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The principle of duality and the theory of simple and semi-simple groups

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1. – In a recent paper, Weinstein ([1]) posed and solved the following problem:

Make an arbitrary matrix X of order n correspond to another matrix X′ of order n according to some rule, in such a way that the product XY of two arbitrary matrices will correspond to the product X′Y′ of the two corresponding matrices.

If one confines oneself to the case in which the n^2 elements of X' are *independent* functions of the n^2 elements of X, and in which the matrices in question have determinants that are equal to 1, then the problem will include two solutions, which are given by the formulas:

$$X' = A^{-1} X A, \tag{1}$$

$$X' = A^{-1} \bar{X}^{-1} A, \tag{2}$$

respectively, in which A denotes an arbitrary, but fixed, matrix, and $\bar{X}$ denotes the matrix that is deduced from X by switching the rows and columns.

If one remarks that the n^2 of a matrix X are the parameters of a linear substitution then Weinstein's problem amounts to finding all parameter transformations that will leave invariant the law of composition of the substitutions of the linear group. A substitution with determinant 1 can be regarded as defining a homography in n – 1-dimensional space, so formulas (1) and (2) will provide all of the transformations that can be performed on the homographies of the projective space that preserve the *structure* of that space. Formulas (1) and (2) show that *the structure of projective space is invariant under the mixed group of homographies* [form. (1)] *and correlations* [form. (2)]. One sees that, from that viewpoint, *the principle of duality* is introduced into projective geometry out of necessity ([2]).

The Weinstein problem is a particular case of the following general problem:

([1]) Math. Zeit. **16** (1923), 78-91.

([2]) One might even say that the structure of ordinary Euclidian space is invariant under the mixed group of displacements and symmetries; it is also invariant under the group of similitudes.

Given a continuous group G with r parameters a_1 , …, a_r , find all of the transformations that leave invariant the law of composition for the transformations of the group when they are performed on those parameters, or rather, *they leave invariant the structure of the group.*

If one lets S_a denote any of the transformations of the group G then one will have a first solution to the problem in the formula:

$$S_{\xi'} = S_a^{-1} S_\xi \, S_a , \tag{3}$$

in which the a are fixed. It makes the transformation $S_{\xi'}$ correspond to the variable transformation S_ξ while obviously respecting the law of composition of the transformations of the group. The transformations of the parameters that are defined by equation (3) generate the *adjoint group* Γ to G, in the sense of S. Lie.

It can happen that the transformations of the adjoint group are the only ones that leave the structure of the group invariant. However, opposite case can also present itself: In that case, the most general transformations will form a group Γ' that contains the adjoint group Γ as an invariant subgroup. If one indeed performs the *same* transformation T of Γ' on the a, ξ, and ξ' in formula (3) then the transformations S_a , S_ξ , $S_{\xi'}$ will change into S_b , S_η , $S_{\eta'}$, and one will have the relation:

$$S_{\eta'} = S_b^{-1} S_\eta \, S_b ,$$

which indeed shows that the transformation T leaves the group Γ invariant. (The transformation of the parameter a of Γ is simply changed into the transformation of the parameters b.)

Any transformation T of Γ' will be determined theoretically if one knows the effect that it produces on the *infinitesimal* transformations of G. Indeed, it is obvious that the identity transformation of G is preserved by T and that every infinitesimal transformation of G will be, in turn, changed into another infinitesimal transformation by T. Having said that, let:

$$X_1, \qquad X_2, \qquad \dots, \qquad X_r$$

be the symbols of r independent infinitesimal transformations of G with the *structure relations:*

$$(X_i \, X_j) = \sum_s c_{ijs} \, X_s \qquad (i, j = 1, \dots, r) . \tag{4}$$

Under the transformation T, the transformations X_i will submit to a linear substitution:

$$X_i' = \sum_k \alpha_{ik} \, X_k \qquad (i = 1, \dots, r) . \tag{5}$$

The coefficients α_{ik} are constrained to satisfy the algebraic relations:

(6) $$\sum_{k,l} \alpha_{ik}\, \alpha_{jl}\, c_{kls} = \sum_{l} c_{ijl}\, \alpha_{ls} \qquad (i, j, s = 1, \ldots, r),$$

which expresses the idea that the substitution (5) preserves the relations (4).

Formulas (6) collectively define the group Γ' then.

2. – The problem is particularly interesting in the case where the group G is simple or semi-simple. In that case, in effect, every *infinitesimal* transformation of Γ' will leave invariant the continuous adjoint group Γ, which is simple or semi-simple and belongs to the adjoint group itself ([1]). Therefore, if the group Γ' does not coincide with Γ then it will be composed of several discrete continuous families of transformation (a *mixed* group, according to S. Lie), only one of which forms a group (namely, the adjoint group). That is what happens in the example that was cited above, in which one has two families that are defined by (1) and (2).

One knows that any *general* infinitesimal transformation Y ([2]) of G belongs to an Abelian subgroup γ whose order is equal to the rank l of the group and which is defined by the set of infinitesimal transformations that commute with Y. That subgroup is not invariant under any subgroup that is greater than G. The subgroups γ depend upon essentially $r - l$ parameters, since Y depends upon r parameters and ∞^l distinct transformations Y will give the same subgroup γ. On the other hand, when one performs the ∞^r transformations of the adjoint group on a subgroup γ, one will obviously get ∞^{r-l} subgroups γ, since there exist ∞^l transformations of the adjoint group that leave γ invariant. One must then presume that the various subgroups γ are all homologous to each other with respect to the adjoint group. However, the preceding argument is not sufficient to prove that, because the various subgroups γ can form several distinct (but not mutually homologous) families *a priori*, each of which nonetheless has dimension $r - l$. *We shall see that the latter possibility cannot present itself.*

Indeed, any subgroup γ is composed of r transformations $\sum e_i X_i$ that are defined by r linear equations (only $r - l$ of which are independent):

$$\sum_{i,k} a_i\, e_k\, c_{iks} = 0 \qquad (s = 1, \ldots, r),$$

in which the arbitrary *parameters* a_i are subject to the single condition that they must not annul a certain integer algebraic polynomial $\psi^{r-l}(a_1, \ldots, a_r)$. It then results that *in the complex domain*, one can always pass from an arbitrary subgroup γ to another subgroup γ *by continuity*. The subgroups γ then form *only one connected domain*. If the hypothesis that was posed is not satisfied exactly then one can find a subgroup γ_0 that will not be homologous with all of the infinitely-close subgroups γ. Now, that contradicts the

([1]) E. CARTAN, *Thése*, Paris, Nony, 1894, pp. 113.

([2]) That means that its *characteristic equation* admits the minimum number (namely, l) of zero roots. The characteristic equation of Y is the one to which one will be led upon seeking the values of λ for which there exists an infinitesimal transformation Z such that one has $(Y\, Z) = \lambda\, Z$.

transitivity of the adjoint group in the infinitesimal domain when one considers it to operate on the subgroups γ.

Having said that, let Y be a general transformation of the group G, and let Y' be the transformation that one deduces from it by a given transformation T of Γ'. Let γ and γ' be two Abelian subgroups that correspond to Y and Y', resp. Finally, let Θ be a transformation of the adjoint group that transforms γ' into γ. *The transformation* $T\,\Theta^{-1}$ *of* Γ' *will leave the subgroup* γ *fixed.*

It will then suffice to determine all of the transformations T that leave invariant a subgroup γ that is fixed once and for all and to then multiply it by an arbitrary transformation of the adjoint group. On the other hand, if two transformations T and T' leave invariant the subgroup γ when it is transformed in the same manner as the infinitesimal transformations of γ then the transformation $T'T^{-1}$ will leave invariant each of those transformations, and one will easily prove that it belongs to the adjoint group then.

Finally, *everything comes down to seeking the transformations of* Γ' *that leave invariant the subgroup* γ *and studying the manner by which they transform the transformations of* γ *amongst themselves.*

3. – Recall that the roots of the characteristic equation of an arbitrary transformation of γ are $r - l$ linear forms $\omega_1, \ldots, \omega_{r-l}$ in $e_1, \ldots, e_l$ and that $r - 2l$ of them are linear combinations with well-defined *integer* coefficients of the other l (which are called *fundamental*). Each of the $r - l$ roots ω_α is *associated* with a well-defined transformation Y_α of the subgroup γ. *Any transformation T will have the effect of performing a substitution* (1) *on the $r - l$ transformations Y_α that are associated with the $r - l$ roots ω_α, and as a result, also on those roots themselves, subject to only the condition that they must leave invariant the linear relations with integer coefficients that couple those $r - l$ roots.* The set of all those substitutions forms a finite group of $r - l$ letters that we shall call $\mathcal{G}$ (2).

It can happen that some of the substitutions of $\mathcal{G}$ are derived from transformations of the adjoint group. Here is how that can happen:

Each root ω_α is associated with a group with three parameters g_α in G that is generated by the infinitesimal transformation Y_α of γ that is associated with ω_α, and with two other transformations (that do not belong to γ) X_α and X'_α. In addition to the identity transformation, the corresponding subgroup of the adjoint group contains another transformation that leaves invariant the subgroup γ, and that transformation will change the transformation Y_λ that is associated with a roots ω_λ into the transformation $Y_\mu = Y_\lambda + a_{\alpha\lambda}\,Y_\alpha$ that is associated with the roots $\omega_\lambda + a_{\alpha\lambda}\,\omega_\alpha$, in which the $a_{\alpha\lambda}$ are well-defined integers (3). The root ω_α then corresponds to a certain substitution Θ_α that acts upon the r

(1) It amounts to a substitution of $r - l$ objects, each of which is replaced by another.

(2) I studied that group in a paper that was entitled: "Sur la réduction à sa forme canonique de la structure d'un groupe de transformations fini et continu, Amer. J. Math. **18** (1896), 1-61.

(3) E. CARTAN, *Thése*, pp. 57.

– l transformations Y_λ and belongs to the adjoint group. Those substitutions Θ_α generate (by themselves and their products) a finite group $\mathcal{G}'$ that is an (invariant) subgroup of $\mathcal{G}$.

If the group $\mathcal{G}'$ is identical to $\mathcal{G}$ then the group Γ' will coincide with the continuous adjoint group Γ. If $\mathcal{G}'$ is a subgroup of index h in $\mathcal{G}$ then the group Γ' will be composed of h discrete continuous families of transformations, and one will immediately have a particular transformation from each of those families.

4. – The comparative study of the two finite groups $\mathcal{G}$ and $\mathcal{G}'$ presents no difficulty in the case where the group $\mathcal{G}$ is simple. The two groups $\mathcal{G}$ and $\mathcal{G}'$ are distinct only for three types of simple groups, namely:

The type A of rank $l \geq 2$,
The type D of rank $l \geq 4$,
The type E of rank $l = 6$.

In the case of type A (viz., the projective group in l variables), the group $\mathcal{G}$ has order $2\,(l + 1)!$, while the group $\mathcal{G}'$ has order only $(l + 1)!$. Here, one then has $h = 2$. The two families of transformation of Γ' are obtained, in the former case, by transforming a transformation of G by a fixed *homography*, and in the latter case, by transforming it with a fixed *correlation*. One sees the role of the *principle of duality* in projective geometry here.

In the case of type D with rank $l > 4$, the group $\mathcal{G}$ has order $2^l\, l\,!$ When the roots ω_α are put into the form:

$$\pm\,\omega_i \pm \omega_j \qquad (i, j = 1, \ldots, l),$$

one will get a substitution of $\mathcal{G}$ upon performing an arbitrary permutation of the indices 1, …, l and changing the signs of an arbitrary number of quantities ω_i. The order of the group $\mathcal{G}'$ is only $2^{l-1}\, l\,!$ It is composed of substitutions of $\mathcal{G}$ that correspond to an *even* number of sign changes.

If one takes G to be the linear group of a non-degenerate quadratic form in $2l$ variables then the group Γ' will be composed of two families that are obtained by transforming either an orthogonal transformation with a determinant equal to + 1 (in the adjoint group) or an orthogonal transformation with a determinant equal to – 1.

In the case of type E with rank 6, the group $\mathcal{G}'$ is isomorphic to the Galois group of the equation that gives the 27 roots of a third-order surface; it contains:

$$27 \times 16 \times 10 \times 6 \times 2$$

substitutions. The order of the group $\mathcal{G}$ is twice that ([1]). If one takes G to be the linear group of the cubic form:

$$J = \sum x_i\, y_k\, z_{ik} - \sum z_{\alpha\beta}\, z_{\gamma\beta}\, z_{\lambda\mu}$$

then the group Γ' will be obtained by transforming the transformations of G by either a homography or a correlation that leaves the variety $J = 0$ invariant.

5. – All that remains is the case of type D with rank 4 (viz., the linear group for a non-degenerate quadratic form in eight variables). That case is the most interesting one. Here, as in the general case, the order of the group $\mathcal{G}'$ is three times greater than in the general case, so it will have order $3 \times 2^4 \times 4!$ The index h is equal to 6. The peculiarity that presents itself here comes from the fact that one can replace the quantities ω_i that enter into the general expression for the roots $\pm\ \omega_i\ \pm\ \omega_j$ with expressions of the form $\frac{1}{2}(\pm\omega_1 \pm \omega_2 \pm \omega_3 \pm \omega_4)$.

Suppose that the quadratic form that is invariant under G is reduced to the canonical form:

$$x_0^2 + x_1^2 + \cdots + x_7^2 .$$

Any infinitesimal transformation of G will have the form:

$$\sum_{i,j} a_{ij} \left(x_i \frac{\partial f}{\partial x_j} - x_j \frac{\partial f}{\partial x_i} \right) \qquad (a_{ij} = -\, a_{ji}) .$$

Having said that, let i be any of the indices 1, 2, …, 7 and consider the seven sets of four components:

$$a_{0,\,i}\,, \quad a_{i+1,\,i+3}\,, \qquad a_{i+4,\,i+6}\,, \qquad a_{i+2,\,i+3}\,,$$

in which one supposes that any index that is greater than 7 will be identical to the same index, minus 7. Consider the substitutions:

$$H = \begin{vmatrix} -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{vmatrix}$$

and

([1]) See E. CARTAN, *loc. cit.*, Amer. J. Math., pp. 35-43.

$$K = \begin{vmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix}.$$

They generate a finite group of order 6 that contains, not only H, K, and the identity substitution, but also the substitutions:

$$H^2 = \begin{vmatrix} -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{vmatrix}$$

$$HK = KH^2 = \begin{vmatrix} \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{vmatrix},$$

$$H^2K = KH = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{vmatrix}.$$

Any transformation of Γ' is obtained by performing one of the six preceding substitutions on the quantities:

$$a_{0,\,i}\,, \quad a_{i+1,\,i+3}\,, \qquad a_{i+4,\,i+6}\,, \qquad a_{i+2,\,i+3}$$

for each value 1, 2, …, 7 of the index i and then performing a transformation from the adjoint group.

The preceding results relate to a system of complex numbers that was invented by Graves and Cayley, which generalize the quaternions and which one calls the *octaves*. Consider seven *units* e_α (α = 1, …, 7) that satisfy the laws of multiplication:

$$e_\alpha^2 = -1,$$

$$e_\alpha = e_{\alpha+1}\, e_{\alpha+5} = -\, e_{\alpha+5}\, e_{\alpha+1} = e_{\alpha+4}\, e_{\alpha+6}\,,$$

$$= -\, e_{\alpha+6}\, e_{\alpha+4} = e_{\alpha+2}\, e_{\alpha+3} = -\, e_{\alpha+3}\, e_{\alpha+2}\,.$$

An octave is a complex number of the form:

$$X = x_0 + x_1\, e_1 + x_2\, e_2 + \ldots + x_7\, e_7\ ;$$

the product $Z = \sum z_i\, e_i$ of two octaves:

$$X = \sum x_i\, e_i\ , \quad Y = \sum y_i\, e_i$$

enjoys the property that is expressed by the formula:

$$z_0^2 + z_1^2 + \cdots + z_7^2 = (x_0^2 + x_1^2 + \cdots + x_7^2)(y_0^2 + y_1^2 + \cdots + y_7^2)\ .$$

Having said that, perform a given orthogonal substitution A (with determinant equal to 1) on the components x_i of the octave. It will be possible to determine an orthogonal substitution B such that for any two octaves X and Y, the product AX, BY will be deduced from the product XY by a suitably-chosen orthogonal substitution C. The passage from the substitution A to the substitution B is given by a transformation of Γ' that is nothing but the one that is provided by the substitution H^2 ([1]), and the passage from A to C is provided by the substitution $H^2 K$.

There are an infinitude of transformations of G that are invariant under the substitutions H and K. All of those transformations leave the variable x_0 invariant and transform the other seven variables according to a simple group with fourteen parameters of type G.

6. – One can also present the preceding results in a geometric form that differs only in appearance from the preceding one. One can take the group G to be the conformal group of six-dimensional space. Suppose that the ds^2 of that space reduces to the form:

$$dx_1\, dx_4 + dx_2\, dx_5 + dx_3\, dx_6\ .$$

There exist two families of three-dimensional planar varieties that are *totally isotropic* – i.e., ones that enjoy the property that two arbitrary points of such a variety that are along the same isotropic line will be contained entirely within the variety. The general equations of the varieties V' of the first family will be:

$$\text{(7)} \qquad \begin{cases} x_4 - a_4 + a_2\, x_3 - a_3\, x_2 = 0, \\ x_5 - a_5 + a_3\, x_1 - a_1\, x_3 = 0, \\ x_6 - a_6 + a_1\, x_2 - a_2\, x_1 = 0, \end{cases}$$

while those of the varieties V'' of the second family are:

([1]) When that transformation of Γ' is applied to an *infinitesimal* transformation of G, it will be uniform. When it is applied to a finite transformation A, it will give two transformations B and – B (which will be identical, moreover, if one regards then as *projective* transformation on seven-dimensional space).

$$(8)\qquad \begin{cases} x_1 - b_1 + b_2\, x_3 - b_3\, x_2 = 0, \\ x_5 - b_5 + b_3\, x_4 - b_4\, x_3 = 0, \\ x_6 - b_6 + b_4\, x_2 - b_2\, x_4 = 0. \end{cases}$$

The varieties V', as well as the varieties V'', depend upon six parameters.

Points, the *varieties* V', and the *varieties* V'' enjoy certain common properties. First, in the same way that ∞^4 isotropic lines pass through a point, there likewise exist ∞^4 isotropic lines on a V' or V''. We agree to say that two points are *united* if the line that joins them is isotropic. It can likewise happen that two distinct varieties V' have an isotropic line in common (and therefore only one), so we say that they are *united*. Two varieties V' that are not united will have no point in common. One defines two varieties V'' to be united similarly.

We agree to say that a variety V' is *incident on* a point M when it contains that point. There will then exist ∞^2 isotropic lines that pass through M and are contained in V'. Similarly, a variety V' and a variety V'' do not have a common (isotropic) line, in general (but only a point). If they do have a common line then there will be ∞^2 of them; we then say that they are *incident*.

The conditions for two points with the coordinates (x) and (x'), two varieties V' with parameters (a) and (a'), and two varieties V'' with parameters (b) and (b') to be united are:

$$\begin{aligned} (x_1' - x_1)(x_4' - x_4) + (x_2' - x_2)(x_5' - x_5) + (x_3' - x_3)(x_6' - x_6) &= 0, \\ (a_1' - a_1)(a_4' - a_4) + (a_2' - a_2)(a_5' - a_5) + (a_3' - a_3)(a_6' - a_6) &= 0, \\ (b_1' - b_1)(b_4' - b_4) + (b_2' - b_2)(b_5' - b_5) + (b_3' - x_3)(b_6' - x_6) &= 0, \end{aligned}$$

respectively. Similarly, equations (7) and (8) express the incidence conditions for a point and a variety (V') and (V''), resp. The incidence conditions for a variety (V') and a variety (V'') are:

$$(9)\qquad \begin{cases} b_4 - a_1 + a_2\, b_3 - a_3\, b_2 = 0, \\ b_5 - a_5 + a_3\, b_4 - a_4\, b_3 = 0, \\ b_6 - a_6 + a_4\, b_2 - a_2\, b_4 = 0. \end{cases}$$

One sees from the preceding that the notions of *union* and *incidence* are common to the three sets of geometric entities or *elements* (viz., points, varieties V', varieties V'') when one either considers two elements of the same type or two elements of different types. We shall endow those three types of elements with the three indices 0, 1, 2, resp.

One easily verifies that a given element of type i is incident with ∞^3 elements of a different type j, and that those ∞^3 elements are pair-wise united. Conversely, if there exist ∞^3 elements of type i that are pair-wise united then they will be incident with the same element of a different type. One also verifies that two united elements of type i are incident with ∞^1 elements of type $j \neq i$, and that conversely two elements of type i that are incident with ∞^1 elements of different types are united.

7. – Having said that, we shall consider some other transformations besides direct and inverse conformal transformations, properly speaking. We seek to make any element of a given type i correspond to an element of another given type i' according to a well-defined law, with the condition that two united elements of type i will correspond to two united elements of type i'. For example, if $i = 1$, $i' = 2$, then one must determine six functions $b_1, b_2, \ldots, b_6$ of $a_1, \ldots, a_6$, in such a manner that the Monge equation:

$$db_1\, db_4 + db_2\, db_3 + db_3\, db_6 = 0$$

is a consequence of the Monge equation:

$$da_1\, da_4 + da_2\, da_3 + da_3\, da_6 = 0\ .$$

The general solution is obtained by starting from an arbitrary conformal transformation, while regarding the b_i are the coordinates of the point that is the transform of the point (a_i). There are even two continuous families of correspondence that satisfy the condition that was imposed.

Take one of those correspondences and let $j \neq i$. Consider an element of type j. There exist ∞^3 elements of type i that are incident with it, and they are pair-wise united. the element in question will correspond to ∞^3 elements of type i' that are pair-wise united, and as a result, they will be incident with the same element of a different type. Let j' be that type. We shall then establish a correspondence between an arbitrary element of type j and an element of type j'. Furthermore, when two united elements of type j are incident with ∞^1 elements of type i, the two corresponding elements of type j' will be incident with ∞^1 elements of type i', and as a result, they will united. Finally, if k is the third index besides i and j, and k' is the index besides i' and j' then we can make any element of type k correspond, as above, to an element of type k' by the intermediary of ∞^3 elements of type i that are incident with it. Two united elements of type k correspond to two united elements of type k'. One easily proves that two incident elements of types j and k will correspond to two incident element of types j' and k'.

Therefore, we have finally made any substitution that acts upon the three indices 0, 1, 2 correspond to a continuous family of transformations that will change two united elements into two united elements and two incident elements into two incident elements. The set of all those transformations forms a mixed group that is composed of discrete families that extends the conformal group in the same way that the group of homographies and correlations extends the group of homographies in projective geometry. One can say that the *principle of duality* in projective geometry is replaced with a *principle of triality* here.

Meanwhile, there is an essential difference between the correlations of projective geometry and the new adjoint transformations of the conformal group of six-dimensional space, namely, that the latter *cannot be defined as contact transformations*. That amounts to the fact that the varieties that correspond to an arbitrary point (in the sense of Lie) have the first-order partial differential equation:

$$p_1\, p_4 + p_2\, p_5 + p_3\, p_6 = 0.$$

It is easy now to define the six families of transformations of the group Γ' that preserve the structure of conformal group. It suffices to make an arbitrary transformation of G correspond to the one that is deduced from it by a transformation of the extended mixed conformal group.

8. – Let us return to the general problem. The case of semi-simple groups is easy to discuss. If such a group G is composed of several simple subgroups with p different structures – namely, α_1 with the first structure, α_1 with the second, etc. – then the number h of discrete families of transformations that constitute the group Γ_i will be obtained immediately if one knows the numbers $h_1, h_2, \ldots, h_p$ that correspond to the p given structures. One has:

$$h = \alpha_1 ! \, \alpha_2 ! \ldots \alpha_p ! \; h_1^{\alpha_1} h_2^{\alpha_2} \cdots h_p^{\alpha_p} .$$

One simple case is that of the group G of orthogonal substitutions (with determinant 1) of four variables, which is a group that is only semi-simple, since it is composed of two simple subgroups with three parameters of rank 1. Here, one has $p = 1$, $h_1 = 1$, $\alpha_1 = 2$, $h = 2$. That agrees with the general result that relates to the orthogonal group with an even number of variables (no. **4**).

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