $x_1, x_2, \ldots, x_r$ is not zero by virtue of (35) then we can take the new variables to be:

$$y_1 = f_1, \qquad y_2 = f_2, \ \ldots, \qquad y_r = f_r, \qquad y_{r+1} = f_{r+1}, \qquad \ldots, \qquad y_n = f_n,$$

and from a remark that was made above, if ω transforms into ϖ then the coefficients of $\omega^{(p)}$ will be annulled by virtue of (35) at the same time as those of $\varpi^{(p)}$, and conversely.

Now, if we form the expression ϖ:

$$\varpi = B_1\, dy_1 + B_2\, dy_2 + \ldots + B_r\, dy_r + B_{r+1}\, dy_{r+1} + \ldots + B_n\, dy_n$$

then, by hypothesis, it is necessary that $B_{r+1}, B_{r+2}, \ldots, B_n$ be annulled at the same time as $y_1, y_2, \ldots, y_r$. Then consider:

$$\varpi^{(2r)} = \frac{1}{r!}\varpi \cdot \varpi'^r,$$

and set $y_1 = y_2 = \ldots = y_r = 0$ in the coefficients. The terms in ϖ whose coefficients are not annulled cannot be the terms in $dy_1, dy_2, \ldots, dy_r$. Likewise, if a term in ϖ' has a coefficient that is non-zero by virtue of (35) then it must contain at least one of the differentials $dy_1, dy_2, \ldots, dy_r$, since otherwise there would be a term of the form:

$$\frac{\partial B_{r+i}}{\partial y_{r+j}} dy_{r+i}\, dy_{r+i},$$

and the value of the coefficient of that term for $y_1 = y_2 = \ldots = y_r = 0$ can obviously be obtained by first setting $y_1 = y_2 = \ldots = y_r = 0$ in B_{r+i} and then differentiating with respect to y_{r+i}, which will necessarily give zero.

Therefore, if one preserves only the terms with non-zero coefficients in the $r + 1$ factors ϖ and ϖ' of $\varpi^{(2r)}$ then each term of each of these factors contains at least one of the r differentials $dy_1, dy_2, \ldots, dy_r$. Since there are more factors than differentials, the coefficients of the total symbolic product will certainly all be zero.

The coefficients of $\omega^{(2r)}$ are annulled by virtue of the expressions (35) at the same time as the coefficients of $\varpi^{(2r)}$, so the theorem is proved.

One proves that all of the coefficients of $\omega^{(2r+1)}$ are annulled in the same fashion.

51. We shall further prove a theorem that is a little more general. *If a Pfaff expression ω is annulled by means of r relations, where h of these relations are given by* (27), *then all of the coefficients of the expression:*

$$\omega^{(2r-2h)}\, df_1\, df_2 \ldots df_h \tag{36}$$

are annulled by virtue of these r relations.

Indeed, the theorem is true if one first has that all of the functional determinants of the left-hand sides $f_1, f_2, \ldots, f_h$ of the h given relations with respect to any h of the variables are annulled by virtue of the relations considered, because the expression $df_1\, df_2 \ldots df_h$ then has all of its coefficients zero by virtue of the relations in question, and consequently expression (36) does as well.

If these functional determinants are not all zero by virtue of the r relations then we can derive h of the variables as holomorphic functions of the $n - h$ other ones from (27) and substitute them in ω; we will then have an expression ϖ. Moreover, the coefficients of $\omega^{(p)}\, df_1\, df_2 \ldots df_h$ are annulled at the same time as those of $\varpi^{(p)}$, and conversely. Now, the expression ϖ can be annulled by means of $r - h$ relations between the variables. As a result, from the preceding theorem, all of the coefficients of the derivative $\varpi^{(2r-2h)}$ are annulled by virtue of these $r - h$ relations. Consequently, all of the coefficients of (36) are annulled by virtue of the r relations in question. The same is true for all of the coefficients of:

$$\omega^{(2r-2h+1)}\, df_1\, df_2 \ldots df_h .$$

52. This being the case, one arrives at the solution of the proposed problem: *Solve the system of equations* (26) *and* (27) *by means of r – h relations that are distinct from* (27).

One forms the differential expression:

$$\omega^{(2r-2h)}\, df_1\, df_2 \ldots df_h , \tag{36}$$

and one equates all of its coefficients to zero. In general, one will obtain equations that are distinct from the given equations, in such a way that the system (27) will be replaced with a new system of $h' > h$ equations. If h' is greater than r then the problem is impossible. If not, then one forms the differential expression for this new system that is analogous to (36), and so on. One concludes by arriving at either a system of more than r relations, in which case, one has the impossibility of a solution, or a system of $k < r$

relations for which the expression $\omega^{(2r-2k)}\, df_1\, df_2\, \dots\, df_k$ will have all of its coefficients equal to zero, by virtue of these k relations. Then, if m is the smallest whole number such that $\omega^{(2r-2h)}\, df_1\, df_2\, \dots\, df_h$ has all of its coefficients equal to zero by virtue of the k relations that were obtained then one will have the general solution of the problem by solving a certain Pfaff equation:

$$dZ - P_1\, dX_1 - \dots - P_{m-1}\, dX_{m-1} = 0,$$

where the X, P, Z will be given by the successive complete systems. One will then have an infinitude of systems of m relations, to each of which, one adds $r - m - k$ *arbitrary* equations.

53. The singular solutions are obtained by adding the relations that one obtains by annulling all of the coefficients of $\omega^{(2r-2)}\, df_1\, df_2\, \dots\, df_k$ to the k relations in question. One will then have a new system of relations, and one will come back to the original problem, but h will be augmented. One sees how one continues, and one indeed accounts for the fact that all of these operations will have a conclusion.

The solution that was just presented includes the one where there is no relation between the variables given *a priori* (i.e., $h = 0$) as a special case.

54. *Example.* – Take the example that was treated before (13):

$$\omega = x_5\, dx_1 + x_1\, dx_2 + x_1\, dx_4 + x_1\, dx_5 .$$

One seeks to annul ω by a system of $r = 3$ relations whose $h = 1$ relation is given:

$$x_4 = 0.$$

Here ($r - h = 2$), so one must form the expression $\omega^{\text{IV}}\, dx_4$. Now, upon referring to the value (14) for ω^{IV}, one finds that:

$$\omega^{\text{IV}}\, dx_4 = 0.$$

Here, there are general solutions then. Since one has:

$$\omega''\, dx_4 = - x_5\, dx_1\, dx_2\, dx_3\, dx_4 + x_1\, dx_2\, dx_3\, dx_4\, dx_5 ,$$

the number m is equal to 2 here, and the general solutions are the ones that do not annul both x_1 and x_5 simultaneously. Upon setting $x_4 = 0$ in (37), one finds that:

$$\varpi = x_5\, dx_1 + x_3\, dx_2 + x_1\, dx_5 = x_3\, dx_2 + d(x_1\, x_5) = 0.$$

The general solution of the problem will then be provided by:

$$x_4 = 0, \qquad x_1\, x_5 = \varphi(x_2), \quad x_3 = - \varphi'(x_2).$$

The singular solutions must consist of the $h = 3$ relations:

$$x_1 = x_4 = x_5 = 0.$$

One must therefore annul the coefficients of $\omega\, dx_1\, dx_4\, dx_5$, since $r - h = 3 - 3 = 0$. Now, one finds that:

$$\omega\, dx_1\, dx_4\, dx_5 = - x_3\, dx_1\, dx_2\, dx_4\, dx_5 .$$

One must then append the equation:

$$x_3 = 0$$

to equations (40), which gives more than three relations. There is therefore no singular solution.

55. *Another solution to the same problem.* – The equations that one add to the given equations (27) in the general case, namely, the ones that one obtains by annulling the coefficients of the differential expression:

$$\omega^{(2r-2h)}\, df_1\, df_2 \ldots df_h , \tag{36}$$

are all very complicated if h is large, since they depend on the partial derivatives of h functions $f_1, \ldots, f_h$. *In a very extended case*, one can substitute other equations that are much simpler to define for them.

We first remark that any solution of the problem will be a solution of the system:

$$\begin{cases} \omega = 0, \\ f_1 = 0, \end{cases} \tag{37}$$

and consequently, since h is equal to 1 here, one must annul all of the coefficients of the expression $\omega^{(2r-2)}\, df_1$. We thus see already that one will have to add the equations that one obtains by annulling the coefficients of h differential expressions:

$$\omega^{(2r-2)}\, df_1 \qquad (i = 1, 2, \ldots, h) \tag{38}$$

to equations (27).

Likewise, is h is greater than 1, one will, by an analogous argument, have to annul all of the coefficients of the differential expressions:

$$\begin{cases} \omega^{(2r-3)} df_i df_j \\ \omega^{(2r-4)} df_i df_j \end{cases} \qquad (i, j = 1, 2, \ldots, h). \tag{39}$$

The expressions (38) and (39) contain only the derivatives of at most two functions f. Here is a theorem that permits one to restrict oneself to the consideration of analogous expressions in three very general cases.

56. Theorem. – *Suppose one is given a system of h relations:*

$$\text{(27)} \qquad \begin{cases} f_1(x_1, x_2, \ldots, x_n) = 0, \\ f_2(x_1, x_2, \ldots, x_n) = 0, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots, \\ f_h(x_1, x_2, \ldots, x_n) = 0, \end{cases}$$

whose left-hand sides always satisfy the same conditions relative to their functional determinants.

If one considers only systems of values for the variables that satisfy (27) *and do not annul all of the coefficients in the expression:*

$$\omega \, df_1 \, df_2 \ldots df_h$$

or, at the same time, those of $\omega^{(r-1)}$ *and* $\omega^{(r-2)}$*, then they do not annul those of the expression:*

$$\omega^{(2r-2h-2)} \, df_1 \, df_2 \ldots df_h$$

either.

Furthermore, under the same conditions:

1. *If these systems of values do not annul* $\omega^{(2r-1)}$ *then the equation:*

$$\text{(36)} \qquad \omega^{(2r-2h)} \, df_1 \, df_2 \ldots df_h = 0$$

is algebraically equivalent to the equations:

$$\text{(40)} \qquad \begin{cases} \omega^{(2r-2)} df_i = 0, \\ \omega^{(2r-3)} df_i df_j = 0 \end{cases} \qquad (i, j = 1, 2, \ldots, h).$$

2. *If these systems of values do not annul* $\omega^{(2r-2)}$ *then the equation:*

$$\text{(36)} \qquad \omega^{(2r-2h)} \, df_1 \, df_2 \ldots df_h = 0$$

is algebraically equivalent to the equations:

$$\text{(41)} \qquad \omega^{(2r-4)} \, df_i \, df_j = 0 \qquad (i, j = 1, 2, \ldots, h).$$

3. *If these systems of values do not annul* $\omega^{(2r-2)}$ *then the equation:*

$$\text{(42)} \qquad \omega^{(2r-2h-1)} \, df_1 \, df_2 \ldots df_h = 0.$$

is algebraically equivalent to the equations:

(43) $$\begin{cases} \omega^{(2r-3)} df_i = 0, \\ \omega^{(2r-4)} df_i df_j = 0 \end{cases} \qquad (i, j = 1, 2, \ldots, h).$$

The integer h is assumed to be equal to at least one in the first and third cases, and equal to at most 2 in the second case. Finally, *if the coefficients of* (36) *and* (37) *are zero then the same thing is true for those of* $\omega^{(2r)}$ *in the three cases, and in addition, of* $\omega^{(2r-1)}$ *in the last case.*

We shall first prove the following lemma:

57. Lemma. – *Suppose one is given a differential expression* ϖ *of second degree and* $h + 1$ *differential expressions* $\omega, \omega_1, \ldots, \omega_h$ *of first degree in n variables* $x_1, x_2, \ldots, x_n$.

1. *If the coefficients of* ϖ^r *are not all zero for a certain system of variables that do not annul* $\omega\, \omega_1 \ldots \omega_h$ *then the coefficients of:*

$$\varpi^{r-h}\, \omega\, \omega_1 \ldots \omega_h$$

are annulled for this system of values only at the same time as those of the expressions:

$$\varpi^{r-1}\, \omega\, \omega_i , \qquad \varpi^{r-1}\, \omega_i\, \omega_j \qquad (i, j = 1, 2, \ldots, h),$$

and conversely.

2. *Under the same conditions, if the coefficients of* $\omega\varpi^{r-1}$ *are not all zero then the coefficients of:*

$$\varpi^{r-h}\, \omega\, \omega_1 \ldots \omega_h$$

are annulled only at the same time as those of the expressions:

$$\varpi^{r-2}\, \omega\, \omega_i\, \omega_j \qquad (i, j = 1, 2, \ldots, h).$$

3. *If the coefficients of* $\omega\varpi^{r-1}$ *are not all zero then the coefficients of:*

$$\varpi^{r-h}\, \omega\, \omega_1\, \omega_2 \ldots \omega_h$$

are annulled only at the same time as those of the expressions:

$$\varpi^{r-1}\, \omega_i , \qquad \varpi^{r-2}\, \omega\, \omega_i\, \omega_j \qquad (i, j = 1, 2, \ldots, h).$$

In any case, the coefficients of the expression:

$$\varpi^{r-h-1}\, \omega\, \omega_1\, \omega_2 \ldots \omega_h$$

can never be annulled simultaneously.

Without changing any of the conditions of the statement, one can suppose that the coefficients of the expressions ϖ, ω, ω_1, …, ω_h keep the constant values that they possess for the system of values for the variables that was considered. This amounts to supposing that ϖ, ω, ω_1, …, ω_h are the differentials of linear forms in x_1, x_2, …, x_h .

This being the case, the hypothesis that was made on the product ω, ω_1, …, ω_h expresses the notion that these $h + 1$ linear forms are independent. Moreover, one can perform an arbitrary linear substitution with a non-zero determinant on the last h of them without changing any of the conditions of the statement. Finally, one can likewise make an arbitrary linear substitution with non-zero determinant on the n variables in such a manner as to have, for example:

$$\omega_1 = dx_1, \qquad \omega_2 = dx_2, \qquad \ldots, \qquad \omega_h = dx_h, \qquad \omega = dx_n .$$

58. This being the case, the set of terms in ϖ that contain dx_n is of the form:

$$dx_n\, du,$$

where u is a certain linear form in x_1, x_2, …, x_{n-1} that can be identically zero. Now, consider the terms that do not contain dx_n . If we let:

$i, j, \ldots$ denote the indices $1, 2, \ldots, h$, and
λ, μ “ $h + 1, h + 2, \ldots, n - 1$

then we see that ϖ is composed of three groups of terms, in addition to $dx_n\, du$:

1. Terms of the form $A_{i,j}\, dx_i\, dx_j$,
2. “ $A_{i,\lambda}\, dx_i\, dx_\lambda$,
3. “ $A_{\lambda,\mu}\, dx_\lambda\, dx_\mu$.

Suppose that the coefficient of:

$$dx_{h+1}\, dx_{h+2}$$

in the third group is non-zero. We the make a change of variables by taking:

$$x'_{h+1} = \sum_{\rho=1}^{n-1} A_{\rho,h+2} x_\rho ,$$

$$A_{h+1,\ h+2}\, x'_{h+1} = \sum_{\rho=1}^{n-1} A_{h+1,\rho} x_\rho .$$

We then see that the product $dx'_{h+1} dx'_{h+2}$ contains all of the terms in dx_{h+1} and dx_{h+2} that are found in $\varpi - dx_n\, du$. Therefore, upon taking x'_{h+1} and x'_{h+2} instead of x_{h+1} and x_{h+2} , $\varpi - dx_n\, du$ no longer contains terms in dx'_{h+1} and dx'_{h+2} .

Upon removing $dx'_{h+1} dx'_{h+2}$, we will have an analogous expression in $n - 3$ variables. In this new expression, there will be terms of the third group, so we can repeat the preceding operation until all of these terms are absent. In other words, we can suppose that the terms of the third group are:

$$dx_{h+1}\, dx_{h+2} + dx_{h+2}\, dx_{h+4} + \ldots + dx_{h+2\alpha-1}\, dx_{h+2\alpha},$$

since the terms of the first and second group do not contain any differentials dx_{h+1}, dx_{h+2}, …, $dx_{h+2\alpha}$.

Now, take the terms in the second group – if they exist – that contain one of the differentials $dx_1, dx_2, \ldots, dx_h$. For example, suppose that the coefficient of $dx_1\, dx_{h+2\alpha-1}$ is non-zero. We can then, as we just did, take new variables in place of the x_1 and $x_{h+2\alpha-1}$:

$$x_1' = \sum_{\rho=1}^{h} A_{\rho,h+2\alpha+1} x_\rho ,$$

$$A_{1,\, h+2\alpha+1}\, x'_{h+2\alpha+1} = \sum_{\rho=1}^{n-1} A_{1,\rho} x_\rho ,$$

in such a manner that dx_1' and $dx'_{h+2\alpha+1}$ do not enter into any terms of the first and second group other than $dx_1'\, dx'_{h+2\alpha+1}$. Finally, upon repeating this operation as many times as necessary, one puts the terms of the second group into the form:

$$dx_1\, dx_{h+2\alpha+1} + dx_2\, dx_{h+2\alpha+2} + dx_\beta\, dx_{h+2\alpha+\beta},$$

so the terms of the first group contain none of the differentials $dx_1, dx_2, \ldots, dx_\beta$.

Finally, the terms of the first group themselves – if there are any – can be put into the form:

$$dx_{\beta+1}\, dx_{\beta+2} + dx_{\beta+3}\, dx_{\beta+4} + \ldots + dx_{\beta+2\gamma-1}\, dx_{\beta+2\gamma}$$

by a process that is identical to the preceding ones.

Finally, upon changing the notations, we can write:

$$(44)\qquad \left\{\begin{aligned} \varpi = {} & dx_1 dx_{h+1} + dx_2 dx_{h+2} + \ldots + dx_\alpha dx_{h+\alpha} \\ & + dx_{\alpha+1} dx_{\alpha+1} + dx_{\alpha+3} dx_{\alpha+4} + \ldots + dx_{\alpha+2\beta-1} dx_{\alpha+2\beta} \\ & + dx_{h+\alpha+1} dx_{h+\alpha+2} + dx_{h+\alpha+3} dx_{h+\alpha+4} + \ldots + dx_{h+\alpha+2\gamma-1} dx_{h+\alpha+2\gamma} \\ & + dx_n du, \end{aligned}\right.$$

where α, β, γ are integers that can be zero, such that:

$$\alpha + 2\beta \leq h, \qquad h + \alpha + 2\gamma \leq n - 1.$$

59. This being the case, we pass on to the proof of the lemma. One can first convert the first two cases into each other. Indeed, if the second case is proved then it will suffice to suppose that ϖ does not depend upon dx_n, and to then replace r with $r + 1$, h with h +

1, so the h expressions $\omega_1, \omega_2, \ldots, \omega_h$ will become $h + 1$ expressions $\omega, \omega_1, \omega_2, \ldots, \omega_h$ in order to the get back to the first case.

We thus have only the last two cases to prove.

60. Second case. – The hypothesis is that $\omega\varpi^{r-1}$ does not have all of its coefficients equal to zero; i.e., that $\varpi - dx_n\, du$ contains at least $r - 1$ terms. One then has:

$$\alpha + \beta + \gamma \geq r - 1.$$

One first sees that $\varpi^{r-h-1}\omega\omega_1\omega_2 \ldots \omega_h$ cannot be annulled, because upon removing the terms in $dx_1, dx_2, \ldots, dx_h, dx_n$ from ϖ, there will remain *at least* $r - 1 - h$ terms.

This being the case, if $\varpi^{r-h}\omega\,\omega_1\omega_2 \ldots \omega_h$ is zero then that signifies that upon removing the terms in $dx_1, dx_2, \ldots, dx_h, dx_n$ from ϖ, there remain at most $r - h - 1$ terms. However, this amounts to removing *at most* h of the $\varpi - dx_n\, du$ that contain *at least* $r - 1$ of them. This must then be true upon removing *exactly* h of them, and $\varpi - dx_n\, du$ contains *exactly* $r - 1$ of them. One then has:

$$\alpha + \beta + \gamma = r - 1$$

and

$$a = h, \qquad b = 0;$$

conversely, if this is true then $\varpi^{r-h}\omega\,\omega_1\omega_2 \ldots \omega_h$ is zero.

We likewise look for the conditions for all of the expressions $\varpi^{r-2}\omega\,\omega_i\,\omega_j$ to be zero. In order for this to be true, it is necessary that upon removing the terms in dx_n, dx_i, dx_j from ϖ there must remain *at most* $r - 3$ of them. Now, this amounts to removing *at most* two terms from $\varpi - dx_n\, du$ that contain *at least* $r - 1$ of them. It is then necessary that $\varpi - dx_n\, du$ must contain exactly $r - 1$ of them and that one removes *exactly* two. If that is true then for any indices i and j it is necessary that each of the differentials $dx_1, dx_2, \ldots, dx_h$ are contained in one and only one of the terms in $\varpi - dx_n\, du$; i.e., that one will have:

$$\alpha + \beta + \gamma = r - 1,$$
$$\alpha = h, \qquad \beta = 0,$$

and conversely, if this is true then the expressions $\varpi^{r-2}\omega\,\omega_i\,\omega_j$ are all zero.

Therefore, if:

$$\varpi^{r-h}\omega\,\omega_1\omega_2 \ldots \omega_h$$

is annulled then the same is true for:

$$\varpi^{r-2}\omega\,\omega_i\,\omega_j \qquad (i, j = 1, 2, \ldots, h),$$

and conversely. The proof is obvious.

61. Third case. – The hypothesis is that $\omega\varpi^{r-1}$ is non-zero, so the situation is the same as in the preceding case. One thus has:

$$\alpha + \beta + \gamma \geq r - 1,$$

and one sees in the same manner that $\varpi^{r-h-1} \omega\, \omega_1\, \omega_2 \ldots \omega_h$ cannot be zero.

This being the case, if $\varpi^{r-h} \omega\, \omega_1\, \omega_2 \ldots \omega_h$ is zero then upon setting:

$$\varpi = \varpi_1 + dx_n\, du$$

one sees that:

$$\varpi^{r-h} = \varpi_1^{r-h} + dx_n\, du\, \varpi_1^{r-h-1} .$$

One thus has:

$$\varpi_1^{r-h}\, dx_1, dx_2, \ldots, dx_h = 0,$$
$$du\ \varpi_1^{r-h-1} dx_1, dx_2, \ldots, dx_h = 0.$$

The first equality shows that upon removing the terms in $dx_1, dx_2, \ldots, dx_h$ from ϖ there remain at most $r - h - 1$ of them. One then deduces, as we just did, that ϖ_1 contain exactly $r - 1$ terms and that each of the differentials must appear in one and only one of the terms in ϖ_1 . One thus has:

$$\alpha + \beta + \gamma = r - 1,$$
$$\alpha = h, \qquad \beta = 0.$$

The second equality is then written:

$$dx_1\, dx_2 \ldots dx_h\ dx_{2h+2} \ldots dx_{2r-2} = 0,$$

which shows that u is a linear combination of $x_1, \ldots, x_h , x_{2h+1}, \ldots, x_{2r-2}$. Conversely, these conditions are sufficient in order for $\varpi^{r-h} \omega\, \omega_1\, \omega_2 \ldots \omega_h$ to be zero.

Now, suppose that the expressions $\varpi^{r-1} \omega_i$ and $\varpi^{r-2} \omega\ \omega_i \omega_j$ are all zero. Upon considering the latter, one confirms, as before, that one must have:

$$\alpha + \beta + \gamma = r - 1,$$
$$\alpha = h, \qquad \beta = 0,$$

and that these conditions are sufficient.

Upon now considering the former, one has:

$$\varpi^{r-1} \omega_i = \varpi_1^{r-1} \omega_i + dx_n du\, \varpi_1^{r-2} \omega_i .$$

The first term in the right-hand side is zero, and what remains is:

$$du\, \varpi_1^{r-2} dx_i = 0;$$

i.e.:

$$du\ dx_1\ dx_2 \ldots dx_h\ dx_{h+1} \ldots dx_{h+i-1}\ dx_{h+i+1} \ldots dx_{2h}\ dx_{2h+1} \ldots dx_{2r-2} = 0.$$

Since this is true for any value of the index $i = 1, 2, \ldots, h$, it is necessary and sufficient that u must be a linear combination of $x_1, \ldots, x_h, x_{2h+1}, \ldots, x_{2r-2}$.

It results from this that the two systems:

$$\varpi^{r-h}\,\omega\,\omega_1\,\omega_2\,\ldots\,\omega_h = 0$$

and

$$\varpi^{r-1}\,\omega_i = \varpi^{r-2}\,\omega\,\omega_i\,\omega_j = 0$$

are equivalent, which was to be proved.

62. Moreover, one easily sees that either one of the two systems entails that:

$$\omega\,\varpi^r = 0$$

in the three cases. Indeed, in the last two cases, when one makes $\omega = dx_n = 0$, ϖ is composed of $r - 1$ terms, and in turn:

$$dx_n\,\varpi^r = \omega\,\varpi^r = 0.$$

In the first case, ϖ is a sum of r terms, where one and only one of them contains ω_1 in such a way that $\omega\,\varpi^r$ is further zero. In the second case, one similarly sees that all of the expressions $\varpi^r\;\omega\,\omega_i$ have zero coefficients.

Naturally, the lemma is meaningless if h is greater than or equal to 1 in the first and third case, and greater than or equal to 2 in the second one.

63. We now return to the theorem that we would like to prove. It is deduced immediately from the preceding lemma upon taking ϖ to be the derived expression ω' and ω_i to be the differential df_i, and upon giving the variables only those numerical values that satisfy (27).

64. We shall now apply this theorem to *the solution of the Pfaff equation by means of r relations, where h of the relations are given by* (27), *by supposing that these r relations do not simultaneously annul the coefficients of* $\omega^{(2r-1)}$ *and* $\omega^{(2r-2)}$.

We shall successively examine the case where one considers only solutions that do not annul $\omega^{(2r-1)}$ and then the one where one considers only solutions that do not annul $\omega^{(2r-2)}$.

65. First case. – *Annul a Pfaff expression* ω *by means of r relations that do not annul* $\omega^{(2r-1)}$, *among which, h of them are given by* (27).

First, suppose that h is equal to at most 1; i.e., that one is effectively given one or more relations between the variables *a priori*. From the general theorem, one must adjoin the relations that are obtained by annulling all of the coefficients in the expressions:

$$\omega^{(2r-2)}\,df_i \qquad (i = 1, 2\ldots, h)$$

to these relations, and if h is greater than 1 then all of the coefficients in the expressions:

$$\omega^{(2r-3)}\, df_i\, df_j \qquad (i, j = 1, 2\ldots, h),$$

since one must annul ω by means of $f_i = 0$ and $r - 1$ other relations, and also by means of:

$$f_i = f_j = 0,$$

and $r - 2$ other relations. If the relations thus obtained are consequences of (27) then the system will be said to be *in involution*. Otherwise, one would have a new system of $h' > h$ relations that one could put into a form that satisfies the conditions that relate to the functional determinants of the left-hand sides. One proceeds with the new systems as one did with the first one, and so on, until one arrives at a system in involution.

66. The problem is thus converted into the case where the system (27) is in involution. If this system then contains more than r independent relations then the problem is impossible.

Suppose then that h of them are less than or equal to r.

If one first has:

$$\omega\, df_1\, df_2 \ldots df_h = 0$$

by virtue of (27), since $df_1\, df_2 \ldots df_h$ is not zero, then equations (27) will constitute a solution to the Pfaff equation. Therefore, if h is less than r then the coefficients of $\omega^{(2h)}$ and, by a stronger argument, those of $\omega^{(2r-2)}$, and also of $\omega^{(2r-1)}$, will all be zero by virtue of (27), which is contrary to the hypothesis that was made on $\omega^{(2r-1)}$. Therefore, in this case, h will be equal to r, and equations (27) will constitute the unique solution to the problem.

Conversely, if the system (27) in involution is formed from r relations then one has:

$$\omega\, df_1\, df_2 \ldots df_r = 0,$$

as one sees upon referring to the preceding lemma that was proved, and equations (27) constitute a solution.

Thus, suppose now that h is less than r. One then has that not all of the coefficients of:

$$\omega\, df_1\, df_2 \ldots df_h$$

are zero by virtue of (27). As a result, one has, always by virtue of (27):

$$\omega^{(2r-2h)}\, df_1\, df_2 \ldots df_h = 0,$$

without having

$$\omega^{(2r-2h-2)}\, df_1\, df_2 \ldots df_h = 0.$$

67. One will get the *general solutions* by seeking a non-constant integral of the complete system:

$$\omega^{(2r-2h-2)}\, df_1\, df_2 \ldots df_h\, df = 0;$$

i.e., from the theorem of no. 56, of the equivalent system:

$$(45)\qquad \begin{cases} \omega^{(2r-2)}df = 0, \\ \omega^{(2r-3)}df_1 df = \omega^{(2r-3)}df_2 df = \cdots = \omega^{(2r-3)}df_{h+1}df = 0, \end{cases}$$

and so on, until one has an integral f_r of the complete system:

$$(47)\qquad \begin{cases} \omega^{(2r-2)}df = 0, \\ \omega^{(2r-3)}df_1 df = \omega^{(2r-3)}df_2 df = \cdots = \omega^{(2r-3)}df_{r-1}df = 0 \end{cases}$$

that is independent of $f_{h+1}, f_{h+2}, \ldots, f_{r-1}$.

One will then have, upon taking (27) into account, along with the derived relations in $dx_1, dx_2, \ldots, dx_n$:

$$\omega = \varphi_{h+1}\, df_{h+1} + \varphi_{h+2}\, df_{h+2} + \ldots + \varphi_r\, df_r\,,$$

and the general solutions are deduced as we said before.

In total, the first complete system (45) admits $2r - 2h - 1$ independent integrals upon taking (27) into account, the second one admits $2r - 2h - 3$ independent integrals of f_{h+1}, and finally, the last one admits one independent integral of $f_{h+1}, f_{h+2}, \ldots, f_{r-1}$, always while taking (27) into account.

One must therefore perform $r - h$ operations of orders:

$$2r - 2h - 1, \quad 2r - 2h - 3, \quad \ldots, 3, 1,$$

respectively.

However, one must not forget that this method is valid only under the condition that one considers only solutions that do not annul all of the coefficients of $\omega^{(2r-1)}$.

68. The *singular solutions* of the system (27) are obtained by equating the coefficients of:

$$\omega^{(2r-2h-2)}\, df_1\, df_2 \ldots df_h$$

to zero; i.e. (always by virtue of the same theorem), by annulling the coefficients of:

$$\omega\, df_1\, df_2 \ldots df_h\,.$$

One will thus equate all of the determinants of degree $h + 1$ in the matrix

$$(48)\qquad \left\| \begin{matrix} A_1 & A_2 & \cdots & A_n \\ \dfrac{\partial f_1}{\partial x_1} & \dfrac{\partial f_1}{\partial x_2} & \cdots & \dfrac{\partial f_1}{\partial x_n} \\ \cdots & \cdots & \cdots & \cdots \\ \dfrac{\partial f_h}{\partial x_1} & \dfrac{\partial f_h}{\partial x_2} & \cdots & \dfrac{\partial f_h}{\partial x_n} \end{matrix} \right\|$$

to zero.

One must distinguish two cases: If the system of relations thus obtained does not annul:

$$df_1\, df_2 \dots df_h ,$$

i.e., *it does not annul all of the functional determinants of f with respect to h of the variables, then this system constitutes a solution of the Pfaff equation.* Moreover, if it does not annul $\omega^{(2r-1)}$ then it contains at least r independent relations, since otherwise $\omega^{(2r-2)}$ and $\omega^{(2r-1)}$ would be zero. If it contains exactly r of them then it gives the singular solution of the problem. If it contains more than r then there is no singular solution.

On the contrary, if the equation:

$$\omega\, df_1\, df_2 \dots df_h = 0$$

entails that:

$$df_1\, df_2 \dots df_h = 0$$

then one can say nothing more. One has a new system of $h' > h$ relations that one treats as one treated the original system, which can be incompatible, and which will admit both general solutions and singular solutions.

69. Up to now, we have assumed that one is given at least one relation between the variables *a priori*. In the contrary case, one must annul ω by means of r unknown relations, but not annul $\omega^{(2r-1)}$. In order to do this, one equates the coefficients of $\omega^{(2r)}$ to zero. If $\omega^{(2r)}$ is not identically zero then one is reduced to the preceding case. If $\omega^{(2r)}$ is identically zero then ω is a Pfaff expression of class $2r$, since, by hypothesis, $\omega^{(2r-1)}$ is not identically zero. Here, the singular solutions are obtained by annulling $\omega^{(2r-2)}$, since one does not wish that $\omega^{(2r-1)}$ be annulled *upon equating all of the coefficients of ω to zero.*

As for the general solutions, they are given by the reduction of the expression ω to its canonical form. One will have to seek an integral f_1 of the complete system:

$$\omega^{(2r-2)}\, df = 0,$$

and then an integral f_2 of the complete system:

$$\begin{aligned} \omega^{(2r-2)}\, df &= 0, \\ \omega^{(2r-2)}\, df_1\, df &= 0, \end{aligned}$$

that is independent of f_1, and so on, up to an integral f_r of the complete system:

$$\begin{aligned} \omega^{(2r-2)}\, df &= 0, \\ \omega^{(2r-3)}\, df_1\, df = \omega^{(2r-3)}\, df_2\, df = \dots = \omega^{(2r-3)}\, df_{r-1}\, df &= 0, \end{aligned}$$

that is independent of $f_1, f_2, \dots, f_{r-1}$, and one will then have:

$$\omega = \varphi_1\, df_1 + \varphi_2\, df_2 + \dots + \varphi_r\, df_r .$$

70. In particular, if the number of variables is equal to the class $2r$ then the expressions $\omega^{(2r-2)} df$, $\omega^{(2r-3)} df_i\, df$ will be of degree $2r$, in such a way that each of them provide an equation of the complete system. The successive complete systems are thus composed of 1, 2, …, r equations, respectively. *In this particular case*, the method is due to Clebsch; with the Clebsch notations, one has:

$$\omega^{(2r-2)} df = (f)\, dx_1\, dx_2 \ldots dx_{2r}\,,$$
$$\omega^{(2r-3)} df\, d\varphi = (f, \varphi)\, dx_1\, dx_2 \ldots dx_{2r}\,.$$

In the case where the number of variables is greater than the class, the method is a natural generalization of that of Clebsch. In practice, in order to write the equations of the $(h + 1)^{\text{th}}$ complete system, one equates the coefficients of the monomials:

$$dx_1\, dx_2 \ldots dx_{2r-1}\, dx_{2r-1+i} \qquad (i = 1, 2, \ldots, n - 2r + 1)$$

in $\omega^{(2r-2)} df$ to zero, upon supposing that the term in $dx_1\, dx_2 \ldots dx_{2r-1}$ in $\omega^{(2r-2)}$ has a non-zero coefficient. One then equates the coefficient of one of the differential monomials in each of the expressions $\omega^{(2r-3)} df_i\, df$ to zero, in such a manner that one then obtains h new equations that are independent of the original ones. These $n - 2r + h + 1$ equations form a complete system that must indeed have effectively $2r - h - 1$ independent integrals.

71. Second case. – *Annul a Pfaff expression* ω *by means of r relations that do not annul all of the coefficients of* $\omega^{(2r-2)}$, *among which h relations are given* (27).

First, suppose that h is equal to at least 1 – i.e., that one is effectively given one or more relations between the variables *a priori*. If h is equal to 1 then one must adjoin the relations that one obtains by annulling all of the coefficients in the expression:

$$\omega^{(2r-2)} df_1$$

to these relations, and if h is greater than 1 then one must add the relations that one obtains by annulling all of the coefficients in the expressions:

$$\omega^{(2r-4)} df_i\, df_j \qquad (i, j = 1, 2, \ldots, h). \tag{41}$$

If the relations thus obtained are not consequences of (27) then one will have a new system, to which one repeats the same operation, until one arrives at a system in *involution*; i.e., such that coefficients of (41) are all annulled by virtue of that system.

72. Therefore, suppose that the system (27) is in involution, with h being equal to at most r, since otherwise this would be impossible. One shows, as in the first case, that the coefficients of:

$$\omega\, df_1\, df_2 \ldots df_h$$

can all be zero only if the system (27) constitutes a solution of the Pfaff equation, and that h is then equal to r; conversely, a system of r independent equations in involution constitutes a solution of the Pfaff equation.

If h is less than r then one will get the *general solutions* by seeking an integral f_{h+1} – which is not constant by virtue of (27) – of the complete system:

(49) $$\omega^{(2r-4)}\, df_1\, df = \omega^{(2r-4)}\, df_2\, df = \ldots = \omega^{(2r-4)}\, df_h\, df = 0,$$

and then an integral df_{h+2} that is independent of df_{h+1} by virtue of (27) of the complete system:

(50) $$\omega^{(2r-4)}\, df_1\, df = \omega^{(2r-4)}\, df_2\, df = \ldots = \omega^{(2r-4)}\, df_{h+1}\, df = 0,$$

and so on, until one gets an integral f_r that is independent of $f_{h+1}, f_{h+2}, \ldots, f_{r-1}$ of the complete system:

(51) $$\omega^{(2r-4)}\, df_1\, df = \ldots = \omega^{(2r-4)}\, df_{r-1}\, df = 0.$$

The system that is obtained by combining (27) with the equations:

$$f_{h+1} = a_{h+1}, \qquad f_{h+2} = a_{h+2}, \qquad \ldots, \qquad f_r = a_r$$

is a system of r equations in involution. It thus constitutes a solution of the Pfaff equation that can, in turn, be put into the form:

$$\varphi_{h+1}\, df_{h+1} + \varphi_{h+2}\, df_{h+2} + \ldots + \varphi_r\, df_r = 0.$$

73. This process can be applied to all of the solutions that do not annul all of the coefficients of $\omega^{(2r-2)}$. In practice, one applies it to only the ones that simultaneously annul all of the coefficients of $\omega^{(2r-1)}$, since in the contrary case the method that was previously presented is simpler. However, a simplification in the general method might be possible. Suppose that the relations (27) of a system in involution annul all of the coefficients of the expressions:

$$\omega^{(2r-2)}\, df_i \qquad (i = 1, 2, \ldots, h).$$

Then, from the theorem of no. 56, since one is naturally dealing with only systems of less than r relations, the coefficients of:

$$\omega\, df_1\, df_2 \ldots df_h$$

are not all zero, and in turn, the coefficients of:

$$\omega^{(2r-2h-1)}\, df_1\, df_2 \ldots df_h$$

are all zero. In other words, *when one takes into account the relations* (27) *between the variables and the derived relations between the differentials, the Pfaff expression* ω *has class* $2r - 2h - 1$.

If h is equal to $r - 1$ then this signifies that ω is an exact differential, and *by a quadrature* one has:

$$\omega = df_r\,,$$

which gives all the solutions of the problem.

If h is less than $r - 1$ then one reduces ω to its canonical form by seeking an integral of the complete system:

$$\omega^{(2r-2h-3)}\, df_1\, df_2 \dots df_h\, df = 0;$$

i.e., of the complete system:

$$(52) \qquad \begin{cases} \omega^{(2r-3)} = 0, \\ \omega^{(2r-4)} df_1 df = \omega^{(2r-4)} df_2 df = \cdots = \omega^{(2r-4)} df_h df = 0, \end{cases}$$

a complete system that is equivalent to it, since one cannot have:

$$\omega\, df_1\, df_2 \dots df_h\, df = 0,$$

as $h + 1$ is less than r. If h is equal to $r - 2$ then ω reduces to an exact differential, upon taking into account (27) and the derived relations between the differentials, as well as:

$$f_{h+1} = a, \qquad df_{h+1} = 0.$$

One will thus have, by quadrature:

$$\omega = df_r + \varphi_{r-1}\, df_{r-1}\, .$$

In the general case, one will have to seek $r - h - 1$ successive integrals of $r - h - 1$ complete systems, the last of which is:

$$(53) \qquad \begin{cases} \omega^{(2r-3)} = 0, \\ \omega^{(2r-4)} df_1 df = \omega^{(2r-4)} df_2 df = \cdots = \omega^{(2r-4)} df_{r-2} df = 0, \end{cases}$$

and, upon deducing $r - 1$ of the variables as functions of $n - r + 1$ other ones from (27) and the equations:

$$f_{h+1} = a_{h+1}, \quad \dots, \quad f_{r-1} = a_{r-1},$$

one will have an exact differential Pfaff expression, in such a way that by a quadrature one will obtain:

$$\omega = df_r + \varphi_{h+1}\, df_{h+1} + \dots + \varphi_{r-1}\, df_{r-1}\, .$$

In total, the operations to be performed are of order:

$$2r - 2h - 2,\ 2r - 2h - 4,\ \dots,\ 6,\ 4,\ 2,\ 0.$$

in which an operation of order 0 is a quadrature.

74. The *singular solutions* are obtained, as before, by annulling the coefficients of:

$$\omega\, df_1\, df_2 \ldots df_h\,.$$

If the coefficients of the expression:

$$df_1\, df_2 \ldots df_h$$

are not simultaneously annulled then if the relations thus obtained, when combined with (27), give r independent relations then they constitute the singular integral. If they give more than r relations then there is no singular integral.

If the coefficients of:

$$df_1\, df_2 \ldots df_h$$

are all zero then one will have a new system of more than h relations on which one can proceed as one did on the given system (27), and so on.

75. In the case where one is not given any relations between the variables *a priori*, it suffices to look for solutions that, while not annulling all of the coefficients of $\omega^{(2r-2)}$, annul all of those of $\omega^{(2r-1)}$. Then, if the coefficients of $\omega^{(2r-1)}$ are not identically zero then one has a certain number of relations between the variables, and one comes back to the discarded hypothesis. If the coefficients of $\omega^{(2r-1)}$ are all identically zero then ω is a Pfaff expression of class $2r - 1$. The singular solutions do not exist here, since one is restricted to considering only solutions that do not annul all of the coefficients of $\omega^{(2r-2)}$.

The search for general solutions amounts to the reduction of ω to its canonical form. From the foregoing, one looks for an integral f_1 of the complete system:

$$\omega^{(r-3)}\, df_1 = 0,$$

and then an integral f_2 of the complete system:

$$\omega^{(2r-3)}\, df = \omega^{(2r-4)}\, df_1\, df = 0,$$

and so on, until one has an integral f_{r-1} of the complete system:

$$\omega^{(2r-3)}\, df = \omega^{(2r-4)}\, df_1\, df = \ldots = \omega^{(2r-4)}\, df_{r-2}\, df = 0,$$

and then, upon deducing $r - 1$ of the variables as functions of the $n - r + 1$ other ones from:

$$f_1 = a_1, \qquad f_2 = a_2, \qquad \ldots, \qquad f_{r-1} = a_{r-1}$$

and substituting then in ω, that expression becomes an exact differential form. One then achieves the reduction:

$$\omega = df_r + \varphi_1\, df_1 + \varphi_2\, df_2 + \ldots + \varphi_{r-1}\, df_{r-1}$$

by a quadrature.

In practice, the complete system that gives f_h admits $2r - h - 1$ independent integrals. It is therefore composed of $n - 2r + h + 1$ linearly independent equations.

One obtains them by equating all of the coefficients of the $n - 2r + 2$ differential monomials:

$$dx_1\, dx_2 \ldots dx_{2r-2}\, dx_i \qquad (i = 2r - 1, 2r, \ldots, n)$$

to zero, upon assuming that the coefficient of $dx_1\, dx_2 \ldots dx_{2r-2}$ in $\omega^{(2r-2)}$ is not zero. One will thus have $n - 2r + 2$ equations that give:

$$\frac{\partial f}{\partial x_{2r-1}},\ \frac{\partial f}{\partial x_{2r}},\ \ldots,\ \frac{\partial f}{\partial x_n}$$

as functions of

$$\frac{\partial f}{\partial x_1},\ \frac{\partial f}{\partial x_2},\ \ldots,\ \frac{\partial f}{\partial x_{2r-2}}.$$

One will have $h - 1$ equations remaining upon annulling the coefficients of one of the differential monomials in each of the expressions:

$$\omega^{(2r-4)}\, df_1\, df, \quad \ldots, \qquad \omega^{(2r-4)}\, df_{h-1}\, df,$$

in such a manner as to obtain equations that are mutually independent and independent of the first $n - 2r + 2$ equations.

If n is equal to $2r - 1$ then the equations are defined by the expressions:

$$\omega^{(2r-3)}\, df, \qquad \omega^{(2r-4)}\, df_1\, df, \quad \ldots$$

themselves, which are of degree $2r - 1$.

This method constitutes the generalization of the second method of Clebsch, which was known only for expressions of class $2r$ in $2r$ variables, to the expressions of odd class.

76. *Example*. – Consider the Pfaff expression (Forsythe):

$$\omega = x_2\, dx_1 + x_3\, dx_2 + x_4\, dx_3 + x_5\, dx_4 + x_6\, dx_5 + x_1\, dx_6 .$$

Here, one has:

$$\omega^{\mathrm{V}} = 0,$$

$$\begin{aligned}\omega^{\mathrm{IV}} = {} & (x_2 + x_4 + x_6)\,(dx_1\, dx_2\, dx_3\, dx_4\, dx_5 + dx_3\, dx_4\, dx_5\, dx_6\, dx_1 + dx_5\, dx_6\, dx_1\, dx_2\, dx_3) \\ & + (x_1 + x_3 + x_5)\,(dx_2\, dx_3\, dx_4\, dx_5\, dx_6 + dx_4\, dx_5\, dx_6\, dx_1\, dx_2 + dx_6\, dx_1\, dx_2\, dx_3\, dx_4).\end{aligned}$$

The expression ω is therefore of class five. In order to make the reduction, one calculates the expressions $\omega'''\, df$, $\omega''\, df\, d\varphi$. One has:

$$\begin{aligned}\omega'''\, df = {} & \left(\frac{\partial f}{\partial x_1} + \frac{\partial f}{\partial x_3} + \frac{\partial f}{\partial x_5}\right)(dx_1\, dx_2\, dx_3\, dx_4\, dx_5 + dx_3\, dx_4\, dx_5\, dx_6\, dx_1 + \ldots) \\ & + \left(\frac{\partial f}{\partial x_2} + \frac{\partial f}{\partial x_4} + \frac{\partial f}{\partial x_6}\right)(dx_2\, dx_3\, dx_4\, dx_5\, dx_6 + dx_4\, dx_5\, dx_6\, dx_1\, dx_2 + \ldots),\end{aligned}$$

and then, upon taking into account the fact that the coefficients of $\omega''' \, df$, $\omega''' \, d\varphi$ must be zero, one has:

$$\omega'' \, df \, d\varphi = \left(\frac{\partial f}{\partial x_1}\frac{\partial \varphi}{\partial x_2} - \frac{\partial f}{\partial x_2}\frac{\partial \varphi}{\partial x_1} + \frac{\partial f}{\partial x_4}\frac{\partial \varphi}{\partial x_5} - \frac{\partial f}{\partial x_5}\frac{\partial \varphi}{\partial x_4}\right)$$
$$\times [(x_1 + x_3 + x_5) \, dx_4 \, dx_5 \, dx_6 \, dx_1 \, dx_2 + (x_2 + x_4 + x_6) \, dx_1 \, dx_2 \, dx_3 \, dx_4 \, dx_5]$$
$$+ \left(\frac{\partial f}{\partial x_2}\frac{\partial \varphi}{\partial x_3} - \frac{\partial f}{\partial x_3}\frac{\partial \varphi}{\partial x_2} + \frac{\partial f}{\partial x_5}\frac{\partial \varphi}{\partial x_6} - \frac{\partial f}{\partial x_6}\frac{\partial \varphi}{\partial x_5}\right)$$
$$\times [(x_1 + x_3 + x_5) \, dx_2 \, dx_3 \, dx_4 \, dx_5 \, dx_6 + (x_2 + x_4 + x_6) \, dx_5 \, dx_6 \, dx_1 \, dx_2 \, dx_3]$$
$$+ \left(\frac{\partial f}{\partial x_3}\frac{\partial \varphi}{\partial x_4} - \frac{\partial f}{\partial x_4}\frac{\partial \varphi}{\partial x_3} + \frac{\partial f}{\partial x_6}\frac{\partial \varphi}{\partial x_1} - \frac{\partial f}{\partial x_1}\frac{\partial \varphi}{\partial x_6}\right)$$
$$\times [(x_1 + x_3 + x_5) \, dx_6 \, dx_1 \, dx_2 \, dx_3 \, dx_4 + (x_2 + x_4 + x_6) \, dx_3 \, dx_4 \, dx_5 \, dx_6 \, dx_1].$$

The complete system is therefore:

$$\frac{\partial f}{\partial x_1} + \frac{\partial f}{\partial x_3} + \frac{\partial f}{\partial x_5} = 0,$$

$$\frac{\partial f}{\partial x_2} + \frac{\partial f}{\partial x_4} + \frac{\partial f}{\partial x_6} = 0.$$

Let $f_1 = x_1 - x_3$ be an integral of this complete system. The other one is obtained by appending:

$$\frac{\partial f_1}{\partial x_1}\frac{\partial f}{\partial x_2} - \frac{\partial f}{\partial x_1}\frac{\partial f_1}{\partial x_2} + \frac{\partial f_1}{\partial x_4}\frac{\partial f}{\partial x_5} - \frac{\partial f_1}{\partial x_5}\frac{\partial f}{\partial x_4} = 0$$

to the preceding equations; i.e.:

$$\frac{\partial f}{\partial x_2} = 0.$$

An integral of this second system is, for example:

$$f_2 = x_1 - x_5 .$$

Set:

$$f_1 = x_1 - x_3 = a_1 , \qquad f_2 = x_1 - x_5 = a_2 ,$$

and derive x_3 and x_4 from these equations. We obtain:

$$\omega = x_2 \, dx_1 + (x_1 - a_1) \, dx_2 + x_4 \, dx_1 + (x_1 - a_2) \, dx_4 + x_6 \, dx_1 + x_1 \, dx_6,$$

or

$$\omega = d(x_1 x_2 + x_1 x_4 + x_1 x_6 - a_1 x_2 - a_2 x_4) = d(x_2 x_3 + x_4 x_5 + x_6 x_1),$$

and, upon taking into account the fact that:

$$f_1 = a_1, \qquad f_2 = a_2,$$

one obtains:

$$\omega = d(x_2 x_3 + x_4 x_5 + x_6 x_1) + (x_2 - x_4)\, d(x_1 - x_3) + (x_4 - x_6)\, d(x_1 - x_5).$$

77. *Remark.* – One sees what sort of simplifications that this method introduces into the calculations made during the reduction of Pfaff expressions when compared to the first method that we discussed. Previously, each function whose differential entered into the reduced form was given by a complete system, each equation of which simultaneously contained the partial derivatives of all the functions that were previously found. Now, the partial derivatives of any of the functions that were already found enter into just one equation of the system, and that equation does not contain the other ones.

V. – First-order, partial differential equations.

78. Given n independent variables $x_1, x_2, \ldots, x_n$, and an unknown function z of these variables, consider a system of h first-order, partial differential equations:

$$
(1) \qquad \begin{cases} f_1\left(x_1, x_2, \ldots, x_n, z, \dfrac{\partial z}{\partial x_1}, \dfrac{\partial z}{\partial x_2}, \ldots, \dfrac{\partial z}{\partial x_n}\right) = 0, \\ f_2\left(x_1, x_2, \ldots, x_n, z, \dfrac{\partial z}{\partial x_1}, \dfrac{\partial z}{\partial x_2}, \ldots, \dfrac{\partial z}{\partial x_n}\right) = 0, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \\ f_h\left(x_1, x_2, \ldots, x_n, z, \dfrac{\partial z}{\partial x_1}, \dfrac{\partial z}{\partial x_2}, \ldots, \dfrac{\partial z}{\partial x_n}\right) = 0. \end{cases}
$$

From the generalized notion that is due to Lie, to integrate this system is to find $2n + 1$ quantities $x_1, x_2, \ldots, x_n, z, p_1, p_2, \ldots, p_n$ that are functions of n parameters that satisfy the system:

$$
(2) \qquad \begin{cases} f_1(x_1, x_2, \ldots, x_n, z, p_1, p_2, \ldots, p_n) = 0, \\ f_2(x_1, x_2, \ldots, x_n, z, p_1, p_2, \ldots, p_n) = 0, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \\ f_h(x_1, x_2, \ldots, x_n, z, p_1, p_2, \ldots, p_n) = 0, \end{cases}
$$

identically, and the total differential equation:

$$
(3) \qquad \omega = dz - p_1\, dx_1 - p_2\, dx_2 - \ldots - p_n\, dx_n = 0,
$$

or furthermore, *to satisfy equations* (2) *and* (3) *by a system of* $n + 1$ *distinct relations between* $x_1, x_2, \ldots, x_n, z, p_1, p_2, \ldots, p_n$.

From this last statement, we thus come back to the problem that was treated in the last section: Annul the Pfaff expression ω by means of a system of $r = n + 1$ relations between the $2n + 1$ variables, among which, h of them are given by (2).

79. *Multiplicities.* – Before applying the principles of the preceding chapter, we shall define some expressions of geometric origin that will be of use to us in the sequel.

A system of values $x_1, x_2, \ldots, x_n, z, p_1, p_2, \ldots, p_n$ will be called an *element*.

An arbitrary system of relations between the x, z, and the p will be said to define a *multiplicity* if this system entails the total differential equation (3) as a consequence. We have seen how one can find all of the multiplicities. If the multiplicity is defined by r relations then it will be said to be $n - r + 1$-dimensional. All of the systems of values for variables that satisfy the equations of the multiplicity define elements of the multiplicity. The elements of an $n - r + 1$-dimensional multiplicity then depend upon $n - r + 1$ parameters. An s-dimensional multiplicity will be denoted by the symbol M_s .

A multiplicity cannot be more than n-dimensional, because it must have at least $n + 1$ relations in order to imply equation (3). An element is, moreover, a multiplicity M_s .

An element is called a *simple* element of a multiplicity M_s if one can express $2n – s + 1$ of the variables as holomorphic functions of the h other ones in the neighborhood of that element. We have (formula 6 of the preceding chapter) determined all of the multiplicities M_s that admit a given element as a simple element.

Given a system of partial differential equations (2), any multiplicity whose elements all satisfy the relations (2) will be called an *integral multiplicity*. To integrate the system (2) is therefore to find all of the integral n-dimensional multiplicities M_n .

80. *Application of general theorems*. – *Bracket of two functions*. – Here is how one must proceed in order to integrate the system (2) by using the method that was presented in the preceding chapter.

The number that we have denoted by r is equal to $n + 1$, here. The $(2r – 2)^{\text{th}}$ derivative of ω is:

$$\omega^{(2r-2)} = \omega^{(2n)} = (dz - p_1\,dx_1 - p_1\,dx_1 - \ldots - p_1\,dx_1)\,(dx_1\,dp_1 - \ldots - dx_n\,dp_n)^n$$
$$= dz\,dx_1\,dp_1 \ldots dx_n\,dp_n\,.$$

Therefore, no integral multiplicity can annul the coefficients of that derivative $\omega^{(2n)}$. As a result, we can surely apply the method that was presented at the end of the previous chapter.

We thus have to form the differential expression:

$$\omega^{(2r-4)}\,df\,d\varphi,$$

in which f and φ denote two arbitrary left-hand sides of the system (2) – i.e.:

$$\omega^{(2r-2)}\,df\,d\varphi = \omega\omega^{(n-1)}\,df\,d\varphi,$$

and equate all of its coefficients to zero. Now, that differential expression in $n + 1$ variables has degree $2n + 1$. It therefore has *just one* coefficient. Thus, if we set:

$$\omega^{(2r-2)}\,df\,d\varphi = (f, \varphi)\,dz\,dx_1\,dp_1 \ldots dx_n\,dp_n \tag{4}$$

then the expression (f, φ) is what one calls the *bracket* of the two functions f and φ, which is a bilinear form in the partial derivatives of f and φ, and the equations that must be added to equations (2) are:

$$(f_i, f_j) = 0 \qquad (i, j = 1, 2, \ldots, h). \tag{5}$$

81. It is easy to form the bracket of two functions f and φ explicitly. Indeed, the differential expression (4) does not change if one replaces df and $d\varphi$ by:

$$d'f = df - \frac{\partial f}{\partial z}\,\omega = \left(\frac{\partial f}{\partial x_1} + p_1\frac{\partial f}{\partial z}\right)dx_1 + \ldots + \frac{\partial f}{\partial p_1}\,dp_1 + \ldots$$

and

$$d'\varphi = d\varphi - \frac{\partial\varphi}{\partial z}\,\omega = \left(\frac{\partial\varphi}{\partial x_1} + p_1\frac{\partial\varphi}{\partial z}\right)dx_1 + \ldots + \frac{\partial\varphi}{\partial p_1}dp_1 + \ldots$$

and that is true by the presence of the factor ω in the differential expression (4). One thus has:

$$\omega'^{(2r-2)}\,df\,d\varphi = \omega\omega'^{(n-1)}\,d'f\,d'\varphi,$$

and the differential dz no longer enters into ω in the right-hand side. As a consequence, the coefficient of $dz\,dx_1\,dp_1 \ldots dx_n\,dp_n$ in (4) is nothing but the coefficient of $dx_1\,dp_1 \ldots dx_n\,dp_n$ in the expression $\omega'^{(n-1)}\,d'f\,d'\varphi$. One thus has, moreover:

(6) $$(f, \varphi)\,dx_1\,dp_1 \ldots dx_n\,dp_n = \omega'^{(n-1)}\,d'f\,d'\varphi,$$

and, upon replacing $\omega'^{(n-1)}$ with its value:

$$\omega'^{(n-1)} = \sum dx_1\,dp_1\,dx_2\,dp_2 \ldots dx_{n-1}\,dp_{n-1},$$

the $\sum$ sign being extended over all combinations of $n - 1$ of the indices 1, 2, …, n, one obtains:

$$(f, \varphi) = \sum\left(\frac{\partial f}{\partial x_n} + p_n\frac{\partial f}{\partial z}\right)\frac{\partial\varphi}{\partial p_n} - \left(\frac{\partial\varphi}{\partial x_n} + p_n\frac{\partial\varphi}{\partial z}\right)\frac{\partial f}{\partial p_n},$$

the $\sum$ sign being extended over all of the indices 1, 2, …, n. Conforming to tradition, we set:

(7) $$(f, \varphi) = \sum_{i=1}^{2n}\left[\frac{\partial f}{\partial p_i}\left(\frac{\partial\varphi}{\partial x_i} + p_i\frac{\partial\varphi}{\partial z}\right) - \frac{\partial f}{\partial p_i}\left(\frac{\partial f}{\partial x_i} + p_i\frac{\partial f}{\partial z}\right)\right].$$

82. The bracket of two functions enjoys the following properties: One has:

$$(f, \varphi) = -(\varphi, f).$$

Moreover, if f and φ depend upon variables by the intermediary of a certain number of functions $u, v, w, \ldots$, one has:

(8) $$(f, \varphi) = \frac{D(f,\varphi)}{D(u,v)}(u, v) + \frac{D(f,\varphi)}{D(u,w)}(u, w) + \ldots + \frac{D(f,\varphi)}{D(v,w)}(v, w) + \ldots$$

Indeed, this results from the identity:

$$df\,d\varphi = \frac{D(f,\varphi)}{D(u,v)}du\,dv + \frac{D(f,\varphi)}{D(u,w)}du\,dw + \ldots + \frac{D(f,\varphi)}{D(v,w)}dv\,dw + \ldots,$$

which gives the identity (8) upon multiplying its two sides by $\omega^{(2n-2)}$.

83. *Systems in involution*. – Now that we have established these properties, we return to the system (2). We append all of equations (5) to them, which will give a new system, in general. We proceed with this new system as we did with the first one, and so on. We thus conclude by arriving at either a system of at most $n + 1$ equations, in which case, one has the impossibility of a solution, or a system such that the brackets of any two of the left-hand sides of this system are zero by virtue of the equations of this system. We then say that this system is *in involution*.

A system in involution is therefore a system of $h \leq n + 1$ *equations* (2), *in the left-hand sides of which, one supposes that:*

1. *They are holomorphic in the neighborhood of an arbitrary element* (x_i^0, z^0, p_i^0) *that satisfies that system*.
2. *The functional determinants of these h left-hand sides with respect to any h of the variables are not all zero for the same element.*
3. *The brackets of any two of these h left-hand sides are zero by virtue of the equations of the system*.

If h is equal to 1 then the latter condition is naturally dropped. In the general case, all of the coefficients of $\omega^{(2r-2)}\, df_1$ must be zero. Here, they are always zero, since $\omega^{(2r-2)}\, df_1$ has degree $n + 2$.

From the preceding, one can always convert the integration of an arbitrary system of first-order, partial differential equations into a system in involution.

84. *General integral of a system in involution*. – Suppose one has to integrate a system in involution of h equations (2). From the general theorem, one must consider a certain number of successive complete systems, and for each of them, it suffices to find one integral. The first of these complete systems is given by the equations:

$$\omega^{(2r-4)}\, df_1\, df = \omega^{(2r-4)}\, df_2\, df = \ldots = \omega^{(2r-4)}\, df_h\, df = 0;$$

i.e., one has:

(9) $$(f_1, f) = 0, \qquad (f_2, f) = 0, \qquad \ldots, \qquad (f_h, f) = 0,$$

here.

Let A_1 be a particular integral of this complete system that does not reduce to a constant by virtue of (2).

One considers the second complete system:

(10) $$(f_1, f) = 0, \qquad (f_2, f) = 0, \qquad \ldots, \qquad (f_h, f) = 0, \qquad (A_1, f) = 0,$$

and one seeks an integral A_2 of this second system that does not reduce to a function of A_1 by virtue of (2). One will then have $n - h$ successive complete systems that give $n - h$ independent functions $A_1, A_2, \ldots, A_{n-h}$, respectively, also upon taking (2) into account, and finally a last complete system:

(11) $$\begin{cases} (f_2, f) = 0, \; \cdots, \quad (f_h, f) = 0, \\ (A_1, f) = 0 \; \cdots, \; (A_{n-h}, f) = 0, \end{cases}$$

which will admit one and only one integral that is independent of $A_1, A_2, \ldots, A_{n-h}$, namely, C.

The equation to solve can then be put into the form:

(12) $$dC - B_1\, dA_1 - B_2\, dA_2 - \ldots - B_{n-h}\, dA_{n-h} = 0,$$

where B are $n - h$ functions that are determined by differentiations. Moreover, from the general theory, if one considers an arbitrary element (x_i^0, z^0, p_i^0) that satisfies the system (2) then one can always choose the integrals $A_1, A_2, \ldots, A_{n-h}, C$ in such a manner that the $2n - 2h + 1$ functions $A_1, A_2, \ldots, A_{n-h}, C, B_1, B_2, \ldots, B_{n-h}$ are holomorphic in the neighborhood of that element. The most general integral multiplicity M_n of the system (2) that admits that element as a simple element is obtained by appending $n - h + 1$ relations between the A, B, and C to (2), relations that can be solved with respect to $n - h + 1$ of these quantities, since the right-hand sides are holomorphic in the neighborhood of $(A_1^0, \ldots, A_{n-h}^0, \ldots, B_{n-h}^0)$. The relations fall into the general type of formulas (7) in the preceding chapter.

85. The complete systems (9), (10), …, (11) admit:

$$2n - h + 1, \qquad 2n - h, \quad \ldots, n + 1$$

independent integrals, respectively; of course, they admit all of the integrals $f_1, f_2, \ldots, f_h$. More than that, one must essentially suppose that the variables are coupled by the relations (2); it is only by means of this condition that one can be sure that the systems (9), (10), … are complete.

Finally, if one remarks that if A_1 is known then the system (10) admits $h + 1$ known integrals, and if A_1 and A_2 is known then the system (10) admits $h + 2$ known integrals, and so on, so one sees that the indicated method demands the search for an integral of $n - h + 1$ successive complete systems in $2n + 1$ variables, but which admit:

$$h, \quad h + 1, \qquad h + 2, \quad \ldots, \qquad n$$

known integrals, respectively. From the Mayer method, this method thus amounts to the search for a particular integral of $n - h + 1$ successive systems of differential equations that have:

$$2n - 2h + 2, \quad 2n - 2h, \qquad \ldots, \qquad 4, \qquad 2,$$

variables, respectively.

86. In particular, if $h = 1$ then the first complete system is formed from just one equation:

$$(f_1, f) = 0.$$

Here, there are n successive systems of differential equations that are in:

$$2n, \qquad 2n - 2, \quad \ldots, \quad 4, \qquad 2$$

variables, respectively.

If h is equal to $n + 1$ then there is no integration to be done. *A system in involution of $n + 1$ equations always defines an n-dimensional integral multiplicity.* The converse is obvious, moreover.

87. *Particular case.* – From the general theory, the integration of a system in involution is simplified if the coefficients of the expressions:

$$\omega^{(2r-3)}\, df_i = \omega^{(2n-1)}\, df_i \qquad (i = 1, 2, \ldots, h)$$

are all annulled by virtue of the equations of this system. Now, one has:

$$\omega^{(2r-1)}\, df = \omega^{(n)}\, df = \frac{\partial f}{\partial z} dz\, dx_1\, dp_1\, dx_2\, dp_2 \ldots dx_n\, dp_n\, .$$

The simplification is then provided if the quantities $\partial f_i / \partial z$ are all zero by virtue of (2); i.e., if the system (2) does not contain z explicitly, or contains it only *formally*.

Therefore, if one is dealing with the integration of a system of first-order, partial differential equations in involution that do not contain the unknown function z explicitly:

$$(2)' \qquad \begin{cases} f_1(x_1, x_2, \ldots, x_n, p_1, p_2, \ldots, p_n) = 0, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \\ f_h(x_1, x_2, \ldots, x_n, p_1, p_2, \ldots, p_n) = 0 \end{cases}$$

then one seeks an integral A_1 of the complete system:

$$\omega^{(2n-1)}\, df = \omega^{(2n-2)}\, df_1\, df = \ldots = \omega^{(2n-2)}\, df_h\, df = 0;$$

i.e., the complete system:

$$\frac{\partial f}{\partial z} = 0, \qquad (f_1, f) = 0, \qquad (f_2, f) = 0, \qquad \ldots, \qquad (f_h, f) = 0,$$

and then an integral A_2 of the complete system:

$$\frac{\partial f}{\partial z} = 0, \qquad (f_1, f) = 0, \qquad (f_2, f) = 0, \qquad \ldots, \qquad (f_h, f) = 0, \qquad (A_1, f) = 0$$

that is independent of A_1, and so on, up to an integral A_{n-h} of the complete system:

$$\frac{\partial f}{\partial z} = 0, \qquad (f_1, f) = 0, \qquad \ldots, \qquad (A_{n-h-1}, f) = 0$$

that is independent of $A_1, A_2, \ldots, A_{n-h-1}$. Then, upon deriving $n - 1$ of the variables x_i, p_k as functions of the remaining variables, other than z, from equations (2)′ and the equations:

$$A_1 = a_1, \qquad A_2 = a_2, \qquad \ldots, \qquad A_{n-h-1} = a_{n-h-1} ,$$

the expression:

$$\omega = dz - p_1\, dx_1 - p_2\, dx_2 - \ldots - p_n\, dx_n$$

becomes an exact differential, and by a quadrature one obtains, upon taking (2)′ into account:

$$\omega = dz - dC - B_1\, dA_1 - B_2\, dA_2 - \ldots - B_{n-h-1}\, dA_{n-h-1} ,$$

where C is a function of x and p.

Moreover, we remark that one has:

$$(f, \varphi) = \sum_i \left(\frac{\partial f}{\partial p_i} \frac{\partial \varphi}{\partial x_i} - \frac{\partial f}{\partial x_i} \frac{\partial \varphi}{\partial p_i} \right)$$

for the functions that enter into the complete systems to be integrated.

88. *Singular integrals*. – The expression $\omega^{(2r-2)} = \omega^{(2n)}$ can never have all of its coefficients equal to zero, so the singular integrals of the system in involution (2) are obtained by annulling all of the coefficients in the expression:

(13) $$\omega\, df_1\, df_2 \ldots df_h = dz\, d'f_1\, d'f_2 \ldots d'f_h ,$$

i.e., upon annulling all of the coefficients of the expression:

(14) $$d'f_1\, d'f_2 \ldots d'f_h ,$$

into which dz does not enter. One will thus consider the matrix:

(15) $$\left\| \begin{matrix} \dfrac{\partial f_1}{\partial x_1} + p_1 \dfrac{\partial f_1}{\partial z} \cdots \dfrac{\partial f_1}{\partial x_n} + p_n \dfrac{\partial f_1}{\partial z} \dfrac{\partial f_1}{\partial p_1} \cdots \dfrac{\partial f_1}{\partial p_n} \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \dfrac{\partial f_h}{\partial x_1} + p_1 \dfrac{\partial f_h}{\partial z} \cdots \dfrac{\partial f_h}{\partial x_n} + p_n \dfrac{\partial f_h}{\partial z} \dfrac{\partial f_h}{\partial p_1} \cdots \dfrac{\partial f_h}{\partial p_n} \end{matrix} \right\| ,$$

and one will annul all of its determinants with h rows and h columns. From the general theory, two cases can be presented:

If the coefficients of:

(16) $$df_1\, df_2\, \dots\, df_h$$

are not annulled at the same time as those of (14) then the equations obtained, when combined with equations (2), constitute the singular integral if they are $n + 1$ in number (they cannot be less in number). If they are more in number than $n + 1$ then there is no singular integral.

On the contrary, if the coefficients of (16) are annulled at the same time as those of (14) then one can say nothing further. The equations obtained, when combined with (2), form a new system to integrate. This system certainly contains more than h equations (h is assumed to be less than $n + 1$), but it cannot be in involution. One thus completes it, as needed, in such a manner as to have a system of more than $n + 1$ equations, in which case, a solution is impossible, or a system in involution of $h' \leq n + 1$ equations. One integrates this new system like the first one. It will admit general integrals, and in turn, it can admit singular integrals that one finds by means of a third system in involution of h'' > h' equations, and so on. It is indeed clear that these operations will have a conclusion.

In particular, if the system (2) is composed of just one equation then the singular integrals will satisfy the system:

$$f = 0, \qquad \frac{\partial f}{\partial x_1} + p_1 \frac{\partial f}{\partial z} = \dots = \frac{\partial f}{\partial x_n} + p_n \frac{\partial f}{\partial z} = \frac{\partial f}{\partial p_1} = \dots = \frac{\partial f}{\partial p_n} = 0.$$

If these equations do not annul $\partial f / \partial z$ then they will or will not give a singular integral according to whether they can or cannot be reduced to $n + 1$ independent equations, respectively. If they annul $\partial f / \partial z$ then one has a new system that can be composed of less than $n + 1$ equations, and that one can integrate directly.

89. *Example*. – In the case of $n = 2$, consider the partial differential equation:

(17) $$f = p_1^3 + (z - p_2^2)^2 = 0,$$

and look for its singular integrals. They satisfy the equations:

$$p_1(z - p_2^2) = p_2(z - p_2^2) = p_1^2 = p_2(z - p_2^2) = 0;$$

i.e., the system:

(18) $$\begin{cases} f_1 = p_1 = 0, \\ f_2 = z - p_2^2 = 0, \end{cases}$$

which is in involution, as is easy to verify. In order to have the general integrals, we solve for p_1 and p_2, and substitute them in the equation:

$$dz - p_1\, dx_1 - p_2\, dx_2 = 0.$$

We find:

$$p_2\,(2\, dp_2 - dx_2) = 0.$$

For a general integral depending upon an arbitrary constant a, we thus have:

(19) $$\begin{cases} z = \left(\dfrac{x_2 - a}{2}\right)^2, \\ p_1 = 0, \\ p_2 = \dfrac{x_2 - a}{2}, \end{cases}$$

and for a singular integral:

(20) $$\begin{cases} z = 0, \\ p_1 = 0, \\ p_2 = 0. \end{cases}$$

90. *Contact transformations*. – Following Lie, a contact transformation is defined by $2n + 1$ functions $Z, X_1, X_2, \ldots, P_1, P_2, \ldots, P_n$ of $2n + 1$ variables $z, x_1, x_2, \ldots, x_n, p_1, p_2, \ldots, p_n$, and is such that one has:

(21) $$\Omega = dZ - P_1\, dX_1 - \ldots - P_n\, dX_n = \rho(dz - p_1\, dx_1 - \ldots - p_n\, dx_n) = \rho\omega$$

identically, where ρ denotes a function of the variables z, x_i, p_k that is not identically zero.

We first show that these $2n + 1$ functions are independent. From the identity:

$$\Omega = \rho\omega$$

one indeed deduces that:

$$\Omega' = \rho\omega' + d\rho \cdot \omega,$$

and upon raising this to the n^{th} power:

$$\Omega'^n = \rho^n \omega'^n + \rho^{n-1} \omega'^{n-1}\, d\rho \cdot \omega,$$

and finally:

(22) $$\Omega\Omega'^n = \Omega^{(2n)} = \rho^{n+1} \omega\omega'^n = \rho^{n+1} \omega^{(2n)}.$$

Upon replacing $\omega^{(2n)}$ and $\Omega^{(2n)}$ with their values one obtains:

(23) $$dZ\, DX_1\, dP_1 \ldots dX_n\, dP_n = \rho^{n+1} dz\, dx_1\, dp_1 \ldots dx_n\, dp_n,$$

or finally:

$$\frac{D(Z, X_1, P_1, \ldots, X_n, P_n)}{D(z, x_1, p_1, \ldots, x_n, p_n)} = \rho^{n+1}.$$

The $2n + 1$ functions Z, X_i, P_k are thus indeed independent by virtue of the hypothesis that was made on ρ.

Similarly, let F and Φ denote two arbitrary functions of Z, X_i, P_k , and let f and φ denote what these functions become when one replaces the functions Z, X_i, P_k with their values. One has:

$$\Omega^{(2n-1)}\, dF\, d\Phi = (\rho\omega)^{(2n-2)}\, df\, d\varphi,$$

but

$$\Omega^{(2n-2)} = (\rho\omega)^{(2n-2)} = \rho^n\, \omega\omega'^{n-1} = \rho^n\, \omega^{(2n-2)}.$$

One thus has:

$$\Omega^{(2n-2)}\, dF\, d\Phi = \rho^n\, \omega^{(2n-2)}\, df\, d\varphi.$$

Let:

$$[F, \Phi] = \frac{\partial F}{\partial P_1}\left(\frac{\partial \Phi}{\partial x_1} + P_1\frac{\partial \Phi}{\partial z}\right) - \left(\frac{\partial F}{\partial x_1} + P_1\frac{\partial F}{\partial z}\right)\frac{\partial \Phi}{\partial P_1} + \dots$$

denote the bracket of the two functions F and Φ, which are regarded as functions of Z, X_i, P_h, and, as before, let (f, φ) denote the bracket that relates to the variables z, x_i, p_k . One then has:

$$[F, \Phi]\, dZ\, dX_1 \dots dP_n = \rho^n\, (f, \varphi)\, dz\, dx_1 \dots dp_n ,$$

or, upon replacing the differential monomial in the left-hand side by its value (23):

$$(f, \varphi) = \rho\, [F, \Phi]. \tag{24}$$

This fundamental equality is written explicitly as:

$$\tag{24$'$} \left\{\begin{aligned} &\sum\left[\frac{\partial f}{\partial p_i}\left(\frac{\partial \varphi}{\partial x_i} + p_i\frac{\partial \varphi}{\partial z}\right) - \frac{\partial \varphi}{\partial p_i}\left(\frac{\partial f}{\partial x_i} + p_i\frac{\partial f}{\partial z}\right)\right] \\ &\quad = \rho\sum\left[\frac{\partial F}{\partial P_i}\left(\frac{\partial \Phi}{\partial x_i} + P_i\frac{\partial \Phi}{\partial z}\right) - \frac{\partial \Phi}{\partial P_i}\left(\frac{\partial F}{\partial x_i} + P_i\frac{\partial F}{\partial z}\right)\right], \end{aligned}\right.$$

in which f denotes what F becomes and φ denotes what Φ becomes under substitution of the values of Z, X_i, P_k .

One applies that identity to all of the pairs of functions Z, X_i, P_k . One then has:

$$\tag{25} \left\{\begin{aligned} &(Z, X_i) = (X_i, X_k) = (X_i, P_k) = (P_i, P_k) = 0, \\ &\qquad (Z, P_i) = -\rho P_i, \qquad (P_i, X_i) = \rho. \end{aligned}\right.$$

91. *Conversely, given $n + 1$ independent functions Z, X_1, X_2, …, X_n that satisfy the relations:*

$$(Z, X_i) = (X_i, X_k) = 0,$$

there exist n other functions P_1, P_2, …, P_n such that one has:

$$dZ - P_1\, dX_1 - \dots - P_n\, dX_n = \rho\,(dz - p_1\, dx_1 - \dots - p_n\, dx_n),$$

ρ being a function that is not identically zero.

Indeed, the equations that are obtained by equating $Z, X_1, \ldots, X_n$ to arbitrary constants form a system in involution of $n + 1$ equations; i.e., they determine a multiplicity. One can thus determine $n + 1$ functions λ such that:

$$dz - p_1\, dx_1 - \ldots - p_n\, dx_n = \lambda_1\, dX_1 + \lambda_2\, dX_2 + \ldots + \lambda_n\, dX_n + \lambda_{n+1}\, dZ\,,$$

so one deduces the identity to be proved by setting:

$$P_i = -\frac{\lambda_i}{\lambda_{n+1}}\,, \qquad \rho = \frac{1}{\lambda_{n+1}}\,.$$

92. One can, moreover, obtain the brackets of ρ and the functions Z, X_i, P_k. Indeed, one has, while preserving the same notations:

$$\Omega^{(2n-1)}\, dF = (\rho\omega)^{(2n-1)}\, df,$$

i.e.:

$$\Omega^{(2n-1)}\, dF = \rho^n\, \omega^{(2n-1)}\, df - \rho^{n-1}\, \omega^{(2n-2)}\, d\rho\, df.$$

However, one has:

$$\Omega^{(2n-1)}\, dF = \frac{\partial F}{\partial Z}\, dZ\, dX_1\, dP_1 \ldots dX_n\, dP_n$$

$$= \rho^{n+1} \frac{\partial F}{\partial Z}\, dz\, dx_1\, dp_1 \ldots dx_n\, dp_n\,,$$

so

$$\omega^{(2n-1)} df = \frac{\partial f}{\partial z}\, dz\, dx_1\, dp_1 \ldots dx_n\, dp_n\,,$$

$$\omega^{(2n-2)}\, d\rho\, df = -(\rho, f)\, dz\, dx_1\, dp_1 \ldots dx_n\, dp_n\,,$$

Finally, upon dividing by ρ^{n-1}, one thus has the identity:

$$\rho^2 \frac{\partial F}{\partial Z} = \rho \frac{\partial f}{\partial z} + (\rho, f),$$

i.e.:

(26) $$(\rho, f) = \rho^2 \frac{\partial F}{\partial Z} - \rho \frac{\partial f}{\partial z}\,.$$

Applying this identity to the functions Z, X_i, P_k, one obtains:

(26)′
$$\begin{cases} (\rho, Z) = \rho^2 - \rho \dfrac{\partial Z}{\partial z}, \\ (\rho, X_i) = -\rho \dfrac{\partial X_i}{\partial z}, \\ (\rho, P_k) = -\rho \dfrac{\partial P_k}{\partial z}. \end{cases}$$

93. *Homogeneous partial differential equations*. – Given $2n$ variables $x_1, \ldots, x_n$; $p_1, \ldots, p_n$, one satisfies the total differential equation:

(27)
$$\omega = p_1\, dx_1 + p_2\, dx_2 + \ldots + p_n\, dx_n = 0$$

by means of relations between the variables and derived relations between the differentials, upon being given a certain number h of these relations *a priori*.

The expression ω has class $2n$, and its $(2n - 1)^{\text{th}}$ derivative is:

$$\omega^{(2n-1)} = dp_1\, dx_1\, dp_2\, dx_2 \ldots dp_n\, dx_n .$$

One must thus have at least n relations between the variables in order to satisfy (27), and the singular solutions of (27) are obtained, since $\omega^{(2n-1)}$ has its coefficients essentially non-zero upon annulling the coefficients of ω, i.e., by annulling $p_1, p_2, \ldots, p_n$ at the same time.

Since a *multiplicity* is a system of relations that satisfy (27), it will be called *non-singular* if it does not entail that:

$$p_1 = p_2 = \ldots = p_n = 0.$$

It is at most n-dimensional, and one obtains it in the most general fashion if, for example, p_1 is non-zero, by solving:

$$dx_1 + \frac{p_2}{p_1} dx_2 + \ldots + \frac{p_n}{p_1} dx_n = 0.$$

Given a system of h relations:

(28)
$$\begin{cases} f_1(x_1, x_2, \ldots, p_1, \ldots, p_n) = 0, \\ f_2(x_1, x_2, \ldots, p_1, \ldots, p_n) = 0, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \\ f_h(x_1, x_2, \ldots, p_1, \ldots, p_n) = 0, \end{cases}$$

one integrates that system, and then one must find all of the n-dimensional multiplicities whose elements satisfy these relations. One must therefore annul ω by means of n relations, among which, are h given relations.

94. Here, if $\omega^{(2n-1)}$ does not have all of its coefficients equal to zero then we can apply the theorem of the preceding chapter. In order to apply it, one must form the expressions:

$$\omega^{(2n-2)}\,df, \qquad \omega^{(2n-3)}\,df\,d\varphi.$$

One easily has:

$$\omega^{(2n-2)}\,df = -\left(p_1\frac{\partial f}{\partial p_1} + p_2\frac{\partial f}{\partial p_2} + \cdots + p_n\frac{\partial f}{\partial p_n}\right) dp_1\,dx_1\,dp_2\,dx_2\,\ldots\,dp_n\,dx_n\,,$$

$$\omega^{(2n-3)}\,df\,d\varphi = \sum_{i=1}^{n}\left(\frac{\partial f}{\partial p_i}\frac{\partial \varphi}{\partial x_i} - \frac{\partial f}{\partial x_i}\frac{\partial \varphi}{\partial p_i}\right) dp_1\,dx_1\,dp_2\,dx_2\,\ldots\,dp_n\,dx_n\,.$$

We set:

$$H(f) = p_1\frac{\partial f}{\partial p_1} + p_2\frac{\partial f}{\partial p_2} + \cdots + p_n\frac{\partial f}{\partial p_n}\,, \tag{29}$$

$$(f, \varphi) = \sum_{i=1}^{n}\left(\frac{\partial f}{\partial p_i}\frac{\partial \varphi}{\partial x_i} - \frac{\partial f}{\partial x_i}\frac{\partial \varphi}{\partial p_i}\right). \tag{30}$$

This being done, (28) will be said to be in involution if the equations:

$$H(f_i) = 0, \qquad (f_i, f_k) = 0 \qquad (i, k = 1, 2, \ldots, h) \tag{31}$$

are consequences of this system. The first equations express the idea that the system (28) is *homogeneous* in $p_1, p_2, \ldots, p_n$ – i.e., that it is equivalent to the system that one obtains by replacing $p_1, p_2, \ldots, p_n$ with $\lambda p_1, \lambda p_2, \ldots, \lambda p_n$ – or furthermore, that it can be put into a form such that the left-hand sides are all homogeneous in $p_1, p_2, \ldots, p_n$.

As a result, if the system (28) is not in involution then we append equations (31) to it. We have a new system that, if it is not in involution, can be extended by the same procedure, and so on, until one arrives at a system in involution (at least, one arrives at either an incompatible system or a system of more than n equations in the sequence of calculations).

95. Therefore, suppose that the system (28) is in involution. One will get its general integral by seeking an integral f_{h+1} of the complete system:

$$H(f) = 0, \qquad (f_1, f) = \ldots = (f_h, f) = 0,$$

and then an integral f_{h+2} of the complete system:

$$H(f) = 0, \qquad (f_1, f) = (f_2, f) = \ldots = (f_{h+1}, f) = 0$$

that is independent of f_{h+1}, and so on, until one has an integral f_n of the complete system:

$$H(f) = 0, \qquad (f_1, f) = (f_2, f) = \ldots = (f_{n-1}, f) = 0$$

that is independent of $f_{h+1}, f_{h+2}, \ldots, f_{n-1}$.

Upon taking (28) into account, ω can then be put into the form:

$$\omega = \varphi_{h+1}\, df_{h+1} + \ldots + \varphi_n\, df_n ,$$

and the solution is achieved as usual.

In particular, a system of n equations in involution provides a multiplicity.

96. The *singular integrals* are obtained by annulling all of the coefficients of the expression:

$$\omega\, df_1\, df_2 \ldots df_n ;$$

i.e., all of the determinants with $h + 1$ rows and $h + 1$ columns in the matrix:

$$(32) \qquad \left\| \begin{matrix} p_1 & p_2 & \cdots & p_n & 0 & \cdots & 0 \\ \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} & \frac{\partial f_1}{\partial p_1} & \cdots & \frac{\partial f_1}{\partial p_n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{\partial f_h}{\partial x_1} & \frac{\partial f_h}{\partial x_2} & \cdots & \frac{\partial f_h}{\partial x_n} & \frac{\partial f_h}{\partial p_1} & \cdots & \frac{\partial f_h}{\partial p_n} \end{matrix} \right\| ,$$

and if all of the determinants that are formed from any of the last h rows and h columns are not zero then the system obtained constitutes a singular integral if it contains only n independent equations. In the contrary case, one has a system that one treats as an ordinary system.

In particular, if h is equal to 1 – i.e., if one has an equation that is homogeneous in p_1, $p_2, \ldots, p_n$:

$$(33) \qquad f_1(x_1, \ldots, x_n, p_1, \ldots, p_n) = 0,$$

then the singular integrals satisfy the equations:

$$(34) \qquad \begin{cases} \dfrac{\partial f_1}{\partial p_1} = \dfrac{\partial f_1}{\partial p_2} = \cdots = \dfrac{\partial f_1}{\partial p_n} = 0, \\[2ex] \dfrac{\frac{\partial f_1}{\partial x_1}}{p_1} = \dfrac{\frac{\partial f_1}{\partial x_2}}{p_2} = \cdots = \dfrac{\frac{\partial f_1}{\partial x_n}}{p_n}, \end{cases}$$

and if the ratios in the second row are not zero then equations (33) and (34) furnish the singular integral in the case where they reduce to n. One can, moreover, limit oneself to equations (34), because (33) is a consequence of it by virtue of:

$$H(f_1) = 0.$$

97. *Example*. – In the case of $n = 2$, consider the equation:

(35) $$f_1 = p_1^2 + p_2^2 - (p_1 x_1 + p_2 x_2)^2 = 0.$$

Here, equations (34) become:

$$p_1 - x_1 (p_1 x_1 + p_2 x_2) = 0,$$
$$p_2 - x_2 (p_1 x_1 + p_2 x_2) = 0,$$

$$\frac{-p_1(p_1 x_1 + p_2 x_2)}{p_1} = \frac{-p_2(p_1 x_1 + p_2 x_2)}{p_2}.$$

Since the quantities p_1 and p_2 are assumed to not both be zero, the last two ratios are equal to each other, and the three equations that determine the singular integral reduce to two of them:

$$p_1 = x_1 (p_1 x_1 + p_2 x_2),$$
$$p_2 = x_2 (p_1 x_1 + p_2 x_2).$$

Moreover, by eliminating p_1 and p_2 these equations entail that:

$$x_1^2 + x_2^2 - 1 = 0.$$

98. *Homogeneous contact transformations*. – a homogeneous contact transformation is defined by $2n$ functions $X_1, X_2, \ldots, X_n$; $P_1, P_2, \ldots, P_n$ in $2n$ variables $x_1, x_2, \ldots, x_n$; $p_1, p_2, \ldots, p_n$ that imply the identity:

(36) $$P_1\, dX_1 + P_2\, dX_2 + \ldots + P_n\, dX_n = p_1\, dx_1 + p_2\, dx_2 + \ldots + p_n\, dx_n .$$

If one denotes the left-hand side of that identity by Ω then one first has:

$$\Omega^{(2n-1)} = \omega^{(2n-1)},$$

i.e.:

(37) $$dP_1\, dX_1\, dP_2\, dX_2 \ldots dP_n\, dX_n = dp_1\, dx_1\, dp_2\, dx_2 \ldots dp_n\, dx_n ,$$

which shows that the $2n$ functions X_i, P_k are independent, and that their functional determinant is equal to unity.

If f denotes what an arbitrary function of X_i, P_k becomes when one replaces these quantities with their values then one has, in turn, that:

$$\Omega^{(2n-2)}\, df = \omega^{(2n-2)}\, df ;$$

i.e., upon taking (37) into account:

(38) $$P_1\frac{\partial F}{\partial P_1}+P_2\frac{\partial F}{\partial P_2}+\cdots+P_n\frac{\partial F}{\partial P_n} = p_1\frac{\partial f}{\partial p_1}+p_2\frac{\partial f}{\partial p_2}+\cdots+p_n\frac{\partial f}{\partial p_n}.$$

This identity, when applied to the functions X_i, P_k, gives:

(38)′ $$\begin{cases} H(X_i)=0, \\ H(P_i)=0, \end{cases}$$

which shows that *the X are homogeneous functions of degree zero in* $p_1, p_2, \ldots, p_n$ *, and the P are homogeneous functions of degree one.*

If F and Φ denote two arbitrary functions of the big symbols and f and φ denote the functions of the small symbols that they become after substitution then one finally has:

$$\Omega^{(2n-2)}\, dF\, d\Phi = \omega^{(2n-2)}\, df\, d\varphi;$$

i.e., upon taking (37) into account:

(39) $$\sum_{i=1}^{n}\left(\frac{\partial F}{\partial P_i}\frac{\partial \Phi}{\partial X_i}-\frac{\partial F}{\partial X_i}\frac{\partial \Phi}{\partial P_i}\right) = \sum_{i=1}^{n}\left(\frac{\partial f}{\partial p_i}\frac{\partial \varphi}{\partial x_i}-\frac{\partial f}{\partial x_i}\frac{\partial \varphi}{\partial p_i}\right).$$

When this identity is applied to the functions X_i, P_k , that gives:

(39)′ $$\begin{cases} (X_i, X_k) = (P_i, P_k) = (P_i, X_k) = 0, \\ \qquad\qquad\qquad\qquad (P_i, X_i) = 1 \qquad (i \neq k). \end{cases}$$

The $2n$ functions X_i, P_k therefore satisfy equations (38)′ and (39)′.

99. *Conversely, if one given n independent functions* $X_1, X_2, \ldots, X_n$ *of* $x_1, \ldots, x_n$, $p_1, p_2, \ldots, p_n$ *that satisfy the relations:*

$$(X_i, X_k) = 0 \qquad (i, k = 1, 2, \ldots, n)$$

then there exist n other functions $P_1, P_2, \ldots, P_n$ *that define a homogeneous contact transformation, along with the first ones.*

This is obvious, because, by hypothesis, the n functions X_i , when equated to constants, define a system in involution, in such a way that one can determine n quantities $P_1, P_2, \ldots, P_n$ in such a manner that one has:

$$p_1\, dx_1 + p_2\, dx_2 + \ldots + p_n\, dx_n = P_1\, dX_1 + P_2\, dX_2 + \ldots + P_n\, dX_n .$$

100. *Partial differential equations in homogeneous coordinates.* – Given $2n$ variables $x_1, x_2, \ldots, x_n$; $u_1, u_2, \ldots, u_n$ that are coupled by the relation:

(40) $$\varphi = p_1 x_1 + p_2 x_2 + \ldots + p_n x_n = 0,$$

one solves the total differential equation:

(41) $$\omega = u_1 dx_1 + u_2 dx_2 + \ldots + u_n dx_n$$

by establishing a certain number of relations between the variables, among which, a certain number are given.

If one refers to the previous problem then one sees that one must establish at least $n - 1$ relations besides (40). The singular solutions of the systems are given, moreover, by annulling all of the coefficients of $\omega\, d\varphi$ – i.e., all of the quantities $u_i\, x_k$. One thus has either:

$$u_1 = u_2 = \ldots = u_n = 0$$

or

$$x_1 = x_2 = \ldots = x_n = 0.$$

We thus exclude these two singular solutions.

We have to form (φ, f), where φ is given by (40). One has:

$$(\varphi, f) = x_1 \frac{\partial f}{\partial x_1} + \cdots + x_n \frac{\partial f}{\partial x_n} - \left(u_1 \frac{\partial f}{\partial u_1} + \cdots + u_n \frac{\partial f}{\partial u_n} \right).$$

We set:

(42) $$\begin{cases} H(f) = u_1 \dfrac{\partial f}{\partial u_1} + \cdots + u_n \dfrac{\partial f}{\partial u_n}, \\ K(f) = x_1 \dfrac{\partial f}{\partial x_1} + \cdots + x_n \dfrac{\partial f}{\partial x_n}, \end{cases}$$

and

(43) $$(f, f_1) = \sum_{i=1}^{n} \left(\frac{\partial f}{\partial u_i} \frac{\partial f_1}{\partial x_i} - \frac{\partial f}{\partial x_i} \frac{\partial f_1}{\partial u_i} \right).$$

101. Given a system of h relations:

(44) $$\begin{cases} f_1(x_1 \ldots, x_n, u_1, \cdots, u_n) = 0, \\ f_2(x_1 \ldots, x_n, u_1, \cdots, u_n) = 0, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \\ f_h(x_1 \ldots, x_n, u_1, \cdots, u_n) = 0, \end{cases}$$

this system is in involution if the equations:

(45) $$\begin{cases} H(f_i) = K(f_i) = 0, \\ (f_i, f_k) = 0 \qquad (i, k = 1, 2, \ldots, h) \end{cases}$$

are consequences of (40) and (44). Any system can be converted into a system in involution. A system of $n - 1$ equations in involution constitutes a solution to equation (41).

If h is less than $n - 1$ then one integrates (44) by seeking an integral f_{h+1} of the complete system:

$$H(f) = K(f) = 0, \qquad (f_1, f) = (f_2, f) = \ldots = (f_h, f) = 0,$$

and so on, up to an integral f_{n-1} of the complete system:

$$H(f) = K(f) = 0, \qquad (f_1, f) = (f_2, f) = \ldots = (f_{n-2}, f) = 0.$$

One then has, upon taking (40) and (44) into account, that:

$$\omega = \varphi_{h+1}\, df_{h+1} + \varphi_{h+2}\, df_{h+2} + \ldots + \varphi_{n-1}\, df_{n-1}\,.$$

102. The *singular integrals* are obtained by annulling all of the coefficients of the expression:

$$\omega\, d\varphi\, df_1 \ldots df_h\,;$$

i.e., upon annulling all of the determinants with $h + 2$ rows and $h + 2$ columns in the matrix:

$$(46) \qquad \left\| \begin{matrix} u_1 & u_2 & \cdots & u_n & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & x_1 & x_2 & \cdots & x_n \\ \dfrac{\partial f_1}{\partial x_1} & \dfrac{\partial f_1}{\partial x_2} & \cdots & \dfrac{\partial f_1}{\partial x_n} & \dfrac{\partial f_1}{\partial u_1} & \dfrac{\partial f_1}{\partial u_2} & \cdots & \dfrac{\partial f_1}{\partial u_n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \dfrac{\partial f_h}{\partial x_1} & \dfrac{\partial f_h}{\partial x_2} & \cdots & \dfrac{\partial f_h}{\partial x_n} & \dfrac{\partial f_h}{\partial u_1} & \dfrac{\partial f_h}{\partial u_2} & \cdots & \dfrac{\partial f_h}{\partial u_n} \end{matrix} \right\| .$$

In the case where h equals 1, the singular integrals are given by the equations:

$$\frac{\dfrac{\partial f}{\partial x_1}}{u_1} = \frac{\dfrac{\partial f}{\partial x_2}}{u_2} = \ldots = \frac{\dfrac{\partial f}{\partial x_n}}{u_n},$$

$$\frac{\dfrac{\partial f}{\partial u_1}}{x_1} = \frac{\dfrac{\partial f}{\partial u_2}}{x_2} = \ldots = \frac{\dfrac{\partial f}{\partial u_n}}{x_n},$$

and if these ratios are not all equal to each other then these equations define the singular integral in the case where they reduce to just n of them.

103. *Particular case*. – When n is equal to 3, one obtains ordinary differential equations in the two variables x and y. Indeed, if we denote the homogeneous coordinates of a point by x_1, x_2, x_3 and the homogeneous coordinates of a line in the plane by u_1, u_2, u_3 then one has:

$$u_1\, x_1 + u_2\, x_2 + u_3\, x_3 = 0$$

for an element, and:

$$\frac{x_1}{x} = \frac{x_2}{y} = \frac{x_3}{1}, \qquad \frac{u_1}{y'} = \frac{u_1}{-1} = \frac{u_1}{y - xy'}.$$

In order to integrate an equation:

$$F(x, y, y') = 0,$$

i.e.:

$$F\left(\frac{x_1}{x_3}, \frac{x_2}{x_3}, -\frac{u_1}{u_2}\right) = 0,$$

it is necessary to integrate the complete system:

$$H(f) = K(f) = 0, \qquad (F, f) = 0,$$

i.e., to find an integral f of the system of differential equations:

$$\frac{dx_1}{\frac{\partial F}{\partial u_1}} = \frac{dx_2}{\frac{\partial F}{\partial u_2}} = \frac{-du_1}{\frac{\partial F}{\partial x_1}} = \frac{-du_2}{\frac{\partial F}{\partial x_2}} = \frac{-du_3}{\frac{\partial F}{\partial x_3}}$$

that is homogeneous and of degree zero in x_1, x_2, x_3, on the one hand, and in u_1, u_2, u_3 , on the other.
