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On the integration of systems of total differential equations

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The problem of the existence of integrals for a given system of s total differential equations in r variables when that system is not completely integrable has not especially been the object of any research that would extend the memoir of Biermann "Ueber n simultane Differentialgleichungen der Form $\sum X_\mu \, dx^\mu = 0$" that was published in 1885 in vol. XXX of *Schlöm. Zeitschrift.* Furthermore, he proposed to only look for the maximum number of independent variables that one must take in order for there to exist a family of integral multiplicities that filled up all of space. He then found that *when the coefficients are arbitrary* this number is equal to the quotient, up to a unit, by default, of the total number r of variables by the number s of equations, augmented by 1. Moreover, the remainder of that division indicates the number of independent variables that one can take arbitrarily without the problem ceasing to be possible. Since then, there have hardly been any presentations of the proof of the same results in another form that would ever attain a state of perfect rigor, moreover, and there has been nothing done regarding the case in which the coefficients of the differential system are not arbitrary.

One can arrive at some precise and general results by taking into account the bilinear covariants of the left-hand sides of the equations of the given system, whose introduction by Frobenius and Darboux has proved to be fruitful in the theory of just one Pfaff equation. In summary, if one limits oneself to considering the given equations then – to employ a geometric language – one says that each tangent at a given point A of an integral multiplicity M that passes through that point is contained in a certain $r-s$-dimensional planar multiplicity (P) that is associated with that point. However, if one introduces the bilinear covariants then one finds that not only is every planar multiplicity (T) of dimension 1, 2, … that is tangent to an integral multiplicity contained in (P), but, in addition, two arbitrary lines of that planar multiplicity (T) satisfy certain bilinear relations with respect to their director parameters. Furthermore, if one represents a line that issues from A by a point in an $r-1$-dimensional space R then *the image of a tangent to M is required to be in a planar multiplicity* (H) *of R, but also the line that joins the images of two tangents to the same integral multiplicity M is required to belong to a certain number of linear complexes that are associated with A.*

In summary, one makes each point A of the space correspond to not only a planar multiplicity (H), but also a set of linear complexes in that planar multiplicity. It is clear that the nature of these linear complexes must influence the existence and degree of indeterminacy of the integral multiplicities.

Upon denoting the set consisting of a point A and a p-dimensional multiplicity that passes through that point by E_p and agreeing to say that E_p is *integral* whenever its image in R is situated entirely in (H), and, in addition, contains only lines that belong to the linear complexes that belong to A, one sees that the necessary and sufficient condition for a multiplicity to be integral is that all of its elements must be integral.

If one then seeks to make an m-dimensional integral multiplicity pass through a known m – 1-dimensional integral multiplicity then one finds that this is possible whenever an integral element E_m passes through an arbitrary integral element E_{m-1}. The solution is given by a system that is due to Kowalewsky, and it is unique if only one E_m passes through an arbitrary E_{m-1}.

This being the case, one is led to define an integer n in the following manner:

At least one integral element E_1 passes through an arbitrary point A.

At least one integral element E_2 passes through an arbitrary integral element E_1, etc.

At least one integral element E_n passes through an arbitrary integral element E_{n-1}.

Finally, no integral element E_{n+1} passes through an arbitrary integral element E_n.

The integer n thus defined can be called the *genus* of the system.

One can infer some precise conclusions from this on the existence of integrals of the given system. In order to do this, suppose, in a general manner, that the integral elements E_{i+1} that pass through an arbitrary integral element E_i depend upon r_{i+1} parameters. (If the element is unique, we agree to give r_{i+1} the value zero). Here, then, is a system of geometric conditions that determine the n-dimensional integral completely:

Given an arbitrary point μ_0, *an arbitrary multiplicity* μ_{r-r_1} *that passes through that point, an arbitrary multiplicity* μ_{r-r_2} *that passes through* μ_{r-r_1}, *etc., an arbitrary multiplicity* μ_{r-r_n} *that passes through* $\mu_{r-r_{n-1}}$ *m there exists one and only one integral multiplicity* M_n *that passes through* μ_0 *that has in common with* μ_{r-r_1} *a* 1-*dimensional, …,resp., multiplicity such that* μ_{r-r_2} *is an i-dimensional multiplicity that is contained in* μ_{r-r_n}.

Upon interpreting this statement analytically and specializing the manner by which one obtains all of the integral multiplicities once and only once, one proves that the general n-dimensional integral is determined, and in a unique manner, by a system of:

s_n	arbitrary functions of	n	arguments,
s_{n-1}	“	$n - 1$	“
…………………………………………			
s_1	“	1	“

and

s arbitrary constants,

upon setting:

$$
\begin{aligned}
s_n &= r_n, \\
s_{n-1} &= r_{n-1} - r_n - 1, \\
&\ldots\ldots, \\
s_1 &= r_1 - r_2 - 1, \\
s &= r - r_1 - 1.
\end{aligned}
$$

Moreover, these integers s are all positive, and *they increase – or, at least, they do not decrease – from* s_n *to* s.

Furthermore, one can give a precise definition to the word *arbitrary* that is found in these statements.

One thus sees the important role that is played by these integers s and the simple manner by which they depend upon the planar multiplicity (H) and the system of linear complexes that we spoke of above.

In particular, if the coefficients of the given equations are not subject to any specialization, which is the case that was studied by Biermann, then the genus n is the quotient, up to a unit, of r by $s + 1$, and if one denotes the remainder by σ then one has:

$$s_n = s, \qquad s_{n-1} = s_{n-2} = \ldots = s_1 = s,$$

in such a way that the general integral depends upon σ arbitrary functions of n arguments, s arbitrary functions of $n - 1$ arguments, etc., and s arbitrary constants. This is the result that was proved by Biermann, but obviously with much more precision.

The differential systems for which the integer s_n is zero enjoy some particularly interesting properties; one can call them *systems of the first kind.*

In a general manner, the integration can be simplified if several of the numbers s are zero. If s_ν is that one of these zero numbers that has the smallest index then one has:

$$s_\nu = s_{\nu+1} = \ldots = s_n = 0.$$

For these systems, one and only one integral element E_n passes through an arbitrary integral element $E_{\nu+1}$. Likewise, it suffices to give the multiplicities μ_0, μ_{r-r_1}, …, $\mu_{r-r_{\nu-1}}$ that we spoke of above in order to *determine* the integral M_n and *to see whether that integral can be converted into that of a system of genus* ν. It suffices to make an arbitrary, but well-defined, multiplicity $\mu_{r-r_{\nu-1}-1}$ pass through $\mu_{r-r_{\nu-1}}$, and make a family of multiplicities μ_{r-r_ν} that depend upon $r_\nu = n - \nu$ parameters and fill all of space pass through it. An integral multiplicity M_ν corresponds to each of them. The locus of these muliplicities M_ν when one varies the $n - \nu$ parameters that they depend upon is the desired multiplicity $r - r_\nu$. By definition, one is reduced to a system of $r - r_\nu$ variables of genus ν, but whose coefficients depend upon $n - \nu$ arbitrary constants. In the case where ν is equal to 1, this is the Lie-Mayer method for the integration of completely integrable systems. One can call ν the *true genus* of the system.

Along a different line of reasoning, there is a case where the integration can simplify further, which is the one where a *characteristic* element passes through each point A. One thus calls an element E_p integral when any other element that is formed from E_p and an integral linear element is also integral, or, as one can say, when E_p is associated with an arbitrary integral linear element. One can then prove that the system of total differential equations that defines the characteristic elements is *completely integrable*. In other words, there exists a family of p-dimensional multiplicities that admits the corresponding characteristic element E_p at each of their points. These multiplicities, which one calls *characteristic*, depend upon $r - p$ parameters, and one and only one of them passes through each arbitrary point of space. For the systems of genus n where there exist characteristic elements E_p, any non-singular characteristic multiplicity M_n is generated by characteristic multiplicities that depend upon $n - p$ parameters, and if two integral multiplicities M_n and M'_n have a common point then they have the characteristic multiplicity that issues from that point in common.

Finally, if one takes the new variables to be the $r - p$ parameters that the characteristics depend upon and p other arbitrary functions then the system can be put into a form such that it only contains the first $r - p$ variables. Furthermore, the search for these $r - p$ variables – in other words, the integration of the characteristic differential system – can, in general, be simplified by taking into account some properties of the linear complexes that are associated with the given system.

In particular, if one has a system of the first kind of genus n for which s_1 is equal to 1, which is the case of just one equation, then there are always $n - \nu + 1$-dimensional characteristic multiplicities, where n denotes the true genus of the system. Once these characteristics have been found by operations whose order decreases by two units each time, one only has to integrate a system of $r - n + \nu - 1$ variables and genus $\nu - 1$.

There likewise exist very simple theorems in the case where s_1 is equal to 2, but the study of these theorems enters into the theory of the *classification* of total differential systems.

It is hardly necessary to remark that there are links between all of this theory and the theory of systems of partial differential equations. I will content myself to pointing out the agreement in form between the results that are found for the degree of indeterminacy of the general integral of a Pfaff system and the ones that were found by Delassus ([1]) for the degree of indeterminacy of the general integral of a system of partial differential equations that is in involution. However, whereas Delassus put the system into a particular form by differentiating the dependent variables of the unknown functions completely, moreover, here, there is no difference between the two types of variables, and the origin itself of the numbers $s, s_1, \ldots, s_n$ shows their invariance with regard to any change of dependent or independent variables.

The first two paragraphs of this memoir introduce integral elements, along with the linear complexes that I have already spoke of. § III treats the problem that consists of making a multiplicity M_{m+1} pass through an integral multiplicity M_m . §§ IV and V give some theorems that one might call arithmetic on the genus n and the numbers r_i and s_i . § VI contains the presentation of the Cauchy problem and the degree of indeterminacy of the general integral of a system of genus n. § VII is dedicated to the systems of the first kind and the generalized Lie-Mayer method. Finally, § VIII is occupied with systems that admit characteristics in the sense that was given to that word above and gives some indications on the search for these characteristics.

This research can be extended in many directions, and, as one sees, the problem of the *classification* of the differential systems can already start with a first preliminary problem that takes the form of *the search for systems of linear complexes of genus n.* Another very important question will be the study of *singular* integral multiplicities. It is not difficult to define them, but what will be interesting is the study of the new differential

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([1]) "Extension du théorème de Cauchy aux systèmes les plus généraux d'équations aux dérivées partielles" Ann. de l'Éc. Norm. (3), t. XIII, pp. 421-467.

systems that they define. As for the first classification problem, one can, without too much difficulty, prove a certain number of interesting results, but I will not insist upon them.

I.

Consider a system of total differential equations in r variables $x_1, x_2, \ldots, x_r$:

$$(1) \qquad \begin{cases} \omega \equiv a_1 dx_1 + a_2 dx_2 + \cdots + a_r dx_r = 0, \\ \varpi \equiv b_1 dx_1 + b_2 dx_2 + \cdots + b_r dx_r = 0, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \\ \chi \equiv l_1 dx_1 + l_2 dx_2 + \cdots + l_r dx_r = 0, \end{cases}$$

in which the coefficients $a, b, \ldots, l$ are functions of the variables x. One can regard a certain number n of the variables x as independent and the $r - n$ other ones as functions of them. The system (1) then establishes linear relations between the total differentials of the $r - n$ functions and the n independent variables, which is collectively equivalent to a system of (linear) partial differential equations that these $r - n$ functions must satisfy ([1]). To use a geometric language, one can say that the equations that define the $r - n$ dependent variables as functions of the n independent ones represent an *n-dimensional multiplicity* M_n in r-dimensional space, and the system (1) can be regarded as establishing the conditions that the differentials of the coordinates $x_1, \ldots, x_r$ of a point of the multiplicity must satisfy under an arbitrary displacement on that multiplicity. However, if one remarks that these differentials (or their ratios, which is all that enters in) are nothing but the *director parameters* of the *tangent* to the multiplicity under the displacement considered, one can say that the system (1) expresses the idea that *the tangents to a multiplicity* M_n *that passes through an arbitrary point of space satisfy certain conditions that depend only upon the point considered*, and the form of equations (1) shows that these tangents are required to be in a certain planar multiplicity ([2]) that is determined by the point.

To integrate the system (1), where one supposes that the number of independent variables is equal to n, is therefore to solve the following problem:

One makes a planar multiplicity ([3]) *that passes through any point of space correspond to that point; determine an n-dimensional multiplicity* M_n *such that at each of its points all of the tangents to that multiplicity are situated in the planar multiplicity that corresponds to that point.*

([1]) One knows, moreover, that any system of partial differential equations can be converted into a system of total differential equations by regarding, if needed, some of the partial derivatives of the unknown functions as new dependent variables.

([2]) As one knows, a planar multiplicity is defined by linear equations; a line is a *one*-dimensional planar multiplicity.

([3]) Naturally, the dimension of the that multiplicity is the same for all points of space. It is equal to the difference between r and the number of equations (1).

Any multiplicity M_n that satisfies that condition will be called an *integral* multiplicity. The condition, thus stated, that integral multiplicities must satisfy is *independent* of the dimension n of these multiplicities.

Call the set that consists of a point and a line that passes through that point a *linear element.* In addition, agree to say that the set that consists of a point of a multiplicity and a tangent to the multiplicity at that point constitutes a linear element of that multiplicity. Finally, call any linear element that satisfies equations (1) (where $dx_1, dx_2, \ldots, dx_r$ will be regarded as the director parameters of the line of the element) an *integral linear element*. We can then state the following proposition:

In order for a multiplicity to be integral, it is necessary and sufficient that all of its linear elements be integral.

II.

Along with the linear elements, we shall consider the ones that we call 2, 3, …*dimensional elements.* In a general manner, we refer to *the set that consists of a point and a p-dimensional planar multiplicity that passes through that point as a p-dimensional element*, and we denote such an element with the general symbol E_p . We say that the element E_p contains the element E_q ($p > q$) if the two elements are at the same point and the planar multiplicity of the first one contains the entire second planar multiplicity. In particular, a linear element will be denoted by the symbol E_1 .

We call the p-dimensional elements E_p such that all of the linear elements that are contained in them belong to a multiplicity M the *elements E_p of a multiplicity M*, or, more briefly, the elements that are formed from linear elements of M. If the multiplicity M is n-dimensional then it admits 2, 3, …, n-dimensional elements, but it does not admit n+1-dimensional elements. It admits only one n-dimensional element at each point, which is the locus of linear elements that contain that point.

Any element E_p of an *integral* multiplicity obviously enjoys the property that of containing only *integral* linear elements; *however, it also satisfies other conditions that can be established independently of any particular integral multiplicity.*

In order to arrive at these conditions, imagine that the coordinates of a point of an integral multiplicity M_n are expressed by means of n parameters $u, v, \ldots$, and consider the two displacements on that multiplicity that are obtained, in the first case, by varying only the parameter u to the exclusion of the other ones, and in the second case, by varying only the parameter v. Denote the differentials that relate to these two displacements by the symbols d and δ. From (1), we will obviously have:

$$\omega_d \equiv a_1\, dx_1 + a_2\, dx_2 + \ldots + a_r\, dx_r = 0,$$
$$\omega_\delta \equiv a_1\, \delta x_1 + a_2\, \delta x_2 + \ldots + a_r\, \delta x_r = 0,$$

and as a result:

$$\omega' \equiv \delta\omega_d - d\,\omega_\delta = 0.$$

Upon forming this expression and remarking that the symbols d and δ are *commutable* ($d\delta = \delta d$), and then proceeding analogously for all of the equations of system (1), one arrives at the following system:

$$
(2) \qquad \begin{cases}
\omega' \equiv \sum_{i,k} \left(\frac{\partial a_i}{\partial x_k} - \frac{\partial a_k}{\partial x_i} \right) (dx_i \delta x_k - dx_k \delta x_i) = 0, \\
\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots, \\
\chi' \equiv \sum_{i,k} \left(\frac{\partial l_i}{\partial x_k} - \frac{\partial l_k}{\partial x_i} \right) (dx_i \delta x_k - dx_k \delta x_i) = 0.
\end{cases}
$$

The system (2) *is verified by any arbitrary pair of two displacements on the integral multiplicity*, or further by the set that consists of an arbitrary point of the multiplicity and two arbitrary tangents to that point, and in a general manner, *by two integral linear elements that issue from the same point and belong to the same integral multiplicity.*

Call an element that is *formed from integral elements and is such that any two of them satisfy system* (2), *moreover,* a 2, 3, …-dimensional *integral* element; we then have the following proposition:

All of the 1, 2, 3, *…-dimensional elements of an arbitrary integral multiplicity are integral elements, and conversely.*

In order to simplify the language, we agree to say that two integral linear elements that issue from the same point and satisfy the system (2) are *associated* ([1]). A 2, 3, …-dimensional *integral* element is then *an element that is formed from integral linear elements that are pair-wise associated.* From the bilinear form of equations (2), in order for an element E_p to be integral it *suffices* that p independent linear elements ([2]) of E_p should be integral and pair-wise associated. (Moreover, any element E_p can be defined by p independent linear elements that issue from the same point.)

The expressions ω', ϖ', …, χ' that are the left-hand sides of equations (2) are called the *bilinear covariants* ([3]) of the Pfaff expressions ω, ϖ, …, χ. From the manner itself by which they are obtained, and conforming to their name, one sees that they are covariants under an arbitrary change of variables.

One can give the system (2) a geometric interpretation. Consider the various integral linear elements that issue from a given point A of space, and project them onto an r–1-dimensional planar multiplicity (P) that does not pass through A, where the point of view is the point A itself. Each element is then defined by the *trace* of its line on the planar multiplicity of projection – i.e., by a point of that multiplicity (P) – and with our notations the quantities dx_1, dx_2, …, dx_r are the homogeneous coordinates of that point in (P). Say that the linear element is integral – i.e., that the coordinates of its projection satisfy equations (1), so they are contained in a certain planar multiplicity (Q) that is situated in (P). If we now take two integral linear elements that are associated and their projections onto (P) then the quantities $dx_i\ \delta x_k - \delta x_k\ dx_i$ are precisely the Plückerian

([1]) Two linear elements that are associated with a third one are not necessarily associated with each other.

([2]) One says that p linear elements are *independent* when they do not belong to the same p–1-dimensional element.

([3]) Their introduction into the Pfaff problem is due to Frobenius ("Ueber das Pfaff'sche Problem," J. de Crelle, t. LXXXII, 1877) and Darboux, ("Sur le problème de Pfaff," Bull. Soc. Math. (2) t. VI (1882).

coordinates of the line that joins these two projections. The first of expressions (2) expresses a linear and homogeneous relation between these coordinates – i.e., the idea that this line belongs to a certain linear complex – and the same thing is true for the other equations (2).

In summary, *to say that two linear elements that issue from the same point A are integral and associated is to say that upon projecting those elements from that point A onto an* $r-1$*-dimensional planar multiplicity* (*P*) *the line that joins the traces of the two elements is entirely situated in a certain planar multiplicity* (*Q*) *and furthermore, simultaneously belongs to a certain number of linear complexes.*

Moreover, in turn, *to say that an element* E_p *that issues from A is integral is to say that the planar multiplicity that is traced from that elements on* (*P*) *is situated entirely on* (*Q*), *and, in addition, that each of the lines of that multiplicity belong to a certain number of linear complexes.*

In summary, each point *A* of the given system corresponds to a planar multiplicity (*Q*) and a set of linear complexes in that multiplicity (*Q*) in an arbitrarily chosen $\sigma-1$-dimensional planar multiplicity (*P*).

If one makes a change of variables then the elements that issue from a point *A* are linked homographically with the corresponding elements that issue from the corresponding point *A′*, and *the set of linear complexes that corresponds to A is also subjected to a simple homographic transformation* ([1]).

The important consequence already results from this very simple remark that if two systems of total differential equations (in the same number of variables) do not correspond to the points of space of the planar multiplicities (*Q*) and sets of linear complexes that are reducible to each other under a homographic transformation then it is impossible to reduce one of the two systems to the other one by a change of variables. In a more precise manner, if one denotes the variables of the second system of total differential by $y_1, y_2, \ldots, y_r$, and we denote the systems that are analogous to (1) and (2) by (1)′ and (2)′ then one seeks to express the idea that one can pass from the system [(1), (2)] to the system [(1)′, (2)′] by a linear transformation that acts on $dx_1, \ldots, dx_r$, as well as on $\delta x_1, \ldots, \delta x_r$.

Three cases can present themselves: Either this is not possible for any system of values of *x* and *y*, and then no change of variables can transform the one of the two given systems into the other one, or it will be possible on the condition that certain finite relations between the *x* and *y* are verified, and then any change of variables that effects the desired transformation – *if it is possible* – must respect these relations, or finally that it is possible for any values of *x* and *y*, and then one can say nothing about the change of variables, *if it is possible*.

Finally, one perceives, without having to insist upon the fact, that the classification of systems of total differential equations demands the prior classification of all systems of linear complexes, while not regarding two systems of linear complexes as distinct when one is reducible to the other by a homographic transformation, i.e., in other words, the search for all *types* of systems of linear complexes.

([1]) It is obvious that if one simply changes the plane of projection then one obtains two equivalent systems of complexes under a homographic transformation, since they are the *projection* of each other. If one replaces equations (1) with other ones that form an equivalent system then it is likewise obvious that neither (*Q*) nor the set of linear complexes in (*Q*) are changed.

In order to apply the preceding to an example, consider the system:

$$\text{(3)} \qquad \begin{cases} \omega \equiv dz - p\,dx - q\,dy = 0, \\ \varpi \equiv dp - u\,dq - a\,dx - b\,dy = 0, \end{cases}$$

where the variables are x, y, z, p, q, u, and a and b denote two given functions of these six variables. The integration of this system, when considered as having two independent variables x and y, amounts to the integration of one second-order, partial differential equation that admits a system of first-order characteristics, and, with the usual notations, that equation is obtained by eliminating u from the two relations:

$$r - us - a = 0,$$
$$s - ut - b = 0.$$

Here, the planar multiplicity (Q) is three-dimensional, since the homogeneous coordinates of one of its points are defined when one is given dx, dy, dq, du. We can thus locate (Q) in ordinary space. Here, there are *two* linear complexes. Now, a system of two linear complexes in space is always reducible to one of the three following ones by a homographic transformation:

(α) $$p_{12} = p_{24} = 0,$$
(β) $$p_{12} = p_{13} + p_{24} = 0,$$
(γ) $$p_{12} = p_{13} = 0,$$

in which the p_{ik} are the Plückerian coordinates of the line. Case (α) gives the set of lines that meet two fixed lines that are not situated in the same plane. Case (β) gives the set of tangents to a fixed quadric at the various points of a fixed generator of that quadric. Finally, case (γ) gives the set of lines that are situated in a fixed plane, along with the set of lines that issue from a fixed point of that plane.

Each of these cases corresponds to a type of second-order equation of the indicated form. Case (α) corresponds to equations whose two systems of second-order characteristics are distinct. Case (β) corresponds to equations whose characteristics coincide, and is obtained by expressing the idea that the equation:

$$r + 2us + u^2 t + 2\varphi(u, x, y, z, p, q) = 0$$

must admit a double root in u, where the function φ is *arbitrary*. Finally, case (γ) corresponds to those of these latter equations for which the function φ satisfies a certain second-order partial differential equation, and which were the object of Goursat's research. Their interest is based in the fact that one can integrate them by means of systems of ordinary differential equations, as we will confirm in paragraph VIII.

III.

Having posed these preliminary notions, we shall occupy ourselves with what one can call *the first Cauchy problem.* The problem to which we thus refer is the following one:

Given an integral p-dimensional multiplicity M_p of a system of total differential equations, pass from M_p to a p+1-dimensional integral multiplicity M_{p+1} .

An obvious remark to make is that if the problem is possible then at least one *integral* element E_{p+1} passes through any element E_p of M_p, namely, an element E_{p+1} of M_{p+1} . One thus arrives immediately at a first necessary condition.

In order for the Cauchy problem to be possible, one must have that at least one integral element E_{p+1} must pass through each element E_p of the given integral multiplicity M_p .

Without investigating whether this condition is sufficient, which it is not, moreover, we shall limit ourselves to a special case, which nonetheless presents great generality. *We shall suppose in the sequel that the given system is such that at least one integral element E_{p+1} passes through every integral E_p in space.* In other words, we suppose that the property that belongs to the elements E_p of M_p belongs to all of the integral elements E_p in space.

With that hypothesis, *the Cauchy problem is always possible.* However, before commencing the proof of that proposition, it will be useful to present some geometric remarks on the integral elements E_{p+1} that contain a given integral element E_p . If one defines the element E_p by means of p linearly independent elements $\varepsilon^{(1)}, \varepsilon^{(2)}, \dots, \varepsilon^{(p)}$ then one can define an element E_{p+1} that contains E_p by means of a new linear element ε that is independent of the first p. We will have the desired element E_{p+1} by expressing the idea that ε is an *integral* linear element, and that it is *associated* with each of the linear elements $\varepsilon^{(1)}, \varepsilon^{(2)}, \dots, \varepsilon^{(p)}$. It results from this that *the locus of integral elements E_{p+1} that contain an integral element E_p is a planar element* (which is not necessarily integral), because if ε and ε' provide two distinct solutions E_{p+1} and E'_{p+1} then the $p + 2$ linear elements $\varepsilon^{(1)}, \varepsilon^{(2)}, \dots, \varepsilon^{(p)}, \varepsilon, \varepsilon'$ determine an element E_{p+2}, and any linear element of E_{p+2} is integral and associated with $\varepsilon^{(1)}, \varepsilon^{(2)}, \dots, \varepsilon^{(p)}$; in other words, all of the elements E_{p+1} that are contained in E_{p+2} and contain E_{p+1} are integral.

Analytically, the elements E_{p+1} that contain E_p depend upon $r - p$ homogeneous parameters ([1]). The equations that express the idea that E_{p+1} is integral are *linear* with respect to these parameters.

([1]) For example, if the equations for E_p are:

$$P_1 = P_2 = \dots = P_{r-p} = 0,$$

where the P are linear forms in $dx_1, \dots, dx_r$ then the equations of E_{p+1} are:

$$\frac{P_1}{\lambda_1} = \frac{P_2}{\lambda_2} = \dots = \frac{P_{r-p}}{\lambda_{r-p}} .$$

If we suppose that these equations reduce to $r - p - s - 1$ independent ones for an *arbitrary* E_p , where s is zero or positive, then at least one integral element E_{p+1} passes through each arbitrary integral element E_p . If s is zero then one and only one will pass through it, and if s is positive then an infinitude of them that depend upon s arbitrary constants will pass through it. In both cases, we say that ∞^s of them pass through it. The locus of all these elements is an element E_{p+s+1} .

It can happen that for some particular integral element E_p there will be a great degree of indeterminacy, moreover; we then say that the integral element E_p is *singular*. An integral multiplicity M_p whose elements E_p are all singular will be called a *singular* integral multiplicity.

We now arrive at the solution to the Cauchy problem. We shall prove the following theorem:

Given a non-singular integral multiplicity E_p *, at least one integral multiplicity* M_{p+1} *passes through that multiplicity. One and only one passes through it if each non-singular integral element* E_p *belongs to one and only integral element* E_{p+1} *. There are an infinitude of them that depend upon s arbitrary functions of* $p + 1$ *arguments if each non-singular integral element* E_p *belongs to* ∞^s *integral elements* E_{p+1} .

More precisely, take a particular non-singular element of M_p, say, E_p^0. Let $(x_1^0, x_2^0, \ldots, x_r^0)$ be the coordinates of the point from which that element issues. We suppose that the multiplicity M_p is *analytic* – i.e., that in a neighborhood of the point (x_i^0), $r - p$ of the x coordinates – say, $x_{p+1}, \ldots, x_r$ – are expressed as holomorphic functions of $x_1 - x_1^0$, $x_2 - x_2^0$, …, $x_r - x_r^0$. The $r - p$ equations of the element E_p^0 can then be solved for $dx_{p+1}, \ldots, dx_r$. Take a particular integral element E_{p+1}^0 that passes through E_p^0. The $r - p - 1$ linear equations that define it can be solved for $r - p - 1$ of the differentials $dx_{p+1}, \ldots, dx_r$, say, $dx_{p+2}, dx_{p+3}, \ldots, dx_r$. If an integral multiplicity M_{p+1} admits the element E_{p+1}^0 then this signifies that $x_{p+2}, \ldots, x_r$ are expressed as holomorphic functions of $x_1, \ldots, x_{p+1}$ in a neighborhood of the point considered. For the sake of convenience in what follows, we shall change the notations while preserving $x_1, x_2, \ldots, x_p$, and replace x_{p+1} with x and the other variables $x_{p+2}, \ldots, x_r$ with $z_1, z_2, \ldots, z_m$ $(m = r - p - 1)$.

With these notations, the equations for the multiplicity M_p are:

$$
(4) \qquad \begin{cases} x = \varphi(x_1, x_2, \ldots, x_p), \\ z_1 = \varphi_1(x_1, x_2, \ldots, x_p), \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ z_m = \varphi_m(x_1, x_2, \ldots, x_p), \end{cases}
$$

and the desired multiplicity M_{p+1} can be defined by giving $z_1, z_2, \ldots, z_m$ as holomorphic functions of $x, x_1, \ldots, x_p$ in a neighborhood of x_0, x_1^0, …, x_p^0.

Finally, we shall make a change of variables by preserving the variables $x, x_1, \ldots, x_p$; $z_1, z_2, \ldots, z_m$ and taking the variable x to be the quantity $x - \varphi$, which obviously changes none of the preceding conventions that were made. This amounts to supposing that $\varphi \equiv 0$ and $x^0 = 0$ in formulas (4).

To conclude the statement of these preliminary conventions, we suppose that the coefficients $a, b, \ldots, l$ of the system (1) are holomorphic in the neighborhood of $(x_1^0, x_2^0, \ldots, z_m^0)$.

The desired multiplicity M_{p+1} is defined by m functions $z_1, z_2, \ldots, z_m$ of $p + 1$ variables, $x, x_1, \ldots, x_p$ that are holomorphic in a neighborhood of $(0, x_1^0, \ldots, x_p^0)$ and required to reduce to m functions $\varphi_1, \varphi_2, \ldots, \varphi_m$ of $x_1, x_2, \ldots, x_p$ that are given in advance for $x = 0$.

The equations that determine these functions are deduced from equations (1) by replacing $dz_1, \ldots, dz_m$ with their values and identifying them. However, *we shall replace the system thus obtained with another system that contains a larger number of equations, and which quite simply expresses the idea that the elements E_{p+1} of the multiplicity M_{p+1} are integral*.

In order to do this, we remark that each element E_{p+1} of M_{p+1} can be defined by $p + 1$ independent linear elements, namely, the ones that one obtains by varying just one of the independent variables $x, x_1, \ldots, x_p$. These $p + 1$ elements, which we call $\varepsilon^{(1)}, \ldots, \varepsilon^{(p)}$, are defined by:

$$(\varepsilon) \qquad \frac{dx}{1} = \frac{dx_1}{0} = \ldots = \frac{dx_p}{0} = \frac{dz_1}{\frac{\partial z_1}{\partial x}} = \ldots = \frac{dz_m}{\frac{\partial z_m}{\partial x}},$$

$$(\varepsilon^{(1)}) \qquad \frac{dx}{0} = \frac{dx_1}{1} = \ldots = \frac{dx_p}{0} = \frac{dz_1}{\frac{\partial z_1}{\partial x_1}} = \ldots = \frac{dz_m}{\frac{\partial z_m}{\partial x_1}},$$

…………………………………………..,

$$(\varepsilon^{(p)}) \qquad \frac{dx}{0} = \frac{dx_1}{1} = \ldots = \frac{dx_p}{1} = \frac{dz_1}{\frac{\partial z_1}{\partial x_p}} = \ldots = \frac{dz_m}{\frac{\partial z_m}{\partial x_p}}.$$

We divide the equations that express the idea that E_{p+1} is integral into two groups: The first group expresses the idea that the element E_p that is defined by $\varepsilon^{(1)}, \varepsilon^{(2)}, \ldots, \varepsilon^{(p)}$ is integral. The second group expresses the idea that ε is integral and associated with $\varepsilon^{(1)}$, $\varepsilon^{(2)}, \ldots, \varepsilon^{(p)}$.

If one of the equations of the system is:

$$\omega \equiv a\,dx + a_1\,dx_1 + \ldots + a_p\,dx_p + b_1\,dz_1 + \ldots + b_m\,dx_m = 0$$

then we set:

$$\Omega \equiv a + b_1 \frac{\partial z_1}{\partial x} + \ldots + b_m \frac{\partial z_m}{\partial x},$$

$$\Omega_i \equiv a_i + b_1 \frac{\partial z_1}{\partial x_i} + \ldots + b_m \frac{\partial z_m}{\partial x_i} \qquad (i = 1, 2, \ldots, p).$$

With these notations, the equations of the first group are, as is easy to see:

$$\text{(I)} \qquad \begin{cases} \Omega_i = 0, \quad \dfrac{\partial \Omega_i}{\partial x_j} - \dfrac{\partial \Omega_j}{\partial x_i} = 0 \quad (i, j = 1, 2, \ldots, p), \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{cases}$$

and those of the second group are, for example:

$$\text{(II)} \qquad \begin{cases} \Omega = 0, \quad \dfrac{\partial \Omega}{\partial x_j} - \dfrac{\partial \Omega_i}{\partial x} = 0 \quad (i, j = 1, 2, \ldots, p), \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{cases}$$

where the ellipses refer to the other equations $\varpi = 0, \ldots, \chi = 0$ of the given system. The symbol $\partial f / \partial x_i$ refers to a derivation with respect to x_i, while regarding $z_1, z_2, \ldots, z_m$ as functions of x_i .

Equations (I) do not contain $\partial z_1 / \partial x, \ldots, \partial z_m / \partial x$, while those of the second group are *linear* in these quantities. One can, moreover, modify them by taking equations (I) into account.

Now, take the hypotheses that were made into account. From the fact that the element E_p that is defined by $\varepsilon^{(1)}, \varepsilon^{(2)}, \ldots, \varepsilon^{(p)}$ is integral, the equations that φ must satisfy in order for E_{p+1} to be integral are compatible. This signifies that *in order for the equations to be verified, equations* (II), *when considered as equations that are linear in* $\partial z_1 / \partial x, \ldots, \partial z_m / \partial x$, *are algebraically compatible.* More precisely, they reduce to $m - s$ independent linear equations. In particular, this is true, by hypothesis, for the system of values $(0, x_1^0, \ldots, x_m^0)$. We suppose, to fix ideas, that with the initial values these $m - s$ equations are soluble for:

$$\frac{\partial z_1}{\partial x}, \quad \frac{\partial z_2}{\partial x}, \quad \ldots, \quad \frac{\partial z_{m-2}}{\partial x},$$

namely:

$$\text{(II}'\text{)} \qquad \begin{cases} \dfrac{\partial z_1}{\partial x} = \Phi_1\left(x, x_i, z_k, \dfrac{\partial z_k}{\partial x_j}, \dfrac{\partial z_{m-s+1}}{\partial x}, \ldots, \dfrac{\partial z_m}{\partial x} \right), \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \\ \dfrac{\partial z_{m-s}}{\partial x} = \Phi_{m-s}\left(x, x_i, z_k, \ldots, \dfrac{\partial z_m}{\partial x} \right), \end{cases}$$

in which the Φ are holomorphic in a neighborhood of the initial values of their arguments (and linear with respect to $\partial z_{m-s+1} / \partial x, \ldots, \partial z_m / \partial x$).

This being the case, instead of preserving the set of equations (I) and (II), we preserve only equations (II′), while we nevertheless recall that equations (I) and (II′) imply equations (II) algebraically.

We shall now seek to determine a solution to equations (II′) that satisfies the following conditions: $z_1, \ldots, z_m$ are holomorphic functions of $x, x_1, \ldots, x_p$ in a neighborhood of $(0, x_1^0, \ldots, x_p^0)$, and for $x = 0$ they reduce to m given functions $\varphi_1, \ldots, \varphi_m$ of $x_1, \ldots, x_p$.

Now, the system (II′) is a Kowalewski system. From the work that has been done on these systems, there exists one and only one solution that is holomorphic in a neighborhood of $(0, x_1^0, \ldots, x_p^0)$, and is such that $z_{m-s+1}, \ldots, z_m$ are arbitrarily given (holomorphic) functions, and for $x = 0$, $z_1, \ldots, z_m$ reduce to s given functions of $x_1, \ldots, x_p$.

This being the case, one can thus take:

$$z_{m-s+1} = f_{m-s+1}(x, x_1, \ldots, x_p),$$
$$z_m = f_m(x, x_1, \ldots, x_p),$$

where the s functions f are subject to only the condition that for $x = 0$ *they must reduce to s given functions* $\varphi_{m-s+1}, \ldots, \varphi_m$. *Once these s functions have been chosen, the system* (II′) *will admit one and only one solution that satisfies the stated conditions*.

One sees, moreover, that one can always arrange this in such a manner that the s quantities $\frac{\partial z_{m-s+1}}{\partial x}, \ldots, \frac{\partial z_m}{\partial x}$ take arbitrarily fixed values for $x = 0$, $x_i = x_i^0$; i.e., *that the multiplicity* M_{p+1} *thus determined will admit any one of the integral elements* E_{p+1} *that passes through* E_p^0 *that one desires.*

The original problem is still not solved now, since it is clear that the desired integral multiplicities can be found only among the multiplicities that we just determined, thanks to the theorems of Kowalewski, so it does not result that these multiplicities will truly be *integral*. In other words, it still remains for us to prove that these multiplicities satisfy equations (I) and (II). In order to do this, *we shall prove that if a multiplicity* M_{p+1} *that is determined in the way that we said satisfies equations* (I) *and* (II) *for a certain value of* x *then it also satisfies them for the infinitely neighboring value* $x + \delta x$.

If this is proved, as it is for $x = 0$, then equations (I) express the idea that the multiplicity M_p that M_{p+1} reduces to is integral, which is nothing but the hypothesis, and that equations (II′) are assumed to be verified by the multiplicity M_{p+1}, and in turn, equations (II). It will then result that equations (I) and (II) will be verified for any value of x.

Now, suppose that equations (I) and (II) are verified for a certain value of x. One then has, in particular, that for that value of x:

$$\Omega = 0, \qquad \Omega_i = 0, \qquad \frac{\partial \Omega_i}{\partial x} - \frac{\partial \Omega}{\partial x_i} = 0.$$

However, if Ω is zero then the same is true for its derivative $\partial\Omega / \partial x_i$ when it is taken with respect to the variable x_i , *independently of x*. Therefore, $\partial\Omega_i / \partial x$ is zero for the value of x considered. Now, to say that Ω_i and $\partial\Omega_i / \partial x$ are annulled for the value x is to say that Ω_i is annulled for the infinitely close value $x + \delta x$. The same thing is true for $\partial\Omega_i / \partial x_j$ and analogous quantities for $x + \delta x$. Therefore, *equations* (I) *are verified for* $x + \delta x$. By hypothesis, the same thing is true for equations (II′), and as an algebraic consequence for equations (II), which are equivalent to (II′), upon taking (I) into account. Therefore, *all of equations* (I) *are verified for* $x + \delta x$.

The theorem is thus proved. We given it the name of *Cauchy's theorem*, by analogy with a well-known theorem in the theory of first-order partial differential equations, and of which it is only a special case.

If we refer to the system (3) as an application then we see that each integral linear element that issues from a given point, that is arbitrary moreover, can be represented by a point in ordinary space, and that an integral element E_2 is then represented by a fixed line that, in the general case, is required to meet two fixed lines that are not situated in the same plane. It results from this in an obvious way that one and only one two-dimensional integral element passes through any integral element (one and only one line that meets two fixed lines passes through a point in ordinary space). Therefore, one and only one integral multiplicity M_2 passes through any non-singular integral multiplicity M_1 . Here, the singular linear elements are the ones that are represented by the various points of two fixed lines. The singular integral multiplicities M_1 thus divide into two distinct series; they are nothing but what one calls the *characteristics* in the theory of second-order equations.

We return to the general case. An integral multiplicity M_1 of system (3) will be obtained, for example, by taking x, y, z, p, q to be five functions of the same parameter variable that are required to verify the equation:

$$dz - p\,dx - q\,dy = 0,$$

and upon determining u by the equation:

$$p' - a\,q' - ax' - by' = 0.$$

In geometric language, one thus obtains the set in the space of (x, y, z) that consists of a curve and a developable that is circumscribed by that curve, and Cauchy's theorem shows that *the second-order partial differential equation that is equivalent to system* (3) *always admits one and only one surface integral in the space of* (x, y, z) *that passes through an arbitrarily given curve and is inscribed along that curve by an arbitrarily given developable.*

IV.

Cauchy's theorem exhibits the importance of the property of the system (I) from which each integral element E_p belongs to at least one integral element E_{p+1} . This legitimizes the following definition:

We say that a system of total differential equations is of GENUS *n if the integral elements with respect to that system satisfy the following conditions:*

At least one integral element E_1 passes through an arbitrary point. At least one integral element E_2 passes through an arbitrary integral element E_1, etc.

At least one integral element E_n passes through an arbitrary integral element E_{n-1} .

Finally, no integral element E_{n+1} passes through an arbitrary integral element E_n .

More precisely, we suppose that:

∞^{r_1} integral elements	E_1	pass through	an arbitrary	point,
∞^{r_2} “	E_2	“	“	integral E_1 ,
∞^{r_n} “	E_n	“	“	integral E_{n-1} ,

where some of the numbers $r_1, r_2, \ldots, r_n$ can be zero, and we continue to let r denote the number of variables, which amounts to saying that there are ∞^r points.

We also sometimes say that the system, *when considered as having $i \leq n$ independent variables,* is *in involution.*

A system of genus zero will necessarily imply that:

$$dx_1 = dx_2 = \ldots = dx_r = 0;$$

one can omit such systems.

One recognizes the following properties of a system of genus n immediately from the preceding and from Cauchy's theorem:

A system of genus n always admits at least one integral multiplicity M_1 that passes through an arbitrary point, an integral multiplicity M_1 passes through an arbitrary integral multiplicity, etc., an integral multiplicity M_n passes through an arbitrary integral multiplicity M_{n-1} .

We agree to say that an integral element E_n is *singular* if it belongs to at least one integral element E_{n+1} , that an integral element E_{n-1} is singular if it belongs to at most ∞^{r_n} integral elements E_n , or if the ∞^{r_n} integral elements to which it belongs are all singular, etc., and finally that a point is singular if it belongs to at most ∞^{r_1} integral elements E_1, or if the ∞^{r_1} integral linear elements that issue from it are all singular.

Since the conditions that a singular integral element must satisfy are conditions of *equality*, one neatly sees that one can always, and in an infinitude of ways, find a sequence of integral elements:

$$E_0, E_1, E_2, \ldots, E_n ,$$

where E_0 denotes a point such that each element of the sequence belongs to the one that follows it, and where none of them are singular. One can then confirm the existence of an integral multiplicity that passes through the point E_0 and admits the element E_1, of an integral multiplicity M_2 that passes through M_1 and admits the element E_2, …, of an integral multiplicity M_n that passes through M_{n-1} and admits the element E_n ; *however, by contrast, one can confirm that no integral multiplicity M_{n+1} passes through M_n*, since the element E_n does not belong to any integral element E_{n+1} .

Therefore, *a system of genus n admits no integral multiplicity M_{n+1} that passes through an ordinary integral multiplicity.*

These propositions show the importance of the *genus* of a system of total differential equations.

V.

The numbers r, r_1, r_2, …, r_n play a big role in the study of the indeterminacy of the most general n-dimensional integral multiplicity. Also, before beginning that study, we shall prove some remarkable properties of these numbers.

We first prove the following theorem:

Each number in the sequence:

$$r, r_1, r_2, \ldots, r_n ,$$

is greater than the following one by at least one unit.

Indeed, first of all, the linear elements that issue from a point in space depend upon r-1 parameters. Now, not all of these linear elements are necessarily integral; therefore:

$$r - 1 \geq r_1 .$$

In a general manner, take a non-singular integral element E_{p-1} . By hypothesis, that element belongs to ∞^{r_p} integral elements E_p, at least one of which is not singular. Each of them can be defined by an (integral) linear element that is independent of E_{p-1}, which gives us $r_p + 1$ linear elements:

$$\varepsilon, \varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{r_p}$$

that are independent of each other and of E_{p-1}, and we may suppose that the integral element (E_{p-1}, ε), for example, is not singular. That element, in turn, belongs to $\infty^{r_{p+1}}$ integral elements E_{p+1}, each of which can be defined by means of a linear element that is independent of (E_{p+1}, ε), but which necessarily depends upon E_{p-1}, ε, ε_1, ε_2, …, ε_{r_p} . It is therefore necessary that one must find $r_{p+1} + 1$ such independent elements. One then has:

$$r_p \geq r_{p+1} + 1.$$

Q. E. D.

It results from this that each of the numbers:

$$r,\ r_1 + 1,\ r_2 + 1,\ \ldots,\ r_i + i,\ \ldots,\ r_{n-1} + n - 1, \qquad r_n + n$$

is equal to at least n, since these numbers cannot be increasing, and the last of them is equal to at least n.

Here is a second proposition:

$\infty^{r_p + r_{p+1} - 1}$ *integral elements* E_{p+1} ($p \leq n - 1$) *pass through any non-singular integral element* E_{p-1} .

Indeed, take a non-singular integral element E_{p-1}. Let:

$$\varepsilon, \varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{r_p}$$

be $r_p + 1$ linear elements that are independent of each other and independent of E_{p-1}, and define $r_p + 1$ independent integral elements E_p. Suppose, to fix ideas, that the element (E_{r-1}, ε_1), which we denote by E_p^0, is non-singular. Finally, suppose that the integral element E_{p+1} that passes through E_p^0 is $(E_{r-1}, \varepsilon, \varepsilon_1)$, which is always permissible; let E_{p+1}^0 be that element. Any integral element E_{p+1} that passes through E_{p-1} will be obtained by appending two linear elements ε', ε'' that depend upon $\varepsilon, \varepsilon_1, \ldots, \varepsilon_{r_p}$ to E_{p-1}. In general, there will exist just one element that is a linear combination of ε', ε'' that depends upon $(\varepsilon_1, \ldots, \varepsilon_{r_p})$ (because this is also true for the particular element E_{p+1}^0). We thus see that any integral element E_{p+1} that passes through E_{p-1} can be obtained, and in only one way, by taking a linear element ε' that depends upon $r_p - 1$ parameters; thus, the same thing is true for E_p. Moreover, exactly $\infty^{r_{p+1}}$ integral elements E_{p+1} pass through E_p (because this is true for the particular non-singular element E_p^0). Therefore, one finally has that E_{p+1} depends upon:

$$r_p - 1 + r_{p+1}$$

parameters.

The proof persists just the same for $n = 1$.

We shall prove in the same manner that if $p \leq n - 2$ then $\infty^{r_p - 2 + r_{p+1} - 1 + r_{p+2}}$ *integral elements E_{p+2} pass through a non-singular integral element E_{p-1}.*

We always preserve the same notations. Let E_p^0 denote a non-singular integral element that passes through E_{p-1}, namely, (E_{p-1}, ε_2), let E_{p+1}^0 denote a non-singular integral element that passes through E_p^0, namely, $(E_{p-1}, \varepsilon_1, \varepsilon_2)$, and finally let E_{p+2}^0 denote an integral element that passes through E_{p+1}^0, namely, $(E_{p-1}, \varepsilon, \varepsilon_1, \varepsilon_2)$. Any integral element E_{p+2} can then be obtained, *and in only one way*, by appending a linear element ε' that depends upon $(\varepsilon_2, \varepsilon_3, \ldots, \varepsilon_{r_p})$ to E_{p-1} and making an integral element E_{p+2} pass through the integral element E_p that was thus determined. Indeed, the particular integral element E_{p+1}^0 contains just one integral element E_p that satisfies that condition, namely, E_p^0. Now, the element ε' depends upon r_{p-2} parameters; the same is then true for E_p. Moreover, $\infty^{r_{p+1} + r_{p+2} - 1}$ integral elements E_{p+2} pass through E_p, which is not singular (since E_p^0, in particular, is not). Therefore, the number of parameters that E_{p+2} depends upon is indeed equal to:

$$(r_p - 2) + (r_{p+1} - 1) + r_{p+2}.$$

Q. E. D.

One sees how the theorem is generalized step-by-step. In a general manner, *if $p \leq n - i$ then integral elements E_{p+i} pass through a non-singular element E_{p-1} that depend upon:*

$$(r_p - i) + (r_{p+1} - i - 1) + \ldots (r_{p+i-1} - 1) + r_{p+i} = r_p + \ldots + r_{p+i} - \frac{i(i+1)}{2}$$

arbitrary constants.

Of course, the locus of all these elements is not, in general, a planar element, except when i is zero.

In particular, an infinitude of integral elements E_n pass through a non-singular point of space that depend upon:

$$r_1 + r_2 + \ldots + r_n - \frac{n(n-1)}{2}$$

arbitrary constants. If $n = r$ then $r_1 = n - 1, \ldots, r_n = 0$, and there is just one integral element E_n .

Finally, here is one last very important theorem on the sequence of numbers r:

Each number in the sequence of positive or zero integers:

$$r - r_1 - r, \qquad r_1 - r_2 - 1, \ldots, r_{n-1} - r_n - 1$$

is equal to at least the following one.

The fact that the numbers considered are positive or zero results from the first theorem that was proved about the sequence:

$$r, r_1, \ldots, r_n .$$

To prove the stated theorem, consider a non-singular integral element E_{p-1} . It is possible to make a non-singular integral element E_p^0 pass through E_p, and then make a non-singular integral element E_{p+1}^0 pass through it, and finally make an integral element E_{p+2}^0 pass through the latter. (We suppose that $p \leq n - 2$.) Let ε, ε_1, ε_2 be three independent linear elements of E_{p-1}, and define E_{p+2}^0. These three elements are thus integral, associated with E_{p-1} , and associated with each other. Now, there exist $r_p + 1$ independent integral linear elements that are associated with E_{p-1} . We can thus denote them by:

$$\varepsilon, \varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{r_p} .$$

We seek all of the integral elements E_{p+2} that contain E_{p-1} . Each of them will contain at least one linear element ε'' that is deduced linearly from:

$$\varepsilon_2, \varepsilon_3, \ldots, \varepsilon_{r_p} ,$$

and, in general, it will contain just one (like E_{p+2}^{0}). Likewise, it will contain one, and, generally only one, linear element ε' that is deduced linearly from:

$$\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{r_p},$$

and finally one and only one ε'' that is deduced linearly from:

$$\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{r_p}.$$

One thus sees that, in general, a desired element E_{p+2} will be *defined* by the three linear elements $\varepsilon, \varepsilon', \varepsilon''$. Each of them depends upon $r_p - 2$ parameters, which makes:

$$3(r_p - 2)$$

parameters, in all. In order for the element to be integral, it is necessary and sufficient that these three elements be pair-wise associated. Now, an arbitrary element ε that is deduced linearly from $\varepsilon, \varepsilon_1, \ldots, \varepsilon_{r_p}$ is associated with $r_{p+1} + 1$ other independent elements of the same form. In other words, in order to express the idea that an arbitrary element that is deduced linearly from $\varepsilon, \ldots, \varepsilon_{r_p}$ and, in turn, depends upon r_p parameters is associated with a particular element of the same form, it is necessary that these r_p parameters satisfy $r_p - r_{p+1} - 1$ relations. Upon returning to our three elements $\varepsilon, \varepsilon', \varepsilon''$, we then see that in order to express the idea that two of them are associated, *at most* $r_p - r_{p+1}$ relations between their parameters are necessary, which gives *at most:*

$$3(r_p - r_{p+1} - 1)$$

relations, in all. Since there are:

$$3(r_p - 2)$$

parameters, one sees that *the integral elements E_{p+2} that pass through a non-singular integral element E_{p-1} depend upon* at most:

$$3(r_p - 2) - 3(r_p - r_{p+1} - 1) = 3\, r_{p+1} - 3$$

parameters.

Now, from a preceding theorem, this number of parameters is equal to:

$$r_p + r_{p+1} + r_{p+2} - 3,$$

so one thus has:

$$r_p + r_{p+1} + r_{p+2} - 3 \geq 3\, r_{p+1} - 3;$$

i.e.:

$$r_p + r_{p+1} \geq r_{p+1} - r_{p+2}.$$

Q. E. D.

The proof applies just the same if p is equal to 1.

One can complete this theorem with the following remark:

If n is the genus of the system then one has:

$$r_{n-1} - r_n - 1 \geq r_n .$$

Indeed, let E_{n-2} be a non-singular integral element. Let (E_{n-2}, ε) or E_{n-1}^0 denote a non-singular integral element that passes through E_{n-2} and let $(E_{n-2}, \varepsilon, \varepsilon_1)$ or E_n^0 denote a non-singular integral element that passes through E_{n-1}^0. One can find $r_{n-1} + 2$ independent integral linear elements that are associated with E_{n-2}, and, like ε and ε_1, are already two of them; one can denote them by:

$$\varepsilon, \varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{r_{n-1}} .$$

Exactly ∞^{r_n} integral elements E_n pass through E_{n-1}^0. We can thus suppose that they are all deduced from:

$$(E_{n-2}, \varepsilon, \varepsilon_1), \quad (E_{n-2}, \varepsilon, \varepsilon_2), \quad \ldots, \quad (E_{n-2}, \varepsilon, \varepsilon_{r_{n+1}}).$$

Now, take the integral element $(E_{n-2}, \varepsilon, \varepsilon_1)$; it also belongs to (at least) ∞^{r_n} integral elements E_n . One can obtain each of them by means of a linear element that is deduced from:

$$\varepsilon, \varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{r_{n-1}}$$

and associated with ε_1 . Now, if we first take the ones that are deduced from:

$$\varepsilon, \varepsilon_1, \ldots, \varepsilon_{r_{n+1}}$$

then *there is only* ε, since otherwise one would have that the element ε_2, for example, i.e.:

$$(E_{n-2}, \varepsilon, \varepsilon_1, \varepsilon_2),$$

will be integral, which is contrary to hypothesis, since it passes through the *non-singular* element E_n^0. There thus exist at least r_n independent linear elements that can be deduced from:

$$\varepsilon_{r_n+2}, \ldots, \varepsilon_{r_n-1} ;$$

one then necessarily has ([1]):

$$r_{n-1} - r_n - 1 \geq r_n .$$

Q. E. D.

The sequence of inequalities:

([1]) The proof does not persist when $n = 1$. However, the theorem does not cease to be true, and it is pointless to give the proof.

$$(5) \qquad r - r_1 - 1 \geq r_1 - r_2 - 1 \geq r_{n-1} - r_n - 1 \geq r_n$$

results from these various theorems.

The numbers in this sequence play a very big role. We denote them by:

$$s, s_1, s_2, \ldots, s_n$$

by setting:

$$(6) \qquad \begin{cases} s \;\; = r - r_1 - 1, \\ s_1 = r_1 - r_2 - 1, \\ \ldots\ldots\ldots\ldots\ldots\ldots, \\ s_{n-1} = r_{n-1} - r_n - 1, \\ s_n \;\; = r_n . \end{cases}$$

An interesting special case is the one in which there is a zero term in the sequence of s. Suppose that s_ν ($\nu < n$) is the first one that enjoys that property. One will then necessarily have, from the inequalities (5):

$$s_\nu = s_{\nu+1} = \ldots = s_n = 0.$$

The following considerations permit one to account for this result in another way, and, at the same time, lead to some new and important properties of these systems.

Consider a non-singular integral element $E_{\nu-1}$. Let $(E_{\nu-1}, \varepsilon)$ be a non-singular integral element that issues from $E_{\nu-1}$, where ε denotes an integral linear element that is independent of $E_{\nu-1}$ and associated with it. $\infty^{r_{\nu+1}}$ $\nu+1$-dimensional integral elements pass through that element $(E_{\nu-1}, \varepsilon)$; i.e., since $r_{\nu+1}$ is equal to $r_{\nu-1}$, from the fact that $s_\nu = 0$, one can have r_ν and only r_ν mutually independent integral linear elements that are independent of $(E_{\nu-1}, \varepsilon)$ and associated with $E_{\nu-1}$ and ε, namely:

$$\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{r_\nu} .$$

Now, one cannot find more than $r_\nu + 1$ mutually independent integral linear elements that are independent of $E_{\nu-1}$ and associated with it. Therefore, *any integral linear element that is associated with* $E_{\nu-1}$ *is deduced linearly from:*

$$E_{\nu-1}, \varepsilon, \varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{r_\nu} .$$

It results from this that any two elements ε are associated – for example, ε_1 and ε_2 ; because the integral element $(E_{\nu-1}, \varepsilon_1)$ that belongs to at least $\infty^{r_\nu+1} = \infty^{r_{\nu-1}}$ $(n + 1)$-dimensional integral elements is associated with *at least* r_ν mutually independent integral linear elements that are independent of $(E_{\nu-1}, \varepsilon_1)$, and since there are *at least* r_ν of them that enjoy that property, namely:

$$\varepsilon, \varepsilon_2, \ldots, \varepsilon_{r_\nu},$$

one sees, in particular, that $(E_{\nu-1}, \varepsilon_1)$ is associated with ε_2. One see, moreover, that the element $(E_{\nu-1}, \varepsilon_1)$ belongs to exactly $\infty^{r_\nu+1}$ $(\nu + 2)$-dimensional integral elements. One thus has:

$$r_{\nu+2} = r_\nu - 2,$$

and so on: A $(\nu - 1 + r_\nu)$-dimensional element that passes through $E_{\nu-1}$, namely, $(E_{\nu-1}, \varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{r_\nu})$, belongs to exactly *one* = $\infty^{r_\nu - r_\nu}$ ν-dimensional integral element, and finally, one and only one $\nu + r_\nu$-dimensional integral element, namely, $(E_{\nu-1}, \varepsilon, \varepsilon_1, \ldots, \varepsilon_{r_\nu})$, and no $\nu + r_\nu + 1$-dimensional integral element passes through that element. Finally, it results from this that all of the integral elements that pass through $E_{\nu-1}$ are non-singular elements.

In summary, *if one has:*

$$s_\nu = r_\nu - r_{\nu+1} - 1 = 0$$

then the genus of the system is:

$$n = \nu - r_\nu .$$

One and only one integral element E_n passes through a non-singular integral element $E_{\nu-1}$. The locus of integral elements that through $E_{\nu-1}$ is the element E_n. Moreover, one has the equalities:

$$r_\nu = r_{\nu+1} + 1 = r_{\nu+2} + 2 = \ldots = n - \nu,$$

which implies that:

$$s_\nu = s_{\nu+1} = \ldots = s_{n-1} = s_n = 0.$$

In particular, if ν is equal to 1 then any two integral linear elements that issue from a non-singular point are associated. An integral element is simply an element that is formed from integral linear elements.

To conclude this paragraph, we shall determine the numbers $r, r_1, \ldots, r_n$ for a system of h total differential equations in r variables, *while supposing that the coefficients are not subject to any specialization.*

One will obviously first have:

$$r_1 = r - (h + 1).$$

In a general manner, suppose that the genus n is greater than p and that one knows r_p. If E_{p-1} then denotes an arbitrary integral element then any integral linear element that is associated with E_{p-1} can be deduced linearly from E_{p-1} and $r_p + 1$ other linear elements:

$$\varepsilon, \varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{r_p}.$$

We seek to discover how many integral elements E_{p+1} pass through the integral element (E_{p-1}, ε). In order to do this, one must append a linear element ε' to ε that can be deduced from:

$$\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{r_p}$$

and is associated with ε. Now, that element ε' depends upon $r_p - 1$ parameters, and one needs h equations to express the fact that the element is associated with ε. If $r_p - 1 \geq h$, then one has that:

$$r_{p+1} = r_p - 1 - h,$$

and if $r_p - 1 < h$ then there is no p + 1-dimensional integral element. One thus sees that *one passes from the number r to the following one by subtracting* h + 1, and repeating that as many times as possible:

$$r_1 = r - (h + 1),$$
$$r_2 = r - 2(h + 1),$$
$$\ldots\ldots\ldots\ldots\ldots\ldots$$

As a consequence, the genus n is the quotient, up to a unit, of r by h + 1, and r_n is equal to the remainder of the division.

$$r_n = r - (h + 1) = k.$$

The genus of a system whose coefficients are not specialized is therefore equal to the quotient, up to a unit, of the number of variables by the number of equations plus one.

Here, the sequence of number s is:

$$s = s_1 = \ldots = s_{n-1} = h, \qquad s_n = k.$$

In particular, if there is only one equation then the genus is the mean of the number of variables; if there are $2n$ or $2n$ + 1 variables then it is n. In the first case, an integral multiplicity M_{n-1} belongs to one and only integral multiplicity M_n . This result is well known.

VI.

We shall now look for a system of conditions that *determine* any integral multiplicity M_n that is required to satisfy these conditions, where n denotes the *genus* of the system of total differential equations (1).

We first make the obvious remark that all of the results that were proved up to here persist if one adjoins a certain number of *finite* equations:

$$(1)' \qquad \begin{cases} f_1(x_1, x_2, \ldots, x_r) = 0, \\ \ldots\ldots\ldots\ldots\ldots\ldots, \\ f_h(x_1, x_2, \ldots, x_r) = 0, \end{cases}$$

to equations (1).

Indeed, it suffices to append to equations (1) the ones that one obtains by taking the total differentials of equations (1)′ and considering only the points in space that satisfy equations (1)′ in the new system that is obtained, which we will refer to by the name of *integral points.*

We seek what sort of variations will preserve the genus and the integers r_i when one then adds h *arbitrary* finite equations. In summary, one obtains a new system of total differential equations whose integral multiplicities are those integral multiplicities of the original system that are required to be completely contained in the arbitrary multiplicity m that is represented by equations (1)′. It is obvious, first of all, that the number r is reduced by h units; in other words, there are now only ∞^{r-h} points to consider. *We suppose that these points are not all singular* (with respect to the original system), since otherwise the multiplicity μ would *not* be called *arbitrary*.

Now, take a *non-singular* point E_0 of μ. ∞^{r_1} integral elements E_1 pass through that point; i.e., one can find $r_1 + 1$ independent integral linear elements:

$$\varepsilon, \quad \varepsilon_1, \; \varepsilon_2, \; \ldots, \; \varepsilon_{r_1} .$$

On the other hand, the element e_{r-h}, which is the locus of all linear elements of μ, also contains $r - h$ independent linear elements. If one has:

$$r_1 + 1 + r - h \leq r$$

then suppose that e_{r-h} contains no integral linear element, which is the general case. The second system then has genus zero:

$$r' = r - h, \qquad r_1 - h < 0.$$

If, on the contrary:

$$r_1 + 1 + r - h > r$$

then e_{r-h} contains *at least* $r_1 + 1 - h$ integral linear elements. We suppose, and this is obviously the general case, that e_{r-h} contains exactly $r_1 + 1 - h$ of them. One will then have:

$$r' = r - h, \qquad r_1' = r_1 - h.$$

In addition, we suppose that the ∞^{r_1-h} integral linear elements of e_{r-h} *are not all singular*.

Then, let ε be a non-singular integral linear element of e_{r-h}. ∞^{r_1} integral elements E_2 of the system (1) pass through ε; i.e., there exist $r_2 + 1$ independent integral linear elements that are associated with ε, namely:

$$\varepsilon, \quad \varepsilon_1, \; \varepsilon_2, \; \ldots, \; \varepsilon_{r_2+1} .$$

On the other hand, e_{r-h} contains $r - h - 1$ independent linear elements along with ε. If one has:

$$(r_2 + 2) + (r - h - 1) \leq r,$$

i.e.:

$$r_2 < h,$$

then e_{r-h} will not, in general, contain the integral element E_2 ; that is what we suppose. In that case, one thus has:

$$r_2 < h, \qquad n' = 1, \qquad r' = r - h, \qquad r_1' = r_1 - h.$$

However, if $r_2 \geq h$ then e_{r-h} contains *at least* $r_2 + 1$ independent integral linear elements that are associated with ε. We suppose, and this is obviously the general case, that e_{r-h} contains *exactly* $r_2 + 1 - h$ of them; i.e., that ∞^{r_2-h} integral elements that are contained in e_{r-h} pass through ε. Moreover, we suppose that at least one of then is not singular. One will then have:

$$r' = r - h, \qquad r_1' = r_1 - h, \qquad r_2' = r_2 - h, \qquad n \geq 2.$$

One sees how one can proceed and what the properties are that one supposes for the multiplicity μ in order that the line should have the qualifier *arbitrary*. *In this case, if r_m denotes the last number r that is greater than or equal to h then the genus becomes equal to m, and one has:*

$$r' = r - h, \qquad r_1' = r_1 - h, \qquad \ldots, \qquad r_m' = r_m - h.$$

It is clear that the conditions that a multiplicity μ must satisfy in order to not be arbitrary are conditions of *equality*. In particular, on any multiplicity one can find a non-singular point E_0, a non-singular integral element E_1^0 that issues from E_0, a non-singular integral element E_2^0 that issues from E_1^0, …, and a non-singular integral element E_m^0 that issues from E_{m-1}^0. However, no integral element E_{m+1} that belongs to the multiplicity passes through E_m^0, and the number of integral elements E_i that belong to μ that pass through the integral element E_{i-1}^0 $(i \leq m)$ must be exactly ∞^{r_i-h}.

This having been established, we shall consider an arbitrary non-singular point μ_0 . Make an arbitrary $r - r_1$-dimensional multiplicity μ_{r-r_1} pass through this point, an arbitrary $r - r_2$-dimensional multiplicity μ_{r-r_2} pass through μ_{r-r_1}, etc., and finally, an arbitrary $r - r_n$-dimensional multiplicity μ_{r-r_n} pass through $\mu_{r-r_{n-1}}$ ([1]). Each of these

---

([1]) This is always possible. Indeed, consider a non-singular integral element E_1^0 that issues from E_0, a non-singular integral element (E_1^0, ε_1), or E_2^0, that contains E_1^0, …, a non-singular integral element (E_{n-1}^0, ε_{n-1}), or E_n^0, that contains E_{n-1}^0. Then, let e_{r-r_1} denote an element that is formed from E_1^0 and $(r - r_1 - 1)$ other non-integral linear elements, let e_{r-r_2} denote an element that is formed from e_{r-r_1}, e_1, and $r_1 - r_2 - 1$ other non-integral linear elements that are not associated with E_1^0, …, and finally let e_{r-r_n} denote an element that is formed from $e_{r-r_{n-1}}$, ε_{n-1}, and $r_{n-1} - r_n - 1$ other non-integral linear elements that are not multiplicities corresponds to a certain system of total differential equations. For the multiplicity μ_{r-r_1}, one has $h = r_1$, in such a way that:

$$n' = 1, \qquad r' = r - r_1, \qquad r_1' = 0.$$

For μ_{r-r_2}, one has $h = r_2$, and in turn:

$$n'' = 2, \qquad r'' = r - r_2, \qquad r_1'' = r_1 - r_2, \quad r_2'' = 0,$$

and so on.

It results from this that the given system admits one and only one integral multiplicity M_1 that passes through μ_0 and is contained in μ_{r-r_1} (since the system that gives the integral multiplicities that are contained in μ_{r-r_1} has genus 1 and r_1' is zero). Moreover, that multiplicity is not singular, because it admits (*see* the note) a non-singular linear element.

Likewise, since the integral multiplicities that are contained in μ_{r-r_2} are given by a system of genus 2 with $r_2' = 0$ and M_1 is a non-singular integral of that system, it results, from a theorem of Cauchy, that there exists one and only integral multiplicity M_2 that passes through M_1 and is contained in μ_{r-r_1}. Moreover, that multiplicity is not singular.

One can continue step-by-step, until one has an integral M_{n-1} that is contained in $\mu_{r-r_{n-1}}$. There then exists one and only one integral M_n that passes through M_{n-1} and is contained in μ_{r-r_n}, and that multiplicity is not singular. Therefore, there finally exists no integral multiplicity M_{n+1} that passes through M_n.

In summary, upon applying Cauchy's theorem several times, one arrives at the following result:

Given:

an arbitrary point μ_0 ,
an arbitrary multiplicity μ_{r-r_1} *that passes through* μ_0,
“ μ_{r-r_2} “ μ_{r-r_1} ,
…………………………………………………………..,
“ μ_{r-r_n} “ $\mu_{r-r_{n-1}}$,

there exists one and only one integral multiplicity M_n *that passes through* μ_0 *and:*

has a multiplicity M_1 *in common with* μ_{r-r_1} ,

associated with E_{n-1}^0. It suffices to take μ_{r-r_1} to be a multiplicity that admits the element e_{r-r_1}, μ_{r-r_2} to be a multiplicity that admits the element e_{r-r_2}, etc.

“ M_2 “ μ_{r-r_2},

…………………………………………,

“ M_{n-1} “ $\mu_{r-r_{n-1}}$,

and is contained entirely in μ_{r-r_n}.

Moreover, no integral multiplicity M_{n+1} *passes through the multiplicity* M_n ([1]).

The problem that consists of finding M_n from the stated conditions will be called the *Cauchy problem*. The *general integral* will be the set of integral multiplicities M_n that can be obtained by the preceding process.

We shall now seek to formulate the Cauchy problem in an *analytical manner,* or rather, by appealing to the preceding statement of that problem we shall determine the general integral M_n *by a set of analytical conditions that exhibit its degree of indeterminancy.* In order to do this, we start from a non-singular point E_0, let ε_1 denote a non-singular integral element issues from that point, let $(\varepsilon_1, \varepsilon_2)$ denote a non-singular integral element E_2 that passes through ε_1, …, and let (E_{n-1}, ε_n) denote a non-singular integral element E_n that passes through E_{n-1}, in such a way that:

$$\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n$$

are n independent integral linear elements that are all associated with each other.

The element E_n can be defined by a system (Σ) of $r - n$ independent linear equations in $dx_1, dx_2, \ldots, dx_r$. We suppose that the indices are chosen in such a manner that these equations are soluble for $dx_{n+1}, \ldots, dx_r$. The element E_{n-1}, in turn, will be defined by the system (Σ), to which, one must adjoin a linear equation in $dx_1, dx_2, \ldots, dx_n$. Suppose that it is soluble for dx_n, namely:

$$(E_{n-1}) \qquad dx_n = \alpha_{n,1}\, dx_1 + \ldots + \alpha_{n,n-2}\, dx_{n-2} + \alpha_{n,n-1}\, dx_{n-1}\,.$$

Likewise, one will get E_{n-2} by adjoining an equation to the preceding equations that is linear in $dx_1, \ldots, dx_{n-1}$ and soluble for dx_{n-1}, for example, namely:

$$(E_{n-2}) \qquad dx_{n-1} = \alpha_{n-1,1}\, dx_1 + \ldots + \alpha_{n-1,n-2}\, dx_{n-2}\,,$$

and so on, until one gets the element E_1 that one obtains by adjoining an equation to the equations that define E_2 that is linear in dx_1, dx_2 and soluble for dx_2, for example, namely:

$$(E_1) \qquad dx_2 = \alpha_{2,1}\, dx_1\,.$$

([1]) The statement persists if the genus is *greater than n,* but then the last part must be suppressed, from the fact that no integral multiplicity M_{n+1} passes through M_n.

Now, let denote (P_0) denote the planar multiplicity that is the locus of the integral linear elements that pass through the point E_0 . It obviously contains E_n and is $(r_1 + 1)$-dimensional. It is thus defined by $r - r_1 - 1 = s$ linear equations that are soluble for s of the differentials dx_{n+1} , …, dx_r . We call these s differentials:

$$dz_1, \quad dz_2, \quad \ldots, \quad dz_s .$$

Moreover, we remark that these s equations are nothing but the given equations (1) themselves. Now, let (P_1) denote the planar multiplicity that is the locus of the integral linear elements that are associated with E_1; it is obviously contained in (P_0) and contains E_n . Moreover, it is r_2 + 2-dimensional. It is thus defined by $r - r_2 - 2 = s + s_1$ equations, among which, one finds the s equations of (P_0). One thus obtains it by adjoining to these s equations, s_1 other ones that are soluble for s_1 and some differentials other than dz_1, …, dz_s ; dx_1, …, dx_n . Upon changing the notation, let:

$$dz_1^{(1)},\ dz_2^{(1)},\ \ldots,\ dz_{s_1}^{(1)}$$

be these differentials. Likewise, the planar multiplicity (P_2) that is the locus of integral linear elements that are associated with E_2 is obtained by adjoining to the $s + s_1$ equations of (P_1), s_2 other ones that are soluble for:

$$dz_1^{(2)},\ dz_2^{(2)},\ \ldots,\ dz_{s_2}^{(2)},$$

in which the $z^{(2)}$ are s_2 variables other than x_1, …, x_n, the z and the $z^{(1)}$, and so on. The planar multiplicity (P_{n-1}) that is the locus of integral linear elements that are associated with E_{n-1} will introduce s_{n-1} variables:

$$z_1^{(n-1)},\ \ldots,\ z_{s_{n-1}}^{(n-1)},$$

and finally, the element E_n will be defined by the equations that define (P_{n-1}) and $r - s - s_1 - \ldots - s_{n-1} = r_n = s_n$ new equations that are soluble with respect to s_n variables other than x_1, …, x_n, the z, $z^{(1)}$, …, $z^{(n-1)}$, and that we call:

$$z_1^{(n)},\ z_2^{(n)},\ \ldots,\ z_{s_n}^{(n)}.$$

Finally, we can summarize the equations that define (P_0), (P_1), …, (P_{n-1}), E_n , E_{n-1} , …, E_1 in the following table:

$$
(E_{n-2})\left\{\begin{array}{l}
(E_{n-1})\left\{\begin{array}{l}
(E_n)\left\{\begin{array}{l}
(P_{n-1})\left\{\begin{array}{l}
(P_1)\left\{\begin{array}{ll}(P_0): & dz \;\;\;= [dz^{(1)}, dz^{(2)}, \ldots, dz^{(n)}, dx],\\ & dz^{(1)} = [dz^{(2)},\ldots,dz^{(n)},dx],\end{array}\right.\\
\qquad\qquad\qquad \ldots\ldots\ldots\ldots\ldots\ldots,\\
\qquad\qquad\qquad dz^{(n-1)} = [dz^{(n)}, dx],
\end{array}\right.\\
\qquad\qquad\qquad\qquad dz^{(n)} = [dx],
\end{array}\right.\\
\qquad\qquad\qquad\qquad dx_n = \alpha_{n,1}dx_1 + \alpha_{n,2}dx_2 + \cdots + \alpha_{n,n-1}dx_{n-1},
\end{array}\right.\\
\qquad\qquad\qquad\qquad dx_{n-1} = \alpha_{n-1,1}dx_1 \cdots + \alpha_{n-1,n-2}dx_{n-2},
\end{array}\right.
$$

$$
\begin{aligned}
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,\\
&dx_2 = \alpha_{2,1}dx_1.
\end{aligned}
$$

The first row expresses the idea that each of the differentials dz_1, dz_2,…, dz_s is expressed as a linear combination of the differentials $dz_1^{(1)}$, …, dx_n .

Having made these conventions, we make the following transformation of coordinates: Without changing the variables z, $z^{(1)}$, …, $z^{(n-1)}$, we take the new variables:

$$
\begin{aligned}
x_1' &= x_1,\\
x_2' &= x_2 - \alpha_{21}\, x_1,\\
x_3' &= x_3 - \alpha_{31}\, x_1 - \alpha_{32}\, x_2,\\
&\ldots\ldots\ldots\ldots\ldots\ldots,\\
x_n' &= x_n - \alpha_{n1}\, x_1 - \alpha_{n2}\, x_2 - \ldots - \alpha_{n,\,n-2}\, x_{n-1} .
\end{aligned}
$$

In other words, we suppose that the coefficients α_{ij} are all zero.

Finally (once that coordinate transformation is performed), we denote the coordinates of the point E_0 by:

$$
a_1, \ldots, a_n\,; \qquad c_1, \ldots, c_s\,;\; c_1^{(1)}, \ldots, \qquad c_{s_1}^{(1)}; \quad \ldots, \qquad c_1^{(n)}, \quad \ldots,\; c_{s_n}^{(n)} .
$$

We remark that in the last case any integral multiplicity M_n that admits the element E_n can be defined by $r - n$ equations that are soluble for z, $z^{(1)}$, …, $z^{(n)}$ (from the form itself of the equations for E_n). The same will be true for any integral multiplicity M_n that admits a sufficiently close element of E_n . One can thus take x_1, x_2, …, x_n to be the independent variables of these multiplicities.

This being the case, in order to be sure of obtaining *arbitrary* multiplicities μ_{r-r_1}, μ_{r-r_2}, …, we seek to make an element e_{r-r_1} that admits one and only one integral linear element E_1 pass through E_0 , i.e., to *cut the element* (P_0) *along* E_1 , and to make an element e_{r-r_2} that admits just one two-dimensional integral element that issues from E_1 – namely, E_2 – pass through e_{r-r_1}, i.e., *cut the element* (P_1) *along* E_2, etc., and make an element e_{r-r_n} that that admits just one n-dimensional integral element that issues from E_{n-1} – namely, E_n – pass through $e_{r-r_{n-1}}$, i.e., *cut the element* (P_{n-1}) *along* E_n. Any multiplicity μ_{r-r_i} that admits the element e_{r-r_i}, *or a sufficiently close element*, will obviously satisfy the conditions that were imposed upon *arbitrary* multiplicities. Now, it is indeed easy to find elements e_{r-r_n}, $e_{r-r_{n-1}}$, …, e_{r-r_1} that enjoy the properties that we just stated. It suffices to take e_{r-r_n} to be the system:

$$dz^{(n)} = [dx],$$

and one takes $e_{r-r_{n-1}}$ to be the system that is obtained by adjoining to the preceding equations, the following ones:

$$dz^{(n-1)} = [dz^{(n)}, dx],$$
$$dx_n = 0,$$

and one takes $e_{r-r_{n-2}}$ to be the system that is obtained by adjoining to the preceding equations, the equations:

$$dz^{(n-2)} = [dz^{(n-1)}, dz^{(n)}, dx],$$
$$dx_{n-1} = 0,$$

and so on. The brackets in the right-hand sides denote the same linear combinations as in the equations that define (P_0), (P_1), …, (P_{n-1}), E_n.

This being the case, we are justified in defining μ_{r-r_n} by the equations:

$$(A_n) \qquad \begin{cases} z_1^{(n)} = \varphi_1^{(n)}(x_1, x_2, \ldots, x_n), \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \\ z_{s_n}^{(n)} = \varphi_{s_n}^{(n)}(x_1, x_2, \ldots, x_n), \end{cases}$$

defining $\mu_{r-r_{n-1}}$ by the preceding equations and the following ones:

$$(A_{n-1}) \qquad \begin{cases} z_1^{(n-1)} = \varphi_1^{(n-1)}(x_1, x_2, \ldots, x_{n-1}), \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \\ z_{s_{n-1}}^{(n-1)} = \varphi_{s_{n-1}}^{(n-1)}(x_1, x_2, \ldots, x_{n-1}), \end{cases}$$

$$(B_n) \qquad x_n = a_n,$$

and so on, until μ_{r-r_1} is defined by the equations that were already written, and:

$$(A_1)\qquad \begin{cases} z_1^{(1)} = \varphi_1^{(1)}(x_1), \\ \dots\dots\dots\dots\dots\dots, \\ z_{s_1}^{(1)} = \varphi_{s_1}^{(1)}(x_1), \end{cases}$$

$$(B_2)\qquad x_2 = a_2 ,$$

and finally the point μ is defined by all of the equations that were already written, along with:

$$(A_0)\qquad \begin{cases} z_1 = \varphi_1, \\ z_2 = \varphi_2, \\ \dots\dots\dots, \\ z_s = \varphi_s, \end{cases}$$

$$(B_1)\qquad x_1 = a_1 .$$

In these formulas, *the quantities $\varphi_1, \varphi_2, \dots, \varphi_s$ are some arbitrary constants that are sufficiently close to $c_1, c_2, \dots, c_s$. As for the functions $\varphi^{(1)}, \varphi^{(2)}, \dots, \varphi^{(n)}$, they are arbitrary functions that are holomorphic in a neighborhood of:*

$$x_1 = a_1, \qquad x_2 = a_2, \qquad \dots, \qquad x_n = a_n ,$$

and are such that for this system of values these functions and their first-order partial derivatives take values that are sufficiently close to certain fixed values.

With these hypotheses, there will exist one and only one integral multiplicity M_n that passes through μ_0 and has a one-dimensional multiplicity in common with μ_{r-r_1}, a two-dimensional multiplicity in common with μ_{r-r_2}, etc., an $n-1$-dimensional multiplicity in common with $\mu_{r-r_{n-1}}$, and finally, it is completely contained in μ_{r-r_n}. On the other hand, that multiplicity is defined by:

$$r - n = s_n + s_{n-1} + \dots + s$$

functions $z^{(n)}, z^{(n-1)}, \dots, z$ of the independent variables $x_1, x_2, \dots, x_n$. To say that M_n is contained in μ_{r-r_n} is to say that the first s_n functions $z_1^{(n)}, \dots, z_{s_n}^{(n)}$ are equal to the given functions $\varphi_1^{(n)}, \dots, \varphi_{s_n}^{(n)}$. On the other hand, if M_n has an $n-1$-dimensional multiplicity in common with $\mu_{r-r_{n-1}}$ then this multiplicity can be obtained by setting $x_n = a_n$ in the expressions for the functions $z, z^{(1)}, \dots$; it is then necessary that the s_{n-1} functions $z_1^{(n-1)}, \dots, z_{s_{n-1}}^{(n-1)}$ must reduce to the given functions $\varphi_1^{(n-1)}, \dots, \varphi_{s_{n-1}}^{(n-1)}$ for $x_n = a_n$, and so on.

It results from this that *in the indicated limits, system* (1), *when considered as defining $z_1, \dots, z_{s_n}^{(n)}$ as functions of $x_1, \dots, x_n$, admits one and only one solution for which the unknown functions are holomorphic in a neighborhood of $x_1 = a_1, \dots, x_n = a_n$, and such that the s_n functions $z^{(n)}$ satisfy:*

$$
\begin{array}{lll}
z_1^{(n)} \textit{ is identical to the arbitrary function} & & \varphi_1^{(n)}(x_1, \dots, x_n), \\
\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots, & & \\
z_{s_n}^{(n)} & \text{“} & \varphi_{s_n}^{(n)}(x_1, \dots, x_n),
\end{array}
$$

and for $x_n = a_n$ *the* s_{n-1} *functions* $z^{(n-1)}$ *reduce as follows:*

$$
\begin{array}{lll}
z_1^{(n-1)} \textit{ reduces to the arbitrary function} & & \varphi_1^{(n-1)}(x_1, \dots, x_n), \\
\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots, & & \\
z_{s_{n-1}}^{(n-1)} & \text{“} & \varphi_{s_{n-1}}^{(n-1)}(x_1, \dots, x_n),
\end{array}
$$

and so on, so for $x_2 = a_2, \dots, x_n = a_n$ *the* s_1 *functions* $z^{(1)}$ *reduced as follows:*

$$
\begin{array}{lll}
z_1^{(1)} \textit{ reduces to the arbitrary function} & & \varphi_1^{(1)}(x_1), \\
\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots, & & \\
z_{s_1}^{(1)} & \text{“} & \varphi_{s_1}^{(1)}(x_1),
\end{array}
$$

and finally, for $x_1 = a_1, \dots, x_n = a_n$ *the* s *functions* z *reduce as follows:*

$$
\begin{array}{lll}
z_1 \textit{ reduces to the arbitrary constant} & & \varphi_1, \\
\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots, & & \\
z_s & \text{“} & \varphi_s .
\end{array}
$$

On the other hand, it is indeed clear that *any integral multiplicity* M_n *that admits an element that is close to the particular element* E_n*, or* $(\varepsilon_1, \dots, \varepsilon_n)$*, as previously defined, can be obtained by the preceding process, since the functions and constants* φ *are perfectly determined, and in a unique manner.*

One can thus say that *any integral multiplicity* M_n *that admits an n-dimensional integral element that is sufficiently close to a given non-singular integral element is completely defined by a set of:*

$$
\begin{array}{llll}
s_n \textit{ arbitrary functions of} & n \textit{ arguments} & & x_1, x_2, \dots, x_n, \\
s_{n-1} \qquad \text{“} & n-1 & \text{“} & x_1, x_2, \dots, x_{n-1}, \\
\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots, & & & \\
s_1 \qquad \text{“} & 1 & \text{“} & x_1, \\
s \textit{ arbitrary constants,} & & &
\end{array}
$$

under the single condition that for certain given values of the independent variables, the arbitrary elements take values that are sufficiently close to certain fixed constants, as well as their first-order derivatives.

It is in this sense that one can say that the general integral M_n depends upon s arbitrary constants, s_1 arbitrary functions of one argument, etc., and s_n arbitrary functions of n arguments.

One can say that *the numbers of the sequence:*

$$(S) \qquad s, s_1, s_2, \ldots, s_n$$

measure the indeterminacy of the general integral M_n . The geometric origin of these numbers shows that *the measure of the indeterminacy does not change if one performs an arbitrary change of variables*, because this amounts to performing a simple homographic transformation on the integral elements that issue from a point, which obviously changes none of the values of the numbers r, and in turn, the numbers s.

Moreover, recall the property of the sequence (S) that is expressed by the inequalities:

$$s \geq s_1 \geq s_2 \geq \ldots \geq s_{n-1} \geq s_n ,$$

and finally the values of r in terms of the s:

$$\begin{aligned}
r_n &= s_n , \\
r_{n-1} &= s_n + s_{n-1} + 1, \\
r_{n-2} &= s_n + s_{n-1} + s_{n-2} + 1, \\
r_1 &= s_n + s_{n-1} + \ldots + s_1 + n - 1, \\
r &= s_n + s_{n-1} + \ldots + s + n.
\end{aligned}$$

As a particular case, if we take a system of h total differential equations in r variables with arbitrary coefficients then we have seen that the genus n is equal to the quotient, up to a unit, of r by $h + 1$, and upon denoting the remainder by k, one has:

$$s = s_1 = \ldots = s_{n-1} = h, \qquad s_n = k.$$

One thus has the following theorem:

The general integral M_n *of a system of h total differential equations in r variables whose coefficients are arbitrary functions, and where n denotes the quotient, up to a unit, of r by h* + 1, *and k denotes the remainder, depends upon:*

k arbitrary functions of		*n arguments,*		
h	"	$n - 1$	"	,
h	"	$n - 2$	"	,
………………………………………				,
h	"	1	"	,

and h arbitrary constants.

This is, with much more precision, the result that was proved by Biermann. *One can add that there is no n* + 1*-dimensional integral, in general.*

If h is equal to 1 and r is even, and consequently equal to $2n$, then there is no arbitrary function of n arguments. If r is odd, and consequently equal to $2n + 1$, then there is an arbitrary function of n arguments.

We return to the general case. The stated results persist, *even if the genus is greater than n*, with the condition that one take s_n to be the value r_n and the other s_i equal to the values $r_i - r_{i+1} - 1$. *It suffices that the given system, when considered as having n independent variables, should be in involution.* However, if the genus is greater than n then s_n can be *greater than* s_{n-1} .

The preceding results simplify if s_n is zero. The general integral then depends upon only arbitrary functions of at most $n - 1$ arguments.

The analytic search for the integral M_n amounts to the integration of n successive Kowalewski systems. The first one gives the s functions of x_1 to which the $z_1, z_2, \ldots, z_s$ reduce when one makes:

$$x_2 = a_2 , \qquad \ldots, \qquad x_n = a_n .$$

It is a system of ordinary differential equations that one obtains by replacing the $z^{(1)}$ with the $\varphi^{(1)}(x_1)$, the $z^{(2)}$ with the $\varphi^{(2)}(x_1, a_2)$, …, and the $z^{(n)}$ with the $\varphi^{(n)}(x_1, a_2, \ldots, a_n)$ in the equations of the given system.

The second Kowalewski system gives the $s + s_1$ functions of x_1, x_2 to which the $z_1, \ldots, z_s$, reduce $z_1^{(1)}, \ldots, z_{s_1}^{(1)}$ when one makes:

$$x_3 = a_3 , \qquad \ldots, \qquad x_n = a_n ,$$

where these functions reduce to known functions of x_1 for $x_2 = a_2$, and so on. The last system gives the $s + s_1 + \ldots + s_{n-1}$ functions of $x_1, x_2, \ldots, x_{n-1}$ that $z_1, \ldots, z_{s_{n-1}}^{(n-1)}$ reduces to when one makes:

$$x_n = a_n ,$$

where these functions reduce to known functions of $x_1, \ldots, x_{n-2}$ for $x_{n-1} = a_{n-1}$.

In order to clarify all of the preceding results by means of a very simple example, take the system that is formed from the single equation:

$$dz - p\,dx - q\,dy = 0, \tag{1}$$

where x, y, z, p, q are five variables. Here, there is one equation that expresses the idea that two integral linear elements are associated. It is:

$$dx\,\delta p - dp\,\delta x + dy\,\delta q - dq\,\delta y = 0. \tag{2}$$

Here, $r = 5$ and $r_1 = 3$. As for r_2, the equations that define the integral linear elements that are associated with a given integral linear element ($\delta x, \delta y, p\,\delta x + q\,\delta y, \delta p, \delta q$) are two independent ones in number, namely:

$$dz - p\,dx - q\,dy = 0,$$
$$\delta p\,dx + \delta q\,dy - \delta x\,dp - \delta y\,dq = 0;$$

As a result, $r_2 = 1$. One thus has:

$$s = 1, \qquad s_1 = 1, \qquad s_2 = 1.$$

A non-singular point E_0 is, for example:

$$x = y = z = p = q = 0.$$

An integral element E_2 that passes through this point is, for example:

$$(E_2) \qquad dz = dp = dq = 0,$$

and a non-singular integral element E_1 that is contained in E_2 is, for example:

$$(E_1) \qquad dz = dp = dq = dy = 0.$$

Here, the element (P_0) is given by (1), where one makes $p = q = 0$:

$$(P_0) \qquad dz = 0,$$

so from (2), the element (P_1) is given by:

$$(P_1) \qquad dz = dp = 0.$$

There thus exists one and only one integral that if formed from three functions z, p, q of x and y that are holomorphic in a neighborhood of $x = y = 0$, and are such that:

q is identical to $f(x, y)$,
p reduces to $\varphi(x)$ for $y = 0$,
z reduces to c for $x = y = 0$,

where c is a very small constant, f and φ are arbitrary functions that are holomorphic in a neighborhood of $x = 0$, $y = 0$, and take on very small values for $x = y = 0$, along with their first-order derivatives.

Here, there are two Kowalewski systems. The first one gives a function z of x that reduces to c for $x = 0$ when $p = \varphi(x)$ and $q = f(x, 0)$. It is obviously given by:

$$\frac{dz}{dx} = p = \varphi(x),$$

so:

$$z = c + \int_0^r \varphi(x)\,dx\,.$$

The second Kowalewski system gives the functions p and z of x, y that reduce to $\varphi(x)$ and $c + \int_0^r \varphi(x)\,dx$, respectively, when one makes $q = f(x, y)$. This system is [*see* the formulas (II) of paragraph IV]:

$$\frac{\partial f}{\partial y} - f(x, y) = 0,$$

$$\frac{\partial p}{\partial y} - \frac{\partial f}{\partial x} = 0,$$

and give:

$$z = c + \int_0^x \varphi(x)\,dx + \int_0^y f(x, y)\,dy,$$

$$p = \varphi(x) + \int_0^y \frac{\partial f}{\partial x}\,dy,$$

$$q = f(x, y).$$

We shall conclude this paragraph by giving some definitions. In the sequence:

$$s, \quad s_1, \ \ldots, \quad s_n,$$

which measures the indeterminacy of the general integral M_n of a system (1) of genus n, the first number s is nothing but the number of independent equations in $dx_1, \ldots, dx_r$ in that system (1); i.e., upon preserving the notations of § 1, it is the degree of the principal minor of the matrix:

$$(\Delta) \qquad \begin{Vmatrix} a_1 & a_2 & \cdots & a_r \\ b_1 & b_2 & \cdots & b_r \\ \ldots & \ldots & \ldots & \ldots \end{Vmatrix}.$$

The following number s_1 gets a special name: One calls it the *character* (1) of the system. We remark that $s + s_1$ is nothing but the number of independent equations that express the idea that a linear element $(dx_1, \ldots, dx_r)$ is integral and associated with an arbitrary integral linear element $(\delta x_1, \ldots, \delta x_r)$. Now, upon setting:

$$a_{ik} = \frac{\partial a_i}{\partial x_k} - \frac{\partial a_k}{\partial x_i}, \qquad \ldots, \qquad l_{ik} = \frac{\partial l_i}{\partial x_k} - \frac{\partial l_k}{\partial x_i},$$

these equations become:

$$(1) \qquad \begin{cases} a_1 dx_1 + a_2 dx_2 + \ldots + a_r dx_r = 0, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \\ l_1 dx_1 + l_2 dx_2 + \ldots + l_r dx_r = 0, \end{cases}$$

(1) This terminology is due, I believe, to H. von WEBER, "Zur Invariantentheorie der Systeme Pfaff'scher Gleichungen," Leipz. Ber. (1898), 207-229.

$$(2) \qquad \begin{cases} \sum_i a_{1i}\delta x_i dx_1 + \cdots + \sum_i a_{ri}\delta x_i dx_r = 0, \\ \dotfill, \\ \sum_i l_{1i}\delta x_i dx_1 + \cdots + \sum_i l_{ri}\delta x_i dx_r = 0. \end{cases}$$

Therefore, if one considers the matrix:

$$(\Delta_1) \qquad \left\| \begin{matrix} a_1 & a_2 & \cdots & a_r \\ \cdots & \cdots & \cdots & \cdots \\ l_1 & l_2 & \cdots & l_r \\ \sum a_{1r}\delta x_i & \sum a_{2r}\delta x_i & \cdots & \sum a_{ri}\delta x_i \\ \cdots & \cdots & \cdots & \cdots \\ \sum l_{1r}\delta x_i & \sum l_{2r}\delta x_i & \cdots & \sum l_{ri}\delta x_i \end{matrix} \right\|,$$

where $\delta x_1, \ldots, \delta x_r$ are arbitrary, but uniquely required to verify the equations:

$$\begin{aligned} &a_1\,\delta x_1 + \ldots + a_r\,\delta x_r = 0, \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \\ &l_1\,\delta x_1 + \ldots + l_r\,\delta x_r = 0, \end{aligned}$$

then the character s_1 *of the system is the difference between the degree of the principal minor of the matrix* (Δ_1) *and the degree of the principal minor of the matrix* (Δ).

One can give the other numbers $s_2, s_3, \ldots$, the names of the *2nd, 3rd, … character* of the system (1). They are calculated from the degrees of the principal minors, like s and s_1. However, *instead of saying that a system of genus n has the number* s_n *for its* n^{th} *character, we say that the system is of the* $(s_n + 1)^{\text{th}}$ *kind*. A system of the first kind is thus a system for which $s_n = 0$. It enjoys the property that *one and only one* integral multiplicity M_n passes through an integral multiplicity M_{n-1}.

VII.

In this paragraph, we shall occupy ourselves with systems of the first kind for which the $(n-1)^{\text{th}}$ character s_{n-1} is zero. Suppose, in a general manner, that s_n is the first number that is zero in the sequence:

$$s, \quad s_1, \; s_2, \qquad \ldots, \qquad s_n,$$

where ν is less than n. In § V, we saw some properties of these systems, which we recall:

One and only one integral element E_n passes through a non-singular integral element $E_{\nu-1}$. That element E_n is the locus of all integral elements that pass through $E_{\nu-1}$, and none of these elements is singular.

One has, moreover:

$$r_n = 0, \qquad r_{n-1} = 1, \qquad r_{n-2} = 2, \qquad \ldots, \qquad r_\nu = n - \nu, \qquad r_{n-2} \leq n - n - 2.$$

As a corollary to the property of the integral elements that pass through a non-singular integral element $E_{\nu-1}$, we shall prove the following theorem:

One and only one integral multiplicity M_n passes through a non-singular integral multiplicity $M_{\nu-1}$.

In order to prove this, we make an arbitrary multiplicity μ_{r-r_ν} pass through $M_{\nu-1}$, which is always possible, since the integral multiplicity $M_{\nu-1}$ is not singular. In particular, if $E_{\nu-1}$ is a non-singular integral element of $M_{\nu-1}$ then the multiplicity μ_{r-r_ν} will admit one and only one integral element E_ν that passes through $E_{\nu-1}$. This being the case, let M_n be an arbitrary integral multiplicity that passes through $E_{\nu-1}$. On the other hand, since the sum of the dimensions of M_n and μ_{r-r_ν} is:

$$r + n - r_\nu = r + n,$$

these two multiplicities have a multiplicity in common that is *at least* ν-dimensional, and that multiplicity is necessarily integral. However, since μ_{r-r_ν} does not admit a $\nu + 1$-dimensional integral element that passes through $E_{\nu-1}$, the integral multiplicity that is common to M_n and μ_{r-r_ν} is *exactly* ν-dimensional, namely M_ν .

This being the case, we known that *one and only one* ν-dimensional integral multiplicity that is required to be contained in the arbitrary multiplicity μ_{r-r_ν} passes through a non-singular integral multiplicity $M_{\nu-1}$. *Therefore, the multiplicity M_ν is determined in a unique manner when one is given μ_{r-r_ν} .* In other words, *if two n-dimensional integral multiplicities M_n and M'_n pass through $M_{\nu-1}$ then these two multiplicities cut μ_{r-r_ν} along the same multiplicity M_ν, and that is true for any μ_{r-r_ν} that passes through $M_{\nu-1}$.*

It results from this that the two multiplicities M_n and M'_n are identical, because if A is an arbitrary point of the first one then one can always make a multiplicity μ_{r-r_ν} pass through A and $M_{\nu-1}$. That multiplicity corresponds to an integral multiplicity M_ν that is situated on M_n and then passes through A. However, it is also situated on M'_n. Therefore, the point A belongs M'_n, and the two multiplicities coincide.

In a more precise and rigorous manner, make an arbitrary multiplicity $\mu_{r-r_\nu-1}$ pass through $M_{\nu-1}$; i.e., one that does not admit any integral element that passes through $E_{\nu-1}$ other than $E_{\nu-1}$ itself, which is always possible. Then make a family of multiplicities μ_{r-r_ν} that depend upon $r_\nu = n - \nu$ parameters and *fill up all of space* ([1]) pass through that well-defined multiplicity $\mu_{r-r_\nu-1}$. These multiplicities are all *arbitrary*, because they obviously have just one integral element E_ν that passes through $E_{\nu-1}$, and we know that any integral element that passes through $E_{\nu-1}$ is non-singular. Each of them thus contains one and only one integral multiplicity M_ν that passes through $M_{\nu-1}$, and all of these multiplicities M_ν belong to an arbitrary integral multiplicity M_ν that passes through $M_{\nu-1}$. One can add that M_n *is the locus of these multiplicities* M_ν *when the* $n - \nu$ *parameters that they depend upon are varied*, because each of them is contained in M_n, and, on the other hand, one of the multiplicities μ_{r-r_ν} (that fills up all of space) passes through an arbitrary point of M_n , and, as a result, the corresponding multiplicity M_ν . As a result, M_ν is determined in a unique manner.

We summarize the results that we just obtained in the following manner:

One and only one integral multiplicity M_n *passes through a non-singular integral multiplicity* $M_{\nu-1}$. *In order to obtain it, one makes an arbitrary multiplicity* $\mu_{r-r_\nu-1}$ *pass through* $M_{\nu-1}$ *, and then makes a family of multiplicities* μ_{r-r_ν} *that depend upon* $r_\nu = n - \nu$ *parameters and fill up all of space pass through the latter multiplicity. For each of these multiplicities* μ_{r-r_ν} *, one determines the integral multiplicity* M_ν *that passes through* $M_{\nu-1}$*, and which is contained entirely in* μ_{r-r_ν} *. The geometric locus of these multiplicities* M_ν *when one varies the* $n - \nu$ *parameters upon which they depend, is the desired integral multiplicity* M_ν *.*

Moreover, *that multiplicity* M_ν *is the integral of a system of total differential equations in* $r - r_\nu$ *variables of genus* ν*, although its coefficients depend upon* $n - \nu$ *parameters.*

One deduces the following theorem from this, which refers to the Cauchy problem, properly speaking:

Let one be given a system of total differential equations of genus n *for which the character* s_ν *is zero* ($\nu < n$). *If one is then given an arbitrary point* μ_0 *, an arbitrary*

([1]) If:

$$f_1 = f_2 = \ldots = f_{r_\nu+1} = 0$$

are the equations of $\mu_{r-r_\nu-1}$ then it obviously suffices to take:

$$f_1 - t_1\, f_{r_\nu+1} = f_2 - t_2\, f_{r_\nu+1} = \ldots = f_{r_\nu} - t_{r_\nu}\, f_{r_\nu+1} = 0.$$

multiplicity μ_{r-r_1} *that passes through that point, etc., an arbitrary multiplicity* $\mu_{r-r_\nu-1}$ *that passes through* $\mu_{r-r_\nu-2}$, *then there exists one and only one integral multiplicity* M_n *that passes through* μ_0 *and has a 1-dimensional multiplicity in common with* μ_{r-r_1}, *etc., and a* $\nu - 1$*-dimensional multiplicity in common with* $\mu_{r-r_\nu-1}$. *In order to obtain it, one makes an arbitrary multiplicity* $\mu_{r-r_\nu-1}$ *pass through* $\mu_{r-r_{\nu-1}}$ *and a family of multiplicities* μ_{r-r_ν} *that depend upon* $r_\nu = n - \nu$ *parameters and fill up all of space pass through the latter. Each of the multiplicities contain one and only integral multiplicity* M_ν *that passes through* μ_0 *and has a 1-dimensional multiplicity in common with* μ_{r-r_1}, *etc., and finally, a* $\nu - 1$*-dimensional multiplicity in common with* $\mu_{r-r_{\nu-1}}$. *The geometric locus of these multiplicities* M_ν *when one varies the* $n - \nu$ *parameters that they depend upon is the desired integral multiplicity* M_n.

Indeed, it suffices to remark that the $\nu - 1$-dimensional integral multiplicity that is situated on $\mu_{r-r_{\nu-1}}$ is the same for M_n and for all the M_ν. One then only needs to apply the preceding theorem to that $\nu - 1$-dimensional integral multiplicity.

The last theorem shows that *the Cauchy problem for the given system of genus n is converted into the Cauchy problem for a new system of genus ν; however, the coefficients of this new system depend upon n – ν parameters.* The numbers $s, s_1, s_2, \ldots, s_n$ have the same values for the two systems, moreover.

We say that the integer ν is the *true genus* of the system.

We shall now interpret the preceding results analytically. We keep the notations of § VI. Here, a simplification presents itself, due to the fact that $s_n, s_{n-1}, \ldots, s_\nu$ are zero. As a result, there are no variables $z^{(n)}, z^{(n-1)}, \ldots, z^{(\nu)}$.

The multiplicity $\mu_{r-r_{\nu-1}}$ is defined by:

$$(A_{\nu-1})\qquad \begin{cases} z_1^{(\nu-1)} = \varphi_1^{(\nu-1)}(x_1, x_2, \ldots, x_{\nu-1}), \\ \dots\dots\dots\dots\dots\dots, \\ z_{s_\nu-1}^{(\nu-1)} = \varphi_{s_\nu-1}^{(\nu-1)}(x_1, x_2, \ldots, x_{\nu-1}), \end{cases}$$

$$(B_\nu)\qquad x_n = a_n, \qquad x_{n-1} = a_{n-1}, \qquad \ldots, \qquad x_\nu = a_\nu .$$

The multiplicity $\mu_{r-r_{\nu-2}}$ is defined by the preceding equations, and in addition:

$$(A_{\nu-2})\qquad \begin{cases} z_1^{(\nu-2)} = \varphi_1^{(\nu-2)}(x_1, x_2, \ldots, x_{\nu-2}), \\ \dots\dots\dots\dots\dots\dots, \\ z_{s_\nu-2}^{(\nu-2)} = \varphi_{s_\nu-2}^{(\nu-2)}(x_1, x_2, \ldots, x_{\nu-2}), \end{cases}$$

$$(B_{\nu-1})\qquad x_{\nu-1} = a_{\nu-1} ,$$

and so on, as in the general case.

We can now take the multiplicity $\mu_{r-r_\nu-1}$ to be the one that is defined by the $n - \nu + 1 = r_\nu + 1$ equations (B_ν):

$$x_n = a_n, \qquad x_{n-1} = a_{n-1}, \qquad \ldots, \qquad x_\nu = a_\nu,$$

and the multiplicities μ_{r-r_ν} to be the ones that are defined by:

$$(\mu_{r-r_\nu}) \qquad \begin{cases} x_{\nu+1} - a_{\nu+1} = t_1(x_\nu - a_\nu), \\ x_{\nu+2} - a_{\nu+2} = t_2(x_\nu - a_\nu), \\ \ldots\ldots\ldots\ldots\ldots\ldots, \\ x_n - a_n = t_{n-\nu}(x_\nu - a_\nu). \end{cases}$$

The preceding results that were stated in a geometric manner can now be expressed in the following manner:

The given system admits one and only one integral for which $z_1, \ldots, z_{s_{\nu-1}}^{(\nu-1)}$ *are functions of* $x_1, x_2, \ldots, x_n$ *that are holomorphic in a neighborhood of:*

$$x_1 = a_1, \qquad x_2 = a_2, \qquad \ldots, \qquad x_n = a_n,$$

and reduce as follows:

$$\left.\begin{array}{lll} z_1^{(\nu-1)} & \textit{reduces to the arbitrary function} & \varphi_1^{(\nu-1)}(x_1, x_2, \ldots, x_{\nu-1}) \\ \ldots\ldots & \ldots\ldots\ldots\ldots & \ldots\ldots\ldots\ldots \\ z_{s_{\nu-1}}^{(\nu-1)} & \text{"} & \varphi_{s_{\nu-1}}^{(\nu-1)}(x_1, x_2, \ldots, x_{\nu-1}) \end{array}\right\} \quad \begin{array}{c} \textit{for} \\ x_\nu = a_\nu, \\ x_{\nu+1} = a_{\nu+1} \\ \ldots\ldots\ldots, \\ x_n = a_n, \end{array}$$

$$\left.\begin{array}{lll} z_1^{(\nu-2)} & \text{"} & \varphi_1^{(\nu-2)}(x_1, x_2, \ldots, x_{\nu-1}) \\ \ldots\ldots & \ldots\ldots\ldots\ldots & \ldots\ldots\ldots\ldots \\ z_{s_{\nu-2}}^{(\nu-2)} & \text{"} & \varphi_{s_{\nu-2}}^{(\nu-2)}(x_1, x_2, \ldots, x_{\nu-1}) \end{array}\right\} \quad \begin{array}{c} \textit{for} \\ x_{\nu-1} = a_{\nu-1}, \\ \ldots\ldots\ldots, \\ x_n = a_n. \end{array}$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,$$

$$\left.\begin{array}{lll} z_1^{(1)} & \text{"} & \varphi_1^{(1)}(x_1) \\ \ldots\ldots & \ldots\ldots\ldots\ldots & \ldots\ldots \\ z_{s_1}^{(1)} & \text{"} & \varphi_{s_1}^{(1)}(x_1) \end{array}\right\} \quad \begin{array}{c} \textit{for} \\ x_2 = a_2, \ldots, x_n = a_n, \end{array}$$

$$\left.\begin{array}{lll} z_1 \textit{ reduces to the arbitrary constant} & & \varphi_1 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & & \cdots \\ z_s & '' & \varphi_s \end{array}\right\} \quad \begin{array}{c} \textit{for} \\ x_1 = a_1, \ldots, x_n = a_n, \end{array}$$

respectively.

In order to obtain these functions, in the equations of the given differential system, one replaces:

$$\begin{array}{lll} x_{\nu+1} & \textit{with} & a_{\nu+1} + t_1(x_\nu - a_\nu), \\ x_{\nu+2} & \textit{with} & a_{\nu+2} + t_2(x_\nu - a_\nu), \\ \multicolumn{3}{l}{\ldots\ldots\ldots\ldots\ldots\ldots,} \\ x_n & \textit{with} & a_n + t_{n-\nu}(x_\nu - a_\nu), \end{array}$$

where one regards the t as constants. The new system that is obtained then admits one and only one integral for which $z_1, \ldots, z_{s_{\nu-1}}^{(\nu-1)}$ *are functions of* $x_1, x_2, \ldots, x_n$ *that are holomorphic in a neighborhood of:*

$$x_1 = a_1, \qquad x_2 = a_2, \qquad \ldots, \qquad x_\nu = a_\nu,$$

and reduce as follows:

$$\begin{array}{llllll} z_i^{(\nu-1)} & \textit{to} & \varphi_i^{(\nu-1)}(x_1, x_2, \ldots, x_{\nu-1}) & \textit{for} & x_\nu = a_\nu, \\ z_j^{(\nu-2)} & \textit{to} & \varphi_j^{(\nu-2)}(x_1, x_2, \ldots, x_{\nu-2}) & \textit{for} & x_{\nu-1} = a_{\nu-1}, x_\nu = a_\nu, \\ \ldots\ldots & \ldots & \ldots\ldots\ldots\ldots\ldots\ldots & \ldots & \ldots\ldots\ldots\ldots\ldots, \\ z_k^{(1)} & \textit{to} & \varphi_k^{(1)}(x_1) & \textit{for} & x_1 = a_1, \ldots, x_\nu = a_\nu, \\ z_h & \textit{to} & \varphi_h & \textit{for} & x_1 = a_1, \ldots, x_\nu = a_\nu, \end{array}$$

$(i = 1, 2, \ldots, s_{\nu-1};\ j = 1, 2, \ldots, s_{\nu-2};\ k = 1, 2, \ldots, s_1;\ h = 1, 2, \ldots, s)$.

If one replaces the n – ν parameters t upon which the functions thus found depend with:

$$t_1 = \frac{x_{\nu+1} - a_{\nu+1}}{x_\nu - a_\nu},$$
$$\ldots\ldots\ldots\ldots\ldots,$$
$$t_{n-\nu} = \frac{x_n - a_n}{x_\nu - a_\nu}$$

then one obtains the desired integral of the original system.

In summary, one sees that *if the general integral* M_n *of a system in involution (i.e., of genus greater than or equal to n) depends upon arbitrary functions of ν –* 1 *arguments,* *but not upon arbitrary functions of v arguments, then one converts its search into that of the integral M_v of a system of genus n, and consequently, into a problem in v independent variables. However, the coefficients of the new system depend upon n – v parameters.*

Indeed, it suffices to remark that since M_n does not depend upon arbitrary functions of n arguments, one necessarily has $r_n = 0$, and in turn, the genus of the system is exactly n.

In particular, if the general integral M_n of a system in involution depends upon only *arbitrary constants* then one has $v = 1$; the *character* s_1 of the system is zero, and the system is completely integrable. One converts the search for its integral to that of a system of genus 1, i.e., to a system of ordinary differential equations. *The method reduces to that of Lie-Mayer for the integration of completely integrable system.* If the systems of the system are soluble for $dz_1, dz_2, \ldots, dz_s$ then since the other differentials are $dx_1, dx_2, \ldots, dx_n$, one replaces:

$$\begin{array}{lll} x_2 & \text{with} & a_2 + t_2(x_1 - a_1), \\ .. & \ldots & \ldots\ldots\ldots\ldots\ldots\ldots, \\ x_n & \text{with} & a_n + t_{n-1}(x_1 - a_1), \end{array}$$

and one seeks the integral of the new system in order for $z_1, z_2, \ldots, z_s$ to reduce to reduce to given arbitrary constants $\varphi_1, \varphi_2, \ldots, \varphi_s$ for $x_1 = a_1$. One then replaces t_i with $\dfrac{x_{i+1} - a_{i+1}}{x_1 - a_1}$ in the functions thus obtained.

As an example, we take the simplest one possible, in such a manner that we nonetheless do not have a completely integrable system. We choose:

$$v = 2, \qquad n = 3, \qquad s_1 = 1, \qquad s = 1,$$

so:

$$r_3 = 0, \qquad r_2 = 1, \qquad r_1 = 3, \qquad r = 5.$$

The following equation corresponds to this case:

$$x_4\, dx_1 - x_4\, x_5\, dx_2 - (x_2\, x_4 + x_3 + x_1\, x_5)\, dx_5 = 0.$$

We shall not carry out the verification, and we shall simply apply the generalized Lie-Mayer method in order to find the general integral of this equation. One easily sees that the point:

$$x_1 = 0, \qquad x_2 = 0, \qquad x_3 = 0, \qquad x_5 = 0$$

is not singular, and that the linear element E_1 that issues from this point:

$$dx_1 = dx_2 = dx_3 = dx_4 = 0$$

is integral. Here, (P_0) has the equation:

(P_0) $\qquad dx_1 = 0;$

as for (P_1), one easily finds:

(P_1) $\qquad dx_1 = dx_3 = 0,$

and one confirms that the element E_3 is integral. Moreover, E_1 is not singular.

Here, one can thus take the variables that are denoted by z, $z^{(1)}$, x_3, x_2, x_1 in the general theory to be x_1, x_3, x_4, x_2, x_5 , respectively.

There will then be one and only one integral such that:

x_3 reduces to $f(x_5)$ for $x_2 = 0$, $x_4 = 1$,
x_1 " c for $x_2 = 0$, $x_4 = 1$, $x_4 = 0$.

In order to obtain it, it will suffice to replace:

$$x_4 - 1 \quad \text{with} \quad t\, x_2$$

in the equation, which gives:

$$(t\, x_2 + 1)\, dx_1 - (t\, x_2 + 1)\, x_5\, dx_2 - (t\, x_2^2 + x_2 + x_3 + x_1\, x_5)\, dx_5 = 0.$$

One can first look for the function x_1 of x_5 that reduces to c for $x_5 = 0$ when one makes $x_3 = f(x_5)$, $x_2 = 0$. That function is given by:

$$\frac{dx_1}{dx_5} = f(x_5) + x_1\, x_5 ,$$

from which, one infers that:

$$x_1 = ce^{x_5^2/2} + e^{x_5^2/2}\int_0^{x_5} e^{-x_5^2/2} f(x_5)\, dx_5 = \varphi(x_5).$$

One must then look for two functions x_3 and x_1 of x_2, x_5 that reduce to $f(x_5)$ and $\varphi(x_5)$ for $x_2 = 0$. They are given by:

$$\frac{\partial x_1}{\partial x_2} = x_5 ,$$

$$1 = 1 + \frac{\partial}{\partial x_2} \frac{x_3 + x_1 x_5}{t\, x_2 + 1}.$$

The last one gives:

$$x_3 + x_1\, x_5 = (t\, x_2 + 1)\left[f(x_5) + ce^{x_5^2/2} + x_5 e^{x_5^2/2}\int_0^{x_5} e^{-x_5^2/2} f(x_5)\, dx_5 \right],$$

and the first one gives:

$$x_3 = x_2\, x_5 + ce^{x_5^2/2} + x_5 e^{x_5^2/2}\int_0^{x_5} e^{-x_5^2/2} f(x_5)\, dx_5 .$$

Upon replacing $tx_2 + 1$ with x_4 in the first formula, one obtains the general integral, which can be further written:

$$x_1 - x_2\, x_5 = F(x_5),$$

$$x_3 + x_1\, x_5 = x_4\, F'(x_5),$$

upon setting:

$$F(x_5) = c e^{x_5^2/2} + e^{x_5^2/2} \int_0^{x_5} e^{-x_5^2/2} f(x_5)\, dx_5 \,.$$

VIII.

In this last paragraph, we shall occupy ourselves with certain systems for which the Kowalewski system that determines the integral multiplicity M_{p+1} that passes through a given integral multiplicity M_p presents certain simple properties that make the integration easy. Upon preserving the notations of § III, this system is, if we limit ourselves to the case where $s_{p+1} = 0$, solved for:

$$\frac{\partial z_1}{\partial x}, \frac{\partial z_2}{\partial x}, \ldots, \frac{\partial z_m}{\partial x},$$

where the right-hand sides depend upon the variables and the first-order derivatives of the unknown functions z with respect to the independent variables $x_1, x_2, \ldots, x_p$ other than x.

If one solves the Cauchy problem for a first-order partial differential equation in one unknown function then, by a change of independent variables, one is reduced to a Kowalewski system precisely, *but in which the right-hand sides do not depend upon the derivatives* $\partial z_i / \partial x_k$. In summary, one is then reduced to a system of ordinary differential equations.

We seek to find in which case this fact will be produced. Upon letting, as we did in § III, $\varepsilon, \varepsilon^{(1)}, \varepsilon^{(2)}, \ldots, \varepsilon^{(p)}$ denote the $p + 1$ linear elements:

$$(\varepsilon) \qquad \frac{dx}{1} = \frac{dx_1}{0} = \ldots = \frac{dx_p}{0} = \frac{dz_1}{\dfrac{\partial z_1}{\partial x}} = \ldots \frac{dz_m}{\dfrac{\partial z_m}{\partial x}},$$

$$[\varepsilon^{(1)}] \qquad \frac{dx}{0} = \frac{dx_1}{1} = \ldots = \frac{dx_p}{0} = \frac{dz_1}{\dfrac{\partial z_1}{\partial x_1}} = \ldots \frac{dz_m}{\dfrac{\partial z_m}{\partial x_1}},$$

……………………………………………,

$$[\varepsilon^{(p)}] \qquad \frac{dx}{0} = \frac{dx_1}{0} = \ldots = \frac{dx_p}{1} = \frac{dz_1}{\dfrac{\partial z_1}{\partial x_p}} = \ldots \frac{dz_m}{\dfrac{\partial z_m}{\partial x_p}},$$

the Kowalewski equations express the idea that the element ε *is integral and associated with the element* E_p $[\varepsilon^{(1)}, \varepsilon^{(2)}, \ldots, \varepsilon^{(p)}]$, *under the single condition that the element* E_p *should be integral.*

This being the case, suppose that the Kowalewski equations do not depend upon $\partial z_i / \partial x_k$, i.e., on $\varepsilon^{(1)}, \varepsilon^{(2)}, \ldots, \varepsilon^{(p)}$, collectively. The values of $\partial z_1 / \partial x, \ldots, \partial z_m / \partial x$ that are determined provide an integral linear element ε that depends uniquely upon the point that it issues from, *and which is associated with all of the linear elements* E_p *that pass through that point*, because an arbitrary element E_p can always be linearly deduced from e and an element of the form $[\varepsilon^{(1)}, \varepsilon^{(2)}, \ldots, \varepsilon^{(p)}]$.

In summary, *an integral linear element passes through each point that enjoys the property of being associated with an arbitrary integral linear element that issues from the same point.*

That linear element is necessarily *singular*, because it belongs to $\infty^{r_1 - 1}$ integral elements E_2; we say that it is *characteristic.*

In general, *an integral element* E_p *that issues from a non-singular point of space is called* characteristic *if it is associated with an arbitrary integral linear element that issues from the same point.*

All of the linear elements that are contained in a characteristic element E_h ($h > 1$) are themselves characteristic, and the locus of characteristic linear elements is necessarily a characteristic planar element that is the largest characteristic element that issues from the point.

In order to obtain the characteristic linear elements that issue from a given non-singular point analytically, we let:

$$\delta x_1, \quad \delta x_2, \quad \ldots, \quad \delta x_r$$

denote the coordinates of such an element, and let:

$$dx_1, \quad dx_2, \quad \ldots, \quad dx_r$$

denote the coordinates of a variable integral linear element that issues from the same point. In order to determine the δx_i, one will have the equations:

$$\left\{\begin{array}{c} a_1\,\delta x_1 + \ldots + a_r\,\delta x_r = 0, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \\ l_1\,\delta x_1 + \ldots + l_r\,\delta x_r = 0, \\ \sum a_{1i}\,dx_i\,\delta x_1 + \ldots + \sum a_{ri}\,dx_i\,\delta x_r = 0, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \\ \sum l_{1i}\,dx_i\,\delta x_1 + \ldots + \sum l_{ri}\,dx_i\,\delta x_r = 0, \end{array}\right.$$

where the notations are the same as in § I ([1]). Moreover, these equations can be true for any:

$$dx_1, \quad dx_2, \quad \ldots, \quad dx_r\,,$$

([1]) We have set simply:

$$a_{ik} = \frac{\partial a_i}{\partial x_k} - \frac{\partial a_k}{\partial x_i}, \quad \ldots, \quad l_{ik} = \frac{\partial l_i}{\partial x_k} - \frac{\partial l_k}{\partial x_i}.$$

with the single condition that these quantities must verify equations (1), which are s in number:

$$
(1) \qquad \begin{cases} a_1\,dx_1 + \ldots + a_r\,dx_r = 0, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \\ l_1\,dx_1 + \ldots + l_r\,dx_r = 0. \end{cases}
$$

As a result, the equation in $dx_1, \ldots, dx_r$:

$$\sum a_{1i}\,\delta x_i\,dx_1 + \ldots + a_{ri}\,\delta x_i\,dx_r = 0,$$

must be a consequence of equations (1). In other words, *all of the minors with* $s + 1$ *columns in the matrix:*

$$
(A) \qquad \left\| \begin{matrix} a_1 & a_2 & \cdots & a_r \\ \cdots & \cdots & \cdots & \cdots \\ l_1 & l_2 & \cdots & l_r \\ \sum a_{1i}\,\delta x_i & \sum a_{2i}\,\delta x_i & \cdots & \sum a_{ri}\,\delta x_i \end{matrix} \right\|
$$

must be zero. The same is true if one replaces the last row in this matrix with the $s - 1$ analogous rows that are deduced from the last $s - 1$ equations (1), which gives the matrices $(B), \ldots, (L)$.

By definition, *the equations that determine the characteristic linear elements are of two kinds: first, one has the s equations:*

$$
(1') \qquad \begin{cases} a_1\,\delta x_1 + \ldots + a_r\,\delta x_r = 0, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \\ l_1\,\delta x_1 + \ldots + l_r\,\delta x_r = 0, \end{cases}
$$

which express the idea that the element is integral. Then, one has the equations that are obtained by annulling all of the minors in the matrices $(A), \ldots, (L)$ *with* $s + 1$ *columns:*

$$
(A) \qquad \left\| \begin{matrix} a_1 & \cdots & a_r \\ \cdots & \cdots & \cdots \\ l_1 & \cdots & l_r \\ \sum a_{1i}\,\delta x_i & \cdots & \sum a_{ri}\,\delta x_i \end{matrix} \right\|,
$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,$$

$$(L) \qquad \left\| \begin{matrix} a_1 & \cdots & a_r \\ \cdots & \cdots & \cdots \\ l_1 & \cdots & l_r \\ \sum l_{1i}\,\delta x_i & \cdots & \sum l_{ri}\,\delta x_i \end{matrix} \right\|.$$

If at least r of these equations are independent then there exist characteristic linear elements, and these equations determine their locus; i.e., the largest characteristic element that issues from the point.

If the given system is completely integrable then two arbitrary integral linear elements are associated, and as a result, the equations of the characteristic elements must reduce to equations (1′). The principal minors of the matrices (A), …, (L) are of degree s, if one takes (1′) into account.

Now, here are some simple fundamental properties of the characteristic elements:

If one is given a characteristic element E_p then any non-singular *integral element E_n must contain E_p*, since otherwise, in effect, the smallest element that is contained in E_n and E_p would be at least $n + 1$-dimensional, and it would necessarily be integral since E_n and E_p are associated. Since the integral element E_n belongs to an integral element E_{n+1}, it would then be singular. Of course, n denotes the genus of the given system.

If a characteristic element passes through any singular point in space then the given differential system is of the first kind. This is because if we let ε be a characteristic linear element then any non-singular integral element E_n will contain ε, so there certainly exist certain integral elements E_{n-1} that do not contain ε, and naturally, among these integral elements there are ones that are not singular ([1]); let E_{n-1} be one of them. ∞^{r_n} integral elements E_n pass through E_{n-1}, and at least one of them is not singular; i.e., it contains ε. If r_n is equal to at least 1 then there will be at least one integral element E_n other than (E_{n-1}, ε), namely, (E_{n-1}, ε'). However, the element (E_{n-1}, ε, ε') will then be integral, and the non-singular element (E_{n-1}, ε) will belong to another $n + 1$-dimensional integral element, which is impossible. One must then have that *r_n is zero*; i.e., that the given differential system is of the first kind. *There are thus systems of the first kind for which characteristic elements can exist.*

In the same way, one sees that *if there exists a characteristic integral element E_p then the true genus of the system is at most $n - p + 1$*, because there certainly exists a non-singular integral element E_{n-p} that has no point in common with E_p. Any non-singular integral element E_n that passes through E_{n-p} must contain E_p. It is therefore determined uniquely, and one can denote it by (E_{n-p}, E_p). If E_{n-p} belongs to another integral element E_n then the element (E_{n-p}, E_p) will be integral and at least $n + 1$-dimensional. On the other hand, it will contain (E_{n-p}, E_p), which will be, in turn, singular. Therefore, E_{n-p} belongs to just one integral element E_n. Finally, as a result, the true genus of the system is at most $n - p + 1$.

One can add that *if there exists a non-singular integral element E_{n-1} that contains E_p then the true genus is at most $n - p$*, because there always exists an integral element E_{n-p-1}

([1]) Otherwise, and non-singular integral element E_{n-1} would be subject to a condition of *equality*, namely, that it contain ε.

that is contained in E_{n-1} and has no element in common with E_p . If an n-dimensional integral element passing through E_{n-p-1} contains E_p then it also contains E_{n-1}, and as a result, it is completely determined and unique, since the non-singular integral element E_{n-1} belongs to just one n-dimensional integral element E_n , which is itself non-singular. Now, if another integral element E'_n passes through E_{n-p-1} then the element (E'_n, E_p) will be at least n + 1-dimensional and integral. On the other hand, it will contain (E_{n-p-1}, E_p) – i.e., E_{n-1} – which is impossible, because no integral element that is more than n-dimensional passes through E_{n-1} . Therefore, just one n-dimensional integral element passes through E_{n-p-1} . Finally, the true genus of the system is therefore at most $n - p$.

The following property falls out of these properties, and it will suffice for us to state it, since the proof appears to be obvious:

If a p-dimensional characteristic element passes through each non-singular point of space for a differential system of genus n then all of the non-singular integral multiplicities M_n that pass through a non-singular point have a p-dimensional element that issues from that point in common, and conversely.

We shall now see the role that the existence of characteristic elements plays in the determination of non-singular n-dimensional integral multiplicities.

First, suppose that there exists a characteristic *linear* element. That linear element then makes any point of space correspond to a certain line D that passes through that point. As one knows, there exists a family of curves (i.e., one-dimensional multiplicities) such that each of their points are tangent to the line D that corresponds to that point. These curves depend upon $r - 1$ parameters, and one and only one of them passes through each non-singular point of space. We call them *characteristic curves;* they are obviously integral curves.

This being the case, consider a non-singular integral multiplicity M_n . Each of its non-singular points admits a non-singular integral element E_n that, in turn, contains the characteristic element ε that issues from that point. In other words, at each of its points, the multiplicity M_n is tangent to the line D that corresponds to that point. There thus exists a family of curves on M_n that are tangent to the corresponding line D at each of its points. These curves depend upon $n - 1$ parameters, and one and only one of them passes through each non-singular point of M_n . However, it is obvious that these curves are *characteristic curves.* One therefore arrives at the following result:

Any non-singular integral multiplicity M_n is generated by a family of characteristic curves that depend upon $n - 1$ parameters. One and only one of these curves passes through each non-singular point of M_n . If two non-singular multiplicities M_n have a non-singular point in common then they have the entire characteristic curve that issues from that point in common.

It results from this that *if one is given a non-singular integral multiplicity M_{n-1} that is not generated by characteristic curves then one will have the integral multiplicity M_n that passes through M_{n-1} by making the characteristic curve that issues from each point of M_n pass through that point.*

One thus has the solution to the Cauchy problem when M_{n-1} is not generated by characteristic curves.

We shall now prove these results analytically, which will permit us to neatly see what the integration problem reduces to when one knows the characteristic curves.

In the case where we have placed ourselves, the characteristic curves are given by a system of $r - 1$ total differential equations. They are the equations that were previously found that determine the characteristic element that issues from point of space. Let:

$$y_1 = C_1, \qquad y_2 = C_2, \qquad \ldots, \qquad y_{r-1} = C_{r-1}$$

be $r - 1$ independent first integral of these equations; they determine the characteristic curves. Make a change of variables by taking the new variables to be $y_1, y_2, \ldots, y_{r-1}$, and an r^{th} quantity y_r that depends upon the first $r - 1$ of them. With these new variables, the system of integral linear elements and the associated linear elements does not change. As a result, the system of total differential equations that determines the characteristic elements remains the same. It is therefore:

$$dy_1 = dy_2 = \ldots = dy_{r-1} = 0.$$

The equations of the transformed system must therefore first be verified for $dy_1 = \ldots = dy_{r-1} = 0$. As a result, one can put this system into the form:

$$(1)_1 \qquad \begin{cases} dy_1 + b_{1,s+1}dy_{s+1} + \ldots + b_{1,r-1}dy_{r-1} = 0, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \\ dy_s + b_{s,s+1}dy_{s+1} + \ldots + b_{s,r-1}dy_{r-1} = 0, \end{cases}$$

in which the b depend upon $y_1, y_2, \ldots, y_{r-1}$. Now, write down that the integral element:

$$\frac{dy_1}{0} = \ldots = \frac{dy_{r-1}}{0} = \frac{dy_r}{1}$$

is associated with any other integral element $(dy_1, \ldots, dy_r)$. One will first have:

$$\frac{\partial b_{1,s+1}}{\partial y_r} dy_{s+1} + \ldots + \frac{\partial b_{1,r-1}}{\partial y_r} dy_{r-1} = 0,$$

with the single condition that the dy must satisfy $(1)_1$; i.e., one must have:

$$\frac{\partial b_{1,s+1}}{\partial y_r} = \ldots = \frac{\partial b_{1,r-1}}{\partial y_r} = 0.$$

In other words, one will finally have that *all of the coefficients of b are independent of* y_r .

The transformed system can thus be put into a form such that there remains no trace of the r – 1 variables:

$$y_1, y_2, \ldots, y_{r-1}$$

in either the coefficients or the differentials.

One then indeed sees that the number of variables has been reduced by one unit. In order to find the multiplicities M_n of the original system, it will suffice to find the integral multiplicities M_{n-1} of the new system. *The genus of the new system is diminished by one unit, but the degree of indeterminacy does not change*, except that the new system can no longer be of the first kind.

Therefore, *whenever one has to integrate the differential equations of the characteristics, one is reduced to a new differential system with one less variable, while the genus has also been subjected to a reduction by one unit.* One has:

$$s' = s, \qquad s'_1 = s_1, \qquad \ldots, \qquad s'_{n-1} = s_{n-1}\,,$$

$$n' = n - 1,$$

$$r' = r - 1, \qquad r'_1 = r_1 - 1, \qquad \ldots, \qquad r'_{n-1} = r_{n-1} - 1.$$

Now, pass on to the case where *a characteristic element E_p that is at least two-dimensional passes through each point of space.* The linear equations in $dx_1, \ldots, dx_r$ that determine E_p then consist of $r - p$ independent ones. One can believe that these equations do not determine a completely integral differential system, in general, *but this is not true. The differential system that we call the* characteristic *differential system is always completely integrable.*

In order to account for this, it suffices to choose a particular linear element in each E_p; i.e., it suffices to append p – 1 arbitrary, but well-defined, linear equations to the characteristic differential system. One thus has a system of r – 1 independent equations that is, in turn, completely integrable, and where we let:

$$y_1, y_2, \ldots, y_{r-1}$$

denote a system of r – 1 independent first integrals. As we just saw, by a change of variables, the equations of the system no longer depend upon $y_1, \ldots, y_{r-1}$. The characteristic differential system then changes into a system of $r - p$ equations, *but in r – 1 variables*. One argues with them as one did with the original ones until one has reduced the variables to no more than $r - p$ in number, namely:

(equations missing from the original)

It is then clear that the characteristic differential system is nothing but:

$$dz_1 = dz_2 = \ldots = dz_{r-p} = 0.$$

Therefore, *the characteristic differential system is completely integrable, and one can, by a change of variables, put the given system into a form such that its coefficients and differentials no longer depend upon the* $r - p$ *first integrals of the characteristic system*.

One further sees that *there exists a family of p-dimensional multiplicities that admit the characteristic element* E_p *at each of their points; one calls them* characteristic multiplicities. *They depend upon* $r - p$ *parameters, and one and only one of them passes through each non-singular point of space*.

Any non-singular integral multiplicity M_n *is generated by a family of characteristic multiplicities that depend upon* $n - p$ *parameters. One and only one of these multiplicities passes through any non-singular point of* M_n . *If two non-singular n-dimensional integral multiplicities have a non-singular point in common then they have the characteristic multiplicity that issues from that point in common.*

If a non-singular integral multiplicity M_{n-p} does not have any curve in common with the characteristic multiplicity that issues from each of its points then in order to get the unique integral multiplicity M_n that passes through M_{n-p} , it will suffice to make the characteristic multiplicity that issues from each point of M_{n-p} pass through that point.

Finally, *the general determination of the integral* M_n *amounts to the integration of a new differential system whose genus is reduced by p units, as well as the number of variables, but which has the same degree of indeterminacy as the given system*.

In order to see this last point, it suffices to recall that the true genus of the given system is at most $n - p + 1$. As a result, one has:

$$s_n = s_{n-1} = \ldots = s_{n-p+1} = 0.$$

One then has:

$$n' = n - p,$$

$$s'_{n-p} = s_{n-p}, \quad \ldots, \quad s'_1 = s_1, \quad s' = s,$$

$$r'_{n-p} = s_{n-p} = r_{n-p} - p, \quad \ldots, \quad r'_1 = r_1 - p, \quad r' = r - p.$$

However, one must not forget that the reduction of this new system assumes the prior determination of the characteristic multiplicities. The generalized Lie-Mayer method permits one to convert to a system of genus $n - p + 1$ (instead of $n - p$) *with no prior integration*. However, this system depends upon the particular Cauchy problem that one must solve.

Finally, we remark that if the number of variables in the given differential system can be reduced by p units by a suitable change of variables then the characteristic differential system is necessarily composed of at most $r - p$ independent equations. One thus has the following theorem, which was stated for the first time in a slightly different form by von Weber ([1]), and which is itself a generalization of a theorem of Frobenius for systems with just one equation:

The minimum number of variables that one can make the coefficients and the differentials of a given system depend upon by a change of variables is equal to the

([1]) *Loc. cit.*

number of linearly independent equations in its characteristic differential system. The integration of that characteristic system gives these variables.

Finally, to conclude the subject, we shall prove the existence of characteristic elements *in the differential systems of the first kind whose character is equal to one.*

Take a differential system of genus n for which one has $s = 1$. The numbers $s_2, s_3, \ldots$, cannot exceed s_1 – i.e., the unit – then, and one will have, to fix ideas:

$$s_1 = s_2 = \ldots = s_{\nu-1} = 1, \qquad s_\nu = \ldots = s_n = 0.$$

ν is the true genus (which can be equal to n).

This being the case, consider a non-singular point E_0 and the set of integral linear elements that issue from that point; they form an element E_{r_1+1}. In the sequel, we shall speak only of the elements that are situated in E_{r_1+1}; i.e., of the elements that are composed of integral linear elements. (One has, moreover, $r_1 + 1 = n + \nu - 1$.)

Take an integral element E_n and a linear element ε that is not contained in E_n. The locus of (integral) linear elements that are associated with ε is an element of dimension $r_1 + 1 - s_1 = r_1$. That element thus cuts E_n along an element H_{n-1} (of dimension $n + r_1 - r_1 + 1 = n - 1$). All of the linear elements that are contained in H_{n-1} are then associated with E_n and ε; i.e., with the element $E_{n+1} : (E_n, \varepsilon)$.

Now, take a linear element ε' that is not contained in E_{n+1}. The locus of linear elements that are associated with ε' is, moreover, an r_1-dimensional element that cuts H_{n-1} along an at least $n - 2$-dimensional element H_{n-2}, and all of the linear elements of H_{n-2} are associated with E_{n+1} and ε'; i.e., with the element $E_{n+2} : (E_{n+1}, \varepsilon')$. One can continue in this way step-by-step. One will have an element H_{n-3} whose linear elements are all associated with an element E_{n+3}, and so on, until one finally arrives at an element $H_{n-\nu+1}$ whose elements are all associated with an element $E_{n+\nu-1}$, i.e., an E_{r_1+1}. In other words, there exists an element $H_{n-\nu+1}$ whose linear elements are all linear and associated with an *arbitrary* integral linear element. That element $H_{n-\nu+1}$ is therefore *characteristic*.

It results from this that *the given differential system of genus n, true genus ν, and character* 1 *admits n – ν +* 1*-dimensional characteristic multiplicities. After the determination of these characteristics, it will then be converted into a system of genus ν –* 1.

This result applies to just one Pfaff equation (provided that it is of the first kind). One thus recovers the characteristic multiplicities of the systems of first-order partial differential equations in just one unknown function.

In particular, *if the general integral of a differential system depends upon just one arbitrary function of one argument (and arbitrary constants) then its integration reduces to that of the completely integrable characteristic system and to that of a system of ordinary differential equations* ([1]).

([1]) Beudon has proved this result for a system of partial differential equations in one unknown function. In a series of notes and memoirs, he was occupied with partial differential equations of this nature that admitted characteristic multiplicities in the sense of the term that was given in this paper. In particular, *see,* If the general integral of a system of the first kind depends upon an arbitrary function of 1, 2, …, $v - 1$ arguments (and arbitrary constants), where $v - 1$ is equal to at least 2, then one can prove ([1]) that the system can, with no integration, be put into the following form: First, a system of $s - 1$ completely integrable equations. Then, a s^{th} equation that can be put into the form:

$$dz - p_1\, dx_1 - \ldots - p_{v-1}\, dx_{v-1} = 0$$

by the suitable integration that leads to the characteristic differential system.

The problem of integrating the characteristic differential system is not, indeed, an arbitrary problem of integrating a completely integrable system of total differential equations. In order to account for this, imagine that one has found a first integral y_1, and consider the multiplicity $y_1 = C$ in space, where C is an arbitrary constant.

Then consider, the element E_{r-1}: $dy_1 = 0$ at an arbitrary point A of that multiplicity. The characteristic element E_p that issues from A is necessarily contained in the element E_{r-1}. However, if one seeks the integral linear elements of E_{r-1} then one can, in certain cases, find that they are not contained in E_p, in such a way that one obtains a characteristic element E_q that contains E_p ($q > p$), but which is not characteristic for any sort of element E_{r-1} . In other words, the characteristic differential system of the given system, when one makes $y_1 = C$, $dy_1 = 0$, can contain more than one equation less than the original characteristic system. One seeks a first integral y_2 of this new system, and so on. One arrives at a certain number of first integrals $y_1, y_2, \ldots, y_h$, in such a manner that upon making $y_1 = C_1, \ldots, y_h = C_h$ the differential system thus obtained verifies all of its characteristic equations.

It is clear that the equations of the given system can all be put into the form:

$$\alpha_1\, dy_1 + \ldots + \alpha_h\, dy_h = 0,$$

and one perceives that by a convenient choice of the s linearly independent equations that define the system, those of the coefficients α that are mutually independent and independent of the y define the various integrals of the y of the characteristic differential system.

This is, moreover, also the way that one can proceed with just one Pfaff equation. To fix ideas, take one equation in four variables with arbitrary coefficients. If one represents a linear element by a point in three-dimensional space R_3 then the integral linear elements are represented by points of a certain plane (P) in that space, and the images of the two associated integral linear elements are such that the line that is their join belongs to a certain linear complex. Now, in ordinary space, the lines of a linear complex that are situated in a plane (P) all pass through a fixed point A of the plane. The point A is therefore the image of a characteristic linear element. The characteristic differential system thus admits three independent first integrals. One then seeks a y_1, which will determine a plane (Q) in the space R_3 . The integral linear elements that satisfy $dy_1 = 0$ have points that belong to both (P) and (Q) for their images in R_3; i.e., the points of the line of intersection (D) of these two planes. Now, however, *two arbitrary points of that*

"Sur les systèmes d'équations aux dérivées partielles dont les charactéristiques dépendent d'un nombre fini de constantes arbitraires," Annales de l'École Normale, supplement to **XIII** (1896), 3-51.

([1]) In particular, *see* von WEBER, *loc. cit.*

line are associated, in such a way that one has a second characteristic differential system that is formed from just one equation [the equation of the line (D) in the plane (Q)]. Let y_2 be a first integral. Then:

$$dy_1 = dy_2 = 0$$

are, if one wishes, the equations of the line (D). The equation of the plane (P), which is nothing but the given Pfaff equation, is then of the form:

$$dy_2 - y_3\, dy_1 = 0,$$

and y_3 is the desired third first integral, because it is obvious that the characteristic equations, when put into its new form, can only be:

$$dy_1 = dy_2 = dy_3 = 0.$$

To take another example, consider the case of two equations in six variables. In the general case, the genus of the system is equal to $2 = \frac{6}{2+1}$. One can represent a linear element by a point in five-dimensional space R_5 . The images of the integral linear elements are then situated in a three-dimensional space R_3, and the lines that join two associated points in that space belong to two linear complexes. In § II, we saw that three cases can present themselves. We take the last one, in which the lines of the complex are lines that pass through a fixed point A of R_3, and in addition, the lines that are situated in a certain plane (P) that passes through A. Here, there is therefore a characteristic element whose image is A.

The characteristic differential system will admit five independent first integrals. One first looks for a y_1. Upon replacing y_1 with C, one will define a space R_4 in R_5 that will cut R_3 along the plane (Q). The images of the integral linear element in R_4 are situated in this plane (Q), which naturally passes through A, and the lines that join two associated points in this plane (Q) are the lines that issue from A. Here, there is, moreover, just one characteristic linear element. The new characteristic system is defined by four equations that define the point A in R_4 . Let y_2 a first integral of this new system. It defines a space R_3' in R_4 that cuts (Q) along a line (D) that passes through A; however, all of the points of (D) are then associated with each other. The new characteristic system is thus composed of the *two* equations that define (D) in R_3'. One will only have to look for two independent first integrals y_3 and y_4 of that system.

One will thus have to look for four integrals using operations of order 5, 4, 2, 1, respectively.

In reality, one can further simplify this integration after the first integration and limit oneself to three integrals that are given by operations of order 5, 3, 1, resp. However, in order to do this, one must enter into the consideration of certain covariant equations, which leaves the scope of this memoir.

There is one case in which the integration simplifies: It is the one where the first integral y_1 gives a space R_4 that contains the plane (P); i.e., the case where *the three equations that define* (P) *admit an integrable combination*. In this case, the images of the integral linear elements of the new system are the points of (P), and *these points are all* *associated with each other.* The new characteristic system is composed of *two* equations that define (P) in R_4 . Upon integrating them, one will have two first integrals y_2 and y_3, and the equations of (P) in R_5 are then:

$$dy_1 = dy_2 = dy_3 = 0.$$

The two equations that define the space R_2 that is the locus of the images of the integral linear elements that pass through (P) – i.e., the given equations – are of the form:

$$dy_2 - y_4\, dy_1 = 0,$$
$$dy_3 - y_5\, dy_1 = 0;$$

y_4 and y_5 are two first integrals other than y_1, y_2, y_3 . Here, one has a *canonical form* for the system in this same situation.

The operations that one must perform in this particular case are of order 3, 2, 1, because it suffices, in summary, to integrate the three equations that define the plane (P), since the three equations are found to form a completely integrable system.