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Theory of the balancing foot for the Gervat gyroscope

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1. Description of the apparatus. – The balancing foot *ABCDE* (Fig. 1) is made from a metallic wire of around 1.5 mm in diameter. It is composed (more or less) of a vertical semi-circle *ABC* that is equipped at the base with an appendage *BDE* whose purpose is to make it rest on the horizontal plane along the part *DE*, which is rectilinear and perpendicular to the plane of the circle *ABC*. The wire is doubly hooked at *A* and *C* in such a fashion as to form two bearings that receive the extremities of the axis of the gyroscopic top. In that position, the mean plane of the torus of the top will pass through *DE*.

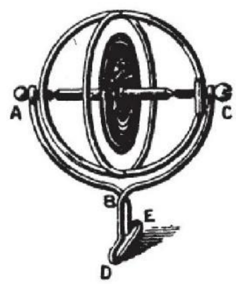

Figure 1.

If the top is not turning then the equilibrium will be unstable and everything will rotate around *DE*. However, if the top turns around itself with a large velocity (around 50 revolutions per second) then the system will seem to be in stable equilibrium. The inventor of that toy gave it the name of *balancing foot*. In reality, the foot executes oscillations about the apparent equilibrium position that are manifested by a sound that is not further than an octave below the *la* of the tuning fork under normal conditions.

Those are the facts that we shall explain theoretically while neglecting friction.

2. Choice of givens and unknowns. – I shall regard all positive rotations to be from right to left. The givens are (Fig. 2):

ω_0 initial angular velocity of the top around its axis Gy'

θ_0	initial angle between the vertical *Oz* and the perpendicular *OG* that is based at the center *G* of the top along the axis *DE*
a	length of that perpendicular *OG*
$A, B, C = A$,	principal moments of inertia of the top. *B* is relative to the axis Gy', *A* and *C* are relative to two lines Gx', Gz', resp., that are perpendicular to Gy'.
P	weight of the top

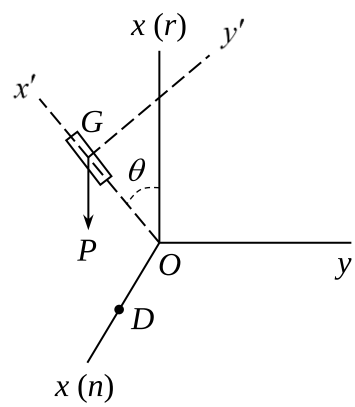

Figure 2.

I shall neglect the weights of the mounting and the base in comparison to that weight, which will simplify the writing without compromising the results.

The unknowns are:

ω	the angular velocity of the top relative to its mounting
$n = \dfrac{d\theta}{dt}$	velocity of rotation of the balancing foot around *ED*
r	angular velocity of the foot around the vertical *Oz*
v	velocity vector of the point *O* on the horizontal plane

3. Exhibiting the equations of the problem. – The external forces reduce to the weight *P*, which is applied at *G*. In order to determine the unknowns, we apply the following principles:

1. The theorem about the quantity of motion projected onto the horizontal plane.

2. Euler's equations for the axis Gy' of the top.

3. The theorem of the moment of quantity of motion with respect to the vertical at the point *G*.

4. The *vis viva* theorem.

I. The first principle tells us that the horizontal projection of the point *G* is fixed, which determines the velocity *v* of the point *O* as a function of the other unknowns. However, experiments (and calculation, as one will see) shows that θ never deviates appreciably from θ_0 . It then results that the point *O* is reasonably fixed. We shall leave aside that motion, which is most strongly perturbed by the friction that was neglected.

II. *The rotation of the top around its axis* Gy' is composed of two parts: Its relative rotation ω and the rotation of the mounting, whose component along Gy' is $+ r \sin \theta$.

On the other hand, the moments of inertia *A* and *C* are equal, and the moment of the force *P* with respect to Gy' is zero, moreover. Euler's equation for the axis Gy' is then:

$$B \frac{d}{dt} (\omega + r \sin \theta) = 0 .$$

That equation is integrated and gives:

(1) $$\omega + r \sin \theta = \omega_0 .$$

III. *The moment of the force P with respect to the vertical is constantly zero, so the moment of the quantities of motion is constant.* Now, the rotations are *n*, *r*, and ω, and their components along the principal axes of inertia at the point *G* are:

$$n, \quad p = \omega + r \sin \theta , \quad q = r \cos \theta .$$

Upon replacing $\omega + r \sin \theta$ with its value ω_0 , the principal moments of inertia will then be:

$$A\, n, \quad B\, \omega_0 , \quad C\, r \cos \theta .$$

Upon replacing *C* with *A*, which it is equal to, the moment of inertia with respect to the vertical at the point *G* will be:

$$B\, \omega_0 \sin \theta + A\, r \cos^2 \theta .$$

Upon expressing the idea that this moment should be constant, that will give:

$$B\, \omega_0 (\sin \theta - \sin \theta_0) + A\, r \cos^2 \theta = 0 ,$$

from which one infers that:

(2) $$r = -\frac{B\omega_0(\sin\theta - \sin\theta_0)}{A\cos^2\theta}.$$

IV. *The increase in the vis viva* since the initial state is equal to the work done by gravity, namely:

$$P\,a\,(\cos\theta_0 - \cos\theta)\,.$$

That *vis viva* is composed of that of the point *G*, which is ([1]):

$$\frac{P}{2g}\,a^2\,n^2\sin^2\theta\,,$$

and that of the three rotations:

$$n, \quad p = \omega_0\,, \qquad q = r\cos\theta\,.$$

The *vis viva* of the resultant rotation is:

$$\tfrac{1}{2}\,[An^2 + B\omega_0^2 + C\,r^2\cos^2\theta]\,.$$

The increase in the total *vis viva* since the initial state is then:

$$\frac{1}{2}\left(\frac{P}{g}\,a^2\,n^2\sin^2\theta + An^2 + C\,r^2\cos^2\theta\right),$$

and after replacing *C* with *A*, which is equal to it, the *vis viva* theorem will give:

$$\left(\frac{P}{g}\,a^2\,n^2\sin^2\theta + A\right)n^2 + A\,r^2\cos^2\theta = 2P\,a\,(\cos\theta_0 - \cos\theta)\,.$$

If I replace *r* with its value (2) in that equation then I will get the formula:

(3) $$n = \pm\sqrt{\frac{2P\,a\,(\cos\theta_0 - \cos\theta) - \dfrac{B^2\,\omega_0^2\,(\sin\theta - \sin\theta_0)^2}{A\cos^2\theta}}{A + \dfrac{P}{g}\,a^2\sin^2\theta}}$$

for determining *n*.

Formulas (1), (2), (3) solve the problem.

([1]) The horizontal projection of the center of gravity is fixed, so the velocity will reduce to its vertical component.

4. Consequences of formula (3). – It can be shown that θ can never differ from θ_0 by more than a very small quantity. That is because if one attributes a value to θ that is noticeably different from θ_0 then the second term in the numerator will become dominant, due to the magnitude of the coefficient $B^2\omega_0^2 / A$, which will be around 7000 times the coefficient $2Pa$ for a normal velocity of 50 revolutions per second. It would then result that n would have an imaginary value, which is inadmissible. One can then account for the motion as follows: At the beginning, n and r are zero, and gravity tends to increase θ. $n = d\theta / dt$ will then take positive values until θ attains the value θ_1 that annuls the radical. θ will no longer increase then. It will decrease with n taking negative values until θ takes the value θ_0 once more, and so on. The top will then oscillate between the initial azimuth θ_0 and a very close azimuth $\theta_1 = \theta_0 + \varepsilon_1$.

5. Approximate integration of equations (2) **and** (3). – Those predictions lead one to look for approximate formulas by taking the unknown to be (1):

$$\varepsilon = \theta - \theta_0 ,$$

from which one infers that:

$$\theta = \theta_0 + \varepsilon ,$$

$$\cos \theta_0 - \cos \theta = \varepsilon \sin \theta_0 + \ldots ,$$

$$\sin \theta - \sin \theta_0 = \varepsilon \cos \theta_0 + \ldots$$

Upon substituting those values in formula (3), one can put them into the form:

$$n = \pm\beta \sqrt{\alpha\,(\varepsilon \sin\theta_0 + \cdots) - \frac{\varepsilon^2 \cos^2\theta_0 + \cdots}{\cos^2\theta_0}} ,$$

in which one sets:

$$\beta = \frac{B\,\omega_0}{\sqrt{A\left(A + \frac{P}{g} a^2 \sin^2\theta\right)}} , \qquad \alpha = \frac{2P\,a\,A}{B^2\,\omega_0^2} .$$

That formula exhibits the fact that the value ε_1 of ε that annuls the radical has the same order of smallness as $\alpha \sin \theta_0$, and from what we said, α is around 1 / 7000. If one limits oneself to the first terms in the development in increasing powers of ε then that will give:

$$\varepsilon_1 = \alpha \sin \theta_0 , \qquad n = \frac{d\varepsilon}{dt} = \pm\beta\sqrt{\varepsilon\,(\varepsilon_1 - \varepsilon)} .$$

(1) I borrowed this transformation from the course that Resal taught at l'École Polytechnique.

That formula is easily integrated if one remarks that since θ varies only slightly, β can be regarded as constant. Upon determining the constant of integration in such a fashion that ε is zero at the origin, one will get:

$$\varepsilon = \frac{\varepsilon_1}{2} - \frac{\varepsilon_1}{2}\cos\beta t = \varepsilon_1 \sin^2\frac{\beta t}{2} .$$

From this, θ will oscillate with a *pendulum motion* between θ_0 and $\theta_0 + \varepsilon_1$. Its mean value is $\theta_0 + \varepsilon_1 / 2$.

Let us see what the rotation r around Oz will become. With the same approximation, the formula (2) will give:

$$r = -\frac{B\,\omega_0}{A\cos\,\theta_0}\varepsilon = \gamma\,\varepsilon ,$$

in which one sets:

$$\gamma = -\frac{B\,\omega_0}{A\cos\,\theta_0} .$$

Upon replacing r with $d\psi / dt$, one will find that:

$$\psi = \gamma\int\varepsilon\,dt = \gamma\left(\frac{\varepsilon_1 t}{2} - \frac{\varepsilon_1}{2\beta}\sin\beta t\right) .$$

That angular motion is composed of a uniform term and a periodic term. The velocity of uniform velocity is:

$$\frac{\gamma\,\varepsilon_1}{2} = -\frac{B\,\omega_0}{A\cos\,\theta_0}\frac{P\,a\,A\sin\theta_0}{B^2\omega_0^2} = -\frac{P\,a}{B\,\omega_0}\tan\,\theta_0 .$$

Under the normal conditions that I have spoken of, $P\,a / B\,\omega_0$ will have a value around 0.1. When the balancing foot is placed almost vertically, the second factor tan θ_0 will also be small, in such a way that the angular velocity $\gamma\,\varepsilon_1 / 2$ will be minor. For example, if θ_0 is 1° then that rotation $\gamma\,\varepsilon_1 / 2$ will be 0.1° per second, which will not be appreciable. It will become noticeable for smaller values of ω_0 and larger values of θ_0. As for the periodic term in ψ, it will have the same period as ε. The phase will change by only 1/4.

6. Conclusions. – In summary, the apparent motion is a rotation around the vertical. That rotation, which is unnoticeable when the foot is almost vertical, is accompanied by two vibrations, one of which is around the vertical, while the other is around *DE*. They are vibrations that maintain the apparent equilibrium by the composite centrifugal forces that come into play.

One can verify that experimentally by placing *DE* in a groove in the floor. That will make it impossible for the angle ψ to vary, and the system will rotate about *DE*.
