INTRODUCTION

TO THE MATHEMATICAL THEORY

OF

ELASTICITY

BY

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PREFACE

This is the first in a series of volumes on *higher mathematics*, in which I propose to attract the attention of young people to the various disciplines that are freely taught to students at the University of Naples. What are published here are the lectures that I had the honor of presenting as a substitute for Prof. G. BATTAGGLINI during the scholastic year 1892-93. They contain nothing new, nor is there any pretense that they constitute a complete course on the mathematical theory of elasticity, but they should only be considered to be a preparation for the reader who wishes to explore the many excellent treatises and study the papers – especially the Italian ones – that have been published on that subject.

Portici, 20 August 1893.

E. CESÀRO

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CHAPTER I

KINEMATICS OF SMALL MOTIONS

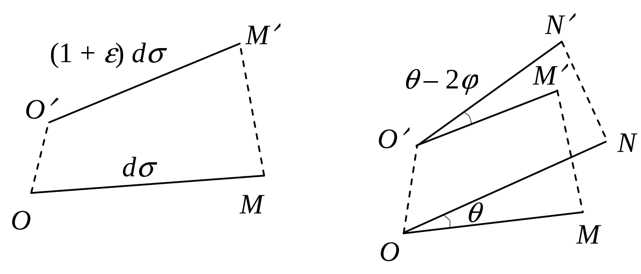
1. – In an initial approximate study of the phenomena that manifest themselves in an arbitrary medium, that medium can be compared to a system of points that are close to each other. Those who propose to study the deformations of such a system confine themselves to the ones that are negligible with respect to the mutual distances between the points, in such a way that if one calls two of those points O and M and represents their positions that result from the deformation O' and M', resp., then it will be legitimate to treat the displacements OO' and MM' as infinitesimal with respect the infinitesimal distance OM. The displacement OO' of an arbitrary point is specified in magnitude and direction by its projections onto three orthogonal axes. Those projections, which are called *the displacements* of point O and are represented by u, v, w, are obviously functions of the coordinates x, y, z of O. Suppose that those functions, which are already required to take very small values, are also continuous, uniform, and at least once differentiable, and that all of those properties also belong to the first and second partial derivatives. In the kinematical study of small deformations, the following combinations of the first partial derivatives are important:

$$a = \frac{\partial u}{\partial x}, \qquad f = \frac{1}{2}\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right), \qquad p = \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right),$$

$$b = \frac{\partial v}{\partial y}, \qquad g = \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right), \qquad q = \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right),$$

$$c = \frac{\partial w}{\partial z}, \qquad h = \frac{1}{2}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right), \qquad r = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right).$$

One soon sees why the functions a, b, c, f, g, h bear the name of *components of deformation*, while p, q, r are what one calls the *components of the rotation* of the medium.



2. – When one analyzes a deformation, one is naturally led to study, first of all, *the alteration of the distances* between neighboring points, and the *alteration of the angles* between two line elements that have a common extreme. If:

$$OM = d\sigma, \qquad O'M' = (1 + \varepsilon)\, d\sigma$$

then the ratio ε of the increment in $d\sigma$ to $d\sigma$ itself is the *coefficient of elongation* in the direction OM. If the elements OM and ON make an angle θ between them before the deformation and if $\theta - 2\varphi$ is the angle between those elements after the deformation (which are transferred to $O'M'$ and $O'N'$, resp.) then one calls 2φ the mutual *shear* of the elements in question. In order to calculate those two important quantities ε and φ, it is necessary to establish some preliminary formulas that will say what the variations $\delta\alpha$, $\delta\beta$, $\delta\gamma$ will be that are felt by the direction cosines of an arbitrary line element as a result of the deformation.

3. – If dx, dy, dz are the projections of OM onto the axes then those of $O'M'$ will be $dx + du$, $dy + dv$, $dz + dw$. Therefore:

$$\alpha = \frac{dx}{d\sigma}, \qquad \alpha + d\alpha = \frac{dx + du}{(1+\varepsilon)\, d\sigma},$$

i.e.:

$$(1 + \varepsilon)(\alpha + \delta\alpha) = \alpha + \frac{du}{d\sigma}.$$

Hence, one neglects $\varepsilon\,\delta\alpha$ in the left-hand side and writes down two more analogous relations in β and γ:

$$\delta\alpha = \frac{du}{d\sigma} - \varepsilon\alpha, \qquad \delta\beta = \frac{dv}{d\sigma} - \varepsilon\beta, \qquad \delta\gamma = \frac{dw}{d\sigma} - \varepsilon\gamma.$$

In other words, if one observes that:

$$\frac{du}{d\sigma} = \frac{\partial u}{\partial x}\frac{dx}{d\sigma} + \frac{\partial u}{\partial y}\frac{dy}{d\sigma} + \frac{\partial u}{\partial z}\frac{dz}{d\sigma} = a\,\alpha + (h - r)\,\beta + (g + q)\,\gamma$$

then one will have:

$$\left.\begin{aligned}
\delta\alpha &= -\varepsilon\,\alpha + (q\gamma - r\beta) + (a\alpha + h\beta + g\gamma),\\
\delta\beta &= -\varepsilon\,\beta + (r\alpha - p\gamma) + (h\alpha + b\beta + f\gamma),\\
\delta\gamma &= -\varepsilon\,\gamma + (p\beta - q\alpha) + (g\alpha + f\beta + c\gamma).
\end{aligned}\right\} \tag{1}$$

4. – Having said that, consider two elements whose directions are defined by the triads of cosines (α, β, γ) and (α', β', γ'). Let θ be the angle between them, in such a way that:

$$\cos\theta = \alpha\,\alpha' + \beta\,\beta' + \gamma\,\gamma'.$$

If 2φ is the angle by which θ diminishes under the effect of the deformation then one will also have:

$$\cos(\theta - 2\varphi) = (\alpha + \delta\alpha)(\alpha' + \delta\alpha') + (\beta + \delta\beta)(\beta' + \delta\beta') + (\gamma + \delta\gamma)(\gamma' + \delta\gamma') ;$$

i.e.:

$$2\varphi\sin\theta = (\alpha'\,\delta\alpha + \beta'\,\delta\beta + \gamma'\,\delta\gamma) + (\alpha\,\delta\alpha' + \beta\,\delta\beta' + \gamma\,\delta\gamma').$$

Now, formulas (1) give:

$$\begin{aligned}\alpha'\,\delta\alpha + \beta'\,\delta\beta + \gamma'\,\delta\gamma \\ &= -\varepsilon\cos\theta + p\,(\beta\,\gamma' - \gamma\,\beta') + q\,(\gamma\,\alpha' - \alpha\,\gamma') + r\,(\alpha\,\beta' - \beta\,\alpha') \\ &\quad + a\,\alpha\alpha' + b\,\beta\beta' + c\,\gamma\gamma' + f\,(\beta\,\gamma' + \gamma\,\beta') + g\,(\gamma\,\alpha' + \alpha\,\gamma') + h\,(\alpha\,\beta' + \beta\,\alpha').\end{aligned}$$

Similarly, if one switches the two terms with each other then:

$$\begin{aligned}\alpha\,\delta\alpha' + \beta\,\delta\beta' + \gamma\,\delta\gamma' \\ &= -\varepsilon'\cos\theta - p\,(\beta\,\gamma' - \gamma\,\beta') + q\,(\gamma\,\alpha' - \alpha\,\gamma') + r\,(\alpha\,\beta' - \beta\,\alpha') \\ &\quad + a\,\alpha\alpha' + b\,\beta\beta' + c\,\gamma\gamma' + f\,(\beta\,\gamma' + \gamma\,\beta') + g\,(\gamma\,\alpha' + \alpha\,\gamma') + h\,(\alpha\,\beta' + \beta\,\alpha').\end{aligned}$$

If one adds the two together then one will get the general formula:

$$\begin{aligned}\varphi\sin\theta + \tfrac{1}{2}(\varepsilon + \varepsilon')\cos\theta \\ &= a\,\alpha\alpha' + b\,\beta\beta' + c\,\gamma\gamma' + f\,(\beta\,\gamma' + \gamma\,\beta') + g\,(\gamma\,\alpha' + \alpha\,\gamma') + h\,(\alpha\,\beta' + \beta\,\alpha').\end{aligned}$$

5. – Suppose that the two directions coincide. One will then have $\alpha = \alpha'$, $\beta = \beta'$, $\gamma = \gamma'$, $\varepsilon = \varepsilon'$, $\theta = 0$, and the preceding formula will become:

$$\varepsilon = a\,\alpha^2 + b\,\beta^2 + c\,\gamma^2 + 2f\,\beta\gamma + 2g\,\gamma\alpha + 2h\,\alpha\beta. \tag{2}$$

If the two directions are mutually-perpendicular, $\theta = \pi / 2$ then the same formula will give:

$$\varphi = a\,\alpha\alpha' + b\,\beta\beta' + c\,\gamma\gamma' + f\,(\beta\,\gamma' + \gamma\,\beta') + g\,(\gamma\,\alpha' + \alpha\,\gamma') + h\,(\alpha\,\beta' + \beta\,\alpha'). \tag{3}$$

In particular, for $\alpha = 1$, $\beta = 0$, $\gamma = 0$, one will have $\varepsilon = a$, ... For $\alpha = 0$, $\beta = 1$, $\gamma = 0$ and $\alpha' = 0$, $\beta' = 1$, $\gamma' = 0$, one will have $\varphi = f$, ... Hence:

a, b, c are the coefficients of elongation of the three line elements that are parallel to the axes, and f, g, h are one-half the mutual shears of those elements.

6. – If the values that the components of the deformation assume at the point O are not all zero then the equality (2), in which one sets $\varepsilon = 0$, will become a homogeneous equation of degree two in α, β, γ. Hence, *the elements that emanate from O that do not elongate or shorten are located along the generators of a quadric cone* with a vertex at O. That surface, which one calls the *shear cone*, can be imaginary or real. If it is real then the line elements around O can be grouped into two classes: viz., the ones that are all elongated and the ones that are all shortened. Hence, ε, which is a continuous function of α, β, γ, cannot change sign without going to zero when α, β, γ vary continuously; i.e., it is not possible to pass from the region of elongation to the region of shortening without crossing the conical surface. If the shear cone is imaginary then one can say that ε will always keep the same sign, and therefore the line elements around the point considered will either all elongate or all shorten.

7. – More generally, the locus of the line elements that suffer a unit elongation will be a quadric cone, because formula (2) can be written as:

$$(a - \varepsilon)\,\alpha^2 + (b - \varepsilon)\,\beta^2 + (c - \varepsilon)\,\gamma^2 + 2f\,\beta\gamma + 2g\,\gamma\alpha + 2h\,\alpha\beta = 0. \tag{4}$$

Any value of ε will correspond to a real or imaginary cone, and all of those cones will have the same axes as the shear cone. If one imagines that the coordinate axes have already been chosen to be parallels to the axes of the cone then the preceding equation must have the form:

$$(a - \varepsilon)\,\alpha^2 + (b - \varepsilon)\,\beta^2 + (c - \varepsilon)\,\gamma^2 = 0, \tag{5}$$

which is to say that the shears f, g, h must be zero for a particular choice of axes. Therefore, *there always exists an orthogonal triad of elements* (and only one, in general) *that remains orthogonal after the deformation.* The lines along which those elements are located are called the *principal lines*, relative to the point considered.

8. – In order for equation (5) to represent a real cone, it is necessary that $a - \varepsilon$, $b - \varepsilon$, $c - \varepsilon$ cannot have the same sign, and therefore ε will always be found between the smallest and largest of the quantities a, b, c. If, to fix ideas, we suppose that $a > b > c$ then the minimum value of ε will be c and the maximum will be a. For $\varepsilon = a$, as for $\varepsilon = c$, equation (5) will not be satisfied for an infinitude of real values of α, β, γ, which is why one must have $\beta = 0$, $\gamma = 0$, $\alpha = 1$ in the first case and $\alpha = 0$, $\beta = 0$, $\gamma = 1$ in the second one. Hence, around any point, *there will exist two elements that suffer the minimum and* *maximum elongations*, and those elements *will always be mutually-orthogonal*. As for the elements that suffer the unit elongation $\varepsilon = b$, they will belong to a pair of planes that intersect along the third principal line ($\alpha = 0$, $\beta = 1$, $\gamma = 0$). One notes that the cone that corresponds to each of the three coefficients of elongation relative to the principal lines will degenerate. Hence, if one reverts to an arbitrary choice of axes then those coefficients must annul the discriminant of the quadratic form (4). They will then be the (always real) roots of the equation:

$$\begin{vmatrix} a-\varepsilon & h & g \\ h & b-\varepsilon & f \\ g & f & c-\varepsilon \end{vmatrix} = 0\ ;$$

i.e.:

$$\varepsilon^2 - (a + b + c)\,\varepsilon^2 + (bc + ca + ab - f^2 - g^2 - h^2)\,\varepsilon - (abc + 2fgh - af^2 - bg^2 - ch^2) = 0.$$

One also observes that since the coefficients of that equation are functions of only the roots, which have a significance that is independent of the choice of axes, they will also be independent of that choice; i.e., they will be *invariant*. It follows, in particular, that *the sum of the coefficients of elongation of the three orthogonal elements will not vary when the elements, which will remain mutually orthogonal, rotate around the common extremity*. One will soon see that this sum is mechanically important.

9. – The variance of ε can also be discussed more simply by recalling the following geometric representation: One carries each element OM over a distance $OP = \dfrac{1}{\sqrt{\pm\varepsilon}}$.

The coordinates of P (when O is assumed to be the origin) are:

$$x = \frac{\alpha}{\sqrt{\pm\varepsilon}}, \qquad y = \frac{\beta}{\sqrt{\pm\varepsilon}}, \qquad x = \frac{\gamma}{\sqrt{\pm\varepsilon}}.$$

If one substitutes this in (2) then one will find that the locus of the points P is the surface that is represented by the equation:

$$a\,x^2 + b\,y^2 + c\,z^2 + 2f\,yz + 2g\,zx + 2h\,xy = \pm 1. \tag{6}$$

One will then see that *the absolute value of the coefficients of elongation will vary* around each of those points *in inverse proportion to the square of the diameter of a quadric* that has its center at the point considered and is asymptotic to the shear cone. If that quadric is imaginary then the representative surface will be an ellipsoid. One can also see that by observing that in this case, ε will always keep the same sign, in such a way that the sign of the right-hand side of (6) will necessarily be that of the coefficients a, b, c. If the shear cone is real then one will need to take the + sign in the right-hand side of (6) for one spatial region and the – sign for the other one. The representative surface will then be composed of two hyperboloids with one and two sheets that have common centers, axes, and asymptotic cones.

10. – Take a *particle* around O, and let M be one of its points. If one continues to take the origin at O then one knows that during the passage from O to M the displacement u will become:

$$u' = u + x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z}.$$

That equality will be rigorously exact when one gives the values to $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$, $\frac{\partial u}{\partial z}$ that those functions will assume at a conveniently-chosen point within the segment OM. However, since those functions are assumed to be continuous, it is legitimate to take the values at the point O and neglect the higher-order infinitesimals in u'. One then observes that:

$$\frac{\partial u}{\partial x} = a, \qquad \frac{\partial u}{\partial y} = h - r, \qquad \frac{\partial u}{\partial z} = g + q.$$

One can also write out the first of the following formulas:

$$\left.\begin{aligned} u' &= (u + qz - ry) + (ax + hy + gz) &&= u_1 + u_2, \\ v' &= (v + rx - pz) + (hx + by + fz) &&= v_1 + v_2, \\ w' &= (w + py - qx) + (gx + fy + cz) &&= w_1 + w_2. \end{aligned}\right\} \tag{7}$$

The displacements u_1, v_1, w_1 refer to the hypothesis of a rigid particle that has been subjected to a translation (u, v, w) and a rotation (p, q, r). In order to see the character of the displacements u_2, v_2, w_2 then it will be enough to orient the axes along the principal lines. One will then have $u_2 = ax$, $v_2 = by$, $w_2 = cz$; viz., simple dilatations along the axes. Those three special motions (translations, rotations, triples of dilatations) can also be regarded as *consecutive* if one observes that the displacements and their derivatives will suffer negligible variations from one part of the particle to the other. Hence, *the deformation of a particle can always be considered to be the result of three dilatations along the principal lines.*

11. – We shall soon see that the absence of the third type of motion in all of the particle (so that motion can then be called a *pure* deformation) characterizes the rigidity of the entire system. The lack of the second type of motion defines a special deformation that one calls a *potential deformation*. One should notice here that the vanishing of p, q, r for all of the system is necessary and sufficient for $u\,dx + v\,dy + w\,dz$ to be an exact differential. Therefore, the deformation potential characterizes the existence of a function whose first partial derivatives will yield the displacements at any point of the system. If a, b, c, f, g, h are constants, in addition, then one will have the deformation that Thomson and Tait ([*]) call *homogeneous*.

12. – Turning to the study of the general deformation of a particle, observe that, by virtue of formulas (7), the coordinates $x + u'$, $y + v'$, $z + w'$ of M' are *linearly* coupled to those of M, and therefore any planar or rectilinear element of a particle will remain planar or rectilinear after deformation, and two parallel elements will remain parallel, etc. Therefore, if one considers an *elementary parallelepiped* – i.e., a parallelepiped that is constructed with a vertex at O with the edges dx, dy, dz that are parallel to the axes – then it will transform into another (generally oblique) parallelepiped whose edges are $(1+a)\,dx$, $(1 + b)\,dy$, $(1 + c)\,dz$, while the planar angles around O will become $\pi / 2 - 2f$, $\pi / 2 - 2g$, $\pi / 2 - 2h$. Among the infinitude of elementary parallelepipeds that one can consider, only one of them will remain rectangular after the deformation. One can use it to calculate the *coefficient of cubic dilatation* Θ; i.e., the ratio of the increase in volume of the particle to the initial volume dS. Since it is clear that, from its significance, Θ will be independent of the form of the particle, one can imagine that it is the elementary parallelepiped that is constructed from the principal lines, in such a way that:

$$dS = dx\,dy\,dz, \qquad (1 + \Theta)\,dS = (1 + a)(1 + b)(1 + c)\,dx\,dy\,dz,$$

and therefore:

$$1 + \Theta = (1 + a)(1 + b)(1 + c)\,;$$

i.e., if one neglects higher-order infinitesimals then $\Theta = a + b + c$, and since that expression is invariant, one can write:

$$\Theta = a + b + c = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \tag{8}$$

for any choice of axes. In order to prove that formula in a different way, a brief digression is in order.

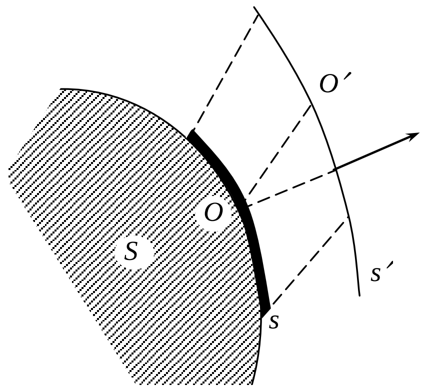
13. – We shall take this opportunity to appeal to a theorem that permits one to transform an integral that is extended over a space S into an integral that is extended over only its surface s, which bounds S. The proof of the theorem that we shall address is included in that of a more general theorem that we shall discuss later. For now, we shall confine ourselves to stating the relation:

$$\int \frac{\partial F}{\partial x} dS = -\int F \frac{dx}{dn} ds\,, \tag{9}$$

in which F is a finite, continuous, and uniform function of x, y, z. In addition, dx / dn represents the cosine of the angle that normal to the surface s (which is considered to be *positive* when it points towards the *interior* of S) will make with the x-axis. One observes that the conditions that are imposed upon the

([*]) *Natural Philosophy*, § 190.

function F, which are indispensible for the proof of the theorem, are not strictly necessary, in the sense that if any of them is absent then it might still be possible for formula (9) to be true. For example, one proves that this formula will also be valid when F becomes infinite at a point O, since if r denotes the distance from O to the point at which F is calculated and μ is a constant between 0 and 2 then the product Fr^{μ} will remain finite as r tends to zero.



14. – Let us return to the calculation of Θ. We propose to evaluate the total dilatation that S experiences when one considers it to be the sum of volumes that surface elements ds generate when one goes to the new surface s'. Take a point O in ds. The volume that ds generates is measured by the product of ds times the projection of the displacement OO' onto the *outward*-pointing normal to the surface S. That projection is $-u\dfrac{dx}{dn} - v\dfrac{dy}{dn} - w\dfrac{dz}{dn}$, and therefore the total dilatation will be given by:

$$-\int\left(u\frac{dx}{dn} + v\frac{dy}{dn} + w\frac{dz}{dn}\right) ds ,$$

or, if one adopts (9), by:

$$\int\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) dS .$$

Since that calculation is applicable to any portion of the space considered, it is clear that the last result includes formula (8). It is enough to imagine that the space S reduces to the single particle dS. More rigorously, we can argue as follows: Since Θ is a continuous function, from the nature of the deformations that we propose to study, we can just as well say that:

$$\vartheta = \Theta - \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right).$$

Meanwhile, since $\int \Theta\, dS$ obviously represents the total dilatation, one will have:

$$\int \vartheta\, dS = 0$$

for any portion of the space considered. Now, if one has $\vartheta > 0$, for example, at a point then one can circumscribe a space a around it, inside of which one always has $\vartheta > 0$, by virtue of continuity, in such a way that $\int \vartheta \, dS$ will be positive when it is extended over all space. That cannot happen. It would then be absurd to suppose that the difference ϑ cannot be zero, even at just one point.

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CHAPTER II

THE COMPONENTS OF DEFORMATION (*)

1. – *The conditions:*

$$a = 0, \quad b = 0, \quad c = 0, \quad f = 0, \quad g = 0, \quad h = 0$$

are necessary and sufficient for rigidity.

If the system moves rigidly then ε must be zero at any point, no matter what the direction (α, β, γ), and therefore a, b, c, f, g, h must vanish identically. Conversely, one can show that if the conditions:

$$(1') \qquad \frac{\partial u}{\partial x} = 0, \qquad\qquad (1'') \qquad \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} = 0,$$

$$(2') \qquad \frac{\partial v}{\partial y} = 0, \qquad\qquad (2'') \qquad \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} = 0,$$

$$(3') \qquad \frac{\partial w}{\partial z} = 0, \qquad\qquad (3'') \qquad \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} = 0$$

are satisfied then the functions u, v, w will necessarily have the characteristic form that is well-known in rational mechanics of the displacements of a rigid system. If one differentiates (3″) with respect to y and observes (2′) then one will get:

$$0 = \frac{\partial^2 u}{\partial y^2} + \frac{\partial}{\partial x}\frac{\partial v}{\partial y} = \frac{\partial^2 u}{\partial y^2}.$$

If one differentiates (2″) with respect to z and (2″) with respect to y then upon summing and taking (1″) into account, one will get:

$$0 = 2\frac{\partial^2 u}{\partial y\,\partial z} + \frac{\partial}{\partial x}\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right) = 2\frac{\partial^2 u}{\partial y\,\partial z}.$$

Therefore, if one observes (1′) then:

$$\frac{\partial}{\partial x}\frac{\partial v}{\partial y} = 0, \qquad \frac{\partial}{\partial y}\frac{\partial u}{\partial y} = 0, \qquad \frac{\partial}{\partial z}\frac{\partial u}{\partial y} = 0.$$

(*) This chapter can be omitted on a first reading.

It will then follow that $\partial u / \partial y$ is constant. Similarly, one proves that $\partial u / \partial z$ is constant. In addition, because of (1′), u will not depend upon x. Therefore:

$$u = l + r'y + qz, \qquad v = m + p'z + rx, \qquad w = n + q'x + py .$$

The nine constants reduce to six. Indeed, when one substitutes the last result in (1), (2), (3), one will get $p + p' = 0$, $q + q' = 0$, $r + r' = 0$. Hence:

$$u = l + qz - ry, \qquad v = m + rx - pz, \qquad w = n + py - qx.$$

2. – *A deformation is determined completely when one knows the components in all of the body and one is given the values at a point of the displacements and three linear relations between the first derivatives of the displacements* (*).

Indeed, suppose that the stated conditions can be satisfied, not only by one system (u', v', w') of displacements, but also by some other system (u'', v'', w''), and consider the residual displacements $u' - u'' = u$, $v' - v'' = v$, $w' - w'' = w$. Since a, b, c, f, g, h have assigned values at each point, one must have:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = \frac{\partial w}{\partial z} = 0, \qquad \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} = 0,$$

and therefore, as was proved before:

$$u = l + qz - ry, \qquad v = m + rx - pz, \qquad w = n + py - qx,$$

in which l, m, n, p, q, r represent quantities that are constant in all of the body. However, u', v', w' and u'', v'', w'' must take the same values at one point (which one can assume to be the origin); i.e., u, v, w must vanish. Therefore, $l = m = n = 0$. Now, three relations such as the following ones:

$$\alpha_1 \frac{\partial u'}{\partial x} + \alpha_2 \frac{\partial w'}{\partial y} + \alpha_3 \frac{\partial v'}{\partial z} + \alpha_4 \frac{\partial v'}{\partial y} + \alpha_5 \frac{\partial u'}{\partial z} + \alpha_6 \frac{\partial w'}{\partial x} + \alpha_7 \frac{\partial w'}{\partial z} + \alpha_8 \frac{\partial v'}{\partial x} + \alpha_9 \frac{\partial u'}{\partial y} + \alpha_{10} = 0$$

must be satisfied at that point. If one writes down the same relation for u'', v'', w'' and subtracts then one will get:

$$\alpha_2 \frac{\partial w}{\partial y} + \alpha_3 \frac{\partial v}{\partial z} + \alpha_5 \frac{\partial u}{\partial z} + \alpha_6 \frac{\partial w}{\partial x} + \alpha_8 \frac{\partial v}{\partial x} + \alpha_9 \frac{\partial u}{\partial y} = 0;$$

i.e., with the two analogous relations:

(*) BETTI, *Teoria della elasticità*, pp. 8.

$$
\begin{aligned}
(\alpha_2 - \alpha_3)\, p + (\alpha_5 - \alpha_6)\, q + (\alpha_8 - \alpha_9)\, r &= 0, \\
(\beta_2 - \beta_3)\, p + (\beta_5 - \beta_6)\, q + (\beta_8 - \beta_9)\, r &= 0, \\
(\gamma_2 - \gamma_3)\, p + (\gamma_5 - \gamma_6)\, q + (\gamma_8 - \gamma_9)\, r &= 0.
\end{aligned}
$$

If, as one supposes, those relations are all distinct then they cannot coexist unless $p = 0$, $q = 0$, $r = 0$, and consequently $u = 0$, $v = 0$, $w = 0$; i.e., $u' = u''$, $v' = v''$, $w' = w''$.

3. – The preceding theorem can be applied to homogeneous deformations. In order to do that, one must know the *constant* values of a, b, c, f, g, h and be given three relations that define the absence of rotation; viz.:

$$
\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} = 0, \quad \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} = 0, \quad \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0. \tag{1}
$$

If one fixes a point, in addition (which one can assume to be the origin), then one must have $u = v = w = 0$ for $x = y = z = 0$. One can attribute arbitrary expressions to u, v, w, and to abandon those expressions would make it impossible to satisfy the imposed conditions. However, if it happens that those conditions are satisfied then the expressions that one will find are *the only possible ones.* In the present case, one must have:

$$
\frac{\partial u}{\partial x} = a, \qquad \frac{\partial v}{\partial y} = b, \qquad \frac{\partial w}{\partial z} = c,
$$

$$
\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} = 2f, \qquad \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} = 2g, \qquad \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} = 2h,
$$

and one will see directly that these conditions, along with the other ones $u = v = w = 0$ for $x = y = z = 0$, can be satisfied if one takes:

$$
u = ax + hy + gz, \qquad v = hx + by + fz, \qquad w = gx + fy + cz,
$$

in such a way that (1) will also be satisfied, not only at the origin, but in all space. The linearity of the last formulas shows that the planes and the lines in the system will remain planes and lines. That would not be true for the most general deformations. However, one could say that, except for the rigid motions, any deformation will be homogeneous *in each particle,* and only the *constants* of the deformation will vary from one particle to the other.

4. – Any deformation is characterized by a particular system of functions a, b, c, f, g, h. Conversely, if those functions are taken arbitrarily then will they correspond to some possible deformation? One deduces immediately from the defining formulas that:

$$\frac{\partial g}{\partial y}+\frac{\partial h}{\partial z}=\frac{\partial^2 u}{\partial y\,\partial z}+\frac{1}{2}\left(\frac{\partial^2 w}{\partial x\,\partial y}+\frac{\partial^2 v}{\partial x\,\partial z}\right)=\frac{\partial^2 u}{\partial y\,\partial z}+\frac{\partial f}{\partial x}.$$

Hence:

$$\frac{\partial^2 a}{\partial y\,\partial z}=\frac{\partial}{\partial x}\frac{\partial^2 u}{\partial y\,\partial z}=\frac{\partial}{\partial x}\left(\frac{\partial g}{\partial y}+\frac{\partial h}{\partial z}-\frac{\partial f}{\partial x}\right).$$

Similarly:

$$\frac{\partial^2 f}{\partial y\,\partial z}=\frac{1}{2}\left(\frac{\partial^3 w}{\partial y^2\,\partial z}+\frac{\partial^3 v}{\partial y\,\partial z^2}\right)=\frac{1}{2}\left(\frac{\partial^2 c}{\partial y^2}+\frac{\partial^2 b}{\partial z^2}\right).$$

Therefore, the conditions:

$$(2)\quad\left\{\begin{aligned}
\frac{\partial^2 a}{\partial y\,\partial z}&=\frac{\partial}{\partial x}\left(\frac{\partial g}{\partial y}+\frac{\partial h}{\partial z}-\frac{\partial f}{\partial x}\right),\\
\frac{\partial^2 b}{\partial z\,\partial x}&=\frac{\partial}{\partial y}\left(\frac{\partial h}{\partial z}+\frac{\partial f}{\partial x}-\frac{\partial g}{\partial y}\right),\\
\frac{\partial^2 c}{\partial x\,\partial y}&=\frac{\partial}{\partial z}\left(\frac{\partial f}{\partial x}+\frac{\partial g}{\partial y}-\frac{\partial h}{\partial z}\right),
\end{aligned}\right.
\qquad
(3)\quad\left\{\begin{aligned}
\frac{\partial^2 f}{\partial y\,\partial z}&=\frac{1}{2}\left(\frac{\partial^2 c}{\partial y^2}+\frac{\partial^2 b}{\partial z^2}\right),\\
\frac{\partial^2 g}{\partial z\,\partial x}&=\frac{1}{2}\left(\frac{\partial^2 a}{\partial z^2}+\frac{\partial^2 c}{\partial x^2}\right),\\
\frac{\partial^2 h}{\partial x\,\partial y}&=\frac{1}{2}\left(\frac{\partial^2 b}{\partial x^2}+\frac{\partial^2 a}{\partial y^2}\right)
\end{aligned}\right.$$

are necessary for the existence of u, v, w. Are they sufficient?

5. – This is how Prof. Beltrami (*) showed that *the conditions* (2) *and* (3) *are necessary and sufficient for a, b, c, f, g, h to be able to represent the components of a deformation:* Suppose that the three components of rotation are given, along with the aforementioned six functions. One must have:

$$\left.\begin{aligned}
&\frac{\partial u}{\partial x}=a, &&\frac{\partial v}{\partial x}=h+r, &&\frac{\partial w}{\partial x}=g-q,\\
&\frac{\partial u}{\partial y}=h-r, &&\frac{\partial v}{\partial y}=b, &&\frac{\partial w}{\partial y}=f+p,\\
&\frac{\partial u}{\partial z}=g+q, &&\frac{\partial v}{\partial z}=f-p, &&\frac{\partial w}{\partial z}=c.
\end{aligned}\right\}\tag{4}$$

Consider the three equations that refer to u. It is known that in order for a function u to exist that satisfies those equations, it is necessary and sufficient that one must have:

$$\frac{\partial a}{\partial y}=\frac{\partial(h-r)}{\partial x},\qquad \frac{\partial a}{\partial z}=\frac{\partial(g+q)}{\partial x},\qquad \frac{\partial(g+q)}{\partial y}=\frac{\partial(h-r)}{\partial z}.\tag{5}$$

(*) "Sulla interpretazione meccanica delle formole di Maxwell," Memorie di Bologna, 1885. (Footnote).

Consider the other two equations that are analogous to the last of (5). One has:

$$\frac{\partial q}{\partial y}+\frac{\partial r}{\partial z}=\frac{\partial h}{\partial z}-\frac{\partial g}{\partial y}, \quad \frac{\partial r}{\partial z}+\frac{\partial p}{\partial x}=\frac{\partial f}{\partial x}-\frac{\partial h}{\partial z}, \quad \frac{\partial p}{\partial x}+\frac{\partial q}{\partial y}=\frac{\partial g}{\partial y}-\frac{\partial f}{\partial x};$$

summing will give:

$$\frac{\partial p}{\partial x}+\frac{\partial q}{\partial y}+\frac{\partial r}{\partial z}=0,$$

and the last three relations will become:

$$\frac{\partial p}{\partial x}=\frac{\partial g}{\partial y}-\frac{\partial h}{\partial z}, \qquad \frac{\partial q}{\partial y}=\frac{\partial h}{\partial z}-\frac{\partial f}{\partial x}, \qquad \frac{\partial r}{\partial z}=\frac{\partial f}{\partial x}-\frac{\partial g}{\partial y}.$$

Hence, if one takes (5) into account then:

$$\left.\begin{aligned}
&\frac{\partial p}{\partial x}=\frac{\partial g}{\partial y}-\frac{\partial h}{\partial z}, \quad \frac{\partial q}{\partial x}=\frac{\partial a}{\partial z}-\frac{\partial q}{\partial x}, \quad \frac{\partial r}{\partial x}=\frac{\partial h}{\partial x}-\frac{\partial a}{\partial y},\\
&\frac{\partial p}{\partial y}=\frac{\partial f}{\partial y}-\frac{\partial b}{\partial z}, \quad \frac{\partial q}{\partial y}=\frac{\partial h}{\partial z}-\frac{\partial f}{\partial x}, \quad \frac{\partial r}{\partial y}=\frac{\partial b}{\partial x}-\frac{\partial h}{\partial y},\\
&\frac{\partial p}{\partial z}=\frac{\partial c}{\partial y}-\frac{\partial f}{\partial z}, \quad \frac{\partial q}{\partial z}=\frac{\partial g}{\partial z}-\frac{\partial c}{\partial x}, \quad \frac{\partial r}{\partial z}=\frac{\partial f}{\partial x}-\frac{\partial g}{\partial y}.
\end{aligned}\right\} \tag{6}$$

Those are necessary and sufficient conditions for the existence of u, v, w when one is given a, b, c, f, g, h, p, q, r. It is then known that in order for (6) to be integrable, it is necessary and sufficient (as far as p is concerned) that the following relations should be satisfied:

$$\frac{\partial}{\partial y}\left(\frac{\partial g}{\partial y}-\frac{\partial h}{\partial z}\right)=\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}-\frac{\partial b}{\partial z}\right), \qquad \frac{\partial}{\partial z}\left(\frac{\partial g}{\partial y}-\frac{\partial h}{\partial z}\right)=\frac{\partial}{\partial x}\left(\frac{\partial c}{\partial y}-\frac{\partial f}{\partial z}\right),$$

$$\frac{\partial}{\partial y}\left(\frac{\partial c}{\partial y}-\frac{\partial f}{\partial z}\right)=\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}-\frac{\partial b}{\partial z}\right),$$

which can then be written:

$$\frac{\partial^2 b}{\partial x\,\partial z}=\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}+\frac{\partial h}{\partial z}-\frac{\partial g}{\partial y}\right), \quad \frac{\partial^2 c}{\partial x\,\partial y}=\frac{\partial}{\partial z}\left(\frac{\partial f}{\partial x}+\frac{\partial g}{\partial y}-\frac{\partial h}{\partial z}\right), \quad \frac{\partial^2 c}{\partial y^2}+\frac{\partial^2 b}{\partial z^2}=2\frac{\partial^2 f}{\partial y\,\partial z}.$$

With that, one sees that the relations (2) and (3) are precisely the necessary and sufficient conditions for the integrability of (6). When they are satisfied, there will exist functions p, q, r that satisfy (6), and therefore u, v, w will exist, as well.

6. – Another proof, which is likewise due to Prof. Beltrami (*), is obtained when one effectively tries to integrate (2) and (3), when they are considered to be third-order partial differential equations in u, v, w. One can always find three functions U, V, W, such that one has:

$$a = \frac{\partial U}{\partial x}, \qquad b = \frac{\partial V}{\partial y}, \qquad c = \frac{\partial W}{\partial z}.$$

The first of (2) will become:

$$\frac{\partial}{\partial x}\left(\frac{\partial g}{\partial y} + \frac{\partial h}{\partial z} - \frac{\partial f}{\partial x} - \frac{\partial^2 U}{\partial y\,\partial z}\right) = 0, \tag{7}$$

and the first of (3) will become:

$$\frac{\partial^2}{\partial y\,\partial z}\left[f - \frac{1}{2}\left(\frac{\partial W}{\partial y} + \frac{\partial V}{\partial z}\right)\right] = 0. \tag{8}$$

Set:

$$f = \frac{1}{2}\left(\frac{\partial W}{\partial y} + \frac{\partial V}{\partial z}\right) + f'.$$

From (8), one must have:

$$\frac{\partial^2 f'}{\partial y\,\partial z} = 0, \qquad \frac{\partial^2 g'}{\partial z\,\partial x} = 0, \qquad \frac{\partial^2 h'}{\partial x\,\partial y} = 0, \tag{9}$$

and from (7), one will next observe that:

$$\begin{aligned}
&\frac{\partial g}{\partial y} + \frac{\partial h}{\partial z} - \frac{\partial f}{\partial x} \\
&\qquad = \frac{\partial g'}{\partial y} + \frac{\partial h'}{\partial z} - \frac{\partial f'}{\partial x} + \frac{1}{2}\left[\frac{\partial}{\partial y}\left(\frac{\partial U}{\partial z} + \frac{\partial W}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{\partial U}{\partial y} + \frac{\partial V}{\partial x}\right) - \frac{\partial}{\partial x}\left(\frac{\partial W}{\partial y} + \frac{\partial V}{\partial z}\right)\right] \\
&\qquad = \frac{\partial g'}{\partial y} + \frac{\partial h'}{\partial z} - \frac{\partial f'}{\partial x} + \frac{\partial^2 U}{\partial y\,\partial z}
\end{aligned}$$

must also give:

$$\frac{\partial}{\partial x}\left(\frac{\partial g'}{\partial y} + \frac{\partial h'}{\partial z} - \frac{\partial f'}{\partial x}\right) = 0.$$

In other words, $\frac{\partial g'}{\partial y} + \frac{\partial h'}{\partial z} - \frac{\partial f'}{\partial x}$ is independent of x, and therefore one can represent it as a second derivative of a function U_x of y and z with respect to y and z. One then sets:

(*) Rendiconti del Circ. mat. di Palermo, **3** (1889). Note fisico-matematiche.

$$\begin{cases} \dfrac{\partial g'}{\partial y} + \dfrac{\partial h'}{\partial z} - \dfrac{\partial f'}{\partial x} = \dfrac{\partial^2 U_x}{\partial y\,\partial z}, \\[2ex] \dfrac{\partial h'}{\partial z} + \dfrac{\partial f'}{\partial x} - \dfrac{\partial g'}{\partial y} = \dfrac{\partial^2 V_y}{\partial z\,\partial x}, \\[2ex] \dfrac{\partial f'}{\partial x} + \dfrac{\partial g'}{\partial y} - \dfrac{\partial h'}{\partial z} = \dfrac{\partial^2 W_z}{\partial x\,\partial y}, \end{cases}$$

in which V_y is independent of y, and W_z is independent of z. When the last two relations are summed, that will give:

$$\frac{\partial f'}{\partial x} = \frac{1}{2}\left(\frac{\partial^2 V_y}{\partial z\,\partial x} + \frac{\partial^2 W_z}{\partial x\,\partial y}\right);$$

i.e.:

$$\frac{\partial}{\partial x}\left[f' - \frac{1}{2}\left(\frac{\partial W_z}{\partial y} + \frac{\partial V_y}{\partial z}\right)\right] = 0.$$

Consequently, one can set:

$$f' = \frac{1}{2}\left(\frac{\partial W_z}{\partial y} + \frac{\partial V_y}{\partial z}\right) + f'',$$

in which f'', g'', h'' are functions that satisfy the equations:

$$\frac{\partial f''}{\partial x} = 0, \qquad \frac{\partial g''}{\partial y} = 0, \qquad \frac{\partial h''}{\partial z} = 0, \tag{10}$$

as well as the other ones:

$$\frac{\partial^2 f''}{\partial y\,\partial z} = 0, \qquad \frac{\partial^2 g''}{\partial z\,\partial x} = 0, \qquad \frac{\partial^2 h''}{\partial x\,\partial y} = 0, \tag{11}$$

which one deduces from (9). The conditions (10) and (11) show that f is the sum of two functions, one of which is a function of only y, while the other one is a function of only z. The former can always be represented as the derivative of a function W_y / 2 of only y, while the other one can always be represented as a derivative of a function V_z / 2 of only z. In other words, one can set:

$$f'' = \frac{1}{2}\left(\frac{dW_y}{dz} + \frac{dV_z}{dy}\right).$$

Hence, in summary:

$$f = \frac{1}{2}\left(\frac{\partial W}{\partial y} + \frac{\partial V}{\partial z}\right) + \frac{1}{2}\left(\frac{\partial W_z}{\partial y} + \frac{\partial V_y}{\partial z}\right) + \frac{1}{2}\left(\frac{dW_y}{dy} + \frac{dV_z}{dz}\right),$$

or:

$$f = \frac{1}{2}\left[\frac{\partial}{\partial y}(W + W_z + W_y) + \frac{\partial}{\partial z}(V + V_z + V_y)\right],$$

$$g = \frac{1}{2}\left[\frac{\partial}{\partial z}(U + U_x + U_z) + \frac{\partial}{\partial x}(W + W_z + W_x)\right],$$

$$h = \frac{1}{2}\left[\frac{\partial}{\partial x}(V + V_y + V_x) + \frac{\partial}{\partial y}(U + U_x + U_y)\right].$$

Now, if one observes that one must have:

$$f = \frac{1}{2}\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right), \qquad g = \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right), \qquad h = \frac{1}{2}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)$$

then one will see that one can take:

$$u = U + U_x + U_y + U_z\,,$$
$$v = V + V_x + V_y + V_z\,,$$
$$w = W + W_x + W_y + W_z\,,$$

since the relations:

$$a = \frac{\partial u}{\partial x}, \qquad b = \frac{\partial v}{\partial y}, \qquad c = \frac{\partial w}{\partial z}$$

will also be satisfied in that way.

7. – The first proof is tied in with some elegant considerations of Prof. Beltrami (*), by means of which he could show that the conditions (2) and (3) can be combined into a single equality by setting a certain part δJ equal to zero of the variation that the integral:

$$J = \tfrac{1}{2}\iiint\left(\frac{\partial x}{\partial p} + \frac{\partial y}{\partial q} + \frac{\partial z}{\partial r}\right)dp\,dq\,dr \tag{12}$$

will experience when one attributes arbitrary variations δa, δb, δc, δf, δg, δh to a, b, c, f, g, h. First, express J in terms of the variables x, y, z. The functional determinant:

(*) Comptes-rendus de l'Académie des Sciences de Paris (1889), pp. 502.

$$\Delta = \begin{vmatrix} \frac{\partial p}{\partial x} & \frac{\partial p}{\partial y} & \frac{\partial p}{\partial z} \\ \frac{\partial q}{\partial x} & \frac{\partial q}{\partial y} & \frac{\partial q}{\partial z} \\ \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} & \frac{\partial r}{\partial z} \end{vmatrix}$$

is non-zero. Indeed, the functions p, q, r are mutually independent, although the constraint:

$$\frac{\partial p}{\partial x} + \frac{\partial q}{\partial y} + \frac{\partial r}{\partial z} = 0 \tag{13}$$

exists between the derivative. Having assumed that, one deduces from the relations:

$$\begin{cases} dp = \frac{\partial p}{\partial x} dx + \frac{\partial p}{\partial y} dy + \frac{\partial p}{\partial z} dz, \\ dq = \frac{\partial q}{\partial x} dx + \frac{\partial q}{\partial y} dy + \frac{\partial q}{\partial z} dz, \\ dr = \frac{\partial r}{\partial x} dx + \frac{\partial r}{\partial y} dy + \frac{\partial r}{\partial z} dz \end{cases}$$

that:

$$\Delta \cdot dx = \begin{vmatrix} \frac{\partial p}{\partial x} & \frac{\partial p}{\partial y} & \frac{\partial p}{\partial z} \\ \frac{\partial q}{\partial x} & \frac{\partial q}{\partial y} & \frac{\partial q}{\partial z} \\ \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} & \frac{\partial r}{\partial z} \end{vmatrix}. \tag{14}$$

Since:

$$dx = \frac{\partial x}{\partial p} dp + \frac{\partial x}{\partial q} dq + \frac{\partial x}{\partial r} dr ,$$

upon equating the coefficients of dp in the two sides of (14), one will get:

$$\Delta \frac{\partial x}{\partial p} = \frac{\partial q}{\partial y}\frac{\partial r}{\partial z} - \frac{\partial q}{\partial z}\frac{\partial r}{\partial y};$$

hence:

$$\Delta\left(\frac{\partial x}{\partial p} + \frac{\partial y}{\partial q} + \frac{\partial z}{\partial r}\right) = \left(\frac{\partial q}{\partial y}\frac{\partial r}{\partial z} - \frac{\partial q}{\partial z}\frac{\partial r}{\partial y}\right) + \left(\frac{\partial r}{\partial z}\frac{\partial p}{\partial x} - \frac{\partial r}{\partial x}\frac{\partial p}{\partial z}\right) + \left(\frac{\partial p}{\partial x}\frac{\partial q}{\partial y} - \frac{\partial p}{\partial y}\frac{\partial q}{\partial x}\right).$$

Therefore, upon substituting that in (12) and representing the spatial element *dx dy dz* by *dS*, as is usual, one will have:

$$J = \tfrac{1}{2}\int\left[\left(\frac{\partial q}{\partial y}\frac{\partial r}{\partial z}-\frac{\partial q}{\partial z}\frac{\partial r}{\partial y}\right)+\cdots\right]dS\,. \tag{15}$$

8. – We shall now attempt to transform J into a surface integral. One has:

$$\frac{\partial q}{\partial y}\frac{\partial r}{\partial z}-\frac{\partial q}{\partial z}\frac{\partial r}{\partial y}=\frac{\partial}{\partial y}\left(q\frac{\partial r}{\partial z}-r\frac{\partial q}{\partial z}\right)+\left(r\frac{\partial^2 q}{\partial y\,\partial z}-q\frac{\partial^2 r}{\partial y\,\partial z}\right).$$

Similarly:

$$\frac{\partial q}{\partial y}\frac{\partial r}{\partial z}-\frac{\partial q}{\partial z}\frac{\partial r}{\partial y}=\frac{\partial}{\partial z}\left(r\frac{\partial q}{\partial y}-q\frac{\partial r}{\partial y}\right)+\left(q\frac{\partial^2 r}{\partial y\,\partial z}-r\frac{\partial^2 q}{\partial y\,\partial z}\right).$$

Hence, upon summing:

$$2\left(\frac{\partial q}{\partial y}\frac{\partial r}{\partial z}-\frac{\partial q}{\partial z}\frac{\partial r}{\partial y}\right)=\frac{\partial}{\partial y}\left(q\frac{\partial r}{\partial z}-r\frac{\partial q}{\partial z}\right)+\frac{\partial}{\partial z}\left(r\frac{\partial q}{\partial y}-q\frac{\partial r}{\partial y}\right).$$

Therefore, upon substituting that in (15) and collecting the terms that are derivatives with respect to x, one will have:

$$J = \tfrac{1}{4}\int\left[\frac{\partial}{\partial x}\left(p\frac{\partial r}{\partial z}-r\frac{\partial p}{\partial z}+p\frac{\partial q}{\partial y}-q\frac{\partial p}{\partial y}\right)\right]dS\,. \tag{16}$$

On the other hand, by virtue of (13), one has:

$$p\frac{\partial r}{\partial z}-r\frac{\partial p}{\partial z}+p\frac{\partial q}{\partial y}-q\frac{\partial p}{\partial y}=-\left(p\frac{\partial p}{\partial x}+q\frac{\partial p}{\partial y}+r\frac{\partial p}{\partial z}\right).$$

Now, if one lets $d\sigma$ denote the line element that is located along the rotational axis (p, q, r), in such a way that:

$$\frac{dx}{p}=\frac{dy}{q}=\frac{dz}{r}=\frac{d\sigma}{\sqrt{p^2+q^2+r^2}},$$

then one can also give the preceding expression the form:

$$-\left(\frac{\partial p}{\partial x}\frac{dx}{d\sigma}+\frac{\partial p}{\partial y}\frac{dy}{d\sigma}+\frac{\partial p}{\partial z}\frac{dz}{d\sigma}\right)\sqrt{p^2+q^2+r^2}=-\frac{dp}{d\sigma}\sqrt{p^2+q^2+r^2}\,,$$

and then if one substitutes this in (16) and makes use of formula (10) in the preceding chapter then one will get:

$$J = \tfrac{1}{4}\int\left(\frac{dp}{d\sigma}\frac{dp}{dn} + \frac{dq}{d\sigma}\frac{dy}{dn} + \frac{dr}{d\sigma}\frac{ds}{dn}\right)\sqrt{p^2 + q^2 + r^2}\,ds\,.$$

9. – The last result has no importance for us. We are only interested in the possibility of transforming J into a surface integral; i.e., we must confirm that J is *only apparently* a triple integral, while in reality it is a double integral. It will then follow that the part of its variation δJ that does not reduce to a surface integral must necessarily be zero identically. Meanwhile, one sees that if one assumes the existence of p, q, r, along with a, b, c, f, g, h, then the relations (6) will be necessary and sufficient for the existence of u, v, w. In that way, the expression inside the integral sign in (15) will become:

$$\left(\frac{\partial h}{\partial z} - \frac{\partial f}{\partial x}\right)\left(\frac{\partial f}{\partial x} - \frac{\partial g}{\partial y}\right) + \left(\frac{\partial f}{\partial x} - \frac{\partial g}{\partial y}\right)\left(\frac{\partial g}{\partial y} - \frac{\partial h}{\partial z}\right) + \left(\frac{\partial g}{\partial y} - \frac{\partial h}{\partial z}\right)\left(\frac{\partial h}{\partial z} - \frac{\partial f}{\partial x}\right)$$
$$- \left(\frac{\partial g}{\partial z} - \frac{\partial c}{\partial x}\right)\left(\frac{\partial b}{\partial x} - \frac{\partial h}{\partial y}\right) - \left(\frac{\partial h}{\partial x} - \frac{\partial a}{\partial y}\right)\left(\frac{\partial c}{\partial y} - \frac{\partial f}{\partial z}\right) - \left(\frac{\partial f}{\partial y} - \frac{\partial b}{\partial z}\right)\left(\frac{\partial a}{\partial z} - \frac{\partial g}{\partial x}\right).$$

Now, suppose that one gives arbitrary variations to a, b, c, f, g, h and attempts to put the resulting variation of J into the form:

$$\delta J = \int(\mathcal{A}\,\delta a + \mathcal{B}\,\delta b + \mathcal{C}\,\delta c + \mathcal{F}\,\delta f + \mathcal{G}\,\delta g + \mathcal{H}\,\delta h)\,dS\,,$$

up to surface integrals. If one varies only a then one will get:

$$\delta J = \tfrac{1}{2}\int\left[\left(\frac{\partial c}{\partial y} - \frac{\partial f}{\partial z}\right)\frac{\partial \delta a}{\partial y} + \left(\frac{\partial b}{\partial z} - \frac{\partial f}{\partial y}\right)\frac{\partial \delta a}{\partial z}\right]dS\,,$$

and upon integrating by parts:

$$\delta J = \tfrac{1}{2}\int\left[\frac{\partial}{\partial y}\left\{\left(\frac{\partial c}{\partial y} - \frac{\partial f}{\partial z}\right)\delta a\right\} + \frac{\partial}{\partial z}\left\{\left(\frac{\partial b}{\partial z} - \frac{\partial f}{\partial y}\right)\delta a\right\}\right]dS$$

$$-\tfrac{1}{2}\int\left[\frac{\partial}{\partial y}\left(\frac{\partial c}{\partial y} - \frac{\partial f}{\partial z}\right) + \frac{\partial}{\partial z}\left(\frac{\partial b}{\partial z} - \frac{\partial f}{\partial y}\right)\right]\delta a\,dS\,.$$

The first part is reducible to a double integral by means of formula (10) in the preceding chapter, and the second part is the expression for $\int \mathcal{A}\,\delta a\,dS$. Hence:

$$\mathcal{A} = \frac{1}{2}\left[\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial z}-\frac{\partial c}{\partial y}\right)+\frac{\partial}{\partial z}\left(\frac{\partial f}{\partial y}-\frac{\partial b}{\partial z}\right)\right] = \frac{\partial^2 f}{\partial y\,\partial z}-\frac{1}{2}\left(\frac{\partial^2 c}{\partial y^2}+\frac{\partial^2 b}{\partial z^2}\right).$$

If one operates analogously on f then one will first get:

$$\delta J = \tfrac{1}{2}\int\left[-2\frac{\partial f}{\partial y}\frac{\partial \delta f}{\partial x}+\left(\frac{\partial g}{\partial y}+\frac{\partial h}{\partial z}\right)\frac{\partial \delta f}{\partial x}+\left(\frac{\partial g}{\partial x}-\frac{\partial a}{\partial z}\right)\frac{\partial \delta f}{\partial y}+\left(\frac{\partial h}{\partial x}-\frac{\partial a}{\partial y}\right)\frac{\partial \delta f}{\partial z}\right]dS$$

and then, upon integrating by parts:

$$\delta J =$$

$$\tfrac{1}{2}\int\left[-2\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\delta f\right)+\frac{\partial}{\partial x}\left\{\left(\frac{\partial g}{\partial y}+\frac{\partial h}{\partial z}\right)\delta f\right\}+\frac{\partial}{\partial y}\left\{\left(\frac{\partial g}{\partial x}-\frac{\partial a}{\partial z}\right)\delta f\right\}+\frac{\partial}{\partial z}\left\{\left(\frac{\partial h}{\partial x}-\frac{\partial a}{\partial y}\right)\delta f\right\}\right]dS$$

$$-\ \tfrac{1}{2}\int\left[-2\frac{\partial^2 f}{\partial x^2}+\frac{\partial}{\partial x}\left(\frac{\partial g}{\partial y}+\frac{\partial h}{\partial z}\right)+\frac{\partial}{\partial y}\left(\frac{\partial g}{\partial x}-\frac{\partial a}{\partial z}\right)+\frac{\partial}{\partial z}\left(\frac{\partial h}{\partial x}-\frac{\partial a}{\partial y}\right)\right]\delta f\, dS\ .$$

If one neglects the first part (which is, in reality, a double integral) and compares the second part to $\int \mathcal{F}\,\delta f\, dS$ then one will get:

$$\mathcal{F} = \frac{\partial^2 f}{\partial x^2}-\frac{1}{2}\frac{\partial}{\partial x}\left(\frac{\partial g}{\partial y}+\frac{\partial h}{\partial z}\right)+\frac{1}{2}\frac{\partial}{\partial y}\left(\frac{\partial g}{\partial x}-\frac{\partial a}{\partial z}\right)+\frac{1}{2}\frac{\partial}{\partial z}\left(\frac{\partial h}{\partial x}-\frac{\partial a}{\partial y}\right)$$

$$= \frac{\partial^2 a}{\partial y\,\partial z}-\frac{\partial}{\partial x}\left(\frac{\partial g}{\partial y}+\frac{\partial h}{\partial z}-\frac{\partial f}{\partial x}\right).$$

One will then have:

$$\left\{\begin{aligned}
\mathcal{A} &= \frac{\partial^2 f}{\partial y\,\partial z}-\frac{1}{2}\left(\frac{\partial^2 c}{\partial y^2}+\frac{\partial^2 b}{\partial z^2}\right), & \mathcal{F} &= \frac{\partial^2 a}{\partial y\,\partial z}-\frac{1}{2}\left(\frac{\partial g}{\partial y}+\frac{\partial h}{\partial z}-\frac{\partial f}{\partial x}\right),\\
\mathcal{B} &= \frac{\partial^2 g}{\partial z\,\partial x}-\frac{1}{2}\left(\frac{\partial^2 a}{\partial z^2}+\frac{\partial^2 c}{\partial x^2}\right), & \mathcal{G} &= \frac{\partial^2 b}{\partial z\,\partial x}-\frac{1}{2}\left(\frac{\partial h}{\partial z}+\frac{\partial f}{\partial x}-\frac{\partial g}{\partial y}\right),\\
\mathcal{C} &= \frac{\partial^2 h}{\partial x\,\partial y}-\frac{1}{2}\left(\frac{\partial^2 b}{\partial x^2}+\frac{\partial^2 a}{\partial y^2}\right), & \mathcal{H} &= \frac{\partial^2 c}{\partial x\,\partial y}-\frac{1}{2}\left(\frac{\partial f}{\partial x}+\frac{\partial g}{\partial y}-\frac{\partial h}{\partial z}\right),
\end{aligned}\right.$$

and one will see that the functions $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, $\mathcal{F}$, $\mathcal{G}$, $\mathcal{H}$, whose simultaneous vanishing is sufficient and (given the arbitrariness in δa, δb, δc, δf, δg, δh) necessary for the identical vanishing of δJ, are precisely the ones that will yield the necessary and sufficient conditions for a, b, c, f, g, h to be the components of a possible deformation when they are set equal to zero.

CHAPTER III

THE ELASTIC FORCE POTENTIAL

1. – Experiments indicate that when bodies in nature are subjected to suitably-small forces, they will deform, but return to their original form as soon as the deforming forces cease to act. One expresses that notion by saying that the deformation gives rise to *elastic forces* that tend to lead the points of the body back to their old positions. Only the elastic forces will intervene in that return to a state of *stable equilibrium*, and it is known from rational mechanics that in an equilibrated system that is subject to forces that depend upon the relative positions of the points of the system and which are the elastic forces, the *potential* or work done by the forces will be a *maximum* or a *minimum* according to whether that equilibrium is *stable* or *unstable*. One sees, in addition, that the work does not depend upon the infinitude of configurations that the system can take on by reason of equilibrium, but only upon the initial and final configurations. If $\Pi\, dS$ represents the potential that relates to the particle dS then one can assert that Π depends upon only the quantities that characterize the extreme configurations of the particle; i.e., if one computes the work done when one starts from the equilibrium configuration then Π will be a function of the quantities a, b, c, f, g, h that characterizes the pure deformation, since the elastic force will do no work under a rigid motion. One adopts a known formula and generally lets φ_θ represent what the function φ of a, b, c, f, g, h will become when the variables are multiplied by the positive number θ, which is less than unity, so one can express the unit potential Π in the following way:

$$\Pi = \Pi_0 + a\left(\frac{\partial\Pi}{\partial a}\right)_0 + b\left(\frac{\partial\Pi}{\partial b}\right)_0 + \ldots + h\left(\frac{\partial\Pi}{\partial h}\right)_0$$

$$+ \frac{1}{2}\left[a^2\left(\frac{\partial^2\Pi}{\partial a^2}\right)_\theta + 2ab\left(\frac{\partial^2\Pi}{\partial a\,\partial b}\right)_\theta + \cdots + h^2\left(\frac{\partial^2\Pi}{\partial h^2}\right)_\theta\right].$$

By convention, $\Pi_0 = 0$. In addition, since the function Π has its maximum value for vanishing a, b, c, f, g, h, its first variation must be zero and its second variation must be *negative*. Finally, when φ is continuous, one can substitute φ_0 for any φ_θ, and neglect the very small quantities that have the same order as a, b, c, f, g, h. Therefore, Π *is an essentially-negative quadratic form in the components of the deformation.* Let us point out the importance that the unit potential Π has had in all of the theory up to now:

> "It has the distinguished property of representing the *energy* per unit volume that the elastic body possesses around the point considered, and that energy is equivalent to the work done by a unit volume of the body when it returns from its present state to the natural state, which is the work

done by the external forces that took the unit volume from the natural state to the present state of elastic coaction (*)."

2. – If one groups together the terms in the expression for Π that contain only a, b, c and the ones that contain only f, g, h then one can write:

$$\begin{aligned}\Pi = -\tfrac{1}{2}\,(Aa^2 + Bb^2 + Cc^2 + 2A'bc + 2B'ca + 2C'ab) \\ - 2\,(Ff^2 + Gg^2 + Hh^2 + 2F'gh + 2G'hf + 2H'fg) \\ - 2a\,(F_1 f + G_1 g + H_1 h) - 2b\,(F_2 f + G_2 g + H_2 h) - 2c\,(F_3 f + G_3 g + H_3 h).\end{aligned}$$

If the medium is homogeneous then the coefficients A, B, …, H_3 will be constant in all of the body, as long as they have no temperature variations, which has been tacitly assumed up to now, and which we shall continue to assume. Observe that in the most general case there are *twenty-one* of those constants or *elastic coefficients*. Rankine (**) distinguished them with the following terminology:

A, B, C:	*direct* elasticity
F, G, H:	*tangential or rigid* elasticity,
A', B', C':	*lateral* elasticity,
F', G', H'; F_1, G_1, H_1; F_2, G_2, H_2; F_3, G_3, H_3;	*asymmetric* elasticity.

If the medium is endowed with a *symmetry plane* (as far as elasticity is concerned) then one can say that Π does not vary in form when one takes that plane to the yz-plane and changes x into $-x$. That exchange implies changing u into $-u$, and then g and h into $-g$ and $-h$, resp., while a, b, c, f remain unaltered. One must then have:

$$G' = G_1 = G_2 = G_3 = 0, \qquad H' = H_1 = H_2 = H_3 = 0.$$

If the medium is endowed with two orthogonal symmetry planes then *all* of the asymmetric elasticity coefficients will be zero. That reveals the necessary existence of a third symmetry plane that is perpendicular to the first two. In that case, the unit potential will assume the simple form:

$$\Pi = -\tfrac{1}{2}\,(Aa^2 + Bb^2 + Cc^2 + 2A'bc + 2B'ca + 2C'ab) - 2\,(Ff^2 + Gg^2 + Hh^2)\,. \tag{1}$$

3. – If one has a *symmetry axis*, instead of a plane, then the medium will be said to be endowed with *isotropy* around that axis. It is known (***) that if one displaces three

(*) BELTRAMI, "Sulle condizioni di resistenza dei corpi elastici," Rend. dell'Ist. Lombardo, 11 June 1885.
(**) "On axes of Elasticity and crystalline forms," Roy. Soc. London, 21 June 1855.
(***) In order to prove that, it is sufficient to observe that if the orthogonality condition:

$$\lambda\lambda' + \mu\mu' + \nu\nu' = 0$$

orthogonal axes around their origin infinitely little then the direction cosines of the new axes can be represented in the following way:

	x'	y'	z'
x	1	γ	$-\beta$
y	$-\gamma$	1	α
z	β	$-\alpha$	1

The angles α, β, γ are infinitesimal, and one supposes that one neglects the higher-order infinitesimals. Meanwhile, the formulas (2) and (3) of the first chapter will yield the new values of a, b, c, f, g, h. They show that the variations that those quantities experience are:

$$\begin{aligned} \delta a &= 2\,(g\,\beta - h\,\gamma), & \delta f &= (b - c)\,\alpha - h\beta + g\,\gamma, \\ \delta b &= 2\,(h\,\gamma - f\,\alpha), & \delta g &= h\alpha + (c - a)\,\beta - f\,\gamma, \\ \delta c &= 2\,(f\,\alpha - g\,\beta), & \delta h &= -\,g\alpha + f\beta + (b - c)\,\gamma. \end{aligned}$$

If one adopts (1) then it will follow that:

$$\begin{aligned} \tfrac{1}{2}\delta\Pi = &-(g\,\beta - h\,\gamma)\,(Aa + C'b + B'c) - 2\,[(b - c)\,\alpha - h\beta + g\gamma]\,Ff \\ &-(h\,\gamma - f\,\alpha)\,(C'a + Bb + A'c) - 2\,[h\,\alpha - (c - a)\,\beta - f\gamma]\,Gg \\ &-(f\,\alpha - g\,\beta)\,(B'a + A'\,b + Cc) - 2\,[-\,g\alpha + f\beta + (a - b)\,\gamma]\,Hh\,, \end{aligned}$$

or, if one orders this with respect to α, β, γ:

$$\tfrac{1}{2}\delta\Pi = -\sum\{[(B' - C')\,a + (A' - B)\,b + (C - A')\,c + 2(b - c)F\,]\,f + 2(G - H)\,gh\}\,\alpha\,.$$

If one takes the isotropy axes to be the x-axis then Π will remain invariant when the yz-plane turns within itself around the origin. One will then need to have $\delta\Pi = 0$ identically for $\beta = \gamma = 0$; i.e.:

$$B' = C', \quad G = H, \quad B - A' = C - A' = 2F.$$

In the case of two orthogonal isotropy axes, one will have:

$$A = B = C, \quad A' = B' = C', \quad F = G = H, \quad A - A' = 2F.$$

However, $\delta\Pi$ will be identically zero then, no matter what triple of values one attributes to α, β, γ. The medium will then be completely *isotropic*; i.e., its elastic properties will manifest themselves with equal intensity in all directions. Meanwhile, if one introduces

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is preserved when the cosines $\lambda, \lambda', \ldots$ vary by $\delta\lambda, \delta\lambda', \ldots$ then one will have:

$$\sum \lambda\,\delta\lambda' + \sum \lambda'\,\delta\lambda = 0, \qquad \text{or} \qquad \sum \lambda\,(\lambda' + \delta\lambda') + \sum \lambda'(\lambda + \delta\lambda) = 0;$$

that is to say, $\cos(xy') = -\cos(x'y), \ldots$

the usual notations that were proposed by Green (so one changes F into B and observes that $A' = A - 2B$) then formula (1) will become:

$$\Pi = -\tfrac{1}{2} A\,(a + b + c)^2 - 2B\,(f^2 + g^2 - bc - ca - ab)$$

or (*):

$$\Pi = -\tfrac{1}{2}(A - 2B)\,(a + b + c)^2 - B\,(a^2 + b^2 + c^2 + 2f^2 + 2g^2 + 2h^2).$$

The coefficients A and B are the *isotropy constants*, which vary from one medium to another.

4. – A more elegant way of finding the special form that Π must assume in the case of incomplete isotropy, or elastic symmetry with respect to an axis, was given by Prof. Beltrami in his *Note fisico-matematiche*. One saw in § **8** of Chapter I that the functions:

$$a + b + c, \qquad bc + ca + ab - f^2 - g^2 - h^2, \qquad abc + 2fgh - af^2 - bg^2 - ch^2$$

have a significance that is independent of the choice of axes. Now, take the x-axis to be the symmetry axis. If one rotates the yz-plane within itself around the origin then a will remain invariant, but always arbitrary, and therefore the part of it that contains a and the one that does not contain a can remain separately invariant in any invariant function of a, b, c, f, g, h. Therefore, if one observes that the invariant expressions that were obtained before can be written in the following way:

$$a + (b + c), \qquad a\,(b + c) + (bc - f^2 - g^2 - h^2), \qquad a\,(bc - f^2) + (2fgh - bg^2 - ch^2)$$

then one will see that the expressions:

$$a, \quad b + c, \qquad bc - f^2 - g^2 - h^2, \qquad bc - f^2, \qquad 2fgh - bg^2 - ch^2,$$

or

$$a^2, \qquad a\,(b + c), \qquad (b + c)^2, \qquad f^2 - bc, \qquad g^2 + h^2,$$

will remain invariant, if one ignores the last one, which cannot enter into the expression for Π. Π is, in fact, of degree two, so one might try to write it thus:

$$-\Pi = Aa^2 + Ba\,(b + c) + C\,(b + c)^2 + D\,(f^2 - bc) + E\,(g^2 + h^2). \tag{2}$$

This can then be considered to be the general expression for Π, as long as one observes that it contains *five* arbitrary coefficients and that the other part of the expression for Π cannot include more than five coefficients, in the case considered. Indeed, the *nine* coefficients that were found in the case where the medium is endowed with three orthogonal symmetry planes will already reduce to *six* when one supposes only that one can switch two of the axes with each other. It will be clear then that any ultimate

(*) See the mechanical interpretation of $a^2 + b^2 + c^2 + 2f^2 + 2g^2 + 2h^2$ at the end of Chapter Two in BETTI's *Teoria della elasticità*.

specialization must bring about some reduction in the number of elasticity coefficients, which is a number that cannot be greater than five, as a result. One can then pass on to the expression for the potential in the case of completely-isotropic media that renders the expression (2) symmetric with respect to *a*, *b*, *c*, and to *f*, *g*, *h*. One will get:

$$A = C, \qquad D = E, \qquad B = 2C - D,$$

and one will get back to the formulas of the preceding paragraph with A and C changed into $A / 2$, D and E changed into $2B$, and B changed into $A - 2B$.

5. – The elasticity coefficients are subject to some limitations that are imposed by the essentially positive character of the form $-\Pi$. Since the discriminant of that form is:

$$\frac{1}{64}\begin{vmatrix} 4B & 0 & 0 & 0 & 0 & 0 \\ 0 & 4B & 0 & 0 & 0 & 0 \\ 0 & 0 & 4B & 0 & 0 & 0 \\ 0 & 0 & 0 & A & A-2B & A-2B \\ 0 & 0 & 0 & A-2B & A & A-2B \\ 0 & 0 & 0 & A-2B & A-2B & A \end{vmatrix},$$

from a known theorem of algebra, the necessary and sufficient conditions for any system of values of the variables to correspond to a positive value of $-\Pi$ are:

$$B > 0, \quad A > 0, \quad \begin{vmatrix} A & A-2B \\ A-2B & A \end{vmatrix} > 0, \quad \begin{vmatrix} A & A-2B & A-2B \\ A-2B & A & A-2B \\ A-2B & A-2B & A \end{vmatrix} > 0;$$

i.e.:

$$B > 0, \;\; A > 0, \;\; A - 2B > 0, \qquad 3A - 4B > 0,$$

which will reduce to the first and last one, since if those two are satisfied then the other ones will be satisfied *a fortiori*. Therefore:

The isotropy constants A and B are necessarily positive, and the first one is greater than four-thirds of the second one, in addition.

Those limitations, which are important in some research (*), were given by Green (**) in another form and proved by Beltrami (***) in the very simple way that is presented here.

(*) BELTRAMI, "Sull'interpretazione meccanica delle formole di Maxwell."
(**) *Mathematical Papers*, pp. 245, 330.
(***) "Sulle condizioni di resistenza dei corpi elastici," Rend. Inst Lombardo, 1885.

6. – Any system of *constant* values for a, b, c, f, g, h corresponds (Chap. II, § **4**) to a possible deformation. Suppose that $f = g = h = 0$, so the unit potential reduces to $-B\,(a^2 + b^2 + c^2)$ when $a + b + c = 0$ and to $-\frac{3}{2}(3A - 4B)$ when $a = b = c = 1$. One will then recover the conditions:

$$B > 0, \quad 3A - 4B > 0 \tag{3}$$

as *necessary*. In order to show that they are also *sufficient*, it is enough to exhibit the essentially negative character of Π, and one can accomplish that by means of a very simple algebraic transformation. One proposes to determine a real number k such that one will have:

$$\Pi = -B\,[(a - k\Theta)^2 + (b - k\Theta)^2 + (c - k\Theta)^2 + 2f^2 + 2g^2 + 2h^2].$$

If one compares this to:

$$\Pi = -\tfrac{1}{2}(A - 2B)\,\Theta^2 + B^2\,(a^2 + b^2 + c^2 + 2f^2 + 2g^2 + 2h^2)$$

then one will see that one must have:

$$\tfrac{1}{2}(A - 2B) = B\,(3k^2 - 2k),$$

and one will deduce from this that:

$$k = \frac{1}{3} \pm \frac{1}{3}\sqrt{\frac{3A - 4B}{2B}}\,.$$

When the conditions (3) are satisfied, k will be real, and $-\Pi$ will still be expressed as a sum of squares. In addition, it will exhibit the fact that in order for Π to vanish, it is necessary and sufficient that a, b, c, f, g, h should vanish simultaneously, since one successively deduces from the facts that $a - k\Theta = b - k\Theta = c - k\Theta = 0$ that:

$$\Theta - 3k\Theta = 0, \qquad \Theta = 0, \qquad a = b = c = 0,$$

resp.

CHAPTER IV

ELASTIC EQUILIBRIUM

1. – When one applies a system of forces to an elastic body, the points of the body will displace, and consequently the internal actions will change; namely, they will then cease to be in equilibrium and will then tend to equilibrate with the external forces. Due to the new equilibrium, the body will take on a definite form that it will not deviate from, except when one suppresses the external forces, so the internal forces will then tend to equilibrate with each other. Having said that, if one knows the deforming actions that the given body is subjected to then one can propose to determine the new distribution of internal actions and then final configuration of the body.

2. Equations of elastic equilibrium. – Let X, Y, Z be the components along three orthogonal axes of the force *per unit volume* that is applied to the particle dS. In other words, let $X\,dS$, $Y\,dS$, $Z\,dS$ be the components of the force that is applied to the mass that is contained in dS. Other than those forces, which one calls *body forces*, one can have *pressures* on the surface of the body. Let $L\,ds$, $M\,ds$, $N\,ds$ be the components of the pressure that is applied to the surface element ds. When the deformed body has attained a definite configuration, its points will be found to be in equilibrium under the action of three groups of forces:

1. Body forces.
2. Surface pressures.
3. Elastic forces.

By virtue of *Lagrange's principle*, the work done by all of those forces under the virtual motion that the system can perform while in its equilibrium configuration must be zero. Thus, whenever a point goes from the position (x, y, z) to the position $(x + u, y + v, z + w)$, under the virtual passage to the position $(x + u + \delta u, y + v + \delta v, z + w + \delta w)$, an elementary work will be performed that is expressed by $\xi\,\delta u + \eta\,\delta v + \zeta\,\delta\omega$, if ξ, η, ζ are the components of the force that is applied to the point considered. As for the elastic forces, the work that they do will be the variation that their potential experiences during that virtual motion. The principle of virtual work will then lead to the equation:

$$\int (X\,\delta u + Y\,\delta v + Z\,\delta w)\,dS + \int (L\,\delta u + M\,\delta v + N\,\delta w)\,ds + \delta \int \Pi\,dS = 0. \tag{1}$$

Now, eliminate the constraints on the arbitrary quantities du, dv, dw in the third integral in such a way that they will behave linearly, as they do in the first two integrals. Then observe that those quantities are mutually independent, and vary arbitrarily from point to point. Next, equating their coefficients in the spatial integral and the surface integral to zero separately. One will then arrive at the required equations. If one recalls that Π is a function of a, b, c, f, g, h then one will first have:

$$\delta\int \Pi\, dS = \int \delta\Pi\, dS = \int\left(\frac{\partial\Pi}{\partial a}\delta a + \frac{\partial\Pi}{\partial b}\delta b + \cdots + \frac{\partial\Pi}{\partial h}\delta h\right)dS\ .$$

Meanwhile:

$$\int \frac{\partial\Pi}{\partial a}\delta a\, dS = \int \frac{\partial\Pi}{\partial a}\frac{\partial\delta u}{\partial x}dS = \int \frac{\partial}{\partial x}\left(\frac{\partial\Pi}{\partial a}\delta u\right)dS - \int \frac{\partial}{\partial x}\frac{\partial\Pi}{\partial a}\delta u\, dS$$

$$= -\int \frac{\partial\Pi}{\partial a}\frac{d\delta u}{dn}dS - \int \frac{\partial}{\partial x}\frac{\partial\Pi}{\partial a}\delta u\, dS\ .$$

Similarly:

$$\int \frac{\partial\Pi}{\partial f}\delta f\, dS = \int \frac{1}{2}\frac{\partial\Pi}{\partial f}\left(\frac{\partial\delta w}{\partial y} + \frac{\partial\delta v}{\partial z}\right)dS$$

$$= \int \frac{1}{2}\left[\frac{\partial}{\partial y}\left(\frac{\partial\Pi}{\partial f}\delta w\right) + \frac{\partial}{\partial z}\left(\frac{\partial\Pi}{\partial f}\delta v\right)\right]dS - \int \frac{1}{2}\left(\frac{\partial}{\partial y}\frac{\partial\Pi}{\partial f}\delta w + \frac{\partial}{\partial z}\frac{\partial\Pi}{\partial f}\delta v\right)dS$$

$$= -\int \frac{1}{2}\frac{\partial\Pi}{\partial f}\left(\frac{dy}{dn}\delta w + \frac{dz}{dn}\delta v\right)ds - \int \frac{1}{2}\left(\frac{\partial}{\partial y}\frac{\partial\Pi}{\partial f}\delta w + \frac{\partial}{\partial z}\frac{\partial\Pi}{\partial f}\delta v\right)dS\ .$$

Therefore, if one collects the terms that multiply δu, δv, δw into three analogous groups then:

$$\delta\int \Pi\, dS = -\int\left(\frac{\partial\Pi}{\partial a}\frac{dx}{dn} + \frac{1}{2}\frac{\partial\Pi}{\partial h}\frac{dy}{dn} + \frac{1}{2}\frac{\partial\Pi}{\partial g}\frac{dz}{dn}\right)\delta u\, ds$$

$$-\int\left(\frac{\partial}{\partial x}\frac{\partial\Pi}{\partial a} + \frac{1}{2}\frac{\partial}{\partial y}\frac{\partial\Pi}{\partial h} + \frac{1}{2}\frac{\partial}{\partial z}\frac{\partial\Pi}{\partial g}\right)\delta u\, ds$$

$$- \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

One finally substitutes this into the relations (1), which will split into the following six equations, due to the arbitrariness in δu, δv, δw :

$$(1')\begin{cases} X = \dfrac{\partial}{\partial x}\dfrac{\partial\Pi}{\partial a} + \dfrac{1}{2}\dfrac{\partial}{\partial y}\dfrac{\partial\Pi}{\partial h} + \dfrac{1}{2}\dfrac{\partial}{\partial z}\dfrac{\partial\Pi}{\partial g}, \\ Y = \dfrac{\partial}{\partial x}\dfrac{\partial\Pi}{\partial h} + \dfrac{1}{2}\dfrac{\partial}{\partial y}\dfrac{\partial\Pi}{\partial b} + \dfrac{1}{2}\dfrac{\partial}{\partial z}\dfrac{\partial\Pi}{\partial f}, \\ Z = \dfrac{\partial}{\partial x}\dfrac{\partial\Pi}{\partial g} + \dfrac{1}{2}\dfrac{\partial}{\partial y}\dfrac{\partial\Pi}{\partial f} + \dfrac{1}{2}\dfrac{\partial}{\partial z}\dfrac{\partial\Pi}{\partial c}, \end{cases} \qquad (1'')\begin{cases} L = \dfrac{\partial\Pi}{\partial a}\dfrac{dx}{dn} + \dfrac{1}{2}\dfrac{\partial\Pi}{\partial h}\dfrac{dy}{dn} + \dfrac{1}{2}\dfrac{\partial\Pi}{\partial g}\dfrac{dz}{dn}, \\ M = \dfrac{1}{2}\dfrac{\partial\Pi}{\partial h}\dfrac{dx}{dn} + \dfrac{\partial\Pi}{\partial b}\dfrac{dy}{dn} + \dfrac{1}{2}\dfrac{\partial\Pi}{\partial f}\dfrac{dz}{dn}, \\ N = \dfrac{1}{2}\dfrac{\partial\Pi}{\partial g}\dfrac{dx}{dn} + \dfrac{1}{2}\dfrac{\partial\Pi}{\partial f}\dfrac{dy}{dn} + \dfrac{\partial\Pi}{\partial c}\dfrac{dz}{dn}. \end{cases}$$

3. Observations. – The relations (1′) are called the *indefinite equations*, because they are valid at any point of the body. The relations (1″), which are valid only on the surface, are called the *boundary conditions.* The boundary conditions can actually be imposed in an infinitude of ways. They are expressed by the relations (1″) when the *pressures* are given on the surface. However, if one assigns the values that the *displacements* must assume on the surface then the boundary conditions will be precisely the equalities by means of which one fixes those values. How can we use the indefinite equations and the boundary conditions? Observe that $\frac{\partial \Pi}{\partial a}, \frac{\partial \Pi}{\partial b}, \ldots$ are linear functions of $a, b, c, \ldots$, and that they will then contain the first derivatives of the displacements linearly. The ultimate differentiation of $\frac{\partial \Pi}{\partial a}, \frac{\partial \Pi}{\partial b}, \ldots$ that was mentioned in equations (1′) can then include the second derivatives of u, v, w. Therefore, the indefinite equations are second-order partial differential systems. When one integrates them, one will get the expressions for u, v, w that contain arbitrary quantities that can be determined by means of substitution in the boundary conditions. However, that makes one doubtful. Are the indefinite equations always enough to individuate a system of displacements, and are the boundary equations sufficient to complete the determination?

4. – We shall answer that question directly by showing that:

The indefinite equations and the boundary equations are sufficient to determine the displacements up to rigid motions.

Suppose that there exist two systems of displacements, (u', v', w') and (u'', v'', w''), that satisfy equations (1′), and consider the displacements:

$$u' - u'' = u, \qquad v' - v'' = v, \qquad w' - w'' = w.$$

It is clear that $a' - a'' = a$, $b' - b'' = b, \ldots$ In addition, observe that $\frac{\partial \Pi}{\partial a}, \frac{\partial \Pi}{\partial b}, \ldots$ contain $a, b, c, \ldots$ linearly, so one will have $\frac{\partial \Pi}{\partial a'} - \frac{\partial \Pi}{\partial a''} = \frac{\partial \Pi}{\partial a}, \ldots$ If one writes equations (1′) for each system of displacements then upon subtraction, one will get:

$$\begin{cases} 0 = \dfrac{\partial}{\partial x}\dfrac{\partial \Pi}{\partial a} + \dfrac{1}{2}\dfrac{\partial}{\partial y}\dfrac{\partial \Pi}{\partial h} + \dfrac{1}{2}\dfrac{\partial}{\partial z}\dfrac{\partial \Pi}{\partial g}, \\ 0 = \dfrac{1}{2}\dfrac{\partial}{\partial x}\dfrac{\partial \Pi}{\partial h} + \dfrac{\partial}{\partial y}\dfrac{\partial \Pi}{\partial b} + \dfrac{1}{2}\dfrac{\partial}{\partial z}\dfrac{\partial \Pi}{\partial f}, \\ 0 = \dfrac{1}{2}\dfrac{\partial}{\partial x}\dfrac{\partial \Pi}{\partial g} + \dfrac{1}{2}\dfrac{\partial}{\partial y}\dfrac{\partial \Pi}{\partial f} + \dfrac{\partial}{\partial z}\dfrac{\partial \Pi}{\partial c}. \end{cases}$$

Those are precisely the indefinite equations of equilibrium, under the hypothesis that the body forces are zero. If one operates analogously on the boundary conditions then one will find equations (1″), in which $L = L' - L''$, … Therefore, u, v, w can be considered to be displacements that are due to the forces:

$$X = Y = Z = 0, \qquad L = L' - L'', \qquad M = M' - M'', \qquad N = N' - N''.$$

Now, if one takes $\delta u = u$, $\delta v = v$, $\delta w = w$ in the equality (1) then it will become:

$$\int (Lu + Mv + Nw)\, ds + 2\int \Pi \, dS = 0, \tag{2}$$

because by virtue of Euler's theorem on homogeneous functions:

$$\delta \Pi = \frac{\partial \Pi}{\partial a}\frac{\partial \delta u}{\partial x} + \ldots = \frac{\partial \Pi}{\partial a} a + \ldots = 2\Pi.$$

If one assigns *pressures* for the boundary conditions then that will imply that $L' = L''$, …; i.e., $L = M = N = 0$. However, if one gives the *displacements* on the surfaces then that will say that one must have $u' = u''$, … on s; i.e., $u = v = w = 0$. Hence, the first integral of the equality (2) will be zero in any case, and it will then reduce to:

$$\int \Pi \, dS = 0.$$

The left-hand side is a sum of quantities that are essentially-negative or zero. It will then be necessary for each of those quantities to be zero; i.e., one must have $\Pi = 0$ in all of the body. However, it has already been observed that Π cannot be annulled unless it is annulled when $a = b = c = f = g = h = 0$, and it is known that in that case, the relative displacements will take the form:

$$u = l + qz - ry, \qquad v = m + rx - pz, \qquad w = n + py - qx,$$

in which l, m, n, p, q, r are constant over all of the body. If one is given the *displacements* on the surface then one must have $l + qz - ry = 0$, … for an infinitude of values of x, y, z. That cannot happen unless one has $l = m = n = p = q = r = 0$, and consequently, $u = v = w = 0$; i.e., $u' = u''$, $v' = v''$, $w' = w''$ in all of the body. However, if one is given the *pressures* then there will exist an infinitude of systems of displacements that satisfy (1′) and (1″); however, the final configuration of the body will always be the same. After all, it is enough to prescribe the rigid motions of an arbitrary particle, because the equations (1′) and (1″) determine the displacements completely. Indeed, if one takes the origin to be in the particle considered then will have:

$$u' = u'', \qquad \ldots, \qquad \frac{\partial w'}{\partial y} - \frac{\partial v'}{\partial z} = \frac{\partial w''}{\partial y} - \frac{\partial v''}{\partial z}, \qquad \ldots$$

for $x = y = z = 0$; i.e., $u = 0$, …, $\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} = 0$, … It will then follow that $l = m = n = p = q = r = 0$; one will then have $u' = u''$, $v' = v''$, $w' = w''$ in all of the body.

5. Case of isotropic media. – One knows (Chap. III, § **3**) that in the case of isotropy, the unit potential has the form:

$$\Pi = -(A - 2B)(a + b + c)^2 - B\,(a^2 + b^2 + c^2 + 2f^2 + 2g^2 + 2h^2).$$

It will then follow that:

$$\frac{\partial \Pi}{\partial a} = -(A - 2B)\,\Theta - 2Ba, \qquad \frac{1}{2}\frac{\partial \Pi}{\partial f} = -2Bf,$$

and then the first of equations (1′) will become:

$$X + (A - 2B)\frac{\partial \Theta}{\partial x} + 2B\left(\frac{\partial a}{\partial x} + \frac{\partial h}{\partial y} + \frac{\partial g}{\partial z}\right) = 0. \tag{3}$$

Now, one has:

$$2\left(\frac{\partial a}{\partial x} + \frac{\partial h}{\partial y} + \frac{\partial g}{\partial z}\right) = 2\frac{\partial^2 u}{\partial x^2} + \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)$$

$$= \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} + \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) = \Delta^2 u + \frac{\partial \Theta}{\partial x}.$$

Substitute that in (3) and suppose the preceding calculation has been repeated for the other two equations. One will see that the *indefinite equations* for the equilibrium of an isotropic body are:

$$\left\{\begin{aligned} X + (A - B)\frac{\partial \Theta}{\partial x} + B\,\Delta^2 u &= 0, \\ Y + (A - B)\frac{\partial \Theta}{\partial y} + B\,\Delta^2 v &= 0, \\ Z + (A - B)\frac{\partial \Theta}{\partial x} + B\,\Delta^2 w &= 0. \end{aligned}\right.$$

One can give another form to those equations that is useful for integration by introducing four functions that are of paramount important in this theory, namely, the *cubic dilatation* Θ and *twice the rotational components of the particle*, which will be denoted in the following way from now on:

$$\mathcal{T}_1 = \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}, \qquad \mathcal{T}_2 = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}, \qquad \mathcal{T}_3 = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}.$$

One has:

$$\Delta^2 u + \frac{\partial \Theta}{\partial x} = \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} - \frac{\partial^2 v}{\partial x \partial y} - \frac{\partial^2 w}{\partial x \partial z}$$

$$= \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) = \frac{\partial \mathcal{T}_2}{\partial z} - \frac{\partial \mathcal{T}_3}{\partial y},$$

and the equations that were obtained before will become:

$$\begin{cases} X + A\dfrac{\partial \Theta}{\partial x} + B\left(\dfrac{\partial \mathcal{T}_2}{\partial z} - \dfrac{\partial \mathcal{T}_3}{\partial y}\right) = 0, \\[2ex] Y + A\dfrac{\partial \Theta}{\partial y} + B\left(\dfrac{\partial \mathcal{T}_3}{\partial x} - \dfrac{\partial \mathcal{T}_1}{\partial z}\right) = 0, \\[2ex] Z + A\dfrac{\partial \Theta}{\partial z} + B\left(\dfrac{\partial \mathcal{T}_1}{\partial y} - \dfrac{\partial \mathcal{T}_2}{\partial x}\right) = 0. \end{cases}$$

As for the *boundary conditions*, the first of (1″) will become:

$$L + (A - 2B)\,\Theta\,\frac{dx}{dn} + 2B\left(a\frac{dx}{dn} + h\frac{dy}{dn} + g\frac{dz}{dn}\right) = 0. \tag{4}$$

Now, one has:

$$B\left(a\frac{dx}{dn} + h\frac{dy}{dn} + g\frac{dz}{dn}\right) = 2\frac{\partial u}{\partial x}\frac{dx}{dn} + \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\frac{dy}{dn} + \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)\frac{dz}{dn}$$

$$= 2\left(\frac{\partial u}{\partial x}\frac{dx}{dn} + \frac{\partial u}{\partial y}\frac{dy}{dn} + \frac{\partial u}{\partial z}\frac{dz}{dn}\right) + \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\frac{dy}{dn} + \left(\frac{\partial w}{\partial x} - \frac{\partial u}{\partial z}\right)\frac{dz}{dn};$$

i.e.:

$$2\left(a\frac{dx}{dn} + h\frac{dy}{dn} + g\frac{dz}{dn}\right) = 2\frac{du}{dn} + \mathcal{T}_3\frac{dy}{dn} - \mathcal{T}_2\frac{dz}{dn}.$$

If one substitutes this in (4) then one will get the first equation of the triple:

$$\begin{cases} L+(A-2B)\Theta\dfrac{dx}{dn}+2B\dfrac{du}{dn}+B\left(\mathcal{T}_3\dfrac{dy}{dn}-\mathcal{T}_2\dfrac{dz}{dn}\right)=0, \\ M+(A-2B)\Theta\dfrac{dy}{dn}+2B\dfrac{dv}{dn}+B\left(\mathcal{T}_1\dfrac{dz}{dn}-\mathcal{T}_3\dfrac{dx}{dn}\right)=0, \\ N+(A-2B)\Theta\dfrac{dz}{dn}+2B\dfrac{dw}{dn}+B\left(\mathcal{T}_2\dfrac{dx}{dn}-\mathcal{T}_1\dfrac{dy}{dn}\right)=0. \end{cases}$$

6. – One can thank Borchardt (*) for an ingenious *decomposition of the expression for* Π *into two parts, one of which makes no contribution to the indefinite equations.* Recall the process that led to the equations of equilibrium. It will become clear that when one has only the formation of the indefinite equations in mind, it will be legitimate to neglect all terms in Π that give rise in $\int \delta\Pi\, dS$ to spatial integrals that are identically zero or to surface integrals. Now, if one observes that:

$$f=\frac{1}{2}\left(\frac{\partial w}{\partial y}+\frac{\partial v}{\partial z}\right)=\tfrac{1}{2}\mathcal{T}_1+\frac{\partial v}{\partial z},$$

and consequently:

$$f^2=\tfrac{1}{4}\mathcal{T}_1^2+\left(\frac{\partial w}{\partial y}-\frac{\partial v}{\partial z}\right)\frac{\partial v}{\partial z}+\left(\frac{\partial v}{\partial z}\right)^2=\tfrac{1}{4}\mathcal{T}_1^2+\frac{\partial w}{\partial y}\frac{\partial v}{\partial z},$$

then one can give the form $\Pi_0 + \Pi_1$ to the potential:

$$\Pi=-\frac{A}{2}(a+b+c)^2-2B\,(f^2+g^2+h^2-bc-ca-ab),$$

if one takes:

$$\Pi_0=-\frac{1}{2}\left[A\Theta^2+B\left(\mathcal{T}_1^2+\mathcal{T}_2^2+\mathcal{T}_3^2\right)\right] \qquad \Pi_1=-\,2B\sum\left(\frac{\partial w}{\partial y}\frac{\partial v}{\partial z}-\frac{\partial v}{\partial y}\frac{\partial w}{\partial z}\right).$$

In order to insure that Π_1 has no influence on the indefinite equations, observe that $\int\delta\Pi_1\,dS$ splits into three parts that are analogous to the following one:

$$\int\left(\frac{\partial w}{\partial y}\frac{\partial\delta v}{\partial z}+\frac{\partial v}{\partial z}\frac{\partial\delta w}{\partial y}-\frac{\partial w}{\partial z}\frac{\partial\delta v}{\partial y}-\frac{\partial v}{\partial y}\frac{\partial\delta w}{\partial z}\right)dS$$

$$=\int\left[\frac{\partial}{\partial x}\left(\frac{\partial w}{\partial y}\delta v-\frac{\partial v}{\partial y}\delta w\right)-\frac{\partial}{\partial z}\left(\frac{\partial w}{\partial z}\delta v-\frac{\partial v}{\partial z}\delta w\right)\right]dS$$

(*) "Ueber die Transformation der Elasticitätsgleichungen," Crelle's Journal (1873), pp. 45.

$$-\int\left[\left(\frac{\partial^2 w}{\partial y\,\partial z}-\frac{\partial^2 w}{\partial z\,\partial y}\right)\delta v-\left(\frac{\partial^2 v}{\partial y\,\partial z}-\frac{\partial^2 v}{\partial z\,\partial y}\right)\delta w\right]dS\,.$$

The first integral transforms into a surface integral, and the second one is identically zero. Therefore, the elastic potential (to the extent that it influences the indefinite equations of equilibrium) can be regarded as a linear combination of the *square of the cubic dilatation* with the *square of the rotation* of the medium, and the coefficients of the combination are proportional to the isotropy constants.

CHAPTER V

BETTI'S THEOREM

1. – That important theorem (*) establishes a relation between two systems of forces and the associated systems of displacements in an elastic body. Let (u, v, w) and (u', v', w') be the displacements that define the configurations that are taken by the body under the action of the systems (X, Y, Z, L, M, N) and (X', Y', Z', L', M', N'), respectively. For the first configuration, the equilibrium conditions can be summarized (Chap. IV, § **1**) by the relation:

$$\int \delta\Pi \, dS + \int (X\,\delta u + Y\,\delta v + Z\,\delta w)\, dS + \int (L\,\delta u + M\,\delta v + N\,\delta w)\, ds = 0,$$

which must be true for arbitrary variations δu, δv, δw, and in particular, for $\delta u = u'$, $\delta v = v'$, $\delta w = w'$, in which case, the preceding relation will become:

$$\int \left(\frac{\partial\Pi}{\partial a} a' + \cdots + \frac{\partial\Pi}{\partial h} h'\right) dS + \int (Xu' + Yv' + Zw')\, dS + \int (Lu' + Mv' + Nw')\, ds = 0.$$

Similarly, one has:

$$\int \left(\frac{\partial\Pi}{\partial a'} a + \cdots + \frac{\partial\Pi}{\partial h'} h\right) dS + \int (X'u + Y'v + Z'w)\, dS + \int (L'u + M'v + N'w)\, ds = 0.$$

Therefore, from a known property of quadratic forms:

$$\frac{\partial\Pi}{\partial a} a' + \frac{\partial\Pi}{\partial b} b' + \cdots + \frac{\partial\Pi}{\partial h} h' = \frac{\partial\Pi}{\partial a'} a + \frac{\partial\Pi}{\partial b'} b + \cdots + \frac{\partial\Pi}{\partial h'} h,$$

one will also have:

$$\begin{aligned} &\int (Xu' + Yv' + Zw')\, dS + \int (Lu' + Mv' + Nw')\, ds \\ = &\int (X'u + Y'v + Z'w)\, dS + \int (L'u + M'v + N'w)\, ds. \end{aligned} \tag{1}$$

That is *Betti's theorem.*

2. – We shall make a first application of that theorem by taking:

$$u' = l + qz - ry, \qquad v' = m + rx - pz, \qquad w' = n + py - qx,$$

(*) BETTI, *Teoria della elasticità*, Chap. VI. See also a communication of M. LÉVY to the Académie des Sciences in Paris (Comptes-rendus, 13 August 1888).

in which l, m, n, p, q, r are constant in all of the body. One has:

$$a' = b' = c' = f' = g' = h' = 0;$$

hence, $\Pi' = 0$. If one substitutes this in the equilibrium equations then one will get:

$$X' = Y' = Z' = L' = M' = N' = 0,$$

and the relation (1) will become:

$$\int [X(l + qz - ry) + \cdots] dS + \int [L(l + qz - ry) + \cdots] ds = 0.$$

Due to the arbitrariness in l, m, n, p, q, r, the last equation will split into the following six:

$$\left\{\begin{array}{ll} \int X\,dS + \int L\,ds = 0, & \int (Yz - Zy)\,dS + \int (Mz - Ny)\,ds = 0, \\ \int Y\,dS + \int M\,ds = 0, & \int (Zx - XZ)\,dS + \int (Nx - Lz)\,ds = 0, \\ \int Z\,dS + \int N\,ds = 0, & \int (Xy - Yx)\,dS + \int (Ly - Mx)\,ds = 0, \end{array}\right.$$

which say that the external forces are in equilibrium. Hence:

In order for an elastic body to be in equilibrium, it is necessary and sufficient that one should insure its equilibrium as a rigid body (*).

3. – Now, take:

$$u' = a'x + h'y + g'z, \qquad v' = h'x + b'y + f'z, \qquad w' = a'x + f'y + c'z,$$

with a', b', c', f', g', h' constant in all of the body. Hence $\frac{\partial \Pi'}{\partial a'}, \frac{\partial \Pi'}{\partial b'}, \ldots$ are constant, and therefore the indefinite equations will give $X' = Y' = Z' = 0$. If one determines $a', b', \ldots$ by means of the six first-order equations:

$$\frac{\partial \Pi'}{\partial a'} = \frac{\partial \Pi'}{\partial b'} = \frac{\partial \Pi'}{\partial c'} = 1, \qquad \frac{\partial \Pi'}{\partial f'} = \frac{\partial \Pi'}{\partial g'} = \frac{\partial \Pi'}{\partial h'} = 0 \tag{2}$$

then the boundary conditions will give:

$$L' = \frac{dx}{dn}, \qquad M' = \frac{dy}{dn}, \qquad N' = \frac{dz}{dn},$$

(*) BETTI, *loc. cit.* See also CLEBSCH, *Théorie de l'élasticité*, pp. 2.

and the left-hand side of (1) will become:

$$\int\left(u\frac{dx}{dn}+v\frac{dy}{dn}+w\frac{dz}{dn}\right)ds = -\int\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)dS = -\int \Theta\, dS\,.$$

Therefore:

$$\int \Theta\, dS = -\int (Xu'+Yv'+Zw')\,dS + \int (Lu'+Mv'+Nw')\,ds\,. \tag{3}$$

That noteworthy formula yields the total dilatation of an elastic body when the external forces are given.

4. – Apply formula (3) to the case of an isotropic body. Since one has:

$$\Pi = -\tfrac{1}{2}(A-2B)(a+b+c)^2 - B\,(a^2+b^2+c^2+2f^2+2g^2+2h^2),$$

equations (2) will become:

$$\left\{\begin{array}{ll} -(A-2B)(a'+b'+c')-2Ba'=1, & f'=0,\\ -(A-2B)(a'+b'+c')-2Bb'=1, & g'=0,\\ -(A-2B)(a'+b'+c')-2Bc'=1, & h'=0. \end{array}\right.$$

One deduces from the equations on the left, upon summing them, that:

$$-(3A-4B)\,(a'+b'+c') = 3;$$

hence, if one substitutes this in those equations then:

$$a' = b' = c' = -\frac{1}{3A-4B}\,.$$

Finally:

$$\frac{u'}{x}=\frac{v'}{y}=\frac{w'}{z}=-\frac{1}{3A-4B}\,.$$

If one substitutes this in (3) then one will get:

$$\int \Theta\, dS = \frac{1}{3A-4B}\left[\int (Xx+Yy+Zz)\,dS + \int (Lx+My+Mz)\,ds\right]. \tag{4}$$

5. – Suppose, for example, that a uniform pressure is exerted upon the surface of an isotropic body and that the body forces are zero or negligible, and ask what the variation of the volume of the body will be. In the present case:

$$X = Y = Z = 0; \qquad L = p\frac{dx}{dn}, \qquad M = p\frac{dy}{dn}, \qquad N = p\frac{dz}{dn}.$$

Formula (4) will then give:

$$\int \Theta\, dS = \frac{p}{3A-4B}\int\left(x\frac{dx}{dn} + y\frac{dy}{dn} + z\frac{dz}{dn}\right)ds = -\frac{p}{3A-4B}\int 3\, dS;$$

i.e.:

$$\frac{\int \Theta\, dS}{S} = -\frac{3p}{3A-4B}.$$

The left-hand side represents the dilatation per unit volume. For a given body, it will then be proportional to the pressure. In practice, one gives the name of *coefficient of cubic compressibility* to it and represents the reduction per unit volume that follows from a unit pressure by q. One deduces from the preceding formula that:

$$q = \frac{3}{3A-4B}.$$

In practice, one considers another coefficient E that is called the *Young's modulus* or *coefficient of elastic traction.* In what follows, one will see that:

$$Eq = \frac{3B}{A-B},$$

and one will know the means by which one can determine E and q experimentally, and consequently, to calculate the isotropy constants A and B for any body. According to the old theory of Navier and Poisson, one must always have $A = 3B$, and therefore $Eq = 3/2$; however, more recent experiments have shown that if Eq has that value for certain types of crystals then its value might be very far from 3/2 in other bodies, and especially metals.

6. – We further propose to determine the alteration in volume that a homogeneous elastic body will experience *under the action of its own weight.* The body is supposed to be supported by means of forces that are applied vertically to the points of a horizontal plane. Place the origin at the barycenter and direct the z-axis in the opposite sense to that of gravity, and let $z = h$ be the equation of the support plane. By hypothesis, one has $X = Y = L = M = 0$, while the ratio of Z to ρ is a constant that is equal and opposite in sign to the acceleration of gravity. Formula (3) becomes:

$$\int \Theta\, dS = -\, Z\int (g'x + f'y + c'z)\, dS - \int N(g'x + f'y + c'z)\, ds.$$

However, in order to have external equilibrium, it is necessary that one must have:

$$\int N\,ds = P, \qquad \int Nx\,ds = \int Ny\,ds = 0,$$

in which P represents the weight of the body. In addition, from the choice of origin, one will have:

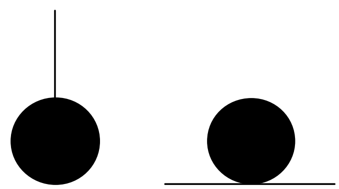
$$\int x\,dS = \int y\,dS = \int z\,dS = 0.$$

Hence:

$$\int \Theta\,dS = -\,c'\int Nz\,ds = -\,c'h\int N\,ds = -\,c'hP.$$

If the body is isotropic then:

$$\int \Theta\,dS = \frac{hP}{3A-4B} = \tfrac{1}{3}\,qhP.$$

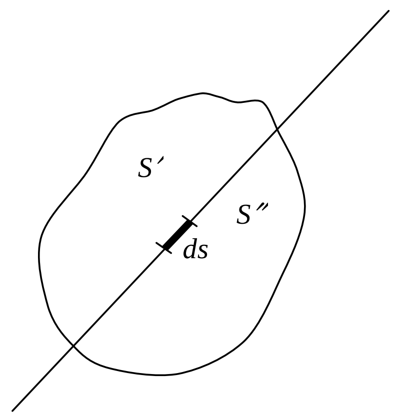


In general, one can say that for a fixed orientation, the total variation in volume will be proportional to the weight and the distance from the center of gravity to the support plane. For example, the variations in volume of a homogeneous, isotropic sphere that is suspended from a rigid wire or supported by a rigid plane are equal opposite and proportional to the fourth power of the radius (*). Observe that for any body, if the support plane contains the center of gravity then the upper part will reduce or increase by the amount that the lower part increases or reduces, resp., in such a way that the total volume will remain invariant.

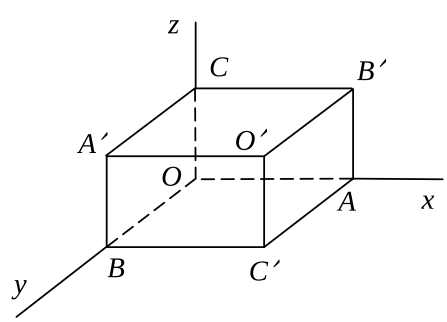
(*) BETTI, *Teoria della elasticità*, Chap. VII.

CHAPTER VI

DISTRIBUTION OF THE INTERNAL ACTIONS



1. – Up to now, we have studied elastic deformations with no concern for the forces that they develop inside the bodies as an effect of those deformations. We shall now take a different viewpoint, because we would like to infer the fundamental formulas for the study of elastic equilibrium from a direct consideration of those internal forces. A comparison of the formulas that were obtained before and the ones that we will get will provide the means for us to study the distribution of the internal actions in deformed elastic bodies. First, let us make an observation: Consider a surface element *ds* inside of an elastic body *S* that has already deformed and equilibrated, and imagine that the plane that contains it is prolonged in such a way that it divides *S* into two parts S' and S''. Among all of the forces that are directed from the points of S' to those of S'', consider only the ones whose lines of action cross *ds*. They will have a certain resultant that one can denote by *p ds*. Similarly, the actions that the points of S'' exert upon the points of S' and that cross *ds* will have a resultant that it equal and opposite in sign to the first one if the body is in equilibrium, as we have supposed. The function *p* will represent (*) the *pressure* on *ds* per unit area.



(*) See the excellent *Cours de physique mathématique* by P. DUHEM (Paris, A. Hermann, 1891, pp. 257) for the various ways of defining *pressure.*

2. Indefinite equations. – Having said that, decompose the body into parallelepiped elements by means of three systems of planes that are parallel to the coordinate planes. Consider one of those parallelepipeds and write down that it is in equilibrium under the action of the internal forces and the body forces. The plane *OYZ* divides the body into two parts. Let p_x denote the unit pressure that is exerted across *OBCA′* by the part that does not contain the parallelepiped onto the one that does, in such a way that $p_x\,dy\,dz$ will be the pressure on *OBCA′*, which will be considered to be positive when it is directed into the interior of the parallelepiped. Similarly, let $p_y\,dz\,dx$, $p_z\,dx\,dy$ be the pressures that act upon the faces *OCAB′*, *OABC′*, resp. Let p_{xx}, p_{xy}, p_{xz} be the components of p_z along *Ox*, *Oy*, *Oz*, resp., and so on. When one passes from the face *OBCA′* to the opposite face *O′B′C′A*, the functions p_{xx}, p_{xy}, p_{xz} will become:

$$p_{xx} + \frac{\partial p_{xx}}{\partial x}dx, \qquad p_{xy} + \frac{\partial p_{xy}}{\partial x}dx, \qquad p_{xz} + \frac{\partial p_{xz}}{\partial x}dx.$$

Hence, the pressure on *O′B′C′A*, when considered to be something that *acts upon the parallelepiped*, will have the components $-\left(p_{xx} + \frac{\partial p_{xx}}{\partial x}dx\right)dy\,dz$, …, and therefore, the internal pressures, when measured along the *x*-axis, will give rise to the sum:

$$p_{xx}\,dy\,dz + p_{yx}\,dz\,dx + \ldots - \left(p_{xx} + \frac{\partial p_{xx}}{\partial x}dx\right)dy\,dz - \ldots = -\left(\frac{\partial p_{xx}}{\partial x} + \frac{\partial p_{yx}}{\partial y} + \frac{\partial p_{zx}}{\partial z}\right)dS.$$

If one writes down that the sum of the forces that act upon the parallelepiped along each axis is equal to zero then one will get the indefinite equations for equilibrium:

$$\left.\begin{aligned} X &= \frac{\partial p_{xx}}{\partial x} + \frac{\partial p_{yx}}{\partial y} + \frac{\partial p_{zx}}{\partial z}, \\ Y &= \frac{\partial p_{xy}}{\partial x} + \frac{\partial p_{yy}}{\partial y} + \frac{\partial p_{zy}}{\partial z}, \\ Z &= \frac{\partial p_{xz}}{\partial x} + \frac{\partial p_{yz}}{\partial y} + \frac{\partial p_{zz}}{\partial z}. \end{aligned}\right\} \tag{1}$$

3. – We further need to write down the equations of the moments. We first assume that the external forces are third-order infinitesimals, so they will give rise to moments that are negligible with respect to those of the internal pressures. For simplicity, we compose them around the center of the parallelepiped and observe that from the continuity that we assume the pressure p to be endowed with (in direction and intensity), it would be legitimate to assume that their resultants are applied to the centers of the respective faces. Recall that the moments that are due to a force (X, Y, Z) that is applied to the point (x, y, z) are $Zy - Yz$, $Xx - Zx$, $Yx - Xy$. Having said that, if we neglect the infinitesimals of order higher than two in the forces then we will have:

a force whose components are	$p_{xx}\,dy\,dz$	$p_{xy}\,dy\,dz$	$p_{xz}\,dy\,dz$	at the point	$(-\tfrac{1}{2}dx, 0, 0)$
″ ″	$p_{yx}\,dz\,dx$	$p_{yy}\,dz\,dx$	$p_{yz}\,dz\,dx$	″	$(0, -\tfrac{1}{2}dy, 0)$
″ ″	$p_{zx}\,dx\,dy$	$p_{zy}\,dx\,dy$	$p_{zz}\,dx\,dy$	″	$(0, 0, -\tfrac{1}{2}dz)$
…………………………	……….	………..	……….	………..	…………...

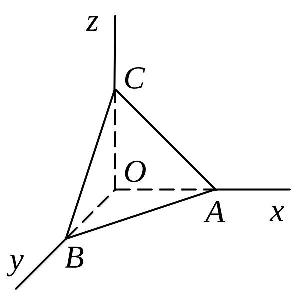
The resulting moment of the couple, which acts parallel to the *yz*-plane, will then be:

$$- dy \cdot p_{yz}\,dz\,dx + dz \cdot p_{zy}\,dx\,dy = (p_{zy} - p_{yz})\,dS,$$

and therefore the equations of the moments are:

$$p_{yz} = p_{yz}\,, \qquad p_{zx} = p_{xz}\,, \qquad p_{xy} = p_{yx}\,.$$

Consequently, the *nine* functions p will always reduce to *six* distinct ones. Once we have introduced the concept of elasticity, we shall see that they will reduce to *three*.



4. Boundary conditions. – On the surface, the triple family of planes will determine tetrahedral elements, such as *OABC*. If one represents the areas of *ABC*, *OBC*, *OCA*, *OAB* by ds, ds_1, ds_2, ds_3, resp., then one must have:

$$L\,ds + p_{xx}\,ds_1 + p_{yz}\,ds_2 + p_{zx}\,ds_3 = 0, \quad \ldots$$

for equilibrium. Now, observe that:

$$ds_1 = -\frac{dx}{dn}\,ds, \qquad ds_2 = -\frac{dy}{dn}\,ds, \qquad ds_3 = -\frac{dz}{dn}\,ds\,.$$

It will then follow that the boundary equations are:

$$\left.\begin{aligned} L &= p_{xx}\frac{dx}{dn} + p_{yx}\frac{dy}{dn} + p_{zx}\frac{dz}{dn}, \\ M &= p_{xy}\frac{dx}{dn} + p_{yy}\frac{dy}{dn} + p_{zy}\frac{dz}{dn}, \\ N &= p_{xz}\frac{dx}{dn} + p_{yz}\frac{dy}{dn} + p_{zz}\frac{dz}{dn}. \end{aligned}\right\} \tag{2}$$

5. – We now pass on to a study of the *variations in the pressures* around each point. Take a tetrahedral element like *OABC inside of* the body, let $p_n\,ds$ be the action that is exerted across *ds* by the part of the body that contains the tetrahedron onto the part that does not. The pressure is computed like something that *acts upon the tetrahedron*, and is $-p_n\,ds$, and for equilibrium, we must have:

$$-p_n\,ds + p_{xx}\,ds_1 + p_{yx}\,ds_2 + p_{zx}\,ds_3 = 0, \qquad \ldots$$

If α, β, γ are the direction cosines of the perpendicular to *ABC*, which is based at *O*, then we will have $ds_1 = \alpha\,ds$, $ds_2 = \beta\,ds$, $ds_3 = \gamma\,ds$; hence:

$$\left.\begin{aligned} p_{nx} &= \alpha\,p_{xx} + \beta\,p_{yx} + \gamma\,p_{zx}, \\ p_{ny} &= \alpha\,p_{xy} + \beta\,p_{yy} + \gamma\,p_{zy}, \\ p_{nz} &= \alpha\,p_{xz} + \beta\,p_{yz} + \gamma\,p_{zz}. \end{aligned}\right\} \tag{3}$$

Those relations, which are independent of the dimensions of the tetrahedron, will obviously persist when the element *ABC*, displaces parallel to itself and concludes by containing the point *O*. One will then have four intersecting elements at *O*, and the relations (3) will show that if one knows the pressure on the three elements then the pressure on a fourth element will be determined in intensity and direction. *How do the direction and intensity of the pressure vary when the surface element that it acts upon rotates around O?*

6. – First of all, we ask whether there exist elements that are subject to only tangential pressures. For brevity, we set:

$$P(\alpha, \beta, \gamma) = \alpha^2\,p_{xx} + \beta^2\,p_{yy} + \gamma^2\,p_{zz} + 2\beta\gamma\,p_{yz} + 2\gamma\alpha\,p_{zx} + 2\alpha\beta\,p_{xy}\,,$$

call the discriminant of that form Δ, represent the reciprocal of Δ by:

$$\begin{vmatrix} q_{xx} & q_{yx} & q_{zx} \\ q_{xy} & q_{yy} & q_{zy} \\ q_{xz} & q_{yz} & q_{zz} \end{vmatrix},$$

and let Q (α, β, γ) be the reciprocal form to P. Now, from (3), the orthogonality condition:

$$\alpha\, p_{nx} + \beta\, p_{ny} + \gamma\, p_{nz} = 0$$

will become $P = 0$, and that equation will represent a quadric cone that is the locus of perpendiculars to the surface elements that are acted upon only tangentially. It is known that the equation of the envelope of the planes that go through the vertex of P perpendicular to the generators will be precisely $Q = 0$. That second cone, when it is real, divides the angular space around the point considered into two regions, and whereas the surface elements that are immersed in one region are subject to only *pressures*, properly speaking, the other ones will support *tensions.* If the cone Q is imaginary then that would say that the surface elements that intersect at the point considered are either all subject to pressures or all subject to tensions. In that case, one would consider the surface $Q = \pm\,\Delta$ and choose the sign on the right-hand side in such a way that one would have a real surface (which would necessarily be an ellipsoid). In the first case, however, one takes the + sign in one region and the – sign in the other, in such a way that the equation $Q = \pm\,\Delta$ will represent a pair of real surfaces; viz., two hyperboloids with one and two sheets that have the asymptotic cone $Q = 0$ in common. In any case, the surface $Q = \pm\,\Delta$ will be called (*) the *directrix surface*, because if one wishes to know the direction of the pressure on each element then it will be enough to know what that surface is. Indeed, if one observes that one has, by virtue of (3):

$$\left.\begin{aligned} p_{nx}\, q_{xx} + p_{ny}\, q_{xy} + p_{nz}\, q_{xz} &= \alpha\,\Delta, \\ p_{nx}\, q_{yx} + p_{ny}\, q_{yy} + p_{nz}\, q_{yz} &= \beta\,\Delta, \\ p_{nx}\, q_{zx} + p_{ny}\, q_{zy} + p_{nz}\, q_{zz} &= \gamma\,\Delta \end{aligned}\right\} \tag{4}$$

then one will see immediately that:

The plane conjugate to the direction of p_n is $\alpha x + \beta y + \gamma z = 0$; viz., the plane of that element.

As for the intensity, if x, y, z are the coordinates of the extremity of the representative segment of p_n then when one squares and sums (4), it will give:

$$(x\, q_{xx} + y\, q_{xy} + z\, q_{xz})^2 + (x\, q_{yx} + y\, q_{yy} + z\, q_{yz})^2 + (x\, q_{zx} + y\, q_{zy} + z\, q_{zz})^2 = \Delta^2.$$

Therefore:

The absolute values of the pressures or tensions around a point vary like the diameters of an ellipsoid. It is called the *ellipsoid of elasticity.*

If one displaces the coordinate axes parallel to the axes of P, in such a way that one will have $p_{yz} = p_{zx} = p_{xy} = 0$ then the equations of the directrix surface and the ellipsoid of elasticity will become:

(*) LAMÉ, *Leçons sur les coordonées curvilignes*, § CXLI.

$$\frac{x^2}{\varpi_1}+\frac{y^2}{\varpi_2}+\frac{z^2}{\varpi_3}=\pm\,1,$$

$$\frac{x^2}{\varpi_1^2}+\frac{y^2}{\varpi_2^2}+\frac{z^2}{\varpi_3^2}=\quad 1,$$

and one will then see more easily that those surfaces have the same axes. In addition, two particular mutually-perpendicular surface elements will support the minimum and maximum tension and will belong to the (generally unique) triple of elements that are not subject to oblique pressures. It will then be clear that regardless of the orientations of the axes, the values ϖ_1, ϖ_2, ϖ_3 of the *principal pressures* will be the roots of the equation:

$$\begin{vmatrix} p_{xx}-\pi & p_{xy} & p_{xz} \\ p_{yx} & p_{yy}-\pi & p_{yz} \\ p_{zx} & p_{zy} & p_{zz}-\pi \end{vmatrix} = 0$$

7. – A comparison of the equations of equilibrium (indefinite and boundary) that were obtained in Chapter IV with equations (1) and (2) will show that *in elastic bodies*, the functions p will depend upon only *three* functions u, v, w, by means of the relations:

$$p_{xx}=\frac{\partial\Pi}{\partial a},\qquad p_{yy}=\frac{\partial\Pi}{\partial b},\qquad \ldots,\qquad p_{yz}=p_{zy}=\frac{1}{2}\frac{\partial\Pi}{\partial f},\qquad \ldots;$$

i.e., if we adopt the notations of Chapter III:

$$\left\{\begin{aligned}
-p_{xx} &= A\,a + C'b + B'c + 2\,(F_1\,f + G_1\,g + H_1\,h),\\
-p_{yy} &= C'a + Bb + A'c + 2\,(F_2\,f + G_2\,g + H_2\,h),\\
-p_{zz} &= B'a + A'b + Cc + 2\,(F_3\,f + G_3\,g + H_3\,h),\\
-p_{yz} &= F_1\,a + F_2\,b + F_3\,c + 2\,(F'\,f + H'g + G'\,h),\\
-p_{zx} &= G_1\,a + G_2\,b + G_3\,c + 2\,(H'\,f + G\,g + F'\,h),\\
-p_{xy} &= H_1\,a + H_2\,b + H_3\,c + 2\,(G'\,f + F'\,g + H\,h).
\end{aligned}\right.$$

Those are the formulas that yield the distribution of the internal forces at any point in a deformed elastic medium. If the medium is endowed with a symmetry plane at the point considered then the asymmetry will disappear, and one will have simply:

$$\left\{\begin{aligned}
-p_{xx} &= Aa + C'b + B'c, & -p_{yz} &= 2Ff,\\
-p_{yy} &= C'a + Bb + A'c, & -p_{zx} &= 2Gg,\\
-p_{zz} &= B'a + A'b + Cc, & -p_{xy} &= 2Hh.
\end{aligned}\right.$$

With those formulas, it is easy (*) to explain the terminology that Rankine proposed in order to distinguish the various coefficients of elasticity.

(*) CLEBSCH, *Théorie de l'élasticité*, pp. 38.

CHAPTER VII

ELASTIC MOTIONS

1. Equations of elastic motion. – Suppose that the points of a body, rather than being in equilibrium, vibrate around a certain fixed position (x, y, z) and are found at the positions ($x + u$, $y + v$, $z + w$) at the end of the time interval t. In that case, the displacements u, v, w will be certain functions of x, y, z, t that will determine the series of configurations that the system assumes as time varies. The determination of u, v, w is provided by the equations of motion, which one deduces from the indefinite equations (Chap. IV, § **2**) of equilibrium by making use of *d'Alembert's principle*, as usual, which says that everything happens as if the body were in equilibrium at any instant under the action of the forces, properly speaking, and the fictitious forces, which are equal and of opposite sign to the ones that produced the effective motion at each point. The latter are measured by the product of the mass $\rho\, dS$ by the acceleration, whose components are, as is known:

$$\frac{\partial^2(x+u)}{\partial t^2} = \frac{\partial^2 u}{\partial t^2}, \qquad \frac{\partial^2(y+v)}{\partial t^2} = \frac{\partial^2 v}{\partial t^2}, \qquad \frac{\partial^2(z+w)}{\partial t^2} = \frac{\partial^2 w}{\partial t^2}.$$

One will then see that the indefinite equations of motion are deduced from the indefinite equations of equilibrium by substituting $X\, dS - \dfrac{\partial^2 u}{\partial t^2}\rho\, dS, \ldots$ for $X\, dS, \ldots$ It will be obvious then that the boundary conditions will always remain the same. The problem of elastic motion will then be solved by the following equations:

$$\left.\begin{aligned} X - \rho\frac{\partial^2 u}{\partial t^2} &= \frac{\partial}{\partial x}\frac{\partial \Pi}{\partial a} + \frac{1}{2}\frac{\partial}{\partial y}\frac{\partial \Pi}{\partial h} + \frac{1}{2}\frac{\partial}{\partial z}\frac{\partial \Pi}{\partial g}, \\ Y - \rho\frac{\partial^2 v}{\partial t^2} &= \frac{1}{2}\frac{\partial}{\partial x}\frac{\partial \Pi}{\partial h} + \frac{\partial}{\partial y}\frac{\partial \Pi}{\partial b} + \frac{1}{2}\frac{\partial}{\partial z}\frac{\partial \Pi}{\partial f} \\ Z - \rho\frac{\partial^2 w}{\partial t^2} &= \frac{1}{2}\frac{\partial}{\partial x}\frac{\partial \Pi}{\partial g} + \frac{1}{2}\frac{\partial}{\partial y}\frac{\partial \Pi}{\partial f} + \frac{\partial}{\partial z}\frac{\partial \Pi}{\partial c}, \end{aligned}\right\} \tag{1}$$

when completed by the equations (Chap. IV, § **2**).

2. – Theorem.

If the external forces do not vary with time then the general problem of elastic motion can always be decomposed into two special problems:

1. *A problem of simple equilibrium.*
2. *A problem of motion under the action of only the elastic forces.*

Indeed, let u', v', w' be the displacements that one needs to give to the points of the body in order for it to remain in equilibrium under the action of the external forces and the elastic forces. The functions u', v', w', which are independent of t, must satisfy equations (1′) and (1″) of Chapter IV. If one subtracts those of equations (1′) and (1″) that relate to the system of displacements (u, v, w) then one will get the relations:

$$
\begin{aligned}
-\rho\frac{\partial^2 u''}{\partial t^2} &= \frac{\partial}{\partial x}\frac{\partial \Pi''}{\partial a''}+\frac{1}{2}\frac{\partial}{\partial y}\frac{\partial \Pi''}{\partial h''}+\frac{1}{2}\frac{\partial}{\partial z}\frac{\partial \Pi''}{\partial g''},\\
0 &= \frac{\partial \Pi''}{\partial a''}\frac{dx}{dn}+\frac{1}{2}\frac{\partial \Pi''}{\partial h''}\frac{dy}{dn}+\frac{1}{2}\frac{\partial \Pi''}{\partial g''}\frac{dz}{dn},\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,
\end{aligned}
\tag{2}
$$

which must be satisfied by the residual displacements $u - u' = u''$, $v - v' = v''$, $w - w' = w''$. The relations (2) are precisely the equations of elastic motion in the case where external forces are absent. It is necessary to suppose that the external forces are independent of time, because if X, for example, is a function of t then since the X that appear in equations (1) are only particular values of that function, the difference between the two X will not always be zero, but will vary with time. Observe that the theorem that was just proved can take on the following noteworthy interpretation:

The points of an elastic body that is subject to forces that are independent of time will vibrate around their corresponding equilibrium positions in the same way that they will vibrate around their natural positions in the absence of external forces (*).

3. – What is the nature of the motion of the points of a vibrating elastic body? By virtue of the last theorem, if one would like to understand the *character* of the elastic vibrations then it would be legitimate to suppose that the body is completely free. Under that hypothesis, the equations of motion will become:

$$
(1')\left\{
\begin{aligned}
-\rho\frac{\partial^2 u}{\partial t^2} &= \frac{\partial}{\partial x}\frac{\partial \Pi}{\partial a}+\frac{1}{2}\frac{\partial}{\partial y}\frac{\partial \Pi}{\partial h}+\frac{1}{2}\frac{\partial}{\partial z}\frac{\partial \Pi}{\partial g},\\
-\rho\frac{\partial^2 v}{\partial t^2} &= \frac{1}{2}\frac{\partial}{\partial x}\frac{\partial \Pi}{\partial h}+\frac{\partial}{\partial y}\frac{\partial \Pi}{\partial b}+\frac{1}{2}\frac{\partial}{\partial z}\frac{\partial \Pi}{\partial f},\\
-\rho\frac{\partial^2 w}{\partial t^2} &= \frac{1}{2}\frac{\partial}{\partial x}\frac{\partial \Pi}{\partial g}+\frac{1}{2}\frac{\partial}{\partial y}\frac{\partial \Pi}{\partial f}+\frac{\partial}{\partial z}\frac{\partial \Pi}{\partial c},
\end{aligned}
\right.
\qquad
(1'')\left\{
\begin{aligned}
0 &= \frac{\partial \Pi}{\partial a}\frac{dx}{dn}+\frac{1}{2}\frac{\partial \Pi}{\partial h}\frac{dx}{dn}+\frac{1}{2}\frac{\partial \Pi}{\partial g}\frac{dx}{dn},\\
0 &= \frac{1}{2}\frac{\partial \Pi}{\partial h}\frac{dx}{dn}+\frac{\partial \Pi}{\partial b}\frac{dx}{dn}+\frac{1}{2}\frac{\partial \Pi}{\partial f}\frac{dx}{dn},\\
0 &= \frac{1}{2}\frac{\partial \Pi}{\partial g}\frac{dx}{dn}+\frac{1}{2}\frac{\partial \Pi}{\partial f}\frac{dx}{dn}+\frac{\partial \Pi}{\partial c}\frac{dx}{dn}.
\end{aligned}
\right.
$$

Let us attempt to satisfy them by taking:

$$u' = u\,\varphi(t), \qquad v' = v\,\varphi(t), \qquad w' = w\,\varphi(t),$$

(*) See CLEBSCH, *Théorie de l'Élasticité*, pp. 58.

in which u', v', w' are independent of time. If we denote everything that relates to u', v', w' by a prime then we will have, after derivation, $a = a'\varphi(t)$, $b = b'\,\varphi(t)$, …, $h = h'\,\varphi(t)$, and consequently:

$$\frac{\partial \Pi}{\partial a} = \frac{\partial \Pi'}{\partial a'}\,\varphi(t), \qquad \frac{\partial \Pi}{\partial b} = \frac{\partial \Pi'}{\partial b'}\,\varphi(t), \qquad \ldots, \qquad \frac{\partial \Pi}{\partial h} = \frac{\partial \Pi'}{\partial h'}\,\varphi(t),$$

in which $\frac{\partial \Pi}{\partial a}, \frac{\partial \Pi}{\partial b}, \ldots, \frac{\partial \Pi}{\partial h}$ is expressed linearly in $a, b, \ldots, h$. Having said that, the first of (1′) will become:

$$-\frac{\rho u'}{\varphi}\frac{d^2\varphi}{dt^2} = \frac{\partial}{\partial x}\frac{\partial \Pi'}{\partial a'} + \frac{1}{2}\frac{\partial}{\partial y}\frac{\partial \Pi'}{\partial h'} + \frac{1}{2}\frac{\partial}{\partial z}\frac{\partial \Pi'}{\partial g'},$$

and since the right-hand side is independent of time, it will be necessary that the same thing will be true for the left-hand side, as well, and one will have, consequently:

$$\frac{d^2\varphi}{dt^2} = -k^2\,\varphi,$$

in which $-k^2$ is intended to represent an arbitrary constant. Therefore, the most general form possible for the function φ is:

$$\varphi(t) = \lambda \cos(kt) + \mu \sin(kt),$$

and equations (1′) and (1″) will become:

$$(2')\begin{cases} \rho k^2 u' = \dfrac{\partial}{\partial x}\dfrac{\partial \Pi'}{\partial a'} + \dfrac{1}{2}\dfrac{\partial}{\partial y}\dfrac{\partial \Pi'}{\partial h'} + \dfrac{1}{2}\dfrac{\partial}{\partial z}\dfrac{\partial \Pi'}{\partial g'}, \\[2ex] \rho k^2 v' = \dfrac{1}{2}\dfrac{\partial}{\partial x}\dfrac{\partial \Pi'}{\partial h'} + \dfrac{\partial}{\partial y}\dfrac{\partial \Pi'}{\partial b'} + \dfrac{1}{2}\dfrac{\partial}{\partial z}\dfrac{\partial \Pi'}{\partial f'}, \\[2ex] \rho k^2 w' = \dfrac{1}{2}\dfrac{\partial}{\partial x}\dfrac{\partial \Pi'}{\partial g'} + \dfrac{1}{2}\dfrac{\partial}{\partial y}\dfrac{\partial \Pi'}{\partial f'} + \dfrac{\partial}{\partial z}\dfrac{\partial \Pi'}{\partial c'}, \end{cases} \quad (2'')\begin{cases} 0 = \dfrac{\partial \Pi'}{\partial a'}\dfrac{dx}{dn} + \dfrac{1}{2}\dfrac{\partial \Pi'}{\partial h'}\dfrac{dy}{dn} + \dfrac{1}{2}\dfrac{\partial \Pi'}{\partial g'}\dfrac{dz}{dn}, \\[2ex] 0 = \dfrac{1}{2}\dfrac{\partial \Pi'}{\partial h'}\dfrac{dx}{dn} + \dfrac{\partial \Pi'}{\partial b'}\dfrac{dy}{dn} + \dfrac{1}{2}\dfrac{\partial \Pi'}{\partial f'}\dfrac{dz}{dn}, \\[2ex] 0 = \dfrac{1}{2}\dfrac{\partial \Pi'}{\partial g'}\dfrac{dx}{dn} + \dfrac{1}{2}\dfrac{\partial \Pi'}{\partial f'}\dfrac{dy}{dn} + \dfrac{\partial \Pi'}{\partial c'}\dfrac{dz}{dn}. \end{cases}$$

Suppose that one succeeds in integrating equations (2′) by some means. u', v', w' will then be certain functions of x, y, z, and also k^2, and when they are substituted in (2″) and one eliminates x, y, z from (2) and the surface equation, that will yield an equation in k^2 that admits all possible values k_1^2, k_2^2, k_3^2, … of k^2 as roots. A special solution $u' = u_i$, $v' = v_i$, $w' = w_i$ of the equations (2′) and (2″) will correspond to any k_i^2, and consequently, a particular solution of equations (1′) and (1″), from which one will get (*) the general solution by linearly combining all possible solutions; i.e., by writing that:

(*) See POINCARÉ, *Leçons sur la théorie de l'élasticité*, Paris, G. Carré, 1892, pp. 112.

$$u = \sum_{i=1}^{\infty} u_i\, \varphi_i(t)\,, \qquad v = \sum_{i=1}^{\infty} v_i\, \varphi_i(t)\,, \qquad w = \sum_{i=1}^{\infty} w_i\, \varphi_i(t)\,, \tag{2}$$

in which:

$$\varphi_i\,(t) = \lambda_i \cos\,(k_i\, t) + \mu_i \sin\,(k_i\, t).$$

In addition, one will see how to determine the constants λ_1, λ_2, …, μ_1, μ_2, … by means of the *initial circumstances* of the motion.

4. – Let i and j distinguish two solutions of equations (2′) that correspond to the values k_i and k_j of k. In order to account for the nature of the vibrations (2), it would be appropriate to demonstrate an important property of the integral:

$$K_{ij} = \int (u_i\, u_j + v_i\, v_j + w_i\, w_j)\, \rho\, dS\,.$$

(2′) and (2″) relate to the displacements u_i, v_i, w_i, so one can consider them to be the equations of equilibrium, in which one has set:

$$X = \rho\, k^2\, u_i\,, \quad Y = \rho\, k^2\, v_i\,, \quad Z = \rho\, k^2\, w_i\,, \quad L = M = N = 0.$$

Hence, for $\delta u = u_j$, $\delta v = v_j$, $\delta w = w_j$, the equality (1) in Chapter IV will become:

$$k_i^2 K_{ij} = - \int \left(a_j \frac{\partial \Pi_i}{\partial a_i} + b_j \frac{\partial \Pi_i}{\partial b_i} + \cdots + h_j \frac{\partial \Pi_i}{\partial h_i} \right) dS\,. \tag{3}$$

Since the right-hand side and K_{ij} will not vary when one switches i and j, it is necessary that one must have $k_i^2 K_{ij} = k_j^2 K_{ij}$, and therefore, by hypothesis, $k_i^2 \neq k_j^2$ when $i \neq j$, so one will also have $K_{ij} = 0$. However, if $i = j$ then the right-hand side of (3) will reduce to $-2\int \Pi_i\, dS$. In summary, one sees that:

$$K_{ij} = \begin{cases} -\dfrac{2}{k_i^2} \displaystyle\int \Pi_i\, dS & \text{if } i = j, \\ \qquad 0 & \text{if } i \neq j. \end{cases} \tag{4}$$

5. – We shall use the last result in order to show that *the constants* k_1, k_2, k_3, … *are all real*. From the process that is followed in order to obtain the equation that admits the roots k_1^2, k_2^2, k_3^2, …, it would seem obvious that this equation *has real coefficients*. Any imaginary root will then correspond to its conjugate root. Let k_i^2, k_j^2 be such a pair of roots. It is clear that when equations (2) are written in terms of k_i^2, in one case, and then in terms of k_j^2, that will yield conjugate expressions for u_i and u_j, v_i and v_j, w_i and w_j, and therefore $u_i\,u_j$, $v_i\,v_j$, $w_i\,w_j$ will be sums of squares. Hence, K_{ij} is composed of essentially non-negative elements. However, one already known that one must have $K_{ij} = 0$. That cannot happen unless one has $u_i = 0$, $u_j = 0$, …, $w_j = 0$. Therefore, it will only be possible for k_1^2, k_2^2, k_3^2, … to be all real. The fact that those numbers are also positive will result immediately from the equality $k_i^2 K_{ii} = -2\int \Pi_i\, dS$, since Π_i is an essentially negative quantity, and K_{ii} is a sum of squares. Therefore, k_1, k_2, k_3, … will be real numbers.

6. – Return to the formulas (2). They say that the vibrations of the points of an elastic body can be considered to be resultants of the superpositions of simpler vibrations, which will be the vibrations $u = u'\,\varphi(t)$, $v = v'\,\varphi(t)$, $w = w'\,\varphi(t)$ for each value of k. If k is imaginary then u, v, w will be expressed exponentially in t, and they can decrease or increase indefinitely in time. However, from the reality of k, that cannot happen, since u, for example, is never greater than the quantity $u'\sqrt{\lambda^2 + \mu^2}$ in absolute value, and the latter is independent of time. In addition, if one increases t by $2\pi / k$ in the preceding expression for u, v, w then they will return to their original values. They will then represent a pendulum motion whose period $2\pi / k$ will be the same for all points of the body and vary only in the amplitude of oscillation from point to point. One will then arrive at the following noteworthy conclusion (*):

The internal motions of an elastic body cannot increase or decrease in time. On the contrary, all of the ***partial*** *motions, which evolve between invariable limits in equal time intervals, will prove to be periodic. Formulas* (2) *show that the vibrations of each point will result from the superposition of an infinitude of pendulum motions that have different periods.*

7. – Saint-Venant's theorem:

The vis viva of a vibrating ***elastic*** *body is equal to the sum of the vis vivas that are due to the individual pendulum motions* (**).

The total *vis viva* of the vibrating body:

$$\Phi = \tfrac{1}{2}\int\left[\left(\frac{\partial u}{\partial t}\right)^2 + \left(\frac{\partial v}{\partial t}\right)^2 + \left(\frac{\partial w}{\partial t}\right)^2\right]\rho\, dS.$$

Now, from (2), one will have:

(*) CLEBSCH, *Théorie de l'élasticité*, pp. 130.
(**) See Comptes-rendus, 2nd sem., 1872, pp. 1176, 1425, 1567.

$$\left(\frac{\partial u}{\partial t}\right)^2 + \left(\frac{\partial v}{\partial t}\right)^2 + \left(\frac{\partial w}{\partial t}\right)^2 = \sum_{i,j} (u_i\, u_j + v_i\, v_j + w_i\, w_j) \frac{d\varphi_i}{dt} \frac{d\varphi_j}{dt};$$

if one multiplies this by $\rho\, dS$ and integrates over all S then one will get:

$$\Phi = \tfrac{1}{2} \sum_{i,j} K_{ij} \frac{d\varphi_i}{dt} \frac{d\varphi_j}{dt}.$$

By virtue of (4), the right-hand side will then reduce to only the terms for which one has $i = j$. Hence:

$$\Phi = \tfrac{1}{2} \sum_{i=1}^{\infty} K_{ii} \left(\frac{d\varphi_i}{dt}\right)^2 .$$

On the other hand, the *vis viva* that is due to only the pendulum motions with the index i will be precisely:

$$\Phi_i = \tfrac{1}{2} \int (u_i^2 + v_i^2 + w_i^2) \left(\frac{d\varphi_i}{dt}\right)^2 \rho\, dS = \tfrac{1}{2} K_{ii} \left(\frac{d\varphi_i}{dt}\right)^2 .$$

It will then be true that one has $\Phi = \Phi_1 + \Phi_2 + \Phi_3 + \ldots$ In other words, everything happens as if the body were animated by an infinitude of pendulum motions that coexist with perfect independence of each other.

8. – Now, returning to the integration of (1′), all that remains is to see how the constants:

$$\lambda_1,\ \lambda_2,\ \lambda_3,\ \ldots,\ \mu_1,\ \mu_2,\ \mu_3,\ \ldots$$

will prove to be determined completely when one fixes the initial circumstances of the motion – i.e., when one supposes that the *displacement* (u_0, v_0, w_0) and the *velocity* (u_0', v_0', w_0') of each point are known at a given instant, which one can always assume to be the time origin. For $t = 0$, one has:

$$\varphi = \lambda, \quad \frac{d\varphi}{dt} = k\mu,$$

and consequently (*) if one sets $t = 0$ in formulas (2) and differentiates once with respect to t then:

$$u_0 = \sum_{i=1}^{\infty} \lambda_i\, u_i\,, \qquad\qquad u_0' = \sum_{i=1}^{\infty} k_i\, \mu_i\, u_i\,.$$

(*) See POINCARÉ, *loc. cit.*, pp. 113.

One multiplies the first equality by $\rho\, u_n\, dS$, and after doing the same thing with v_0 and w_0, one sums and integrates. Obviously, one will get:

$$K_{0n} = \sum_{i=1}^{\infty} \lambda_i\, K_{in} = \lambda_n\, K_{nn}\,;$$

that is:

$$\lambda_n = \frac{\int (u_0\, u_n + v_0\, v_n + w_0\, w_n)\,\rho\, dS}{\int (u_n^2 + v_n^2 + w_n^2)\,\rho\, dS}\,.$$

If one wishes to find μ_n then it will be enough to change λ_n into $k_n\, \mu_n$ and u_0, v_0, w_0 into u_0', v_0', w_0':

$$\mu_n = \frac{\int (u_0'\, u_n + v_0'\, v_n + w_0'\, w_n)\,\rho\, dS}{k_n \int (u_n^2 + v_n^2 + w_n^2)\,\rho\, dS}\,.$$

Here, one should observe that the constants, thus-calculated, are the ones that determine the *amplitudes* of the infinitude of component pendulum motions, while the *periods* of those motions can be determined from k_1, k_2, k_3, … It will then result from the preceding analysis that:

Whereas the amplitudes of the oscillations depend upon the initial circumstances of the motion, their periods will depend upon the geometric form and dimensions of the body.

In other words, varying the initial circumstances of the motion in a given body will influence only the amplitudes of the oscillations, while the time over which it was measured will always remain unaltered for any component oscillation (**).

9. – Some doubt remains in all of the preceding analysis, namely, that the equations that provide the values of the k might not admit an infinitude of roots, and also that they might not admit any, in which case, there would exist no solutions to equations (2′) and (2″) that have the form considered. Now, we can show (***) that *those equations admit an infinitude of distinct solutions.* Among that infinitude of displacements, which are constrained by the condition:

$$\int (u^2 + v^2 + w^2)\,\rho\, dS = 1, \tag{5}$$

we seek the one that will give a minimum value to $-\int \Pi\, dS$. Everything leads us to believe that such a minimum exists, and it will be positive or zero, because the function in question (which is essentially positive) is continuous, by virtue of the hypothesis that

(**) CLEBSCH, *Théorie de l'élasticité.*, pp. 104.

(***) POINCARÉ, *loc. cit.*, pp. 104. In reality, that proof leaves much to be desired in regard to confirming the existence of the minimum of $-\int \Pi\, dS$.

was made on the displacements from the outset. The *calculus of variations* will then lead one to set:

$$\int \delta \Pi \, dS + \lambda \int (u\,\delta u + v\,\delta v + w\,\delta w)\,\rho\, dS = 0, \tag{6}$$

in which λ represents a constant and δu, δv, δw represent the arbitrary variations of u, v, w. In particular, if one sets $\delta u = u$, $\delta v = v$, $\delta w = w$ then one will also have:

$$\delta a = \delta \frac{\partial u}{\partial x} = \frac{\partial \delta u}{\partial x} = \frac{\partial u}{\partial x} = a, \qquad \ldots;$$

hence:

$$\delta \Pi = \frac{\partial \Pi}{\partial a}\,\delta a + \ldots = \frac{\partial \Pi}{\partial a}\,a + \ldots = 2\Pi,$$

and (6) will become:

$$-\int \Pi \, dS = \tfrac{1}{2}\lambda, \tag{7}$$

by virtue of (5). Therefore, λ will be a positive number or zero that is represented by k_1^2. If one compares (6) with the equation that is obtained by applying Lagrange's principle and from which one gets (Chap. IV, § **2**) all of the equations of elastic equilibrium then one will see that due to the arbitrariness in δu, δv, δw, the six equations into which (6) splits can be comfortably deduced from those equations of equilibrium by setting $X = \rho\lambda u$, $Y = \rho\lambda v$, $Z = \rho\lambda w$, $L = M = N = 0$. In that way, one will recover equations (2′) and (2″) precisely, which must then admit a solution (u_1, v_1, w_1) for $k = k_1$ that constitutes precisely the functions that will satisfy (5) and give a minimum to $-\int \Pi \, dS$. In order to establish the existence of those particular solutions then, consider from among the infinitude of solutions that satisfy (5), the ones that also satisfy the condition:

$$\int (u u_1 + v v_1 + w w_1)\,\rho\, dS = 0, \tag{8}$$

and from them, one determines the ones that yield a minimum for $-\int \Pi \, dS$. For those functions, one must have:

$$\delta \Pi \, dS + \lambda \int (u\,\delta u + v\,\delta v + w\,\delta w)\,\rho\, dS + \lambda_1 \int (u_1\,\delta u + v_1\,\delta v + w_1\,\delta w)\,\rho\, dS = 0, \tag{9}$$

no matter what the variations δu, δv, δw are, and for convenient values of λ and λ_1. In particular, for $\delta u = u_1$, $\delta v = v_1$, $\delta w = w_1$:

$$\int \delta \Pi \, dS + \lambda_1 = 0,$$

in which:

$$\delta \Pi = \frac{\partial \Pi}{\partial a}\,\delta a + \ldots = \frac{\partial \Pi}{\partial a}\,a_1 + \ldots$$

On the other hand, (6), which is satisfied by the functions u_1, v_1, w_1, will become:

$$\int \delta \Pi \, dS = 0, \qquad \text{in which} \qquad \delta\Pi = \frac{\partial \Pi}{\partial a_1}\delta a + \ldots = \frac{\partial \Pi}{\partial a} a_1 + \ldots$$

when $\delta u = u$, $\delta v = v$, $\delta w = w$; hence, $\lambda_1 = 0$. With that, (9) will assume the form of (6), λ must then be a number k_2^2 that is *not less than* k_1^2, since the new functions u_2, v_2, w_2, which satisfy (2) for $k = k_2$, are required to verify the relation (8), in addition to (5). One succeeds in determining the functions u_3, v_3, w_3 in an analogous way, and for them $-\int \Pi \, dS$ will attain a value $\frac{1}{2}k_3^2$, which is the minimum of all of the ones that can be assumed for functions u, v, w that satisfy conditions (5), (8), and the new condition:

$$\int (uu_2 + vv_2 + ww_2)\,\rho \, dS = 0.$$

It is obvious that k_3^2 is not less than k_2^2, …

10. – One then obtains numbers $k_1^2 \le k_2^2 \le k_3^2 \le \ldots$, and one needs to show that they are all truly distinct, and in order to do that, we begin by removing all doubt as to whether they can all be zero. If that were the case then (7) would give $\int \Pi \, dS = 0$, and the functions u_1, v_1, w_1 would necessarily have the characteristic form of rigid displacements. The same thing would be true for the successive triples of functions, but the seventh triple could be expressed linearly in terms of the first six, but we shall show that this cannot happen, because *the infinitude of displacements* (u_i, v_i, w_i) *are mutually linearly independent.* Indeed, if one had:

$$u_n = \sum_{i=1}^{n-1} \lambda_i u_i\,, \quad v_n = \sum_{i=1}^{n-1} \lambda_i v_i\,, \quad w_n = \sum_{i=1}^{n-1} \lambda_i w_i\,,$$

in which λ_1, λ_2, λ_3, … are constants, then one would also have:

$$\int (u_n^2 + v_n^2 + w_n^2)\,\rho \, dS = \sum_{i=1}^{n-1} \lambda_i K_{in} = 0,$$

but the left-hand side must be equal to unity, since the functions u_n, v_n, w_n must satisfy (5).

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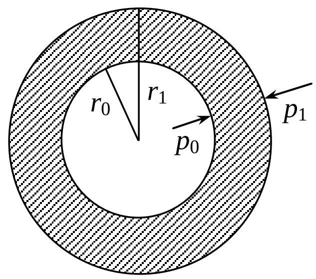
CHAPTER VIII

APPLICATION TO THE SPHERE

1. – The considerations that were developed in the preceding chapter make it possible for one to solve completely the general problem that was posed at the beginning of Chapter IV, as long as one knows how to integrate certain differential equations. As a result, one must address the means to facilitate or effect that integration in general. However, in certain special cases, the very simplicity of what is given can make the form that one must attribute to the displacements intuitive, because they must satisfy the equations of equilibrium or motion. For example, take the sphere, with the sole purpose of making an immediate application of the results that were obtained before.



2. Equilibrium. – Considers a homogeneous, isotropic, spherical shell that is subjected to a pressure of p_0 per unit area internally and a pressure of p_1 externally. Let r_0 be the inner radius and let r_1 be the outer radius. If the pressures are distributed uniformly then one knows that the displacement at a distance of r from the center must depend upon only r and can take place only in the direction of that radius, in such a way that if one calls the unit elongation ε and lets $u = \varepsilon x$, $v = \varepsilon y$, $w = \varepsilon z$ then one will have:

$$\frac{\partial u}{\partial x} = \varepsilon + x\frac{d\varepsilon}{dr}\frac{\partial r}{\partial x} = \varepsilon + \frac{x^2}{r}\frac{d\varepsilon}{dx}, \qquad \frac{\partial u}{\partial y} = x\frac{d\varepsilon}{dr}\frac{\partial r}{\partial y} = \frac{xy}{r}\frac{d\varepsilon}{dx}, \qquad \ldots$$

Hence:

$$\Theta = 3\varepsilon + r\frac{d\varepsilon}{dr}, \qquad \mathcal{T}_1 = \mathcal{T}_2 = \mathcal{T}_3 = 0,$$

and the indefinite equations will become:

$$\frac{\partial \Theta}{\partial x} = \frac{\partial \Theta}{\partial y} = \frac{\partial \Theta}{\partial z} = 0.$$

Hence, Θ will be a constant, which one calls 3λ. One deduces from:

$$3\varepsilon + r\frac{d\varepsilon}{dr} = 3\lambda$$

upon integration, that:

$$\varepsilon = \lambda + \frac{\mu}{r^3}. \tag{1}$$

One now addresses the determination of the constants λ and μ. The first boundary equation, which can relate to either of the two spherical surfaces, will become:

$$p\frac{dx}{dn} + 2B\frac{du}{dn} + (A-2B)\Theta\frac{dx}{dn} = 0,$$

except for the indices 0 and 1 that distinguish the two surfaces from each other; i.e., if one observes that one needs to set n equal to r or $-r$ then:

$$p\frac{x}{r} + 2B\left(\varepsilon\frac{x}{r} + x\frac{d\varepsilon}{dr}\right) + 3\lambda(A-2B)\frac{x}{r} = 0.$$

If one divides this by x / r and adopts (1) then it will follow that:

$$p - \frac{4\mu B}{r^3} + \lambda\,(3A - 4B) = 0.$$

One can arrive at the same result by means of the other two boundary conditions. One will then have the equations:

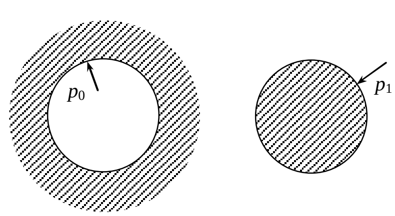
$$p_0\, r_0^3 = 4\mu B - \lambda\,(3A - 4B)\, r_0^3, \qquad p_1\, r_1^3 = 4\mu B - \lambda\,(3A - 4B)\, r_1^3,$$

from which, one can deduce that:

$$\lambda = -\frac{p_1\, r_1^3 - p_0\, r_0^3}{(3A-4B)(r_1^3 - r_0^3)}, \qquad \mu = -\frac{(p_1 - p_0)\, r_1^3\, r_0^3}{4B(r_1^3 - r_0^3)}. \tag{2}$$

If one substitutes this in (1) then one will have the means to know the deformation at each point, as well as the variations of the thickness, volume, etc. For example, the total increase in volume will be:

$$\int \Theta\, dS = 3\lambda\cdot\frac{4}{3}\pi(r_1^3 - r_0^3) = \frac{4\pi\,(p_0\, r_0^3 - p_1\, r_1^3)}{3A-4B}.$$



3. – In the case of a solid sphere or an indefinite medium that is provided with a spherical cavity, one will have only one surface to consider, and therefore only one equation for the determination of λ and μ. However, in that case, one can determine one constant directly by observing that the displacement $\varepsilon\, r$ must remain finite, and therefore one must have $\mu = 0$ in the former case and $\lambda = 0$ in the latter. Observe that formulas (2) also persist in those extreme cases, since one is given that:

$$\lambda = -\frac{p_1}{3A-4B}, \qquad \mu = 0$$

for $r_0 = 0$, and when r_1 is infinite, when one supposes that $p_1 = 0$, in addition:

$$\lambda = 0, \quad \mu = \frac{p_0\, r_0^2}{4B}.$$

It is known that the penultimate formulas are usually suited to the case of an arbitrary spherical shell that is subject to pressures that are equal and opposite per unit area.

4. Pressures and tensions. – For an isotropic body, one has:

$$\begin{cases} -p_{xx} = (A-2B)\Theta + 2B\dfrac{\partial u}{\partial x}, & -p_{yz} = B\left(\dfrac{\partial w}{\partial y} + \dfrac{\partial v}{\partial z}\right), \\ -p_{yy} = (A-2B)\Theta + 2B\dfrac{\partial v}{\partial y}, & -p_{zx} = B\left(\dfrac{\partial u}{\partial z} + \dfrac{\partial w}{\partial x}\right), \\ -p_{zz} = (A-2B)\Theta + 2B\dfrac{\partial w}{\partial z}, & -p_{xy} = B\left(\dfrac{\partial v}{\partial x} + \dfrac{\partial u}{\partial y}\right). \end{cases}$$

In the present case, those formulas will become:

$$-p_{xx} = \lambda\,(3A-4B) + 2\mu B\,\frac{y^2+z^2-2x^2}{r^5}, \qquad \ldots$$

$$p_{yz} = 6\mu B\,\frac{yz}{r^5}, \qquad p_{zx} = 6\mu B\,\frac{zx}{r^5}, \qquad p_{xy} = 6\mu B\,\frac{xy}{r^5}.$$

If one passes the z-axis through the point around which one would like to study the distribution of internal actions then when one takes $x = y = 0$, $z = r$, the last formula will give:

$$\varpi_1 = \varpi_2 = -\lambda\,(3A-4B) - \frac{2\mu B}{r^3}, \qquad \varpi_3 = -\lambda\,(3A-4B) + \frac{4\mu B}{r^3}.$$

In particular, for $\mu = 0$, one will have $\varpi_1 = \varpi_2 = \varpi_3 = p_1$, and the ellipsoid of elasticity at any point will become a sphere. Hence, in a solid sphere that is subjected to a uniform pressure, one can say that the pressure will be transmitted normally to all internal surface elements with equal intensity, as in fluids. However, if one sets $\lambda = 0$ then one will have:

$$\varpi_1 = \varpi_2 = -\tfrac{1}{2}\,\varpi_3 = -\frac{p_0}{2}\left(\frac{r_0}{r_1}\right)^3,$$

and the ellipsoid of rotation will be one of rotation around the radius. Therefore, in an indefinite medium that is homogeneous and isotropic and provided with a spherical cavity, any pressure that is distributed uniformly over the wall of the cavity will be transmitted to the surface elements that are perpendicular to the radii with an intensity that fades away in the region inversely to the cube of the distance from the center of the cavity and will produce tensions in all of the elements that contain the radius. In other words, if one imagines that the medium is subdivided into very thin spherical strata that are concentric to the cavity then one can say that whereas any stratum tends to *tear* with equal intensity along all of its great circles, it will also be subjected to a *crushing* of its thickness that is twice as intense.

5. Vibrations. – For the study of vibrations, one confines oneself to considering the case of a solid sphere of radius a. One will always have $u = \varepsilon\, x$, $v = \varepsilon y$, $w = \varepsilon z$; however, ε will be a function of r and t. Therefore:

$$\Theta = 3\varepsilon + r\frac{\partial \varepsilon}{\partial r}, \qquad \mathcal{T}_1 = \mathcal{T}_2 = \mathcal{T}_3 = 0,$$

and the indefinite equations will become:

$$\rho\frac{\partial^2 u}{\partial t^2} = A\frac{\partial \Theta}{\partial x}, \qquad \rho\frac{\partial^2 v}{\partial t^2} = A\frac{\partial \Theta}{\partial y}, \qquad \rho\frac{\partial^2 w}{\partial t^2} = A\frac{\partial \Theta}{\partial z}.$$

The first equation can also be written:

$$\rho x\frac{\partial^2 \varepsilon}{\partial t^2} = A\frac{\partial \Theta}{\partial r}\frac{x}{r}, \qquad \text{i.e.,} \qquad \rho\frac{\partial^2 \varepsilon}{\partial t^2} = \frac{A}{r}\frac{\partial \Theta}{\partial r},$$

and the other one will lead to the same result:

$$\rho\frac{\partial^2 \varepsilon}{\partial t^2} = A\left(\frac{\partial^2 \varepsilon}{\partial r^2} + \frac{4}{r}\frac{\partial \varepsilon}{\partial r}\right). \tag{3}$$

In addition, on the sphere of radius a, one must have:

$$2B\frac{\partial u}{\partial r}+(A-2B)\,\Theta\frac{\partial x}{\partial r} = 0\ ;$$

i.e.:

$$2B\left(\varepsilon\frac{x}{r}+x\frac{\partial\varepsilon}{\partial r}\right)+(A-2B)\left(3\varepsilon+r\frac{\partial\varepsilon}{\partial r}\right)\frac{x}{r} = 0,$$

or, when one divides by x / r and reduces:

$$Ar\frac{\partial\varepsilon}{\partial r} + (3A - 4B)\ \varepsilon = 0. \tag{4}$$

One now addresses the integration of equation (3) in such a way that equation (4) will be satisfied for $r = a$.

6. – In order to find a particular solution of (3), set:

$$\varepsilon = \mathcal{R}\ [\lambda \cos (kt) + \mu \sin (kt)], \tag{5}$$

in which $\mathcal{R}$ is a function of only r and k is a constant. If one substitutes this (3) and (4) and takes $\rho k^2 = Ah^2$, for simplicity, then one will easily find that $\mathcal{R}$ must satisfy the equation:

$$\frac{d^2\mathcal{R}}{dr^2}+\frac{4}{r}\frac{d\mathcal{R}}{dr} + h^2\ \mathcal{R} = 0, \tag{6}$$

in such a way that for $r = a$, one will have:

$$Ar\frac{d\mathcal{R}}{dr} + (3A - 4B)\ \mathcal{R} = 0. \tag{7}$$

The integration (6) is based upon the property of *Bessel transcendents*, which do not differ substantially from the functions:

$$F_n\,(x) = 1 - \frac{x^2}{2(n+1)}+\frac{x^4}{2\cdot 4(n+1)(n+3)}-\frac{x^6}{2\cdot 4\cdot 6(n+1)(n+3)(n+5)} + \ldots$$

If one differentiates that equality twice then one will easily observe the relation:

$$F_n''(x)+\frac{n}{x}F_n'(x) + F_n\,(x) = 0. \tag{8}$$

Having said that, set $\mathcal{R} = r^{\nu}\,F_n\,(hr)$. Equation (6) will become:

$$F_n''(hr)+\frac{2\nu+4}{hr}F_n'(hr)+\left(1+\frac{\nu(\nu+4)}{h^2r^2}\right)F_n(hr) = 0,$$

and that cannot coincide with (8) unless one has:

$$\nu(\nu+3) = 0, \quad n = 2\nu + 4.$$

One must then have $\nu = 0$, $n = 4$ or $\nu = -3$, $n = -2$, and therefore one will have the following two particular solutions of equation (6):

$$\mathcal{R} = F_4(hr), \quad \mathcal{R} = r^{-3} F_{-2}(hr).$$

That equation will then be linear and second-order, so the general solution will be:

$$\mathcal{R} = \alpha F_4(hr) + \frac{\beta}{r^3} F_{-2}(hr). \tag{9}$$

Meanwhile, the function $\mathcal{R}r$ must remain finite. It will then be necessary for us to have $\alpha = 0$ in the case of an infinite medium that is endowed with a spherical cavity and $\beta = 0$ in the case of a solid sphere. We confine ourselves to the latter case and take into account the fact we can omit the index 4 in the first term in the expression (9), since it will now be useless, and if we take $\alpha = 1$ then we can write:

$$\mathcal{R} = F(hr) = 1 + \sum_{i=1}^{\infty}\frac{(-1)^i(hr)^{2i}}{2\cdot 4\cdots 2i\cdot 5\cdot 7\cdots(2i+3)}. \tag{10}$$

7. – Set $ha = x$, and substitute the last result in equation (7). One will get:

$$Ax\,F'(x) + (3A - 4B)\,F(x) = 0\,;$$

i.e., if one adopts the development (10):

$$(3A - 4B) - \frac{5A-4B}{2\cdot 5}x^2 + \frac{7A-4B}{2\cdot 4\cdot 5\cdot 7}x^4 - \frac{9A-4B}{2\cdot 4\cdot 6\cdot 5\cdot 7\cdot 9}x^6 + \ldots = 0.$$

This transcendental equation has roots that are *all real*, and they will provide the values $k_1, k_2, k_3, \ldots$ of k. If one substitutes a well-defined k_i in the expression (5) then one will get a particular solution, and if one linearly combines the solutions that correspond to the infinite values of the index i then one will find that:

$$\varepsilon = \sum_{i=1}^{\infty}[\lambda_i\cos(k_i\,t) + \mu_i\sin(k_i\,t)]\;F(h_i\,r). \tag{11}$$

All that remains for us to do now is to determine the constants λ_1, λ_2, …, μ_1, μ_2, … that fix the initial conditions of the motion. Suppose that at the initial time, an arbitrary point that is situated at a distance of r from the center in the case of equilibrium is found to be displaced by $\varphi(r)$ and animated with the velocity $\psi(r)$, in such a way that for $t = 0$, one will have:

$$\varphi(r) = \varepsilon r, \qquad \psi(r) = \frac{\partial \varepsilon r}{\partial t} = r\frac{\partial \varepsilon}{\partial t};$$

i.e., if one uses (11):

$$\frac{\varphi(r)}{r} = \sum_{i=1}^{\infty} \lambda_i F(h_i r), \qquad \frac{\psi(r)}{r} = \sum_{i=1}^{\infty} k_i \mu_i F(h_i r). \tag{12}$$

Observe that the functions u_i of the penultimate chapter are the ones that are represented by $r F(h_i r)$, and therefore, if one refers to what was proved in that chapter then one can write:

$$\int_0^a F(h_i r) F(h_j r) r^4 dr = 0$$

for $i \neq j$. Having said that, multiply equations (12) by $F(h_i r)\, r^4\, dr$ and integrate from $r = 0$ and $r = a$. If one takes the last observation into account then one will see that the terms in the right-hand side will go to zero (except for the n^{th} terms), and one will get:

$$\lambda_n = \frac{\int_0^a F(h_n r)\varphi(r) r^3 dr}{\int_0^a F^2(h_n r) r^4 dr}, \qquad \mu_n = \frac{\int_0^a F(h_n r)\psi(r) r^3 dr}{k_n \int_0^a F^2(h_n r) r^4 dr}.$$

8. – To conclude, observe that the function $F_n(x)$ that was considered in § **6** can be expressed in finite form by means of trigonometric functions for any even value of n. One will first have:

$$-\frac{n+1}{x} F_n'(x) = 1 - \frac{x^2}{2(n+3)} + \frac{x^4}{2 \cdot 4 \cdot (n+3)(n+5)} - \ldots = F_{n+2}(x).$$

When one knows $F_n(x)$, the preceding formula will permit one to calculate F_{n+2}. Conversely, if one knows $F_{n+2}(x)$ then one will have:

$$F_n(x) = 1 - \frac{1}{n+1} \int_0^x x F_{n+2}(x)\, dx.$$

Having said that, one knows that:

$$F_0(x) = 1 - \frac{x^2}{1\cdot 2} + \frac{x^4}{1\cdot 2\cdot 3\cdot 4} - \ldots = \cos x.$$

Therefore:

$$F_{-2}(x) = 1 + \int_0^x x\cos x\,dx = x \sin x + \cos x.$$

However:

$$F_2(x) = -\frac{1}{x} F_0'(x) = \frac{\sin x}{x}, \qquad F_4(x) = -\frac{3}{x} F_2'(x) = \frac{3}{x^3}(\sin x - x\cos x).$$

Now, formula (10) will become:

$$\mathcal{R} = \frac{3}{h^3 r^3}(\sin hr - hr \cos hr),$$

and the transcendental equation:

$$Ax\, F'(x) + (3A - 4B)\, F(x) = 0,$$

which must admit an infinitude of real roots and no imaginary ones, will transform into:

$$\frac{1}{x} - \cot x = \frac{Ax}{4B}.$$

The roots are the abscissas of the points at which the curve $y = \cot x$ meets the hyperbola:

$$y = \frac{1}{x} - \frac{Ax}{4B}.$$

The graphical representation of that will show us directly how one and only one root will fall in each interval $(i\pi - \pi, i\pi)$. As the integer i increases, the formula:

$$a\, h_i = i\pi - \frac{4B}{i\pi A}$$

will tend to become exact, and the periods of the infinitude of component vibrations will tend to assume the form $\frac{2a}{i}\sqrt{\frac{\rho}{A}}$ for infinitely-large i.

CHAPTER IX

THE DIRICHLET PROBLEM

1. – Theorem:

If a function U that is finite, continuous, and uniform in all of the space S in which it satisfies the equation $\Delta^2 U = \varphi$ *takes on prescribed surface values then it will be determined completely.*

In other words, no solution U of the differential equation considered can have all of the stated properties without coinciding with U. Indeed, consider the function $V = U - U''$, which takes zero values on the surface and has its second differential parameter (†) equal to zero in all of S. One has:

$$\int \Delta V\, dS = \sum \int \left(\frac{\partial V}{\partial x}\right)^2 dS = \sum \int \frac{\partial}{\partial x}\left(V\frac{\partial V}{\partial x}\right) dS - \sum \int V \frac{\partial^2 V}{\partial x^2} dS\ ;$$

i.e.:

$$\begin{aligned}\int \Delta V\, dS &= -\sum \int V \frac{\partial V}{\partial x}\frac{dx}{dn} ds - \sum \int V \frac{\partial^2 V}{\partial x^2} dS \\ &= -\int V \frac{\partial V}{\partial x} ds - \int V \Delta^2 V\, dS\ .\end{aligned}$$

The last integral is zero because $\Delta^2 V = 0$ at any point of S; the penultimate one will then be zero because $V = 0$ at any point of s. Therefore:

$$\int \Delta V\, dS = 0,$$

and since ΔV, which is the sum of squares, has no negative values, one will necessarily have $\Delta V = 0$ at any point of S, and therefore:

$$\frac{\partial V}{\partial x} = 0, \qquad \frac{\partial V}{\partial y} = 0, \qquad \frac{\partial V}{\partial z} = 0.$$

Therefore, V will be constant, and since it has the value zero on the surface, it will need to keep that value in all of S; viz., $U = U'$.

(†) Translator: i.e., its Laplacian. The first differential parameter amounts to $\| \nabla U \|^2$.

2. Observations. –

a) One gives the name of *Dirichlet problem* (*) to the determination of the function U that satisfies all of the conditions that were imposed in the statement of the preceding theorem. Up to now, it has not been possible to prove rigorously that such a function always exists, and one can assert only that if one does exist then no other one can exist that differs from the first one. One first needs to observe that this would no longer be true if one were to drop one of the conditions. For example, if the function were not required to be *finite* then one would quickly see that it would cease to be unique.

b) Among all of the functions that assume the same values on the surface, the function U, which satisfies the stated conditions, will be the one that gives a minimum to the integral $\int \Delta U\,dS$. Indeed, when one attributes arbitrary variations to U, one will have:

$$\tfrac{1}{2}\delta\int \Delta U\,dS = \sum\int \frac{\partial U}{\partial x}\frac{\partial \delta U}{\partial x}\,dS = -\int \frac{dU}{dn}\,\delta U\,dS - \int \Delta^2 U\,\delta U\,dS\,,$$

and since one will have $\delta U = 0$ on the surface, because $\int \Delta U\,dS$ is a minimum, one must have:

$$\int \Delta^2 U\,\delta U\,dS = 0$$

for any system of variations, and consequently $\Delta^2 U = 0$ at any point of S. That observation will be enough to prove the existence of the function U in any space, as long as one can assume the existence of the *minimum* considered. Unfortunately, one knows only how to prove that the integral $\int \Delta U\,dS$, which is essentially positive, has a *lower limit*, but not that this limit can be necessarily attained.

c) If one gives the values of the first derivative of a function with respect to the normal of a surface on that surface, instead of the values of the function itself, then the preceding theorem will still be true, because dV / dn will be equal to zero over all of the surface.

d) The same theorem can be applied more generally to the differential equation:

$$\sum_{i,j} a_{ij}\,\frac{\partial^2 U}{\partial x_i\,\partial x_j} = \varphi,$$

provided that the quadratic form:

(*) Some of the more recent works on this celebrated problem are:
PICARD, *Traité d'Analyse*, Paris, Gauthier-Villars, 1891, t. I, pp. 141.
DUHEM, *Leçons sur l'électricité et le magnétisme*, Paris, Gauthier-Villars, 1891, t. I, pp. 159.

One can glean some valuable indications from the latter source on the history of the problem and the interesting and incisive research (which is still ongoing) to which it gave rise.

$$\sum_{i,j} a_{ij} \frac{\partial U}{\partial x_i} \frac{\partial U}{\partial x_j}$$

is essentially positive.

3. Green's theorem. – One is given two functions U and V that are continuous and uniform in all of S, and one then considers the integral:

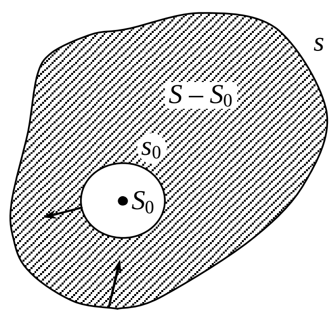
$$\int (U\,\Delta^2 V - V\Delta^2 U)\,dS = \sum \int \left(U \frac{\partial^2 V}{\partial x^2} - V \frac{\partial^2 U}{\partial x^2} \right) dS = \sum \int \frac{\partial}{\partial x} \left(U \frac{\partial V}{\partial x} - V \frac{\partial U}{\partial x} \right) dS \,.$$

It transforms into the surface integral:

$$\sum \int \left(U \frac{\partial V}{\partial x} - V \frac{\partial U}{\partial x} \right) \frac{dx}{dn} dS = \sum \int \left(V \frac{dU}{dn} - U \frac{dV}{dn} \right) ds \,.$$

Green's theorem consists of precisely the equality:

$$\int (U\,\Delta^2 V - V\Delta^2 U)\,dS = \sum \int \left(V \frac{dU}{dn} - U \frac{dV}{dn} \right) ds \,.$$



4. – Now, in order to obtain the solution of $\Delta^2 U = 0$ that will assume a given system of values on the surface, one applies Green's theorem to the functions U and $V = 1 / r$. Let r represent the distance from the variable point at which one calculates V to the fixed, but otherwise arbitrary, point at which one would like to calculate U. In order for Green's theorem to be applicable, it is necessary to exclude the latter point, because the function V will become infinite at it. One can draw a sphere of infinitesimal radius R with its center at the point considered. In the remaining space $S - S_0$, one can write:

$$\int_{S-S_0} \frac{\varphi\, dS}{r} = \int_{s+s_0} \left(U \frac{d(1/r)}{dn} - \frac{1}{r} \frac{dU}{dn} \right) ds \,;$$

i.e.:

$$\int_s \left(U \frac{d(1/r)}{dn} - \frac{1}{r}\frac{dU}{dn} \right) ds - \int_S \frac{\varphi\, dS}{r} = \int_{s_0} \frac{1}{r}\frac{dU}{dn} ds - \int_{s_0} U \frac{d(1/r)}{dn} ds - \int_{S_0} \frac{\varphi\, dS}{r}. \tag{1}$$

Let us see what limits the integrals in the right-hand side will tend to when the sphere vanishes. Let μ denote a convenient mean value of the function φ over all of S_0, and let μ' denote the mean value of dU / dn over all of s_0 ; one will have:

$$\int_S \frac{\varphi\, dS}{r} = \mu \int \frac{dS}{r} = 2\pi\mu R^2, \qquad \int_{s_0} \frac{1}{r}\frac{dU}{dn} ds = \mu' \int \frac{ds}{r} = 4\pi\mu' R.$$

These two integrals will each have a limit of zero then. However, if one lets μ denote the mean value that U assumes on the surface s_0 then one will have:

$$-\int_{s_0} U \frac{d(1/r)}{dn} ds = -\mu \int_{s_0} \frac{d(1/r)}{dr} ds = \mu \int \frac{ds}{r^2} = 4\pi\mu .$$

By virtue of the properties that one assumes the function U to possess, μ will exist and tend to the particular value U that the function has at the point considered. Therefore, in the limit, (1) will give:

$$U = \frac{1}{4\pi} \int \left(U \frac{d(1/r)}{dn} - \frac{1}{r}\frac{dU}{dn} \right) ds - \frac{1}{4\pi} \int \frac{\varphi\, dS}{r}. \tag{2}$$

That formula contains more than is necessary, since one will know U when one is given its values on the surface and those of its first derivative with respect to the normal, while one sees that it is enough to prescribe one or the other set of values in order for U to be determined completely. One then needs to find out how to eliminate one system of values or the other.

5. – One can arrive at an analogous result when one starts from the more general equation that was contemplated in § **2** and substitutes the form:

$$\sum_{i,j} a_{ij} (x_i - \xi_i)(x_j - \xi_j)$$

for r^2 ; viz., a quadratic form in the differences between the coordinates that is *reciprocal* to the form considered.

6. – One calls a finite, continuous, uniform function that is characterized among the infinitude of *harmonic functions* (i.e., ones that satisfy the equation $\Delta^2 = 0$, or *Laplace equation*) by the condition that it must take the values $1 / r$ on the surface a *Green function*, and one represents it by G. Note here that if the function is not required to be *finite* then one can take $1 / r$, which satisfies all of the other conditions. When the Green function is known for a given space S, the solution to the Dirichlet problem will always be possible in that space. Indeed, if one applies Green's theorem to the functions U and G then one will have:

$$\int G\,\Delta^2 U\,dS = \int\left(U\frac{dG}{dn} - \frac{1}{r}\frac{dU}{dn}\right)ds$$

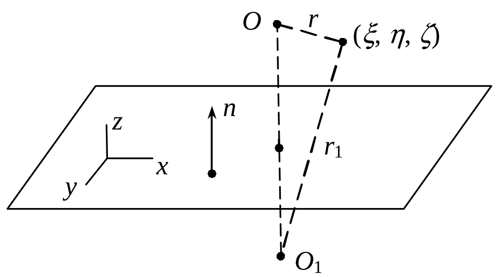
or

$$0 = \frac{1}{4\pi}\int\left(\frac{1}{r}\frac{dU}{dn} - U\frac{dG}{dn}\right)ds + \frac{1}{4\pi}\int G\varphi\,dS\,.$$

If one adds this to (2) then one will get:

$$U = \frac{1}{4\pi}\int U\left(\frac{d(1/r)}{dn} - \frac{dG}{dn}\right)ds + \frac{1}{4\pi}\int\left(G - \frac{1}{r}\right)\varphi\,dS\,.$$

That is the formula that answers the question that was posed.



7. Examples:

a) In the case of an infinite space that is bounded by a plane, the Green function will obviously be obtained by taking the distance r_1 to the point O_1, which is symmetric to the point O with respect to the plane. One has $G_1 = 1 / r_1$, and

$$r^2 = (x - \xi)^2 + (y - \eta)^2 + (z - \zeta)^2, \qquad r_1^2 = (x - \xi)^2 + (y - \eta)^2 + (z + \zeta)^2.$$

Meanwhile:

$$\frac{d(1/r)}{dn} = \frac{\partial(1/r)}{\partial\zeta} = \frac{z-\zeta}{r^3}, \qquad \frac{d(1/r_1)}{dn} = \frac{\partial(1/r_1)}{\partial\zeta} = -\frac{z+\zeta}{r^3},$$

and therefore, one will have:

$$\frac{d(1/r)}{dn} - \frac{dG}{dn} = \frac{z}{r^3}$$

on the surface; i.e., for $\zeta = 0$. Hence, if one requires a function that is finite, continuous, and uniform in the space considered then one can express the second differential parameter in terms of the function φ and have it take prescribed value at the points of the boundary plane, and one will have:

$$U = \frac{z}{2\pi}\int \frac{U\,ds}{r^3} + \frac{1}{4\pi}\int\left(\frac{1}{r_1} - \frac{1}{r}\right)\varphi\,dS$$

for any point (x, y, z).

b) In particular, a harmonic function that is finite, continuous, and uniform in that space will be known from the formula:

$$U = \frac{z}{2\pi}\int \frac{U\,ds}{r^3}$$

when its values are given in all of the plane. If one observes that:

$$\frac{\partial}{\partial z}\int \frac{U\,ds}{r} = -\int U\frac{\partial(1/r)}{\partial\zeta}ds = -\int U\frac{z-\zeta}{r^3}ds = -z\int\frac{U\,ds}{r^3}$$

then one can also write:

$$U = -\frac{1}{2\pi}\frac{\partial}{\partial z}\int\frac{U\,ds}{r}. \tag{3}$$

That result will be very useful in what follows.

c) Suppose, more generally, that one is given the equation $\Delta^2 U = \varphi$, with φ harmonic. What will the value of U be at an arbitrary point (x, y, z)? If that U is harmonic then its value will be given by (3). That is why if one sets:

$$\psi = -\frac{1}{2\pi}\int\frac{U\,ds}{r} \qquad \text{then one will have} \qquad \varphi = \frac{\partial\psi}{\partial z},$$

in which ψ, which is a potential function (*) on the surface, is also harmonic. However, one can always set:

$$U = -\frac{1}{2\pi}\frac{\partial}{\partial z}\int\frac{U\,ds}{r} + U',$$

and the function U' must satisfy the conditions:

$$\Delta^2 U' = \varphi, \qquad U' = 0 \text{ (on the surface)},$$

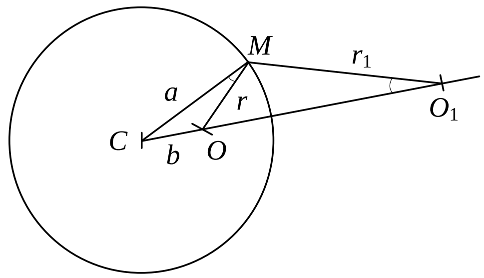
which will be verified if one takes the value $\frac{1}{2}z\psi$ for U', because ψ is always finite, and in addition, from a known formula:

$$\Delta^2(z\,\psi) = z\,\Delta^2\psi + \psi\,\Delta^2 z + 2\frac{\partial\psi}{\partial z} = 2\varphi.$$

(*) We shall discuss those functions at the end of the chapter.

Hence, by virtue of the theorem that was proved in § **1**, one will necessarily have $U = \frac{1}{2} z\psi$; i.e.:

$$U = -\frac{1}{2\pi}\frac{\partial}{\partial z}\int\frac{U\,ds}{r} - \frac{z}{4\pi}\int\frac{\varphi\,ds}{r}.$$



d) The Green function also has a simple form for the space that is contained within a sphere. The point *O* from which one measures the distance that serves to fix the values of the function on the surface is at a distance *b* from the center. If one takes the reciprocal O_1 with respect to that sphere and considers an arbitrary point *M* on the surface. Call the radius of the sphere *a*, so by construction, one will have: $CO \cdot CO_1 = a^2$. It will then follow that the triangles *CMO*, CMO_1 are similar; hence, $r_1 : r = a : b$. Consequently:

$$G = \frac{a}{br_1},$$

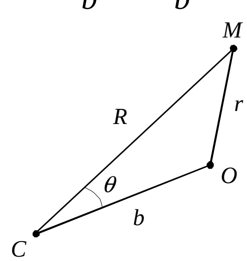
because that function, which is obviously finite inside the sphere, is also harmonic, continuous, and uniform, and it will assume the values $\frac{a}{br_1} = \frac{1}{r}$ on the surface. Having said that, the formula that serves to determine the values of any other function that satisfies the same conditions, but takes different values on the surface that are prescribed arbitrarily, will become:

$$U = \frac{1}{4\pi}\int U\left(-\frac{d(1/r)}{dR} + \frac{a}{b}\frac{d(1/r_1)}{dR}\right)ds$$

in the present case, in which *R* represents the distance to the center of the sphere, in such a way that:

$$r^2 = R^2 + b^2 - 2bR\cos\theta,$$

$$r_1^2 = R^2 + \frac{a^4}{b^2} - 2\frac{a^2}{b}R\cos\theta.$$



Now one has:

$$\frac{d(1/r)}{dR} = -\frac{R - b\cos\theta}{r^3}, \qquad \frac{d(1/r_1)}{dR} = -\frac{R - \dfrac{a^2}{b}\cos\theta}{r_1^3},$$

and on the surface (i.e., $R = a$):

$$\frac{d(1/r)}{dR} = -\frac{a - b\cos\theta}{r^3}, \qquad \frac{a}{b}\frac{d(1/r_1)}{dR} = -\frac{a - \dfrac{a^2}{b}\cos\theta}{\left(\dfrac{a}{b}\right)^2 r^3} = -\frac{(b^2/a) - b\cos\theta}{r^3},$$

so:

$$-\frac{d(1/r)}{dR} + \frac{a}{b}\frac{d(1/r_1)}{dR} = \frac{a - (b^2/a)}{r^3}.$$

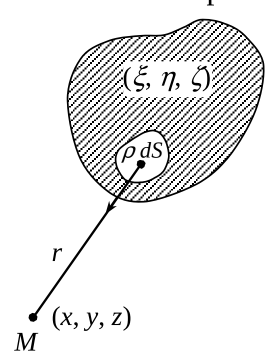
Hence:

$$U = \frac{a^2 - b^2}{4\pi a}\int \frac{U\,ds}{r^3}.$$

That is the desired formula. It will tend to coincide with the one that is obtained in the case of the plane when a and b increase to infinity while the difference $a - b$ is kept constantly equal to z. One can then deduce various interesting consequences from the formula that was found. For example, if $b = 0$ then r will always be equal to a, and one will get the value:

$$\frac{1}{4\pi a^2}\int U\,ds$$

for U. Therefore, the value that U assumes at the center of the sphere is the *arithmetic mean* of the infinitude of arbitrary values that are prescribed on the surface.



8. – A rapid outline of the subject of *potential functions* is indispensible. One imagines that a mass $\rho\,dS$ or $\rho\,ds$ has been concentrated at any particle dS of a well-defined space and at each element ds of a given surface, resp., and that it exerts a Newtonian action upon a unit mass that is concentrated at the point M – i.e., one that is proportional to the masses and inversely proportional to the square of the distance. If one takes the repulsion that is exerted by a unit mass at a unit distance to be unity then the attraction of the particle dS to the point M will be $-\rho\, dS / r^2$, and the entire body will exert an attraction on M whose component along the x-axis will be:

$$-\int \frac{x-\xi}{r} \cdot \frac{\rho dS}{r^2} = \int \frac{\partial(1/r)}{\partial x} \rho\, dS = \frac{\partial}{\partial x} \int \frac{\rho\, dS}{r},$$

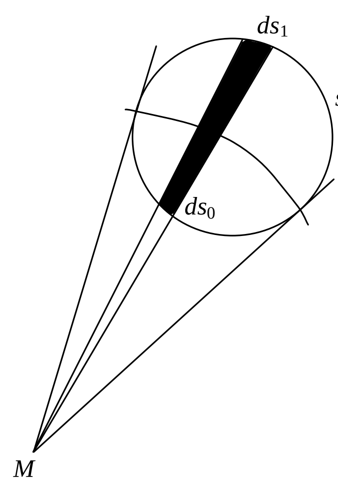
if x, y, z represent the coordinates of the point M and ξ, η, ζ represent the integration variables. Therefore, the first partial derivatives of the functions:

$$V = \int \frac{\rho\, dS}{r}, \quad V = \int \frac{\rho\, ds}{r},$$

will represent the components of the attraction that is experienced by the point M and is produced by the mass that is concentrated in S or distributed over s. One gives the names of the *spatial or surface potential functions* (*), resp., to those functions. Those functions *are harmonic*, although the former acts only in the space that is occupied by the mass. Indeed, as long as M is outside that space, one will have:

$$\Delta^2 V = \int \Delta^2 \frac{1}{r} \cdot \rho\, dS = 0,$$

since $\Delta^2(1 / r) = 0$ at any point of S; however, if M were inside of S then it would not be legitimate to say that, because $1 / r$ will become infinite under the integration sign. In that case, it will necessary to first prove an important formula that includes formula (9) in Chap. I as a limiting case.



9. – Let U be a finite, continuous, and uniform function on a well-defined space S. Locate a pole M outside of S, let r denote the radius vector, and consider the integral:

(*) See BETTI's *Teorica delle forze newtoniane* and PICARD's *Traité d'Analyse* for discussions of these and other potential functions.

$$\int \frac{dU}{dr}\frac{dS}{r^2}.$$

Circumscribe the surface s that bounds S by a cone whose vertex is at M, and which will determine two regions s_0 and s_1 on s. Now draw a cone with its vertex at M and an infinitesimal aperture $d\sigma$ that cuts out the elements ds_0 and ds_1, resp., from those two regions. The integral considered can then be written as:

$$\iint \frac{dU}{dr}\,dr\,d\sigma = \int d\sigma \int \frac{dU}{dr}dr = \int (U_1 - U_0)\,d\sigma = \int_{s_1} U\,d\sigma - \int_{s_0} U\,d\sigma.$$

Since one obviously has:

$$r_0^2\,d\sigma = ds_0 \cdot \cos(n_0, r_0), \qquad r_1^2\,d\sigma = -\,ds_1 \cdot \cos(n_1, r_1),$$

one can also give the following form to the integral:

$$-\int_{s_1} U\cos(n,r)\frac{ds}{r^2} - \int_{s_0} U\cos(n,r)\frac{ds}{r^2} = -\int_s U\cos(n,r)\frac{ds}{r^2} = \int U\frac{d(1/r)}{dn}ds.$$

Hence:

$$\int \frac{dU}{dr}\frac{dS}{r^2} = \int U\frac{d(1/r)}{dn}ds.$$

If M is inside of S (and that is the case that is of interest to us) and one represents the value of U at the point M by U_0 then one will have immediately that:

$$\iint \frac{dU}{dr}\,dr\,d\sigma = \int d\sigma \int \frac{dU}{dr}dr = \int (U - U_0)\,d\sigma = \int_s U\,d\sigma - 4\pi U_0 ;$$

i.e.:

$$\int \frac{dU}{dr}\frac{dS}{r^2} = \int U\frac{d(1/r)}{dn}ds - 4\pi U_0. \tag{4}$$

Here, one should observe that this formula can also be deduced from (2) by adopting formula (9) from Chapter I, which is also applicable when the function under the integration sign becomes infinite like $1 / r$. Therefore, it is legitimate to write:

$$\int \frac{dU}{dr}\frac{dS}{r^2} = -\sum \int \frac{\partial U}{\partial \xi}\frac{\partial(1/r)}{\partial \xi}dS = -\sum \int \frac{\partial}{\partial \xi}\left(\frac{1}{r}\frac{\partial U}{\partial \xi}\right)dS + \sum \int \frac{1}{r}\frac{\partial^2 U}{\partial \xi^2}dS$$

$$= \sum \int \frac{1}{r}\frac{\partial U}{\partial \xi}\frac{d\xi}{dn}ds + \sum \int \frac{1}{r}\frac{\partial^2 U}{\partial \xi^2}dS = \int \frac{1}{r}\frac{dU}{dn}ds + \int \frac{\Delta^2 U}{r}dS.$$

If one substitutes this in (4) then one will get (2).

10. – Now take the potential function:

$$V = \int \frac{\rho\, dS}{r},$$

which is calculated at the point (x, y, z) inside of S, and observe that one will have, in turn:

$$\frac{\partial V}{\partial x} = \int \frac{\partial(1/r)}{\partial x} \rho\, dS = -\int \frac{\partial(1/r)}{\partial \xi} \rho\, dS = -\int \frac{\partial}{\partial \xi} \frac{\rho}{r} dS + \int \frac{1}{r} \frac{\partial \rho}{\partial \xi} dS = \int \frac{\rho}{r} \frac{d\xi}{dn} dS + \int \frac{1}{r} \frac{\partial \rho}{\partial \xi} dS .$$

Therefore:

$$\frac{\partial^2 V}{\partial x^2} = \int \frac{\partial(1/r)}{\partial x} \frac{d\xi}{dn} \rho\, ds + \int \frac{\partial \rho}{\partial \xi} \frac{\partial(1/r)}{\partial x} dS = -\int \frac{\partial(1/r)}{\partial x} \frac{d\xi}{dn} \rho\, ds + \int \frac{\partial \rho}{\partial \xi} \frac{d\xi}{dr} \frac{dS}{r^2},$$

and finally, if one applies (4) then one will get *Poisson's formula:*

$$\Delta^2 V = -\int \frac{d(1/r)}{dn} \rho\, ds + \int \frac{d\rho}{dr} \frac{dS}{r^2} = -4\pi\rho .$$

It will follow from this that when one is given the differential equation $\Delta^2 U = \varphi$, one will directly have a solution by taking U to be a potential function in space and supposing that the density is equal to $-\varphi / 4\pi$. In other words, a solution of the preceding equation is:

$$V(x, y, z) = -\frac{1}{4\pi} \iiint \frac{\varphi(\xi, \eta, \zeta)\, d\xi\, d\eta\, d\zeta}{\sqrt{(x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2}} .$$

It will then follow that the integration of the aforementioned equation can always be reduced to the determination of a *harmonic* function that takes prescribed values on the surface, because if one sets $U = V + U'$ then it will be clear that U' will be a harmonic function whose values on the surface are the amounts by which the *given* values of U exceed the *known* values of V.

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CHAPTER X

SOME PROPERTIES OF ELASTIC DEFORMATIONS

1. – *The general deformation of an elastic body can always be decomposed into two simpler deformations – viz., a deformation that characterizes the absence of* ***rotation*** *and a deformation that characterizes the absence of* ***dilatation*** *at each point of the body – and such that it will always remain bounded by the same surface.*

Let u, v, w be the displacements that define the *given* deformation, and consider a function φ that satisfies the equation:

$$\Delta^2 \varphi = \Theta$$

in all of S, along with the usual properties, while its derivative with respect to the normal takes the values:

$$\frac{d\varphi}{dn} = u\frac{dx}{dn} + v\frac{dy}{dn} + w\frac{dz}{dn}$$

on the surface. Those conditions are not incompatible, since the relation:

$$\int \frac{d\varphi}{dn} ds = \sum \int u\frac{dx}{dn} ds = -\sum \int \frac{\partial u}{\partial x} dS = -\int \Theta\, dS = -\int \Delta^2\varphi\, dS$$

is satisfied. Having said that, one can always set:

$$u = \frac{\partial \varphi}{\partial x} + u', \qquad v = \frac{\partial \varphi}{\partial y} + v', \qquad w = \frac{\partial \varphi}{\partial z} + w', \tag{1}$$

and one will see immediately that one has:

$$\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} + \frac{\partial w'}{\partial z} = 0 \tag{2}$$

in all of S, while:

$$u'\frac{dx}{dn} + v'\frac{dy}{dn} + w'\frac{dz}{dn} = 0 \tag{3}$$

at any point of the surface. The relations (2) and (3) say precisely that the deformation that is defined by the displacements u', v', w' produced no dilatations at any point of the body and that the displacement of any point of the surface will take place tangentially to that surface. It will be obvious then that the other components of the deformation, which are defined by the displacements that admit a function potential φ, produce no rotation.

2. – The decomposition that was indicated in (1) is independent of the mechanical significance of *u*, *v*, *w*. They can be three arbitrary functions that are finite, continuous, and uniform. Now, one sets:

$$U=-\frac{1}{4\pi}\int\frac{u'\,dS}{r},\qquad V=-\frac{1}{4\pi}\int\frac{v'\,dS}{r},\qquad W=-\frac{1}{4\pi}\int\frac{w'\,dS}{r},$$

in such a way that from Poisson's formula, one will have:

$$\Delta^2 U = u',\qquad \Delta^2 V = v',\qquad \Delta^2 W = w'.$$

Meanwhile, if one repeats a transformation that was performed in § **10** of the preceding chapter then one will have:

$$\frac{\partial U}{\partial x}=\frac{1}{4\pi}\int u'\frac{\partial(1/r)}{\partial\xi}dS=\frac{1}{4\pi}\int\frac{\partial}{\partial\xi}\frac{u'}{r}dS-\frac{1}{4\pi}\int\frac{1}{r}\frac{\partial u'}{\partial\xi}dS\ ;$$

i.e.:

$$\frac{\partial U}{\partial x}=-\frac{1}{4\pi}\int\frac{u'}{r}\frac{d\xi}{dn}ds-\frac{1}{4\pi}\int\frac{1}{r}\frac{\partial u'}{\partial\xi}dS\,.$$

Hence, if one takes (2) and (3) into account then:

$$\frac{\partial U}{\partial x}+\frac{\partial V}{\partial y}+\frac{\partial W}{\partial z}=0.$$

It will then follow that:

$$\Delta^2 U=\frac{\partial^2 U}{\partial y^2}+\frac{\partial^2 U}{\partial z^2}-\frac{\partial^2 V}{\partial x\,\partial y}-\frac{\partial^2 W}{\partial x\,\partial z}=\frac{\partial}{\partial y}\left(\frac{\partial U}{\partial y}-\frac{\partial V}{\partial x}\right)+\frac{\partial}{\partial z}\left(\frac{\partial U}{\partial z}-\frac{\partial W}{\partial x}\right).$$

Therefore, if one sets:

$$P=\frac{\partial W}{\partial y}-\frac{\partial V}{\partial z},\qquad Q=\frac{\partial U}{\partial z}-\frac{\partial W}{\partial x},\qquad R=\frac{\partial V}{\partial x}-\frac{\partial U}{\partial y}\tag{4}$$

then one can write:

$$\left.\begin{aligned}u&=\frac{\partial\varphi}{\partial x}-\frac{\partial R}{\partial y}+\frac{\partial Q}{\partial z},\\ v&=\frac{\partial R}{\partial x}+\frac{\partial\varphi}{\partial y}-\frac{\partial P}{\partial z},\\ w&=-\frac{\partial Q}{\partial x}+\frac{\partial P}{\partial y}+\frac{\partial\varphi}{\partial z}.\end{aligned}\right\}\tag{5}$$

3. – In other words, if one is given three functions u, v, w that are endowed with the usual properties then one can find four other ones φ, P, Q, R such that one can put the former three given functions into the form (5). In addition, one knows that once one first observes that by virtue of (4), one will have:

$$\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}+\frac{\partial R}{\partial z}=0,$$

one will easily deduce from (5) that:

$$\Delta^2\varphi=\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z},$$

$$\Delta^2 P=\frac{\partial w}{\partial y}-\frac{\partial v}{\partial z},\qquad \Delta^2 Q=\frac{\partial u}{\partial z}-\frac{\partial w}{\partial x},\qquad \Delta^2 R=\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}.$$

In particular, if u, v, w are the components of a displacement of an arbitrary deformation then the functions φ, P, Q, R will be the ones that provide the values of the cubic dilatation and twice the components of the rotation of the particle by way of their second differential parameter.

4. – Now, by contrast, the decomposition (5) applies to the body forces. We would like to say that there exist four functions Φ, F, G, H such that one can write:

$$\left.\begin{aligned} X &= \frac{\partial\Phi}{\partial x}-\frac{\partial H}{\partial y}+\frac{\partial G}{\partial z},\\ Y &= \frac{\partial H}{\partial x}+\frac{\partial\Phi}{\partial y}-\frac{\partial F}{\partial z},\\ Z &= -\frac{\partial G}{\partial x}+\frac{\partial F}{\partial y}+\frac{\partial\Phi}{\partial z}. \end{aligned}\right\} \tag{6}$$

Therefore, any system of forces (X, Y, Z) can be decomposed into two other ones. The forces in the first component system admit the potential function Φ: Betti called them *forces of dilatation without rotation*. However, the forces of the second system are the *forces of rotation without dilatation* (*), which have the components:

$$\frac{\partial G}{\partial z}-\frac{\partial H}{\partial y},\qquad \frac{\partial H}{\partial x}-\frac{\partial F}{\partial z},\qquad \frac{\partial F}{\partial y}-\frac{\partial G}{\partial x}.$$

(*) BETTI justified this terminology in his *Teoria* …, pp. 29.

5. – *The problem of elastic equilibrium of an isotropic body is always reducible to the case in which the forces act only upon the surface.*

By virtue of (6), the indefinite equations:

$$X + A\frac{\partial \Theta}{\partial x} + B\left(\frac{\partial \mathcal{T}_2}{\partial z} - \frac{\partial \mathcal{T}_3}{\partial y}\right) = 0, \ \ldots$$

will become:

$$\begin{cases} \dfrac{\partial}{\partial x}(\Phi + A\Theta) - \dfrac{\partial}{\partial y}(H + B\mathcal{T}_3) + \dfrac{\partial}{\partial z}(G + B\mathcal{T}_2) = 0, \\ \dfrac{\partial}{\partial x}(H + B\mathcal{T}_3) + \dfrac{\partial}{\partial y}(\Phi + A\Theta) - \dfrac{\partial}{\partial z}(F + B\mathcal{T}_1) = 0, \\ -\dfrac{\partial}{\partial x}(G + B\mathcal{T}_2) + \dfrac{\partial}{\partial y}(F + B\mathcal{T}_1) + \dfrac{\partial}{\partial z}(\Phi + A\Theta) = 0. \end{cases}$$

They will be satisfied when one takes the functions $\Phi + A\,\Theta$, $F + B\,\mathcal{T}_1$, … to be equal to zero; i.e., when one sets:

$$\Phi + A\,\Delta^2\varphi = 0, \qquad F + B\,\Delta^2 P = 0, \qquad H + B\,\Delta^2 R = 0.$$

The theory of potential functions yields the following particular solutions:

$$\varphi = \frac{1}{4\pi A}\int \frac{\Phi\, dS}{r},$$

$$P = \frac{1}{4\pi B}\int \frac{F\, dS}{r}, \qquad Q = \frac{1}{4\pi B}\int \frac{G\, dS}{r}, \qquad R = \frac{1}{4\pi B}\int \frac{H\, dS}{r}.$$

One then gets the displacements:

$$u' = \frac{1}{4\pi A}\frac{\partial}{\partial x}\int \frac{\Phi\, dS}{r} + \frac{1}{4\pi B}\left(\frac{\partial}{\partial z}\int \frac{G\, dS}{r} - \frac{\partial}{\partial y}\int \frac{H\, dS}{r}\right), \ \ldots,$$

which must satisfy the same indefinite equations as the unknown displacements u, v, w. Set:

$$u = u' + u'', \qquad v = v' + v'', \qquad w = w' + w''.$$

The question will then reduce to the determination of the residual displacements u'', v'', w'', which will satisfy the equations:

$$(A - B)\,\frac{\partial \Theta''}{\partial x} + B\,\Delta^2 u'' = 0, \quad \ldots,$$

as one wishes to prove, and additionally assume the values $u - u'$, $v - v'$, $w - w'$ on the surface when one knows the values of u, v, w at any point of the surface. However, if the forces L, M, N are given then the displacements u'', v'', w'' must satisfy the usual boundary equations on the surface when one supposes that the external forces are $L - L'$, $M - M'$, $N - N'$.

CHAPTER XI

THE CANONICAL EQUATION OF SMALL MOTIONS

1. – Recall (Chap. VII, § **1**; Chap. IV, § **5**) the equations of motion in the absence of body forces in the form:

$$\rho\frac{\partial^2 u}{\partial t^2} = A\frac{\partial\Theta}{\partial x} + B\left(\frac{\partial\mathcal{T}_2}{\partial z} - \frac{\partial\mathcal{T}_3}{\partial y}\right), \qquad \ldots$$

From now on, set $A = \rho\, a^2$, $B = \rho\, b^2$, for brevity, and employ the following operator symbols:

$$\mathcal{D}_a = \frac{\partial^2}{\partial t^2} - a^2\,\Delta^2, \qquad \mathcal{D}_b = \frac{\partial^2}{\partial t^2} - b^2\,\Delta^2.$$

The equations of motion will become:

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} = a^2\dfrac{\partial\Theta}{\partial x} + b^2\left(\dfrac{\partial\mathcal{T}_2}{\partial z} - \dfrac{\partial\mathcal{T}_3}{\partial y}\right), \\ \dfrac{\partial^2 v}{\partial t^2} = a^2\dfrac{\partial\Theta}{\partial y} + b^2\left(\dfrac{\partial\mathcal{T}_3}{\partial x} - \dfrac{\partial\mathcal{T}_1}{\partial z}\right), \\ \dfrac{\partial^2 w}{\partial t^2} = a^2\dfrac{\partial\Theta}{\partial z} + b^2\left(\dfrac{\partial\mathcal{T}_1}{\partial y} - \dfrac{\partial\mathcal{T}_2}{\partial x}\right). \end{cases}$$

If one differentiates these equations with respect to x, y, z and sums the results then one will get: $\mathcal{D}_a\Theta = 0$. However, if one differentiates the second equation with respect to z and the third one with respect to y and subtracts then one will find that $\mathcal{D}_b\,\mathcal{T}_1 = 0$, and analogously $\mathcal{D}_b\,\mathcal{T}_2 = 0$, $\mathcal{D}_b\,\mathcal{T}_3 = 0$. Therefore:

The dilatation and components of rotation satisfy a differential equation $\mathcal{D} = 0$.

2. – Let f_0 represent the value of any function $f(x, y, z, t)$ for $t = t_0$ and agree to let $\partial f_0 / \partial t$ represent the value of $\partial f / \partial t$ for $t = t_0$. One will have, in turn:

$$\int_{t_0}^{t_1}\frac{\partial^2 f}{\partial t^2}dt = \frac{\partial f}{\partial t} - \frac{\partial f_0}{\partial t}, \qquad \int_{t_0}^{t} dt\int_{t_0}^{t}\frac{\partial^2 f}{\partial t^2}dt = f - f_0 - (t - t_0)\frac{\partial f_0}{\partial t} = f - f^0,$$

if one sets:

$$f^0 = f_0 + (t - t_0)\frac{\partial f_0}{\partial t}.$$

Now, if one integrates the equations of motion twice (*) over time between the limits t_0 and t and then sets:

$$\varphi' = a^2\int_{t_0}^{t} dt\int_{t}^{t} \Theta\, dt\,, \qquad P' = b^2\int_{t_0}^{t} dt\int_{t}^{t} \mathcal{T}_1\, dt\,, \qquad Q' = b^2\int_{t_0}^{t} dt\int_{t}^{t} \mathcal{T}_2\, dt\,, \qquad \ldots$$

then one will get:

$$\left.\begin{aligned} u - u^* &= \frac{\partial \varphi'}{\partial x} + \frac{\partial Q'}{\partial z} - \frac{\partial R'}{\partial y}, \\ v - v^* &= \frac{\partial \varphi'}{\partial y} + \frac{\partial R'}{\partial x} - \frac{\partial P'}{\partial z}, \\ z - z^* &= \frac{\partial \varphi'}{\partial z} + \frac{\partial P'}{\partial y} - \frac{\partial Q'}{\partial x}. \end{aligned}\right\} \qquad (1)$$

Having said that, imagine that one applies the theorem that was proved in Chap. X (§§ **2**, **3**) to the triples of functions u_0, v_0, w_0 and $\frac{\partial u_0}{\partial t}$, $\frac{\partial v_0}{\partial t}$, $\frac{\partial w_0}{\partial t}$. It is clear that one will get four functions φ'', P'', Q'', R'' that are *linear with respect to time*, and are such that one can write:

$$\left.\begin{aligned} u^* &= \frac{\partial \varphi''}{\partial x} + \frac{\partial Q''}{\partial z} - \frac{\partial R''}{\partial y}, \\ v^* &= \frac{\partial \varphi''}{\partial y} + \frac{\partial R''}{\partial x} - \frac{\partial P''}{\partial z}, \\ z^* &= \frac{\partial \varphi''}{\partial z} + \frac{\partial P''}{\partial y} - \frac{\partial Q''}{\partial x}, \end{aligned}\right\}$$

in which:

$$\Delta^2\varphi'' = \Theta, \qquad \Delta^2 P'' = \mathcal{T}_1^*, \qquad \Delta^2 Q'' = \mathcal{T}_2^*, \qquad \Delta^2 R'' = \mathcal{T}_3^*.$$

Now, if one sets:

$$\varphi = \varphi' + \varphi'', \quad P = P' + P'', \quad Q = Q' + Q'', \quad R = R' + R''$$

then (1) will become:

(*) This procedure was pointed out by SOMIGLIANA (Rend. dell'Acc. dei Lincei, in various places.)

$$\left.\begin{aligned} u &= \frac{\partial\varphi}{\partial x} + \frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}, \\ v &= \frac{\partial\varphi}{\partial y} + \frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}, \\ z &= \frac{\partial\varphi}{\partial z} + \frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}, \end{aligned}\right\} \tag{2}$$

and it is easy to show that the functions φ, P, Q, R satisfy all of the differential equations of the form $\mathcal{D} = 0$. Indeed, if one takes into account the observations of the preceding paragraphs then one will have:

$$\mathcal{D}_a\,\varphi = \frac{\partial^2\varphi'}{\partial t^2} - a^2\,\Delta^2\,\varphi = a^2\,\Theta - a^4\int_{t_0}^{t} dt\int_{t_0}^{t}\Delta^2\Theta\,dt$$

$$= a^2\,\Theta^* + a^2\int_{t_0}^{t} dt\int_{t_0}^{t}\mathcal{D}_a\Theta\,dt = a^2\,\Theta^*,$$

and analogously:

$$\mathcal{D}_b\,P' = b^2\,\mathcal{T}_1^*, \qquad \mathcal{D}_b\,Q' = b^2\,\mathcal{T}_2^*, \qquad \mathcal{D}_b\,R' = b^2\,\mathcal{T}_3^*.$$

Hence:

$$\mathcal{D}_a\,\varphi'' = \frac{\partial^2\varphi''}{\partial t^2} - a^2\,\Delta^2\,\varphi'' = -\,a^2\,\Delta^2\,\varphi'' = -\,a^2\,\Theta^*,$$

and

$$\mathcal{D}_b\,P'' = -b^2\,\mathcal{T}_1^*, \qquad \mathcal{D}_b\,Q'' = -b^2\,\mathcal{T}_2^*, \qquad \mathcal{D}_b\,R'' = -b^2\,\mathcal{T}_3^*.$$

Finally:

$$\mathcal{D}_a\,\varphi = 0, \qquad \mathcal{D}_b\,P = 0, \qquad \mathcal{D}_b\,Q = 0, \qquad \mathcal{D}_b\,R = 0. \tag{3}$$

Hence, one can always give the form (2) to the vibrations of an isotropic elastic body as long as one supposes that the functions φ, P, Q, R are subject to the conditions (3). That is an important *Theorem of Clebsch* (*).

3. – That theorem shows that if one ignores the body forces then any vibratory motion of an elastic body *will decompose into two particular motions*, one of which characterizes the variation of the volume of the particle and depends upon only the constant A and the other of which (the *vorticial* motion), by contrast, characterizes the rotation of the particle and depends upon only the constant B. The first one has no rotation, and the vibrations will admit a potential function φ whose second differential parameters will provide the

(*) See P. DUHEM's *Cours de physique mathématiques*, t. II, pp. 267.

values of the unit cubic dilatation at any time and any point. For the second motion, which has no dilatation, the vibrations have the form:

$$\frac{\partial Q}{\partial z}-\frac{\partial R}{\partial y}, \qquad \frac{\partial R}{\partial x}-\frac{\partial P}{\partial z}, \qquad \frac{\partial P}{\partial y}-\frac{\partial Q}{\partial x},$$

and $\Delta^2 P$, $\Delta^2 Q$, $\Delta^2 R$ are precisely two times the components of rotation. For the first motion, the vibrations satisfy the differential equation $\mathcal{D}_a = 0$, whereas they will satisfy $\mathcal{D}_b = 0$ for the second one. Therefore:

The integration of the equations of elastic motion ***for isotropic bodies*** *will always reduce to the integration of a single equation* $\mathcal{D} = 0$,

which one calls *the canonical equation of small motions.*

4. – Therefore, suppose that one must integrate (*):

$$\frac{\partial^2 V}{\partial t^2} - a^2\,\Delta^2 V = 0. \tag{4}$$

Draw a sphere of radius $r = at$ with the point (x, y, z) for its center, and consider the integral:

$$V = \frac{1}{r}\int F(\xi, \eta, \zeta)\,ds\,, \tag{5}$$

which is extended over the surface of that sphere. We would like to prove that this expression for V is an integral of equation (4). We first calculate $\Delta^2 V$. Varying only x signifies that one is rigidly-displacing the sphere by dx along the x-axis. F will then vary by $\frac{\partial F}{\partial x}dx$, and consequently V will vary by $\frac{1}{r}\int \frac{\partial F}{\partial \xi}dx\,ds$. Therefore:

$$\frac{\partial V}{\partial x} = \frac{1}{r}\int \frac{\partial F}{\partial \xi}\,ds\,.$$

Analogously:

$$\frac{\partial^2 V}{\partial x^2} = \frac{1}{r}\int \frac{\partial^2 F}{\partial \xi^2}\,ds\,.$$

Hence:

$$\Delta^2 V = \frac{1}{r}\int \Delta^2 F \cdot ds\,.$$

(*) See H. POINCARÉ's *Théorie mathématique de la lumière*, pp. 88.

Now, calculate $\frac{\partial^2 V}{\partial t^2}$; i.e., $a^2 \frac{\partial^2 V}{\partial r^2}$ (since $r = at$). Let $d\sigma$ denote the aperture of the infinitesimal cone with its vertex at the center of sphere from which the element ds gets projected, so one has $V = r\int F\, d\sigma$; hence:

$$\frac{\partial^2 V}{\partial r^2} = 2\frac{\partial}{\partial r}\int F\, d\sigma + r\frac{\partial^2}{\partial r^2}\int F\, d\sigma = \frac{1}{r}\frac{\partial}{\partial r}\left(r^2\int \frac{\partial F}{\partial r} d\sigma\right).$$

Meanwhile, it is known from Green's theorem that:

$$\int \Delta^2 F\, dS = -\int \frac{\partial F}{\partial n} ds = r^2 \int \frac{\partial F}{\partial r} d\sigma,$$

in which the first integration extends over all of the space that is contained by the sphere s. Therefore:

$$\frac{\partial^2 V}{\partial r^2} = \frac{1}{r}\frac{\partial}{\partial r}\int \Delta^2 F \cdot dS\ .$$

Now, it is clear that when one varies only r, the center of the sphere will not displace, but one will have only a dilatation of the sphere about the center, and its radius will become $r + dr$. The variation that $\int \Delta^2 F\, dS$ experiences will be represented by that same integral when it is extended over the space that is found between the spherical surfaces of radii r and $r + dr$. One will then have:

$$\iint \Delta^2 F\, ds\, dr = dr \int \Delta^2 F\, ds\,.$$

It will then follow that:

$$\frac{\partial^2 V}{\partial r^2} = \frac{1}{r}\int \Delta^2 F \cdot ds\,.$$

Equation (4) is therefore satisfied.

5. – The fact that formula (5) does not provide the general integral of (4) will result immediately from the fact that it includes *only one* arbitrary function, rather than two. However, one will find another particular integral immediately by observing that if V satisfies (4) then one can say the same thing about $\partial V / \partial r$. The general integral will then be:

$$V = \frac{1}{r}\int F\, ds + \frac{\partial}{\partial r}\left(\frac{1}{r}\int G\, ds\right).$$

The arbitrary functions F and G can be determined by means of the initial conditions; i.e., by prescribing the values V_0 and $\partial V_0 / \partial t$ that V and $\partial V / \partial t$ assume for $t = 0$. Let:

$$V_0 = \Phi\,(x, y, z), \qquad \frac{\partial V_0}{\partial t} = \Psi\,(x, y, z).$$

In order to see which initial conditions correspond to the first particular that was found, observe that one will have:

$$V = r\int F\,d\sigma\,, \quad \frac{\partial V}{\partial r} = \int F\,d\sigma + \frac{1}{r}\int \Delta^2 F \cdot dS\,, \quad \frac{\partial^2 V}{\partial r^2} = r\int \Delta^2 F \cdot d\sigma$$

for that integral. Then observe that for $t = 0$ (and consequently, for $r = 0$), the integral:

$$\frac{1}{r}\int \Delta^2 F \cdot dS \ = \mu\int \frac{dS}{r} = 2\pi\,\mu\, r^2$$

will have zero for its limit, while $\int F\,d\sigma$ will tend to $F\,(x, y, z)\int d\sigma$, so one will have:

$$V = 0, \quad \frac{\partial V}{\partial r} = 4\pi F, \qquad \frac{\partial^2 V}{\partial r^2} = 0.$$

However, it follows directly that one will have $V = 4\pi\, G$, $\partial V / \partial r = 0$ for the other particular integral (viz., $\partial V / \partial r$). Therefore, for the general integral:

$$V_0 = 4\pi\, G\,(x, y, z), \qquad \frac{\partial V}{\partial r} = 4\pi\, a\, F\,(x, y, z),$$

and one will then need to take $F = \Psi / 4\pi\, a$, $G = \Phi / 4\pi$. The general integral, which can be given the form:

$$V = \frac{1}{r}\int F\,ds + \frac{1}{r^2}\int G\,ds + \frac{1}{r}\int \frac{\partial G}{\partial r}\,ds\,,$$

will finally become:

$$V = \frac{1}{4\pi}\int\left(\frac{\Psi}{a} + \frac{\Phi}{r} + \frac{\partial \Phi}{\partial r}\right)\frac{ds}{r}\,. \tag{6}$$

That result is due to Poisson (*).

(*) DUHEM, *loc. cit.*, t. I, pp. 167. KIRCHHOFF did some important research into the differential equation (4) in the Sitzungsberichte of the Berlin Scientific Society (1882). See also the paper by G. A. MAGGI "Sulle propagazione libera e perturbata della onde luminose in una mezzo isotropo," Ann. di Mat. (2) **16**, (1888-89), 21-48 and one by prof. BELTRAMI "Sul principio di Huygens," Rend. dell'Ist. Lombardo (2) **22** (1889), 428-438.

6. – Suppose that V represents a *vibration*, and consequently that $\partial V / \partial t$ represents a *velocity*. If only the points that are contained in an infinitesimal particle that surrounds the point (x_0, y_0, z_0) vibrate at the origin of time then one would like to say that the functions Φ and Ψ have the value 0 at any point in space, except for the points whose coordinates differ from x_0, y_0, z_0 only infinitely-little. Under that hypothesis, formula (6) will generally yield a zero value for V, except at the points (x, y, z) for which the differences $\xi - x_0$, $\eta - y_0$, $\zeta - z_0$ are simultaneously infinitesimal. Since:

$$(x - \xi)^2 + (y - \eta)^2 + (z - \zeta)^2$$

is the square of $r = at$, those points will belong to the sphere:

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = a^2\, t^2$$

or be infinitely close to it. In other words, if a perturbation that satisfies the canonical equation of small motions emanates from a point then at the end of the time interval t one will find that it has been communicated to only those points that are at a distance of at from the starting point, and therefore one can say that a is the *propagation velocity* of the perturbation considered. If one now refers to what was said in § **3** then one will see that:

Any elastic perturbation that is produced at a point in an isotropic medium will split into two particular perturbations, one of which has no rotation and propagates with a velocity of $a = \sqrt{A/\rho}$ *, while the other one, which is purely vorticial, will propagate with a velocity of* $b = \sqrt{B/\rho}$.

CHAPTER XII

CALCULATING THE DILATATION AND ROTATION

1. – The most noteworthy application of Betti's theorem (Chap. V) consists of the *determination of the coefficient of cubic dilatation and twice the components of rotation of the particle* at any point of a homogeneous, isotropic, elastic medium. Take:

$$u' = \frac{1}{r}, \qquad v' = 0, \qquad w' = 0, \tag{1}$$

in which r represents the distance from the point (ξ, η, ζ), which experiences the displacement u', v', w', to a fixed point O with coordinates x, y, z that is taken arbitrarily inside of the body. Since the function u' becomes infinite at that point, Betti's theorem will not be applicable in the space S. However, if one draws an arbitrarily-small sphere with its center at O and then excludes the space S_0 that is inside that sphere from S then Betti's theorem will be applicable in the remaining space $S - S_0$, because u' will be finite, continuous, and uniform there. Since the space $S - S_0$ is bounded by the surfaces s and s_0, one will have:

$$\begin{aligned}\int_{S-S_0} (Xu' + Yv' + Zw')\,dS + \int_{s+s_0} (Lu' + Mv' + Nw')\,ds \\ = \int_{S-S_0} (X'u + Y'v + Z'w)\,dS + \int_{s+s_0} (L'u + M'v + N'w)\,ds,\end{aligned} \tag{2}$$

in which X', Y', Z', L', M', N' are calculated by means of the equilibrium equations. To that end, first note that under the hypothesis (1), one will have:

$$\Theta' = \frac{\partial(1/r)}{\partial\xi}, \qquad \mathcal{T}_1' = 0, \qquad \mathcal{T}_2' = \frac{\partial(1/r)}{\partial\zeta}, \qquad \mathcal{T}_3' = -\frac{\partial(1/r)}{\partial\eta},$$

and it will then result from the indefinite equations that when one observes that one will have $\Delta^2 (1 / r) = 0$ at any point, except for O, one will have:

$$X' = -(A - B)\,\frac{\partial^2(1/r)}{\partial\xi^2}, \qquad Y' = -(A - B)\,\frac{\partial^2(1/r)}{\partial\xi\,\partial\eta}, \qquad Z' = -(A - B)\,\frac{\partial^2(1/r)}{\partial\xi\,\partial\zeta},$$

while the boundary equations will give:

$$\begin{cases} L' = -(A-B)\dfrac{\partial(1/r)}{\partial\xi}\dfrac{d\xi}{dn} - B\dfrac{\partial(1/r)}{\partial\xi}\dfrac{d\xi}{dn} - B\dfrac{d(1/r)}{dn}, \\ M' = -(A-2B)\dfrac{\partial(1/r)}{\partial\xi}\dfrac{d\eta}{dn} - B\dfrac{\partial(1/r)}{\partial\eta}\dfrac{d\xi}{dn}, \\ N' = -(A-2B)\dfrac{\partial(1/r)}{\partial\xi}\dfrac{d\zeta}{dn} - B\dfrac{\partial(1/r)}{\partial\zeta}\dfrac{d\xi}{dn}. \end{cases}$$

2. – Having said that, *in the space* $S - S_0$, upon integrating by parts, one will have:

$$\int (X'u + Y'v + Z'w)\,dS = -(A-B)\int\left(u\frac{\partial^2(1/r)}{\partial\xi^2} + v\frac{\partial^2(1/r)}{\partial\xi\,\partial\eta} + w\frac{\partial^2(1/r)}{\partial\xi\,\partial\zeta}\right)dS$$

$$= -(A-B)\int\left[\frac{\partial}{\partial\xi}\left(u\frac{\partial(1/r)}{\partial\xi}\right) + \frac{\partial}{\partial\eta}\left(v\frac{\partial(1/r)}{\partial\xi}\right) + \frac{\partial}{\partial\zeta}\left(w\frac{\partial(1/r)}{\partial\xi}\right)\right]dS + (A-B)\int\Theta\frac{\partial(1/r)}{\partial\xi}\,dS\,;$$

hence, if one transforms the first integral into a surface integral then:

$$\int (X'u + Y'v + Z'w)\,dS = (A-B)\int\left(u\frac{d\xi}{dn} + v\frac{d\eta}{dn} + w\frac{d\zeta}{dn}\right)\frac{\partial(1/r)}{\partial\xi}\,ds + (A-B)\int\Theta\frac{\partial(1/r)}{\partial\xi}\,dS\,. \tag{3}$$

Similarly, on the surface s:

$$\begin{aligned} \int (L'u + M'v + N'w)\,dS &= -(A-2B)\int\left(u\frac{d\xi}{dn} + v\frac{d\eta}{dn} + w\frac{d\zeta}{dn}\right)\frac{\partial(1/r)}{\partial\xi}\,ds \\ &- B\int u\frac{d(1/r)}{dn}dS - B\int\left(u\frac{\partial(1/r)}{\partial\xi} + v\frac{\partial(1/r)}{\partial\eta} + w\frac{\partial(1/r)}{\partial\zeta}\right)\frac{d\xi}{dn}\,ds\,. \end{aligned} \tag{4}$$

If one observes that:

$$\frac{d(1/r)}{dn} = -\frac{1}{r^2}, \qquad \frac{\partial(1/r)}{\partial\eta} = -\frac{1}{r^2}\frac{\partial r}{\partial\eta}\frac{d\xi}{dn} = -\frac{1}{r^2}\frac{d\eta}{dn}\frac{\partial r}{\partial\xi} = \frac{dy}{dr}\frac{\partial(1/r)}{\partial\xi}, \quad \ldots$$

then the previous formula will reduce to:

$$\int (L'u + M'v + N'w)\,dS = -(A-B)\int\left(u\frac{d\xi}{dn} + v\frac{d\eta}{dn} + w\frac{d\zeta}{dn}\right)\frac{\partial(1/r)}{\partial\xi}\,ds + B\int\frac{u\,ds_0}{r^2} \tag{5}$$

on the surface s_0 . If one sums (3) with (4) and (5) then one will get:

$$\int_{S-S_0} (X'u + Y'v + Z'w)\,dS + \int_{s+s_0} (L'u + M'v + N'w)\,ds$$

$$=$$

$$-(A-B)\int\left(u\frac{d\xi}{dn}+v\frac{d\eta}{dn}+w\frac{d\zeta}{dn}\right)\frac{\partial(1/r)}{\partial\xi}ds + (A-B)\int\left(u\frac{d\xi}{dn}+v\frac{d\eta}{dn}+w\frac{d\zeta}{dn}\right)\frac{\partial(1/r)}{\partial\xi}ds_0$$

$$-(A-2B)\int\left(u\frac{d\xi}{dn}+v\frac{d\eta}{dn}+w\frac{d\zeta}{dn}\right)\frac{\partial(1/r)}{\partial\xi}ds - (A-B)\int\left(u\frac{d\xi}{dn}+v\frac{d\eta}{dn}+w\frac{d\zeta}{dn}\right)\frac{\partial(1/r)}{\partial\xi}ds_0$$

$$-\;B\int\left(u\frac{\partial(1/r)}{\partial\xi}+v\frac{\partial(1/r)}{\partial\eta}+w\frac{\partial(1/r)}{\partial\zeta}\right)\frac{d\xi}{dn}ds + (A-B)\int_{S-S_0}\Theta\frac{\partial(1/r)}{\partial\xi}dS$$

$$-\;B\int u\frac{d(1/r)}{dn}ds + B\int\frac{u\,ds_0}{r^2}\,.$$

When one makes the reductions, one will see that the relation (2) will become:

$$\int_{S-S_0}\frac{X\,dS}{r}+\int_{s+s_0}\frac{L\,ds}{r}$$

$$= B\int\left(u\frac{d\xi}{dn}+v\frac{d\eta}{dn}+w\frac{d\zeta}{dn}\right)\frac{\partial(1/r)}{\partial\xi}ds + (A-B)\int_{S-S_0}\Theta\frac{\partial(1/r)}{\partial\xi}dS$$

$$-\;B\int\left(u\frac{\partial(1/r)}{\partial\xi}+v\frac{\partial(1/r)}{\partial\eta}+w\frac{\partial(1/r)}{\partial\zeta}\right)\frac{d\xi}{dn}ds -\;B\int u\frac{d(1/r)}{dn}ds + B\int\frac{u\,ds_0}{r^2}\,. \tag{6}$$

3. – Now make the space S_0 go to a point. Let ds be the aperture of the infinitesimal cone with its vertex at O from which one projects the contour of a surface particle ds_0 . Obviously, $ds_0 = r^2\,d\sigma$, and therefore if one lets μ denote a convenient value that the finite function u assumes on the surface s_0 then one will have:

$$\int\frac{u\,ds_0}{r^2} = \int u\,d\sigma = \mu\int d\sigma = 4\pi\mu\,.$$

When the sphere tends to disappear, μ will tend to the value that u takes at the center O, because one also supposes that u is a continuous and uniform function. Therefore, for vanishing s_0 :

$$\lim\int\frac{u\,ds_0}{r^2} = 4\pi\,u,$$

in which u represents simply the value of u at the point (x, y, z). One further observes that since $dS_0 = r^2\, dr\, d\sigma$, if R is the radius of S_0 then one will have:

$$\int \frac{dS_0}{r} = \iint r\, dr\, d\sigma = 2\pi R^2, \qquad \int \frac{dS_0}{r^2} = \iint dr\, d\sigma = 4\pi R, \qquad \int \frac{ds_0}{r} = \int r\, d\sigma = 4\pi R,$$

and therefore those integrals will tend to zero along with R, and any other integral will tend to zero that is obtained by multiplying the quantity under the integral by functions that remain *finite* in all of the integration domain, which are X, L, Θ, …, by hypothesis. Consequently:

$$\lim \int \frac{X\, dS_0}{r} = 0, \qquad \lim \int \frac{L\, ds_0}{r} = 0,$$

$$\lim \int \Theta \frac{\partial(1/r)}{\partial \xi} dS_0 = -\lim \int \Theta \frac{d\xi}{dr} \frac{dS_0}{r^2} = 0.$$

Therefore, formula (6) will finally become:

$$4\pi B u = \int \frac{X\, dS}{r} + \int \frac{L\, ds}{r}$$

$$+ B \int \left(u \frac{\partial(1/r)}{\partial \xi} + v \frac{\partial(1/r)}{\partial \eta} + w \frac{\partial(1/r)}{\partial \zeta} \right) \frac{d\xi}{dn} ds + B \int u \frac{d(1/r)}{dn} ds$$

$$- B \int \left(u \frac{d\xi}{dn} + v \frac{d\eta}{dn} + w \frac{d\zeta}{dn} \right) \frac{\partial(1/r)}{\partial \xi} ds - (A - B) \int_{S - S_0} \Theta \frac{\partial(1/r)}{\partial \xi} dS, \tag{7}$$

and one can deduce two more analogous formulas by cyclic permutation.

4. – It is useful to give a more concise form to those relations. Set:

$$\Phi = (A - B) \int \frac{\Theta\, dS}{r} + B \int \left(u \frac{d\xi}{dn} + v \frac{d\eta}{dn} + w \frac{d\zeta}{dn} \right) \frac{ds}{r}. \tag{8}$$

Since the function r is the only one under the integral signs that contains x, y, z, one can write:

$$\frac{\partial \Phi}{\partial x} = (A - B) \int \Theta \frac{\partial(1/r)}{\partial x} dS + B \int \left(u \frac{d\xi}{dn} + v \frac{d\eta}{dn} + w \frac{d\zeta}{dn} \right) \frac{\partial(1/r)}{\partial x} ds$$

$$= -(A - B) \int \Theta \frac{\partial(1/r)}{\partial x} dS - B \int \left(u \frac{d\xi}{dn} + v \frac{d\eta}{dn} + w \frac{d\zeta}{dn} \right) \frac{\partial(1/r)}{\partial x} ds.$$

If one substitutes that result in (7) and sets:

$$(9)\begin{cases} U = \int \frac{X\,dR}{r} + \int \frac{L\,ds}{r} + B\int\left(u\frac{\partial(1/r)}{\partial\xi} + v\frac{\partial(1/r)}{\partial\eta} + w\frac{\partial(1/r)}{\partial\zeta}\right)\frac{d\xi}{dn}ds + B\int u\frac{d(1/r)}{dn}ds, \\ V = \int \frac{Y\,dR}{r} + \int \frac{M\,ds}{r} + B\int\left(u\frac{\partial(1/r)}{\partial\xi} + v\frac{\partial(1/r)}{\partial\eta} + w\frac{\partial(1/r)}{\partial\zeta}\right)\frac{d\eta}{dn}ds + B\int v\frac{d(1/r)}{dn}ds, \\ W = \int \frac{Z\,dR}{r} + \int \frac{N\,ds}{r} + B\int\left(u\frac{\partial(1/r)}{\partial\xi} + v\frac{\partial(1/r)}{\partial\eta} + w\frac{\partial(1/r)}{\partial\zeta}\right)\frac{d\zeta}{dn}ds + B\int w\frac{d(1/r)}{dn}ds, \end{cases}$$

for brevity, then one will finally arrive at the formulas:

$$4\pi B\,u = U + \frac{\partial\Phi}{\partial x}, \qquad 4\pi B\,v = V + \frac{\partial\Phi}{\partial y}, \qquad 4\pi B\,w = W + \frac{\partial\Phi}{\partial z}, \tag{10}$$

which one would like to prove.

5. Calculating Θ, $\mathcal{T}_1$, $\mathcal{T}_2$, $\mathcal{T}_3$. – One deduces immediately from (10) by means of suitable differentiations that:

$$4\pi B\,\mathcal{T}_1 = \frac{\partial W}{\partial y} - \frac{\partial V}{\partial z}, \qquad 4\pi B\,\mathcal{T}_2 = \frac{\partial U}{\partial z} - \frac{\partial W}{\partial x}, \qquad 4\pi B\,\mathcal{T}_3 = \frac{\partial V}{\partial x} - \frac{\partial U}{\partial y}. \tag{11}$$

However, if one differentiates (10) with respect to x, y, z, respectively, and then adds the results together then one will get:

$$4\pi B\,\Theta = \frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} + \frac{\partial W}{\partial z} + \Delta^2\,\Phi\,.$$

However, (8) shows that Φ is the sum of two potential functions, one of which is a function on space and the other of which is a function on the surface, and from the properties of those functions (*), one can assert that:

$$4\pi A\,\Theta = \frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} + \frac{\partial W}{\partial z}\,. \tag{12}$$

Formulas (11) and (12), which are due to Betti (**), will yield the values of Θ, $\mathcal{T}_1$, $\mathcal{T}_2$, $\mathcal{T}_3$ at any point of space when one knows the external forces and the displacements on the

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(*) See BETTI, *Teoria delle forze newtoniane*, pp. 32.
(**) *Teoria della elasticità*, §§ **8**, **9**. See also CERRUTI, Memorie dell'Accademia dei Lincei, v. XIII, pp. 83.

surface. It will therefore contain too many elements, since it must be possible to determine Θ and the $\mathcal{T}$ when one is given only the pressures or only the displacements on the surface, in addition to the body force. However, as a result, one will see how one can proceed to eliminate either the displacements on the surface or the pressures.

6. Calculating *u*, *v*, *w*. – Formula (10) cannot serve, as they are, to calculate *u*, *v*, *w*, because the right-hand sides contain the function Θ in Φ, and the calculation of that function requires that one must know *u*, *v*, *w* in all of *S*, and according to (9), only the functions *U*, *V*, *W* depend exclusively upon the external forces and surface displacements. Nevertheless, one can produce them by substituting the expression for Θ that is provided by (12) in (8). However, one will obtain quintuple and sextuple integrals in that way, which one would prefer to avoid. Rather, one seeks to express *u*, *v*, *w* in a different way that starts from Φ, which contains Θ. To that end, set:

$$u' = \frac{\partial r}{\partial \xi}, \qquad v' = \frac{\partial r}{\partial \eta}, \qquad w' = \frac{\partial r}{\partial \zeta}$$

in Betti's theorem. Observe that:

$$\Theta' = \Delta^2 r = \frac{2}{r}, \qquad \mathcal{T}_1' = \mathcal{T}_2' = \mathcal{T}_3' = 0,$$

so the indefinite equations will give:

$$X' = -2B\frac{\partial(1/r)}{\partial \xi}, \qquad Y' = -2B\frac{\partial(1/r)}{\partial \eta}, \qquad Z' = -2B\frac{\partial(1/r)}{\partial \zeta},$$

and the boundary equations will give:

$$\begin{cases} L' = -2(A-2B)\dfrac{1}{r}\dfrac{d\xi}{dn} - 2B\dfrac{d}{dn}\dfrac{\partial r}{\partial \xi}, \\ M' = -2(A-2B)\dfrac{1}{r}\dfrac{d\eta}{dn} - 2B\dfrac{d}{dn}\dfrac{\partial r}{\partial \eta}, \\ N' = -2(A-2B)\dfrac{1}{r}\dfrac{d\zeta}{dn} - 2B\dfrac{d}{dn}\dfrac{\partial r}{\partial \zeta}. \end{cases}$$

Therefore, if one adopts the usual transformations and recalls that formula (9) of Chap. I is also valid when the function under the integration becomes infinite like $1 / r^n$ at that point, as long as $n < 2$, then:

$$\int (X'u + Y'v + Z'w)\,dS = -2B\int\left(u\frac{\partial(1/r)}{\partial \xi} + v\frac{\partial(1/r)}{\partial \eta} + w\frac{\partial(1/r)}{\partial \zeta}\right)dS$$

$$= -\,2B\int\left(\frac{\partial}{\partial\xi}\frac{u}{r}+\frac{\partial}{\partial\eta}\frac{v}{r}+\frac{\partial}{\partial\zeta}\frac{w}{r}\right)dS+2B\int\left(\frac{\partial u}{\partial\xi}+\frac{\partial v}{\partial\eta}+\frac{\partial w}{\partial\zeta}\right)\frac{dS}{r}$$

$$= 2B\int\left(u\frac{d\xi}{dn}+v\frac{d\eta}{dn}+w\frac{d\xi}{dn}\right)\frac{ds}{r}+2B\int\frac{\Theta\,dS}{r}\,.$$

Similarly:

$$\int(L'u+M'v+N'w)\,ds$$

$$= -\,2\,(A-2B)\int\left(u\frac{d\xi}{dn}+v\frac{d\eta}{dn}+w\frac{d\xi}{dn}\right)\frac{ds}{r}-2B\int\left(u\frac{d}{dn}\frac{\partial r}{\partial\xi}+v\frac{d}{dn}\frac{\partial r}{\partial\eta}+w\frac{d}{dn}\frac{\partial r}{\partial\zeta}\right)ds\,.$$

Therefore, the right-hand side of formula (1) of the preceding chapter will become:

$$2A\int\frac{\Theta\,dS}{r}+4B\int\left(u\frac{d\xi}{dn}+v\frac{d\eta}{dn}+w\frac{d\xi}{dn}\right)\frac{ds}{r}-2B\int\left(u\frac{d}{dn}\frac{\partial r}{\partial\xi}+v\frac{d}{dn}\frac{\partial r}{\partial\eta}+w\frac{d}{dn}\frac{\partial r}{\partial\zeta}\right)ds\,.$$

Consequently:

$$\begin{aligned}\int\frac{\Theta\,dS}{r} &= \frac{1}{2A}\int\left(X\frac{\partial r}{\partial\xi}+Y\frac{\partial r}{\partial\eta}+Z\frac{\partial r}{\partial\zeta}\right)dS\\ &+\frac{1}{2A}\int\left(L\frac{\partial r}{\partial\xi}+L\frac{\partial r}{\partial\eta}+M\frac{\partial r}{\partial\zeta}\right)ds\\ &-2\frac{B}{A}\int\left(u\frac{d\xi}{dn}+v\frac{d\eta}{dn}+w\frac{d\xi}{dn}\right)\frac{ds}{r}\\ &+\frac{B}{A}\int\left(u\frac{d}{dn}\frac{\partial r}{\partial\xi}+v\frac{d}{dn}\frac{\partial r}{\partial\eta}+w\frac{d}{dn}\frac{\partial r}{\partial\zeta}\right)ds\,.\end{aligned}$$

When one finally substitutes that result into (8), one will get:

$$\begin{aligned}\Phi &= \frac{A-B}{2A}\int\left(X\frac{\partial r}{\partial\xi}+Y\frac{\partial r}{\partial\eta}+Z\frac{\partial r}{\partial\zeta}\right)dS\\ &+\frac{A-B}{2A}\int\left(L\frac{\partial r}{\partial\xi}+L\frac{\partial r}{\partial\eta}+M\frac{\partial r}{\partial\zeta}\right)ds\\ &-\frac{B}{A}(A-2B)\int\left(u\frac{d\xi}{dn}+v\frac{d\eta}{dn}+w\frac{d\xi}{dn}\right)\frac{ds}{r}\end{aligned}$$

$$+\frac{B}{A}(A-B)\int\left(u\frac{d}{dn}\frac{\partial r}{\partial\xi}+v\frac{d}{dn}\frac{\partial r}{\partial\eta}+w\frac{d}{dn}\frac{\partial r}{\partial\zeta}\right)ds\,. \tag{13}$$

One now sees that when one expresses U, V, W, and Φ by means of (9) and (13), formulas (10) will represent the displacements at any point of the body by means of external forces and the values that those displacements take on the surface. The effective substitution will then lead to the *Somigliana formulas* (*), which solve the same problem for u, v, w by means of double and triple integrations that the *Betti formulas* solve for Θ, $\mathcal{T}_1$, $\mathcal{T}_2$, $\mathcal{T}_3$.

(*) Annali di Matematica, 1889, pp. 41.

CHAPTER XIII

INTEGRATING THE EQUATIONS OF EQUILIBRIUM FOR ISOTROPIC ELASTIC BODIES

1. – When the *displacements are given* on the surface, the problem of elastic equilibrium consists of the search for three finite, continuous, and uniform functions u, v, w that satisfy the equations:

$$\left.\begin{aligned} X+(A-B)\frac{\partial\Theta}{\partial x}+B\Delta^2 u=0, \\ Y+(A-B)\frac{\partial\Theta}{\partial y}+B\Delta^2 v=0 \\ Z+(A-B)\frac{\partial\Theta}{\partial z}+B\Delta^2 w=0 \end{aligned}\right\} \tag{1}$$

in all of a given space and assume prescribed values on its surface. From a formula by Betti that was proved before, one has:

$$\begin{aligned} -4\pi A\,\Theta = & \int\left(X\,\frac{\partial(1/r)}{\partial\xi}+Y\,\frac{\partial(1/r)}{\partial\eta}+Z\,\frac{\partial(1/r)}{\partial\zeta}\right)dS \\ & +\int\left(L\frac{\partial(1/r)}{\partial\xi}+M\,\frac{\partial(1/r)}{\partial\eta}+N\,\frac{\partial(1/r)}{\partial\zeta}\right)ds \\ & +2B\int\left(u\frac{d}{dn}\frac{\partial(1/r)}{\partial\xi}+v\frac{d}{dn}\frac{\partial(1/r)}{\partial\eta}+w\frac{d}{dn}\frac{\partial(1/r)}{\partial\zeta}\right)ds\,. \end{aligned} \tag{2}$$

The second integral is the only one that is unknown in the present problem. In order to calculate it, one supposes that one comes to know, by any expedient, three finite, continuous, and uniform functions u', v', w' that will satisfy (1) when one sets the body forces equal to the zero, and that they are such that one will have:

$$u'=\frac{\partial(1/r)}{\partial\xi}, \quad v'=\frac{\partial(1/r)}{\partial\eta}, \quad w'=\frac{\partial(1/r)}{\partial\zeta}$$

on the surface. When Betti's theorem is applied to these and the unknown displacements, it will give:

$$\int(Xu'+Yv'+Zw')\,dS+\int\left(L\frac{\partial(1/r)}{\partial\xi}+M\,\frac{\partial(1/r)}{\partial\eta}+N\,\frac{\partial(1/r)}{\partial\zeta}\right)ds$$

$$= \tag{3}$$

$$\int (L'u + M'v + N'w)\, ds\,,$$

in which L, M, N are calculated by means of the known boundary equations:

$$\left.\begin{aligned}
L' &= -(A-2B)\,\Theta'\frac{d\xi}{dn} - 2B\frac{du'}{dn} - B\left(\mathcal{T}_1'\frac{d\eta}{dn} - \mathcal{T}_2'\frac{d\zeta}{dn}\right),\\
M' &= -(A-2B)\,\Theta'\frac{d\eta}{dn} - 2B\frac{dv'}{dn} - B\left(\mathcal{T}_1'\frac{d\zeta}{dn} - \mathcal{T}_3'\frac{d\xi}{dn}\right),\\
N' &= -(A-2B)\,\Theta'\frac{d\zeta}{dn} - 2B\frac{dw'}{dn} - B\left(\mathcal{T}_2'\frac{d\xi}{dn} - \mathcal{T}_1'\frac{d\eta}{dn}\right).
\end{aligned}\right\} \tag{4}$$

Now Θ can be considered to be known by means of formula (2) when one replaces the second integral in it with the value that is provided by (3). If one substitutes the value of Θ in (1) then that will yield $\Delta^2 u$, $\Delta^2 v$, $\Delta^2 w$, and the functions u, v, w (whose values are known on the surface) can then be determined by recalling the considerations of Chap. IX.

2. – When the *external forces were given* on the surface, we previously needed to know, not one, but four systems of auxiliary displacements that were produced by only the surface forces, and which had the following expressions:

$$L' = -\,2B\frac{\partial\varphi}{\partial x}, \qquad M' = -\,2B\frac{\partial\varphi}{\partial y}, \qquad N' = -\,2B\frac{\partial\varphi}{\partial z},$$

$$L_1 = B\frac{\partial\varphi_1}{\partial x}, \qquad M_1 = B\left(\frac{\partial\varphi_1}{\partial y} - \frac{\partial\varphi}{\partial z}\right), \qquad N_1 = B\left(\frac{\partial\varphi_1}{\partial z} + \frac{\partial\varphi}{\partial y}\right),$$

$$L_2 = B\left(\frac{\partial\varphi_2}{\partial x} + \frac{\partial\varphi}{\partial z}\right), \qquad M_2 = B\frac{\partial\varphi_2}{\partial y}, \qquad N_2 = B\left(\frac{\partial\varphi_2}{\partial z} - \frac{\partial\varphi}{\partial x}\right),$$

$$L_3 = B\left(\frac{\partial\varphi_3}{\partial x} - \frac{\partial\varphi}{\partial y}\right), \qquad M_3 = B\left(\frac{\partial\varphi_3}{\partial y} + \frac{\partial\varphi}{\partial x}\right), \qquad N_3 = B\frac{\partial\varphi_3}{\partial z},$$

respectively.

In these equations, one has set $\varphi = \dfrac{d(1/r)}{dn}$ and:

$$\varphi_1 = \frac{\partial(1/r)}{\partial y}\frac{dz}{dn} - \frac{\partial(1/r)}{\partial z}\frac{dy}{dn}, \quad \varphi_2 = \frac{\partial(1/r)}{\partial z}\frac{dx}{dn} - \frac{\partial(1/r)}{\partial x}\frac{dz}{dn}, \quad \varphi_3 = \frac{\partial(1/r)}{\partial x}\frac{dy}{dn} - \frac{\partial(1/r)}{\partial y}\frac{dx}{dn},$$

for brevity. If one applies Betti's theorem to the displacements u', v', w' and to the unknown displacements then one will get:

$$\int (Xu' + Yv' + Zw')\,dS + \int (Lu' + Mv' + Nw')\,ds$$
$$= -2B\int\left(u\frac{\partial\varphi}{\partial\xi} + v\frac{\partial\varphi}{\partial\eta} + w\frac{\partial\varphi}{\partial\zeta}\right) ds\,.$$

(2) will then become:

$$4\pi A\,\Theta = \int\left[X\left(u' - \frac{\partial(1/r)}{\partial\xi}\right) + Y\left(v' - \frac{\partial(1/r)}{\partial\eta}\right) + Z\left(w' - \frac{\partial(1/r)}{\partial\zeta}\right)\right] dS$$
$$+ \int\left[L\left(u' - \frac{\partial(1/r)}{\partial\xi}\right) + M\left(v' - \frac{\partial(1/r)}{\partial\eta}\right) + N\left(w' - \frac{\partial(1/r)}{\partial\zeta}\right)\right] ds\,.$$

Now, Θ can be regarded as known. Analogously, one has:

$$\int (Xu_1 + Yv_1 + Zw_1)\,dS + \int (Lu_1 + Mv_1 + Nw_1)\,ds$$

$$= B\int\left(u\frac{\partial\varphi_1}{\partial\xi} + v\frac{\partial\varphi_1}{\partial\eta} + w\frac{\partial\varphi_1}{\partial\zeta}\right) ds + B\int\left(w\frac{\partial\varphi}{\partial y} - v\frac{\partial_1}{\partial z}\right) ds\,,$$

and since, from some other known formulas, one has:

$$-4\pi B\,\mathcal{T}_1 = \int\left(Z\frac{\partial(1/r)}{\partial\eta} - Y\frac{\partial(1/r)}{\partial\zeta}\right) dS + \int\left(N\frac{\partial(1/r)}{\partial\eta} - M\frac{\partial(1/r)}{\partial\zeta}\right) ds$$

$$+ \int\left(w\frac{\partial\varphi}{\partial\eta} - v\frac{\partial\varphi}{\partial\zeta}\right) ds + B\int\left(u\frac{\partial\varphi_1}{\partial\xi} + v\frac{\partial\varphi_1}{\partial\eta} + w\frac{\partial\varphi_1}{\partial\zeta}\right) ds\,,$$

and also:

$$4\pi B\,\mathcal{T}_1 = \int\left[Xu_1 + Y\left(v_1 - \frac{\partial(1/r)}{\partial\zeta}\right) + Z\left(w_1 + \frac{\partial(1/r)}{\partial\eta}\right)\right] dS$$

$$-\int\left[Lu_1 + M\left(v_1 - \frac{\partial(1/r)}{\partial\zeta}\right) + N\left(w_1 + \frac{\partial(1/r)}{\partial\eta}\right)\right] ds\,.$$

Therefore, $\mathcal{T}_1$, and analogously $\mathcal{T}_2$ and $\mathcal{T}_3$, will also be known.

3. – Having said that, in order to determine u, one will have the conditions:

$$X + (A - B)\,\frac{\partial\Theta}{\partial x} + B\,\Delta^2 u = 0$$

in all of S, and on the surface, one will have:

$$L + (A - 2B)\,\Theta\,\frac{dx}{dn} + 2B\frac{du}{dn} + B\left(\mathcal{T}_3\frac{dy}{dn} - \mathcal{T}_2\frac{dz}{dn}\right) = 0,$$

which will tell one the value of $\Delta^2 u$ at any point of space and that of du / dn on the surface, in such a way that u will prove to be well-defined, as long as it satisfies the condition:

$$\int \frac{du}{dn}\,ds = -\int \Delta^2 u\,dS\,.$$

In order to verify that equality, observe that the boundary equations give:

$$\int L\,ds + 2B\int\frac{du}{dn}\,ds = \int\left[(A-2B)\frac{\partial\Theta}{\partial\xi} + B\left(\frac{\partial\mathcal{T}_3}{\partial\eta} - \frac{\partial\mathcal{T}_2}{\partial\zeta}\right)\right]dS.$$

By virtue of the known identity:

$$\Delta^2 u = \frac{\partial\Theta}{\partial x} - \frac{\partial\mathcal{T}_3}{\partial y} + \frac{\partial\mathcal{T}_2}{\partial z}\,,$$

one can also write:

$$\int L\,ds + 2B\int\frac{du}{dn}\,ds = \int\left[(A-2B)\frac{\partial\Theta}{\partial\xi} - B\,\Delta^2 u\right]dS$$

$$= -\int X\,dS - 2B\int\Delta^2 u\,dS\,.$$

Therefore:

$$\int\frac{du}{dn}\,ds = -\int\Delta^2 u\,dS - \frac{1}{2B}\left(\int X\,dS + \int L\,ds\right) = -\int\Delta^2 u\,dS\,.$$

This method of integration is due to Betti (*): The details of its application are largely presented by prof. Cerruti in his "Ricerche intorno all'equilibrio dei corpi elastici isotropic" (**).

4. – It is easy to imagine other integration procedures that are always based upon a preliminary knowledge of displacements that all satisfy (1) in the absence of body forces,

(*) *Teoria della elasticità.*
(**) *Accademia dei Lincei* (1882), pp. 83, 87, 105. See also two communications by BOUSSINESQ to the Paris Academy in Comptes-rendus 9 and 16 April 1888.

and are characterized by the values that they assume on the surface. For example, if one starts from the formulas:

$$4\pi B\, u = U + \frac{\partial \Phi}{\partial x}, \qquad 4\pi B\, v = V + \frac{\partial \Phi}{\partial y}, \qquad 4\pi B\, w = W + \frac{\partial \Phi}{\partial z},$$

in which:

$$U = \int \frac{X\, dS}{r} + \int \frac{L\, ds}{r} + B \int u \frac{d(1/r)}{dn} ds + B \int \left(u \frac{\partial(1/r)}{\partial \xi} + v \frac{\partial(1/r)}{\partial \eta} + w \frac{\partial(1/r)}{\partial \zeta} \right) \frac{d\xi}{dn} ds,$$

$$V = \int \frac{Y\, dS}{r} + \int \frac{M\, ds}{r} + B \int v \frac{d(1/r)}{dn} ds + B \int \left(u \frac{\partial(1/r)}{\partial \xi} + v \frac{\partial(1/r)}{\partial \eta} + w \frac{\partial(1/r)}{\partial \zeta} \right) \frac{d\eta}{dn} ds,$$

$$W = \int \frac{Z\, dS}{r} + \int \frac{N\, ds}{r} + B \int w \frac{d(1/r)}{dn} ds + B \int \left(u \frac{\partial(1/r)}{\partial \xi} + v \frac{\partial(1/r)}{\partial \eta} + w \frac{\partial(1/r)}{\partial \zeta} \right) \frac{d\zeta}{dn} ds,$$

$$\Phi = (A - B) \int \frac{\Theta\, dS}{r} + B \int \left(u \frac{d\xi}{dn} + v \frac{d\eta}{dn} + w \frac{d\zeta}{dn} \right) \frac{ds}{r},$$

then one will be immediately presented with two integration methods: If you can obtain a system of displacements that assume the values $\frac{\partial r}{\partial \xi}, \frac{\partial r}{\partial \eta}, \frac{\partial r}{\partial \zeta}$ on the surface then the function Φ will be known thanks to Betti's theorem, because one sees elsewhere that it can also be given the form:

$$\begin{aligned}\Phi = {} & \frac{A-B}{2A} \int \left(X \frac{\partial r}{\partial \xi} + Y \frac{\partial r}{\partial \eta} + Z \frac{\partial r}{\partial \zeta} \right) dS \\ & + \frac{A-B}{2A} \int \left(L \frac{\partial r}{\partial \xi} + M \frac{\partial r}{\partial \eta} + N \frac{\partial r}{\partial \zeta} \right) ds \\ & - \frac{B}{A}(A-B) \int \left(u \frac{d\xi}{dn} \frac{\partial r}{\partial \xi} + v \frac{d\eta}{dn} \frac{\partial r}{\partial \eta} + w \frac{d\zeta}{dn} \frac{\partial r}{\partial \zeta} \right) \frac{ds}{r}.\end{aligned}$$

Now U, V, W must satisfy the equations:

$$\Delta^2 U = -4\pi X, \qquad \Delta^2 V = -4\pi Y, \qquad \Delta^2 W = -4\pi Z$$

in all of S and assume the values:

$$4\pi Bu - \frac{\partial \Phi}{\partial x}, \quad 4\pi Bv - \frac{\partial \Phi}{\partial y}, \quad 4\pi Bw - \frac{\partial \Phi}{\partial z}$$

on the surface.

5. – However, one can begin the determination of U, V, W when one succeeds in determining three systems of displacements that take the values:

$$u' = \frac{1}{r}, \qquad v' = 0, \qquad w' = 0,$$

$$u'' = 0, \qquad v'' = \frac{1}{r}, \qquad w'' = 0,$$

$$u''' = 0, \qquad v''' = 0, \qquad w''' = \frac{1}{r}$$

on the surface and verify (1) for $X = Y = Z = 0$ in all of space. Betti's theorem will immediately give:

$$\int \frac{L\,ds}{r} = \int (L'u + M'v + N'w)\,ds - \int (Xu' + Yv' + Zw')\,dS\,,$$

$$\int \frac{M\,ds}{r} = \int (L''u + M''v + N''w)\,ds - \int (Xu'' + Yv'' + Zw'')\,dS\,,$$

$$\int \frac{N\,ds}{r} = \int (L'''u + M'''v + N'''w)\,ds - \int (Xu''' + Yv''' + Zw''')\,dS\,.$$

One can then regard the functions U, V, W, and also Θ, as known, since one has:

$$4\pi A\,\Theta = \frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} + \frac{\partial W}{\partial z}\,.$$

The function Φ is known then. It can also be determined by observing that it satisfies the equation:

$$\Delta^2\Phi = -\,4\pi\,(A - B)\,\Theta,$$

while its first derivative with respect to the normal assumes the value:

$$(4\pi\,Bu - U)\,\frac{dx}{dn} + (4\pi\,Bv - V)\,\frac{dy}{dn} + (4\pi\,Bw - W)\,\frac{dz}{dn}$$

on the surface. However, the method that was described in § **1** is incontestably the simplest of them all.

CHAPTER XIV

APPLICATIONS TO ISOTROPIC ELASTIC FLOORS

1. – In order to clarify the preceding integration process to a greater extent in the case where the displacements are given on the surface, we shall apply it to a homogeneous, isotropic, elastic *floor*; i.e., to an indefinite solid that is bounded by a plane. The first question that presents itself is *the determination of the auxiliary displacements u′, v′, w′*. They must satisfy the equations:

$$\left.\begin{aligned}(A-B)\frac{\partial\Theta'}{\partial x}+B\Delta^2u'=0,\\(A-B)\frac{\partial\Theta'}{\partial y}+B\Delta^2v'=0,\\(A-B)\frac{\partial\Theta'}{\partial z}+B\Delta^2w'=0.\end{aligned}\right\}\tag{1}$$

If one lets r_1 represent the distance from the symmetric point (with respect to the bounding plane) from which one starts to the point that is found at the distance r then one will have:

$$r^2 = (x-\xi)^2+(y-\eta)^2+(z-\zeta)^2, \qquad r_1^2 = (x-\xi)^2+(y+\eta)^2+(z+\zeta)^2,$$

and therefore one will have:

$$\frac{\partial(1/r_1)}{\partial\xi}=\frac{\partial(1/r)}{\partial\xi}, \qquad \frac{\partial(1/r_1)}{\partial\eta}=\frac{\partial(1/r)}{\partial\eta}, \qquad \frac{\partial(1/r_1)}{\partial\zeta}=\frac{\partial(1/r)}{\partial\zeta}$$

on the surface – i.e., for $\zeta = 0$. It will then follow that if the functions u', v', w' must be harmonic, rather than satisfying (1), then they will be equal to:

$$\frac{\partial(1/r_1)}{\partial\xi}, \qquad \frac{\partial(1/r_1)}{\partial\eta}, \qquad \frac{\partial(1/r_1)}{\partial\zeta},$$

because they are harmonic, finite, continuous, and uniform in all of the space considered. Having said that, take one of equations (1) under consideration – for example, the first one. Since the cubic dilatation in the absence of body forces is a harmonic function, from the observations that were made in Chap. IX, one can set:

$$\Theta' = \frac{\partial\vartheta}{\partial\zeta},$$

in which ϑ is also harmonic. The equation considered will become:

$$\Delta^2 u' = -\frac{A-B}{B}\frac{\partial^2 \vartheta}{\partial\xi\,\partial\zeta} = \frac{\partial}{\partial\zeta}\left(-\frac{A-B}{B}\frac{\partial\vartheta}{\partial\xi}\right)$$

at the point (ξ, η, ζ), and therefore, from the same observations, one will have:

$$u' = \frac{\partial(1/r_1)}{\partial\xi} - \frac{A-B}{2B}\zeta\frac{\partial\vartheta}{\partial\xi}.$$

Analogously:

$$v' = \frac{\partial(1/r_1)}{\partial\eta} - \frac{A-B}{2B}\zeta\frac{\partial\vartheta}{\partial\eta}, \qquad w' = \frac{\partial(1/r_1)}{\partial\zeta} - \frac{A-B}{2B}\zeta\frac{\partial\vartheta}{\partial\zeta}.$$

In order to determine ϑ, observe that when one differentiates the last three relations with respect to ξ, η, ζ, one can deduce that:

$$\Theta' = -\,2\frac{\partial^2(1/r_1)}{\partial\zeta^2} - \frac{A-B}{2B}\frac{\partial\vartheta}{\partial\zeta} = \frac{\partial}{\partial\zeta}\left(-2\frac{\partial(1/r_1)}{\partial\zeta} - \frac{A-B}{2B}\vartheta\right).$$

One can then set:

$$\vartheta = -\,2\frac{\partial(1/r_1)}{\partial\zeta} - \frac{A-B}{2B}\vartheta$$

and get:

$$\vartheta = -\frac{4B}{A+B}\frac{\partial(1/r_1)}{\partial\zeta},$$

since that function is harmonic. Hence:

$$\left.\begin{aligned}
u' &= \frac{\partial(1/r_1)}{\partial\xi} + 2\frac{A-B}{A+B}\frac{\partial^2(1/r_1)}{\partial\xi\,\partial\eta},\\
v' &= \frac{\partial(1/r_1)}{\partial\eta} + 2\frac{A-B}{A+B}\frac{\partial^2(1/r_1)}{\partial\eta\,\partial\zeta},\\
w' &= \frac{\partial(1/r_1)}{\partial\zeta} + 2\frac{A-B}{A+B}\frac{\partial^2(1/r_1)}{\partial\zeta^2}.
\end{aligned}\right\} \tag{2}$$

2. – The second question to resolve is the *determination of the cubic dilatation.* First, calculate L', M', N' by means of formulas (4) of the preceding chapter. If one observes that $\frac{d\xi}{dn} = \frac{d\eta}{dn} = 0$ and $\frac{d\zeta}{dn} = 1$ then those equations will assume the form:

$$L' = -\,B\left(\frac{\partial u'}{\partial\zeta} + \frac{\partial w'}{\partial\xi}\right), \qquad M' = -\,B\left(\frac{\partial v'}{\partial\zeta} + \frac{\partial w'}{\partial\eta}\right), \qquad N' = -\,2B\,\frac{\partial w'}{\partial\zeta} - (A-2B)\,\Theta'\,;$$

so when one substitutes that in (2) and takes the values on the surface (viz., $\zeta = 0$), that will give:

$$\left\{\begin{aligned} L' &= -2B\frac{A-B}{A+B}\frac{\partial^2(1/r_1)}{\partial\xi\,\partial\zeta} = 2B\frac{A-B}{A+B}\frac{\partial^2(1/r)}{\partial\xi\,\partial\zeta},\\ M' &= -2B\frac{A-B}{A+B}\frac{\partial^2(1/r_1)}{\partial\eta\,\partial\zeta} = 2B\frac{A-B}{A+B}\frac{\partial^2(1/r)}{\partial\eta\,\partial\zeta},\\ N' &= -2B\frac{A-B}{A+B}\frac{\partial^2(1/r_1)}{\partial\zeta^2} = 2B\frac{A-B}{A+B}\frac{\partial^2(1/r)}{\partial\zeta^2}.\end{aligned}\right.$$

One will then have:

$$\int (L'u + M'v + N'w)\,ds$$

$$= 2B\frac{A-B}{A+B}\int\left(u\frac{\partial^2(1/r)}{\partial\xi\,\partial\zeta} + v\frac{\partial^2(1/r)}{\partial\eta\,\partial\zeta} + w\frac{\partial^2(1/r)}{\partial\zeta^2}\right)ds.$$

Analogously, the last integral in formula (2) of the preceding chapter will reduce to:

$$2B\int\left(u\frac{\partial^2(1/r)}{\partial\xi\,\partial\zeta} + v\frac{\partial^2(1/r)}{\partial\eta\,\partial\zeta} + w\frac{\partial^2(1/r)}{\partial\zeta^2}\right)ds,$$

and therefore, when one substitutes the result (3) in that same chapter, that formula will become:

$$4\pi A\Theta = \int\left[X\left(u' - \frac{\partial(1/r)}{\partial\xi}\right) + Y\left(v' - \frac{\partial(1/r)}{\partial\eta}\right) + Z\left(w' - \frac{\partial(1/r)}{\partial\zeta}\right)\right]dS$$

$$-\frac{4AB}{A+B}\int\left(u\frac{\partial^2(1/r)}{\partial\xi\,\partial\zeta} + v\frac{\partial^2(1/r)}{\partial\eta\,\partial\zeta} + w\frac{\partial^2(1/r)}{\partial\zeta^2}\right)ds.$$

The cubic dilatation is known then.

3. – In order to continue with simple formulas, we ignore the body forces. The last formula will become:

$$\pi\Theta = -\frac{B}{A+B}\int\left(u\frac{\partial^2(1/r)}{\partial\xi\,\partial\zeta} + v\frac{\partial^2(1/r)}{\partial\eta\,\partial\zeta} + w\frac{\partial^2(1/r)}{\partial\zeta^2}\right)ds.$$

Set:

$$P = \int\frac{u\,ds}{r},\qquad Q = \int\frac{v\,ds}{r},\qquad R = \int\frac{w\,ds}{r},$$

and

$$\varphi = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}.$$

Those integrals, which are all known, all satisfy the Laplace equation, since they are surface potential functions. Meanwhile, one has:

$$\frac{\partial^2 P}{\partial x\,\partial y} = \int u \frac{\partial^2 (1/r)}{\partial x\,\partial z} ds = \int u \frac{\partial^2 (1/r)}{\partial \xi\,\partial \zeta} ds, \qquad \ldots,$$

and therefore:

$$\pi\,\Theta = -\frac{B}{A+B}\left(\frac{\partial^2 P}{\partial x\,\partial y} + \frac{\partial^2 Q}{\partial y\,\partial z} + \frac{\partial^2 R}{\partial z^2}\right);$$

i.e.:

$$\pi\,\Theta = -\frac{B}{A+B}\frac{\partial \varphi}{\partial z}.$$

4. – We shall now pass on to the third and final part of the problem: *Determine the displacements.* They must satisfy the equations:

$$\left\{\begin{aligned} \Delta^2 v &= \frac{1}{\pi}\frac{A-B}{A+B}\frac{\partial^2 \varphi}{\partial y\,\partial z}, \\ \Delta^2 u &= \frac{1}{\pi}\frac{A-B}{A+B}\frac{\partial^2 \varphi}{\partial x\,\partial z}, \\ \Delta^2 w &= \frac{1}{\pi}\frac{A-B}{A+B}\frac{\partial^2 \varphi}{\partial z^2}. \end{aligned}\right.$$

If one is to have $u = 0$, $v = 0$, $w = 0$ on the surface then the values of those functions in all of the space considered will be:

$$\frac{1}{2\pi}\frac{A-B}{A+B} z \frac{\partial^2 \varphi}{\partial x}, \qquad \frac{1}{2\pi}\frac{A-B}{A+B} z \frac{\partial^2 \varphi}{\partial y}, \qquad \frac{1}{2\pi}\frac{A-B}{A+B} z \frac{\partial^2 \varphi}{\partial z}.$$

However, if u, v, w must satisfy the Laplace equation and assume assigned values on the surface then their values in all space will be:

$$-\frac{1}{2\pi}\frac{\partial P}{\partial z}, \qquad -\frac{1}{2\pi}\frac{\partial Q}{\partial z}, \qquad -\frac{1}{2\pi}\frac{\partial R}{\partial z}.$$

Hence, one finally has:

$$\left.\begin{aligned} u &= -\frac{1}{2\pi}\frac{\partial P}{\partial z} + \frac{1}{2\pi}\frac{A-B}{A+B}z\frac{\partial \varphi}{\partial x}, \\ v &= -\frac{1}{2\pi}\frac{\partial Q}{\partial z} + \frac{1}{2\pi}\frac{A-B}{A+B}z\frac{\partial \varphi}{\partial y}, \\ w &= -\frac{1}{2\pi}\frac{\partial R}{\partial z} + \frac{1}{2\pi}\frac{A-B}{A+B}z\frac{\partial \varphi}{\partial z}. \end{aligned}\right\} \tag{3}$$

5. – Prof. Cerruti has treated the preceding problem "in order to give a rather simple illustration of the general method" that Betti proposed. When one does not have that goal in mind, but one would only like to arrive at the solution to the problem of elastic floors, it is quite easy to reach the general formulas that Cerruti obtained by a more rapid and direct process that does not abandon "solving the problem in such a way that it will shed some light upon the treatment of analogous problems involving bodies of more complicated forms" (*). In fact, it is enough to temporarily regard the cubic dilatation Θ as known, then calculate the displacements (u, v, w), and deduce the expression for Θ: That function will then be found to be singled out by a relation that serves to determine it. Meanwhile, all of the difficulties in the problem reside in the determination of Θ, but they are no more serious than the ones that must be overcome in order to determine the auxiliary displacements by Betti's method. The difficulties disappeared from the particular problem that Cerruti treated precisely because the function Θ figured linearly in the relations that one had to determine.

6. – The difficulties in integration also disappear from a very favorable situation that continually presents itself in the problem considered; viz., that any harmonic function can be derived from another harmonic function by an arbitrary number of successive partial differentiations with respect to z when one supposes that the z-axis is taken to be perpendicular to the bounding plane. Indeed, it was shown in Chap. IX that if $\Delta^2\varphi = 0$ then one will have:

$$\varphi = \frac{\partial \varphi_1}{\partial z}, \qquad \text{with} \qquad \varphi_1 = -\frac{1}{2\pi}\int \frac{\varphi\, ds}{r}.$$

Meanwhile, observe that:

$$\text{for } \zeta = 0 \qquad \text{one has} \qquad \frac{\partial}{\partial z}\log(z + r) = \frac{1}{r},$$

and consequently:

$$\varphi_1 = \frac{\partial \varphi_2}{\partial z} \qquad \text{with} \qquad \varphi_2 = -\frac{1}{2\pi}\int \varphi \log(z+r)\, ds.$$

Analogously, one will get:

(*) CERRUTI, *loc. cit.*, pp. 81. The problem of elastic floors has been treated since 1878 by BOUSSINESQ. See CLEBSCH's *Théorie*, pp. 375.

$$\varphi_2 = \frac{\partial \varphi_3}{\partial z} \qquad \text{with} \qquad \varphi_3 = -\frac{1}{2\pi}\int \varphi[z \log(z+r) - r]\, ds\,, \qquad \ldots$$

Recall, once more, that in order to satisfy the $\Delta^2 U = \varphi$, when φ is harmonic, one needs to take:

$$U = -\frac{1}{2\pi}\frac{\partial}{\partial z}\int \frac{U\, ds}{r} + \frac{1}{2} z\varphi_1 \tag{4}$$

and prescribe the values of U on the surface – i.e., for $z = 0$. Now suppose that the values of $\partial U / \partial z$ are assigned instead. If U is harmonic then since the derivative of the integral $-\frac{1}{2\pi}\int \frac{V\, ds}{r}$ with respect to z is equal to V, in order to get U, it will be enough to replace V with the values that are prescribed for $\partial U / \partial z$. In any case, one can set:

$$U = -\frac{1}{2\pi}\int \frac{\partial U}{\partial z}\frac{ds}{r} + U',$$

and the function U' will satisfy $\Delta^2 U' = \varphi$, while the values of $\partial U' / \partial z$ will vanish on the surface. It will then follow that if one sets $U = (z\, \varphi_1 - \psi)$ then ψ will be harmonic, and one must have:

$$\frac{\partial}{\partial z}(z\, \varphi_1 - \psi) = 0, \qquad \text{i.e.,} \qquad \varphi_1 = \frac{\partial \psi}{\partial z}.$$

Therefore $\psi = \varphi_2$, and consequently:

$$U = -\frac{1}{2\pi}\int \frac{\partial U}{\partial z}\frac{ds}{r}\,\frac{1}{2}(z\, \varphi_1 - \varphi_2). \tag{5}$$

7. – Having said that, suppose that the problem has already been reduced (as it always can be) to the case in which the external force acts only upon the surface, in such a way that for $z \geq 0$, one must have:

$$\begin{cases} (A-B)\dfrac{\partial \Theta}{\partial x} + B\,\Delta^2 u = 0, \\[2ex] (A-B)\dfrac{\partial \Theta}{\partial y} + B\,\Delta^2 v = 0, \\[2ex] (A-B)\dfrac{\partial \Theta}{\partial z} + B\,\Delta^2 w = 0, \end{cases}$$

and consequently:

$$\Delta^2 \Theta = 0, \qquad \Theta = \frac{\partial \Theta_1}{\partial z} = \frac{\partial^2 \Theta_2}{\partial z^2} = \ldots$$

If one gives the form:

$$\Delta^2 u = -\frac{A-B}{B}\frac{\partial^2\Theta_1}{\partial x\,\partial z}, \quad \Delta^2 v = -\frac{A-B}{B}\frac{\partial^2\Theta_1}{\partial y\,\partial z}, \quad \Delta^2 w = -\frac{A-B}{B}\frac{\partial^2\Theta_1}{\partial z^2}$$

to the preceding equations then one will see immediately that, by virtue of (4), one will have:

$$\left.\begin{aligned} u &= -\frac{1}{2\pi}\frac{\partial P}{\partial z} + \frac{A-B}{2B} z \frac{\partial\Theta_1}{\partial x}, \\ v &= -\frac{1}{2\pi}\frac{\partial Q}{\partial z} + \frac{A-B}{2B} z \frac{\partial\Theta_1}{\partial y}, \\ w &= -\frac{1}{2\pi}\frac{\partial R}{\partial z} + \frac{A-B}{2B} z \frac{\partial\Theta_1}{\partial z}. \end{aligned}\right\}$$

When one differentiates these formulas with respect to x, y, z, resp., and sums, one will deduce that:

$$\Theta = -\frac{1}{2\pi}\frac{\partial\varphi}{\partial z} - \frac{A-B}{2B} z \frac{\partial\Theta_1}{\partial y};$$

hence:

$$\Theta = -\frac{B}{\pi(A+B)}\frac{\partial\varphi}{\partial z}, \qquad \Theta_1 = -\frac{B\varphi}{\pi(A+B)}.$$

If one substitutes these results in the preceding formulas then one will arrive at formulas (3), which Cerruti obtained in the case where the displacements were given on the surface (*).

8. – What if one is given the surface forces (L, M, N)? One will then need to recall formula (5). For $z \geq 0$ and $z = 0$, the component w of the displacement must satisfy the equations:

$$\Delta^2 w = -\frac{A-B}{B}\frac{\partial\Theta}{\partial z}, \qquad N + (A - 2B)\,\Theta + 2B\,\frac{\partial w}{\partial z} = 0,$$

respectively. One deduces from (5) that:

$$w = \frac{1}{4\pi B}\int\frac{N\,ds}{r} + \frac{A-2B}{4\pi B}\int\frac{\Theta\,ds}{r} + \frac{A-B}{2B}\,(\Theta_1 - z\,\Theta);$$

i.e.:

$$w = \frac{1}{4\pi B}\frac{\partial\mathcal{N}}{\partial z} + \frac{1}{2}\Theta_1 - \frac{A-B}{2B} z\,\Theta, \tag{6}$$

after one sets:

(*) CERRUTI, *loc. cit.*, form. (41).

$$\mathcal{L} = \int L \log(z+r)\, ds\,, \qquad \mathcal{M} = \int M \log(z+r)\, ds\,, \qquad \mathcal{N} = \int N \log(z+r)\, ds$$

and observes that:

$$\int \frac{N\, ds}{r} = \frac{\partial \mathcal{N}}{\partial z}\,, \qquad \int \frac{\Theta\, ds}{r} = -\,2\pi\,\Theta_1\,.$$

Now the function u must satisfy the equation:

$$\Delta^2 u = -\,\frac{A-B}{B}\frac{\partial^2 \Theta_1}{\partial x\,\partial z}$$

in all of the space considered, while one must have:

$$L + B\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) = 0$$

on the surface. Therefore, if one adopts formula (5) then:

$$u = \frac{1}{2\pi B}\int \frac{L\, ds}{r} + \frac{1}{2\pi}\int \frac{\partial w}{\partial \xi}\frac{ds}{r} + \frac{A-B}{2B}\frac{\partial}{\partial x}\,(\Theta_2 - z\,\Theta_1). \tag{7}$$

On the other hand, if one gives (6) the form:

$$w = \frac{\partial f}{\partial z} - \frac{A-B}{2B}\, z\,\Theta$$

then one can set:

$$f = \frac{\mathcal{N}}{4\pi B} + \frac{1}{2}\Theta_2\,,$$

and one will also have:

$$\int \frac{\partial w}{\partial \xi}\frac{ds}{r} = \int \frac{\partial^2 f}{\partial \xi\,\partial \zeta}\frac{ds}{r} = -2\pi\frac{\partial f}{\partial x}\,,$$

so (7) will become:

$$u = \frac{1}{2\pi B}\frac{\partial \mathcal{L}}{\partial z} - \frac{1}{4\pi B}\frac{\partial \mathcal{N}}{\partial x} - \frac{1}{2}\frac{\partial \Theta_2}{\partial x} + \frac{A-B}{2B}\frac{\partial}{\partial x}(\Theta_2 - z\,\Theta_1)\,. \tag{8}$$

Analogously, one will have:

$$v = \frac{1}{2\pi B}\frac{\partial \mathcal{M}}{\partial z} - \frac{1}{4\pi B}\frac{\partial \mathcal{N}}{\partial y} - \frac{1}{2}\frac{\partial \Theta_2}{\partial y} + \frac{A-B}{2B}\frac{\partial}{\partial y}(\Theta_2 - z\,\Theta_1)\,,$$

and one can finally write (6) as:

$$w=\frac{1}{2\pi B}\frac{\partial\mathcal{N}}{\partial z}-\frac{1}{4\pi B}\frac{\partial\mathcal{N}}{\partial z}-\frac{1}{2}\frac{\partial\Theta_2}{\partial z}+\frac{A-B}{2B}\frac{\partial}{\partial z}(\Theta_2-z\Theta_1)+\Theta_1 .$$

If one sets:

$$\psi=\frac{\partial\mathcal{L}}{\partial x}+\frac{\partial\mathcal{M}}{\partial y}+\frac{\partial\mathcal{N}}{\partial z}$$

and differentiates the last three formulas then one will deduce that:

$$\Theta=\frac{1}{2\pi B}\frac{\partial\psi}{\partial z}-\frac{A-B}{2B}\Delta^2(z\Theta_1)+\frac{\partial\Theta_1}{\partial z}=\frac{1}{2\pi B}\frac{\partial\psi}{\partial z}-\frac{A-2B}{B}\Theta,$$

i.e.:

$$\Theta=\frac{1}{2\pi(A-B)}\frac{\partial\psi}{\partial z},\qquad \Theta_1=\frac{\psi}{2\pi(A-B)},\qquad \Theta_2=\frac{\chi}{2\pi(A-B)},$$

as long as one sets:

$$\mathfrak{L}=\int L[z\log(z+r)-r]\,ds,$$
$$\mathfrak{M}=\int M[z\log(z+r)-r]\,ds,$$
$$\mathfrak{N}=\int N[z\log(z+r)-r]\,ds,$$

and then takes:

$$\chi=\frac{\partial\mathfrak{L}}{\partial x}+\frac{\partial\mathfrak{M}}{\partial y}+\frac{\partial\mathfrak{N}}{\partial z}.$$

Now, the equality (6) changes into the known formula (*):

$$w=\frac{1}{4\pi B}\frac{\partial\mathcal{N}}{\partial z}+\frac{\psi}{4\pi(A-B)}-\frac{z}{4\pi B}\frac{\partial\psi}{\partial z}. \tag{9}$$

However, (8) will become:

$$u=\frac{1}{2\pi B}\frac{\partial\mathcal{L}}{\partial z}-\frac{1}{4\pi B}\frac{\partial\mathcal{N}}{\partial x}-\frac{1}{4\pi(A-B)}\frac{\partial\chi}{\partial z}+\frac{1}{4\pi B}\frac{\partial}{\partial x}(\chi-z\psi),$$

or

$$u=\frac{1}{4\pi B}\frac{\partial\mathcal{L}}{\partial z}-\frac{1}{4\pi(A-B)}\frac{\partial\chi}{\partial x}-\frac{z}{4\pi B}\frac{\partial\psi}{\partial x}+\frac{1}{4\pi B}\left(\frac{\partial\mathcal{L}}{\partial z}-\frac{\partial\mathcal{N}}{\partial x}+\frac{\partial\chi}{\partial x}\right).$$

If one then observes that:

$$\frac{\partial\mathcal{L}}{\partial z}-\frac{\partial\mathcal{N}}{\partial x}+\frac{\partial\chi}{\partial x}=\frac{\partial^2\mathfrak{L}}{\partial z^2}+\frac{\partial}{\partial x}\left(\frac{\partial\mathfrak{L}}{\partial x}+\frac{\partial\mathfrak{M}}{\partial y}\right)=-\frac{\partial^2\mathfrak{L}}{\partial y^2}+\frac{\partial^2\mathfrak{M}}{\partial x\partial y}$$

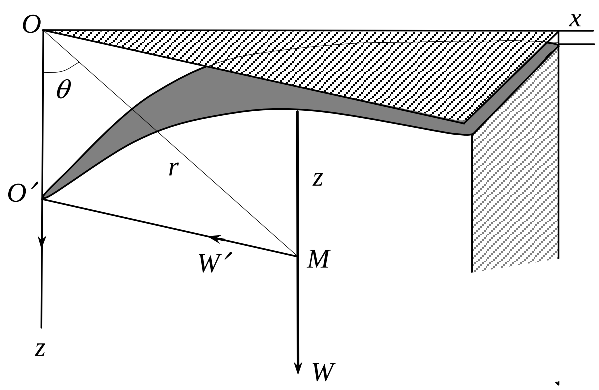
(*) CERRUTI, *loc. cit.*, form. (58).

then one will finally get the formulas ([*]):

$$u = \frac{1}{4\pi B}\frac{\partial \mathcal{L}}{\partial z} - \frac{1}{4\pi(A-B)}\frac{\partial \chi}{\partial x} - \frac{z}{4\pi B}\frac{\partial \psi}{\partial x} + \frac{1}{4\pi B}\frac{\partial}{\partial y}\left(\frac{\partial \mathfrak{M}}{\partial x} - \frac{\partial \mathfrak{L}}{\partial y}\right),$$

$$v = \frac{1}{4\pi B}\frac{\partial \mathcal{M}}{\partial z} - \frac{1}{4\pi(A-B)}\frac{\partial \chi}{\partial y} - \frac{z}{4\pi B}\frac{\partial \psi}{\partial y} + \frac{1}{4\pi B}\frac{\partial}{\partial x}\left(\frac{\partial \mathfrak{M}}{\partial x} - \frac{\partial \mathfrak{L}}{\partial y}\right), \tag{10}$$

and together with (9), that will solve completely the problem of elastic floors that are subject to a known, but arbitrary, system of surface pressures.



9. – Suppose, for example, that a horizontal floor is supported at a point O by a vertical pressure that is assumed to be unity. This must be understood in the sense that when a very small particle is taken at the surface of the floor around O, the pressure on it will be distributed in such a way that when its value per unit area is calculated, it will be very large at the central points of the particle, it will become very small, but vary continuously, in the vicinity of the contour, and it will be zero on that contour. Hence, the continuity of surface pressures that was required for the application of the preceding theory will be respected. However, since we do not suppose that the particle is very small, but in fact infinitesimal, our results will not be valid at the point O, and one must also consider them to be approximate in the points that neighbor O. Having said that, one has:

$$L = \mathcal{L} = \mathfrak{L} = 0, \qquad M = \mathcal{M} = \mathfrak{M} = 0,$$

$$\int N\, ds = 1, \qquad \mathcal{N} = \log(z + r), \qquad \mathfrak{N} = z \log(z + r) - r,$$

$$\psi = \frac{1}{r}, \qquad \chi = \log(z + r), \qquad r^2 = x^2 + y^2 + z^2 .$$

Formulas (9) and (10), in which one sets:

([*]) CERRUTI, *loc. cit.*, form. (63).

$$w' = -\frac{ux + vy}{\sqrt{x^2 + y^2}}, \qquad \sqrt{x^2 + y^2} = r \sin \theta, \quad z = r \cos \theta \qquad \left(0 < \theta < \frac{\pi}{2}\right)$$

will become:

$$w = \frac{1}{4\pi Br}\left(\frac{A}{A-B} + \cos^2 \theta\right),$$

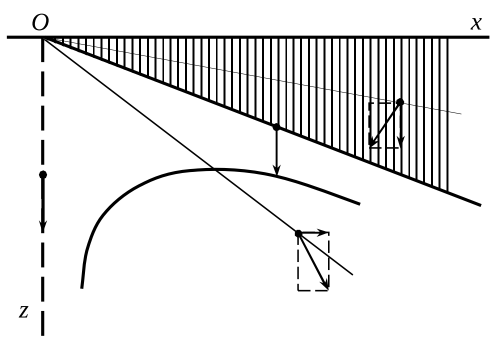
$$w' = \frac{\sin\theta}{4\pi Br}\left(\frac{B}{A-B}\frac{1}{1+\cos\theta} - \cos\theta\right).$$

Therefore, at any point M, the floor will experience a *drop* w and a contraction w' that are inversely proportional to the distance from M to the point of application of the pressure. Actually, one has a *contraction* in the direction of the force only at the points that are external to the cone that is defined by the value of θ for which one has:

$$\frac{B}{A-B}\frac{1}{1+\cos\theta} = \cos \theta,$$

that is to say:

$$\cos \theta = \frac{1}{2}\left(\sqrt{\frac{A+3B}{A-B}} - 1\right).$$



In particular, for those bodies that obey the laws of Navier and Poisson, $\cos \theta$ is roughly 1 / 3. The cone that corresponds to the value of θ that is found will only deform downward and tend to eject – so to speak – the internal particles, while at the same time, the external particles will tend to fill the void that is left by the latter ones. Finally, on the surface of the floor, except for the point O, one will have:

$$\frac{w}{A} = \frac{w'}{B} = \frac{1}{4\pi(A-B)Br},$$

and therefore its surface will take the hyperbolic form that is indicated in the first figure (*).

(*) Other interesting special cases are discussed by BOUSSINESQ in CLEBSCH's *Traité*, pp. 390.

CHAPTER XV

THERMAL DEFORMATIONS

1. – One communicates a very small quantity of heat to a homogeneous, isotropic body in such a way that the temperature of the particle dS will be raised by $\tau\,dS$, where τ is a finite, continuous, and uniform function of the coordinates of the center of the particle. It is known that if k is the coefficient of linear dilatation then any linear element will experience an elongation $k\tau$ per unit length in such a way that one will have:

$$a\,\alpha^2 + b\,\beta^2 + c\,\gamma^2 + 2f\,\beta\gamma + 2g\,\gamma\alpha + 2\,h\,\alpha\beta = k\,\tau$$

in all directions (α, β, γ); i.e.:

$$a = b = c = k\,\tau, \qquad f = g = h = 0.$$

Therefore, the deformation that is produced by the elevation of temperature, when it is considered by itself (i.e., independently of the elastic action that it produces), cannot produce shears, but only dilatations $k\tau$ in all directions.

2. – However, it is clear that, in general, the heat thus-communicated will vary the relative positions of the particles and excite elastic tensions everywhere, and therefore the deformation that is produced (which is characterized by the usual functions a, b, c, f, g, h) can be regarded as the result of the *purely-thermal* deformation:

$$k\tau, \quad k\tau, \quad k\tau, \quad 0, \quad 0, \quad 0$$

and another purely-*elastic* one:

$$a - k\tau, \quad b - k\tau, \quad c - k\tau, \quad f, \quad g, \quad f.$$

Looking for the equilibrium conditions for the latter deformation amounts to determining the values of the displacements, and consequently the pressures, dilatations, … in all of the space considered.

3. – Suppose that the body is devoid of all external forces besides that of heat, so the equilibrium conditions will be:

$$X = \frac{\partial}{\partial x}\frac{\partial \Pi}{\partial a} + \frac{1}{2}\frac{\partial}{\partial y}\frac{\partial \Pi}{\partial h} + \frac{1}{2}\frac{\partial}{\partial z}\frac{\partial \Pi}{\partial g}, \qquad \ldots \tag{1}$$

in space and:

$$L = \frac{\partial \Pi}{\partial a}\frac{dx}{dn} + \frac{1}{2}\frac{\partial \Pi}{\partial h}\frac{dy}{dn} + \frac{1}{2}\frac{\partial \Pi}{\partial g}\frac{dz}{dn}, \ldots \tag{2}$$

on the surface, in which:

$$X = Y = Z = L = M = N = 0, \tag{3}$$

as long as one is careful to reduce a, b, c by $k\,\tau$. Consider only the first equations of each triple. It is known that in the case of isotropy, only $\partial\Pi / \partial a$ will contain a, b, c, and one will have precisely:

$$\frac{\partial \Pi}{\partial a} = -A\Theta + 2B\,(b + c).$$

Therefore, $\partial\Pi / \partial a$ will change into:

$$-A\,(\Theta - 3k\tau) + 2B + 2B\,(b + c - 2k\tau) = \frac{\partial \Pi}{\partial a} + k\,(3A - 4B)\,\tau.$$

If one substitutes this in (1) and (2) then one will arrive at the same results that one would obtain if one left the right-hand sides of those equations intact and took:

$$\left.\begin{aligned} X &= -k(3A-4B)\frac{\partial \tau}{\partial x}, & L &= -k(3A-4B)\,\tau\frac{dx}{dn}, \\ Y &= -k(3A-4B)\frac{\partial \tau}{\partial y}, & M &= -k(3A-4B)\,\tau\frac{dy}{dn}, \\ Z &= -k(3A-4B)\frac{\partial \tau}{\partial z}, & N &= -k(3A-4B)\,\tau\frac{dz}{dn}, \end{aligned}\right\} \tag{4}$$

instead of (3). Therefore, the elevation in temperature that is defined by the function τ produces effects in a homogeneous, isotropic, elastic body that is devoid of the action of any other external forces that are identical to the ones that the forces (4) would produce by acting upon the body when it is supposed to be in equilibrium. One will then see that the temperature behaves like a potential function for a body force in the body and a normal pressure on the surface.

4. – We can now write down the equilibrium conditions immediately. They are:

$$\left.\begin{aligned} k\,(3A-4B)\frac{\partial \tau}{\partial x} &= (A-B)\frac{\partial \Theta}{\partial x} + B\Delta^2 u, \\ k\,(3A-4B)\frac{\partial \tau}{\partial y} &= (A-B)\frac{\partial \Theta}{\partial y} + B\Delta^2 v, \\ k\,(3A-4B)\frac{\partial \tau}{\partial z} &= (A-B)\frac{\partial \Theta}{\partial z} + B\Delta^2 w \end{aligned}\right\} \tag{5}$$

in all of the space considered, and:

$$\left.\begin{aligned}
k\,(3A-4B)\,\tau\frac{dx}{dn} &= (A-B)\,\Theta\frac{dx}{dn}+B\left(\mathcal{T}_3\frac{dy}{dn}-\mathcal{T}_2\frac{dz}{dn}\right),\\
k\,(3A-4B)\,\tau\frac{dx}{dn} &= (A-B)\,\Theta\frac{dx}{dn}+B\left(\mathcal{T}_1\frac{dy}{dn}-\mathcal{T}_3\frac{dz}{dn}\right),\\
k\,(3A-4B)\,\tau\frac{dx}{dn} &= (A-B)\,\Theta\frac{dx}{dn}+B\left(\mathcal{T}_2\frac{dy}{dn}-\mathcal{T}_1\frac{dz}{dn}\right)
\end{aligned}\right\} \tag{6}$$

on the surface. These equations were found by Duhamel and then by F. Neumann (*).

5. – From what was said in § **3**, any question that is concerned with deformations that are due to heat is equivalent to a problem of elastic equilibrium for which the external forces have the expression (4). One can then obtain all of the results that were obtained before in order to deduce some other consequences that relate to equivalent thermal deformations. For example, if one takes the formula:

$$\int\Theta\,dS = \frac{1}{3A-4B}\sum\left(\int Xx\,dS+\int Lx\,ds\right)$$

then the preceding will yield some of the simpler corollaries to Betti's theorem. If one adopts (4) then one will have:

$$\int\Theta\,dS = -k\sum\left(\int x\frac{\partial\tau}{\partial x}dS+\int\tau x\frac{dx}{dn}ds\right).$$

Moreover:

$$\int x\frac{\partial\tau}{\partial x}dS = \int\frac{\partial\tau x}{\partial x}dS-\int\tau\,dS = -\int\tau x\frac{dx}{dn}ds-\int\tau\,dS\,;$$

i.e.:

$$\int x\frac{\partial\tau}{\partial x}dS+\int\tau x\frac{dx}{dn}ds = -\int\tau\,dS\,.$$

Therefore:

$$\int\Theta\,dS = 3k\int\tau\,dS\,.$$

In other words:

The variation of the volume does not depend upon the form of the body or the coefficients of elasticity. It is equal to the total increase in temperature, multiplied by three times the coefficient of linear dilatation.

(*) BETTI, *loc. cit.*, pp. 102.

6. – We now need to know whether it would be possible for the elevation τ in temperature to not provoke elastic forces. For that to be true, it would be necessary and sufficient that the potential Π of those forces should be zero; i.e., that one should have:

$$a = b = c = k\,\tau, \qquad\qquad f = g = h = 0. \tag{7}$$

The known conditions, which are necessary and sufficient because a, b, c, f, g, h represent the components of a possible deformation, will reduce to the simultaneous vanishing of the second derivatives of τ in the present case. Therefore, τ must be a linear function of the coordinates: Let $\tau = \alpha x + \beta y + \gamma z + \delta$. If one integrates the first of (7) then one will get:

$$u = k\frac{\alpha}{2}x^2 + k\,x\,(\tau - \alpha x) + \varphi(y, z)$$

or

$$u = k\tau x + \frac{k}{2}\alpha(x^2 + y^2 + z^2) + u_0\,,$$

in which u_0 is independent of x. v and w will assume analogous forms. Obviously:

$$\frac{\partial u_0}{\partial x} = \frac{\partial v_0}{\partial y} = \frac{\partial w_0}{\partial z} = 0,$$

and in addition, the substitution of the preceding expressions for u, v, w in the remaining equalities (7) will give:

$$\frac{\partial w_0}{\partial y} + \frac{\partial v_0}{\partial z} = \frac{\partial u_0}{\partial z} + \frac{\partial w_0}{\partial x} = \frac{\partial v_0}{\partial x} + \frac{\partial u_0}{\partial y} = 0.$$

Hence, u_0, v_0, w_0 represents (Chap. II, § **1**) a rigid displacement, which one ignores. It will then follow that when the displacements do not have the form:

$$\begin{cases} u = k\tau x - \dfrac{k\alpha}{2}(x^2 + y^2 + z^2), \\ v = k\tau y - \dfrac{k\beta}{2}(x^2 + y^2 + z^2), \\ w = k\tau z - \dfrac{k\gamma}{2}(x^2 + y^2 + z^2), \end{cases}$$

i.e., when τ does not depend linearly upon the coordinates, one can be sure that an elastic deformation will counteract the purely-thermal deformation, and that the elastic equilibrium will be established under conditions that are different from the ones that existed before communicating the heat.

7. – Let us treat the problem in the case of a *solid sphere*, while supposing that the elevation in temperature at each point depends upon only the distance r from that point to the center of the sphere. If we call the unit elongation along the radius ε then we will have $u = \varepsilon x$, $v = \varepsilon y$, $w = \varepsilon z$, and consequently:

$$\Delta^2 u = x\,\Delta^2\varepsilon + 2\frac{\partial\varepsilon}{\partial x}\,2 = x\left(\frac{2}{r}\frac{d\varepsilon}{dr} + \frac{d^2\varepsilon}{dr^2}\right) + 2\frac{x}{r}\frac{d\varepsilon}{dr} = \frac{x}{r}\left(4\frac{d\varepsilon}{dr} + r\frac{d^2\varepsilon}{dr^2}\right).$$

On the other hand, one has:

$$\Theta = 3\varepsilon + r\frac{d\varepsilon}{dr}, \qquad \frac{d\Theta}{dr} = 4\frac{d\varepsilon}{dr} + r\frac{d^2\varepsilon}{dr^2}.$$

Equations (5) will then reduce to the single one:

$$k\,(3A - 4B)\,\frac{d\tau}{dr} = A\frac{d\Theta}{dr},$$

which one can integrate directly by writing down that:

$$\Theta = 3\lambda + \frac{k(3A-4B)}{A}\tau\,.$$

Since $\Theta\, r^2$ is obviously the derivative of $\varepsilon\, r^3$, another integration will give:

$$\varepsilon = \lambda + \frac{\mu}{r^3} + \frac{k(3A-4B)}{Ar^3}\int_0^r \tau\, r^2\, dr\,. \tag{8}$$

In order to determine μ, it is enough to observe that as long as μ differs from zero, the displacement $\varepsilon\, r$ will become infinite at the center of the sphere, but that cannot happen. Meanwhile, it is necessary that one should have $\mu = 0$. In order to determine λ, one recalls (6), which will reduce to the single equation:

$$k\,(3A - 4B) = (A - 2B)\,\Theta + 2B\left(\varepsilon + r\frac{d\varepsilon}{dr}\right) = (3A - 4B)\,\varepsilon + Ar\frac{d\varepsilon}{dr},$$

which must be satisfied when r is equal to the radius a of the sphere. Meanwhile, one deduces from (8) that:

$$\frac{d\varepsilon}{dr} = \frac{k(3A-4B)}{Ar}\left(\tau - \frac{3}{r^3}\int_0^r \tau\, r^2 dr\right).$$

Therefore, one must have:

$$\varepsilon = \frac{3k}{r^3}\int_0^r \tau\, r^2 dr$$

for $r = a$; i.e.,:

$$\lambda = \frac{3k}{a^3}\int_0^a \tau\, r^2 dr - \frac{k(3A-4B)}{Aa^3}\int_0^a \tau\, r^2 dr = \frac{4kB}{Aa^3}\int_0^a \tau\, r^2 dr\,,$$

and finally, with the substitution (8), one will arrive at the formula of F. Neumann and Borchardt (*):

$$\varepsilon = \frac{3k}{r^3}\int_0^r \tau\, r^2 dr + \frac{4kB}{A}\left(\frac{1}{a^3}\int_0^a \tau\, r^2 dr - \frac{1}{r^3}\int_0^r \tau\, r^2 dr\right).$$

8. – To conclude, suppose that one varies the temperature of a homogeneous, isotropic, *elastic floor* very slightly, while keeping the surface in contact with a constant source of heat, in such a way that one will have $\Delta^2\tau = 0$ in all of the body. In order to look for one of the infinitude of possible systems of displacements, one sets:

$$u' = -k\,(3A-4B)\,\frac{\partial\varphi}{\partial x}, \qquad v' = -k\,(3A-4B)\,\frac{\partial\varphi}{\partial y}, \qquad w' = -k\,(3A-4B)\,\frac{\partial\varphi}{\partial z}.$$

The indefinite equations of equilibrium are satisfied when:

$$\tau + A\,\Delta^2\varphi = 0.$$

One can then set:

$$\varphi = -\frac{z\,\tau_1}{2A},$$

after one sets:

$$\tau = \frac{\partial\tau_1}{\partial z} = \frac{\partial^2\tau_2}{\partial z^2} = \frac{\partial^3\tau_3}{\partial z^3} = \dots$$

It will then follow that:

$$u' = \frac{k(3A-4B)}{2A}\,z\,\frac{\partial\tau_1}{\partial x}, \qquad v' = \frac{k(3A-4B)}{2A}\,z\,\frac{\partial\tau_1}{\partial y}, \qquad w' = \frac{k(3A-4B)}{2A}(z\,\tau + \tau_1). \qquad (9)$$

Those displacements will provoke the pressures:

$$L' = -\frac{kB}{A}(3A-4B)\frac{\partial\tau_1}{\partial x}, \qquad M' = -\frac{kB}{A}(3A-4B)\frac{\partial\tau_1}{\partial y}, \qquad N' = -k\,(3A-4B)\,\tau$$

on the surface. Therefore, the displacements:

(*) BETTI, *loc. cit.*, pp. 108.

$$u'' = u - u', \qquad v'' = v - v', \qquad w'' = w - w'$$

will be due to the action of the following forces that are only applied to the surface:

$$L'' = \frac{kB}{A}(3A-4B)\frac{\partial\tau_1}{\partial x}, \qquad M'' = \frac{kB}{A}(3A-4B)\frac{\partial\tau_1}{\partial y}, \qquad N'' = 0.$$

Obviously, one has:

$$\int\frac{L''ds}{r} = \frac{kB}{A}(3A-4B)\int\frac{\partial^2\tau_2}{\partial\xi\,\partial\zeta}\frac{ds}{r} = -2\pi\frac{kB}{A}(3A-4B)\frac{\partial\tau_2}{\partial x},$$

and therefore:

$$\mathcal{L}'' = -2\pi\frac{kB}{A}(3A-4B)\frac{\partial\tau_2}{\partial x}, \qquad \mathcal{M}'' = -2\pi\frac{kB}{A}(3A-4B)\frac{\partial\tau_2}{\partial y},$$

and $\mathcal{N}'' = 0$. Analogously:

$$\mathfrak{L}'' = -2\pi\frac{kB}{A}(3A-4B)\frac{\partial\tau_4}{\partial x}, \qquad \mathfrak{M}'' = -2\pi\frac{kB}{A}\frac{\partial\tau_4}{\partial y}, \qquad \mathfrak{N}'' = 0\,;$$

hence (while always adopting the notation of the preceding chapter):

$$\psi'' = 2\pi\frac{kB}{A}(3A-4B)\;\tau_1\,, \qquad \chi'' = 2\pi\frac{kB}{A}(3A-4B)\;\tau_2\,.$$

Now, formulas (9) and (10) of that same chapter give:

$$u'' = -\frac{k}{2A}(3A-4B)\frac{\partial}{\partial x}\left(\frac{A\tau_2}{A-B}+z\,\tau_1\right),$$

$$v'' = -\frac{k}{2A}(3A-4B)\frac{\partial}{\partial y}\left(\frac{A\tau_2}{A-B}+z\,\tau_1\right),$$

$$w'' = \frac{k}{2A}(3A-4B)\left(\frac{A\tau_1}{A-B}-z\,\tau\right).$$

Therefore, if one takes (9) into account and sets:

$$K = \frac{k(3A-4B)}{4\pi(A-B)},$$

for brevity, then one will finally arrive at the formulas:

$$u = K\frac{\partial}{\partial x}\int \tau \log(z+r)\,ds\,, \qquad v = K\frac{\partial}{\partial y}\int \tau \log(z+r)\,ds\,, \qquad w = -\,K\int \frac{\tau\,ds}{r}\,.$$

For the bodies that behave close to the laws of Navier and Poisson, the constant K is taken to be less than one-fifth of k.

CHAPTER XVI

THE SAINT-VENANT PROBLEM

1. – The study of the deformations of a cylindrical body under the action of a force that is applied to just its base is particularly important in practice. Since the general theory encounters difficulties in calculation that are presently insuperable, one might think of simplifying the question by first treating it in a particular case. The passage to the general case is largely justified (*) by means of theoretical and experimental considerations then. Each element of the base is the base of an infinitely-thin cylinder that belongs to the given body and is called a *fiber*. The cylindrical body is then composed of an infinitude of longitudinal fibers, and we would like to confine ourselves to the study of the deformation that do not provoke lateral tensions between the contiguous fibers, in such a way that they will deform as if they were independent of each other. In addition, suppose that the body is endowed with incomplete isotropy, and that the isotropy axis is parallel to the generators of the cylinder (which will be essentially true for the bodies that one deals with in experimental practice).

2. – Assume that the *xy*-plane is one of the bases, and suppose that they are cross-sections of the cylinder. Hence, the *z*-axis will be parallel to the generators. One has the following formula for expressing the elastic potential:

$$-\Pi = \tfrac{1}{2}(A-2B)\,\Theta^2 + B\,(a^2+b^2+c^2+2f^2+2g^2+2h^2) + Cc^2 + 2A'\,(f^2+g^2) + 2B'\,(h^2-ab),$$

in which A, B, C, A', B' are constant quantities. In order to annul the lateral tensions, it is necessary and sufficient that one must have:

$$p_{xx} = 0, \qquad p_{yy} = 0, \qquad p_{xy} = 0 \tag{1}$$

in all of the body, i.e.:

$$\frac{\partial \Pi}{\partial a} = 0, \qquad \frac{\partial \Pi}{\partial b} = 0, \qquad \frac{\partial \Pi}{\partial h} = 0.$$

The following relations must then be true:

$$(A-2B)\,\Theta + 2Ba = 2B'b, \qquad (A-2B)\,\Theta + 2Bb = 2B'a, \qquad h = 0.$$

The first one implies that:

$$a = b = -\eta\, c, \tag{2}$$

in which one sets:

(*) See CLEBSCH's *Traité*, pp. 175.

$$\eta = \frac{A-2B}{2(A-B-B')}.$$

The conditions $h = 0$ and $a = b$ say that two surface elements that are parallel to the generators and mutually perpendicular will also be perpendicular after the deformation and that any element that is perpendicular to the generators will experience a dilatation or contraction around each point that is the same in all directions. The constant h, which measures the ratio of the transverse contraction to the unit longitudinal elongation, is called the *coefficient of transverse contraction.* Its value is 1/4 in those completely isotropic bodies for which one has $A = 3B$.

3. – As a result of that, one needs to determine the pressures that develop in the elements of the transverse sections of the cylinder. That is an important path of study, because the components p_{xz}, p_{yz}, p_{zz} of those pressures, with the opposite signs, will clearly serve to tell us the forces that one needs to apply to the bases in order to produce a given deformation when one sets $z = 0$ and z equal to the length l of the cylinder (*). One will then immediately have:

$$-p_{zz} = -\frac{\partial \Pi}{\partial c} = (A - 2B)\,\Theta + 2\,(B + C)\,c = Ec,$$

in which one sets:

$$E = (A - 2B)\,(1 - 2\eta) + 2\,(B + C).$$

Hence, for a given unit elongation in the direction of the generators of the cylinder, a tension will develop in that direction that is proportional to the constant E, and for that reason, it is called the *coefficient of longitudinal elasticity*. One then calls the constant $G = B + A'$ the *coefficient of transverse elasticity*, because one has:

$$-p_{xz} = -\frac{1}{2}\frac{\partial \Pi}{\partial g} = 2Gg, \qquad -p_{yz} = -\frac{1}{2}\frac{\partial \Pi}{\partial f} = 2Gf\,;$$

that is to say, for given shears (along the isotropy axis) of the line elements perpendicular to the axis, the tangential components of the tension will be proportional to G. Note that one will have $G = \frac{2}{5}E$ for isotropic crystalline solids ($A = 3B$).

4. – We have seen that u and v necessarily satisfy the conditions:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

(*) One can also see that by writing out the boundary equations that relate to the bases and observing that one has $\frac{dx}{dn} = 0$, $\frac{dy}{dn} = 0$, $\frac{dz}{dn} = 1$ for the free base $z = l$.

It will then follow that $u + iv$ is a function of the complex variable $\zeta = x + iy$, that is to say, only the first of the two of the independent variables x, y, z can enter into the function $u + iv$ of those variables, and only in the combination ζ, in such a way that one will have:

$$u + iv = \varphi(\zeta, z),$$

and the determination of u and v can be accomplished at once when one has determined the function φ. Meanwhile, the indefinite equations of equilibrium will become:

$$\frac{\partial}{\partial z}\cdot\frac{\partial\Pi}{\partial g} = 0, \qquad \frac{\partial}{\partial z}\cdot\frac{\partial\Pi}{\partial f} = 0, \qquad \frac{1}{2}\frac{\partial}{\partial x}\cdot\frac{\partial\Pi}{\partial g} + \frac{1}{2}\frac{\partial}{\partial y}\cdot\frac{\partial\Pi}{\partial f} + \frac{\partial}{\partial z}\cdot\frac{\partial\Pi}{\partial c} = 0. \tag{3}$$

The first one allows one to write, in succession:

$$\frac{\partial^2 u}{\partial z^2} + \frac{\partial^2 w}{\partial x\,\partial z} = 0, \qquad \eta\frac{\partial^2 u}{\partial z^2} = \frac{\partial}{\partial x}\left(-\eta\frac{\partial w}{\partial z}\right) = \frac{\partial^2 u}{\partial x^2}.$$

Therefore, one can give the first two equations in (3) the forms:

$$\frac{\partial^2 u}{\partial x^2} = \eta\frac{\partial^2 u}{\partial z^2}, \qquad \frac{\partial^2 v}{\partial y^2} = \eta\frac{\partial^2 v}{\partial z^2}.$$

Multiply the second one by i and add it to the first one; that will give:

$$\frac{\partial^2 u}{\partial x^2} + i\frac{\partial^2 v}{\partial y^2} = \eta\frac{\partial^2}{\partial z^2}(u+iv) = \eta\frac{\partial^2\varphi}{\partial z^2}.$$

The left-hand side is conjugate to:

$$\frac{\partial^2 u}{\partial x^2} - i\frac{\partial^2 v}{\partial y^2} = \frac{\partial^2 u}{\partial x^2} + i\frac{\partial^2 v}{\partial x^2} = \frac{\partial^2}{\partial x^2}(u+iv) = \frac{\partial^2\varphi}{\partial\zeta^2}.$$

Hence, if one generally represents the number that is conjugate to ζ by $\bar{\zeta}$ then one will see that φ must satisfy the equation:

$$\overline{\frac{\partial^2\varphi}{\partial\zeta^2}} = \eta\frac{\partial^2\varphi}{\partial z^2}. \tag{4}$$

Now, note that the left-hand side does not depend upon ζ, but upon $\bar{\zeta}$, while the right-hand side cannot depend upon $\bar{\zeta}$. Hence, they must both reduce to functions of only z.

It will then follow that $\frac{\partial^2\varphi}{\partial\zeta^2}$ does not contain ζ, and for that to be true, it is necessary that φ must have the form:

$$\varphi = P\,\zeta^2 + 2Q\,\zeta + R,$$

in which P,Q, R are functions of only z. If one substitutes this in (4) then one will get:

$$2\bar{P} = \eta\left(\frac{d^2P}{dz^2}\zeta^2 + 2\frac{d^2Q}{dz^2}\zeta + \frac{d^2E}{dz^2}\right);$$

hence:

$$\frac{d^2P}{dz^2} = 0, \qquad \frac{d^2Q}{dz^2} = 0, \qquad \eta\frac{d^2E}{dz^2} = 2\bar{P}.$$

With a convenient representation of the constants involved, one can then set:

$$P = -\frac{\eta}{2}[(\alpha_1 + \beta_1 z) - i\,(\alpha_2 + \beta_2 z)],$$

$$Q = -\frac{\eta}{2}(\alpha + \beta z) - \frac{i}{2}(\alpha_0 + \beta_0 z)\,.$$

As for the function R, it must satisfy the equation:

$$\frac{d^2R}{dz^2} = -\,(\alpha_1 + \beta_1 z) - i\,(\alpha_2 + \beta_2 z),$$

and upon integrating this, one will then have:

$$R = (\alpha' + i\alpha'') + (\beta' + i\beta'')\,z - (\alpha_1 + i\alpha_2)\frac{z^2}{2} - (\beta_1 + i\beta_2)\frac{z^3}{6}\,.$$

Hence, one will finally obtain the expressions for u and v by taking the real part and the coefficient of i, respectively, in the expression:

$$-\,\eta\,[(\alpha_1 + \beta_1 z) - i\,(\alpha_2 + \beta_2 z)]\left(\frac{x^2 - y^2}{2} + ixy\right)$$
$$-\,\eta\,[(\alpha + \beta z) + i\,(\alpha_0 + \beta_0 z)](x + iy) + R.$$

In that way, one will arrive at the following formulas:

$$u = -\,\eta\left(\alpha x + \alpha_1\frac{x^2 - y^2}{2} + \alpha_2 xy\right) - \eta\,z\left(\beta x + \beta_1\frac{x^2 - y^2}{2} + \beta_2 xy\right)$$

$$+ (\alpha' + \alpha_0\, y) + (\beta' + \beta_0\, y)\, z - \alpha_1 \frac{z^2}{2} - \beta_1 \frac{z^3}{6},$$

$$v = -\,\eta \left(\alpha y + \alpha_1 xy + \alpha_2 \frac{y^2 - x^2}{2} \right) - \eta\, z \left(\beta y + \beta_1 xy + \beta_2 \frac{y^2 - x^2}{2} \right)$$

$$+ (\alpha'' + \alpha_0\, x) + (\beta'' - \beta_0\, x)\, z - \alpha_2 \frac{z^2}{2} - \beta_2 \frac{z^3}{6}.$$

5. – We now need to determine w. If we integrate (2) then we will get:

$$w = F(x, y) + (\alpha + \alpha_1\, x + \alpha_2\, y)\, z + (\beta + \beta_1\, x + \beta_2\, y)\, \frac{z^2}{2}. \tag{5}$$

In addition, we must also satisfy the third indefinite equation – i.e., the last of (3), which takes the form:

$$\frac{\partial g}{\partial x} + \frac{\partial f}{\partial y} + k \frac{\partial c}{\partial z} = 0,$$

in which k represents the constant $E\,/\,2G$, which will reduce to $1 + \eta$ in completely isotropic bodies. If one observes that:

$$2\left(\frac{\partial g}{\partial x} + \frac{\partial f}{\partial y} \right) = \frac{\partial}{\partial x}\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \right) + \frac{\partial}{\partial y}\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} \right) = \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial}{\partial z}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right)$$

$$= \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} - 2\eta \frac{\partial^2 w}{\partial z^2}$$

then the last equation will become:

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + 2(k - \eta) \frac{\partial^2 w}{\partial z^2} = 0.$$

Hence, if one takes w to have the expression in (5) then:

$$\frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2} + 2\,(k - \eta)\,(\beta + \beta_1\, x + \beta_2\, y) = 0. \tag{6}$$

In particular, one can take F to be equal to:

$$\Phi = -\,(k - \eta) \left(\beta \frac{x^2 + y^2}{2} + \beta_1 xy^2 + \beta_2 x^2 y \right),$$

and it is clear that one can add any linear function of x, y to Φ. Hence, if one sets:

$$F = \Omega + \Phi + \gamma - \beta' x - \beta'' y$$

then one will see that Ω is a *harmonic function of the variables* x, y, whose value at a given point can also be assigned arbitrarily, due to the presence of the arbitrary constant γ in the expression for F. If one knows Ω then one will have $w = \Omega + \Phi + \Psi$, in which one sets:

$$\Psi = \gamma - \beta' x - \beta'' y + (\alpha + \alpha_2 x + \alpha_2 y)\, z + (\alpha + \alpha_2 x + \alpha_2 y)\frac{z^2}{2}.$$

Finally, one will know the pressures on the elements of the transverse sections, since one found in § **3** that:

$$-p_{xz} = G\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right), \qquad -p_{yz} = G\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right), \qquad -p_{zz} = E\frac{\partial w}{\partial z},$$

and consequently, if one sets u, v, w equal to the preceding expressions then:

$$-p_{xz} = G\left[-k\beta x + \beta_0\, y - \eta\beta_1\frac{x^2}{2} - (2k - 3\eta)\beta_1\frac{y^2}{2} - (2k - \eta)\beta_2\, xy + \frac{\partial\Omega}{\partial x}\right],$$

$$-p_{xz} = G\left[-k\beta y + \beta_0\, x - \eta\beta_2\frac{y^2}{2} - (2k - 3\eta)\beta_2\frac{x^2}{2} - (2k - \eta)\beta_1\, xy + \frac{\partial\Omega}{\partial y}\right],$$

$$-p_{zz} = E\,[(\alpha + \alpha_1 x + \alpha_2 y) + (\alpha + \alpha_1 x + \alpha_2 y)\, z].$$

6. – The question is then reduced to the determination of Ω. For now, we know only that the function must satisfy Laplace's equation at all points of the cross-section of the cylinder. However, in order to be able to determine it, we also need to see what conditions it must satisfy on the contour of that section. In order to do that, consider the boundary equations that relate to the lateral surface of the cylinder, and first of all observe that, by virtue of (1) and the fact that dz / dn has the value zero on that surface, the first two equations will be satisfied identically, while the third one will reduce to:

$$\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)\frac{\partial x}{\partial n} + \left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right)\frac{\partial y}{\partial n} = 0,$$

and one will get:

$$\frac{dw}{dn} = -\left(\frac{\partial u}{\partial z}\cdot\frac{\partial x}{\partial n} + \frac{\partial v}{\partial z}\cdot\frac{\partial y}{\partial n}\right).$$

Hence, dw / dn can be regarded as known, and therefore the value of:

$$\frac{d\Omega}{dn} = \frac{dw}{dn} - \frac{d\Phi}{dn} - \frac{d\Psi}{dn}$$

will be known at any point of the contour. It will then follow that the function Ω can be considered to be determined completely for a given cross-section (up to an arbitrary additive constant), provided that there is no incompatibility between the indefinite equation $\Delta^2\Omega = 0$ and the set of values that are assigned to $d\Omega / dn$ on the contour. It is known that in order for that to be true it is necessary that one must have:

$$\int \frac{d\Omega}{dn} d\sigma = - \int \Delta^2\Omega \, ds = 0,$$

where the first integral extends over the entire contour and the second one extends over the area that it encloses. Having said that, observe that:

$$\int \frac{d\Omega}{dn} d\sigma = - \int \left(\frac{\partial u}{\partial z}\frac{dx}{dn} + \frac{\partial v}{\partial z}\frac{dy}{dn} \right) d\sigma = \int \left(\frac{\partial^2 u}{\partial x \partial z} + \frac{\partial^2 v}{\partial y \partial z} \right) ds = - 2\eta \int \frac{\partial^2 w}{\partial z^2} ds$$

$$= - 2\eta \int (\beta + \beta_1 x + \beta_2 y) \, ds .$$

Similarly, if one recalls that Φ satisfies (6) then:

$$\int \frac{d\Phi}{dn} d\sigma = - \int \Delta^2\Phi \, ds = 2 (k - \eta) = 2\eta \int (\beta + \beta_1 x + \beta_2 y) \, ds .$$

Finally, observe that Ψ is linear in x and y, so:

$$\int \frac{d\Psi}{dn} d\sigma = - \int \Delta^2\Psi \, ds = 0.$$

Hence:

$$\int \frac{d\Omega}{dn} d\sigma = -2k \int (\beta + \beta_1 x + \beta_2 y) \, ds = - 2ks \, (\beta + \beta_1 x_0 + \beta_2 y_0),$$

in which x_0 and y_0 represent the coordinates of the center of gravity of the section. It will then follow that one must have the relation:

$$\beta + \beta_1 x_0 + \beta_2 y_0 = 0$$

between the constants β, β_1, β_2 in order for the function Ω to exist.

7. – The solution that was obtained previously contains the constants:

$$\alpha,\ \alpha',\ \alpha'',\ \alpha_0,\ \alpha_1,\ \alpha_2,\ \beta,\ \beta',\ \beta'',\ \beta_0,\ \beta_1,\ \beta_2,\ \gamma,$$

which will reduce to *twelve* truly arbitrary ones, by virtue of (7). Six of them will be determined if one prevents the rigid motion of a particle. For example, if one supposes that the center of gravity of a base is kept fixed and that when a surface element is taken from inside of the base, it will remain in the base plane in such a way that one of its line elements does not vary in direction, then one will naturally ask what the conditions would be under which no motion of the body would be allowed if it were rigid. If they are satisfied then one will first need to have that $u = 0$, $v = 0$, $w = 0$ when $x = 0$, $y = 0$, $z = 0$, if the origin is located at the fixed point. Meanwhile, one knows that this choice of origin will reduce the condition (7) to $\beta = 0$. If one then directs the x-axis along that line element (which has been required to not displace laterally) then the displacement v of the point $(dx, 0, 0)$, i.e., $\left(\frac{\partial v}{\partial x}\right)_0 dx$, like the displacements w of all the points $(dx, dy, 0)$, i.e., $\left(\frac{\partial w}{\partial x}\right)_0 dx + \left(\frac{\partial w}{\partial y}\right)_0 dy$, must be zero, and one must then have:

$$\left(\frac{\partial v}{\partial x}\right)_0 = 0, \qquad \left(\frac{\partial w}{\partial x}\right)_0 = 0, \qquad \left(\frac{\partial w}{\partial y}\right)_0 = 0.$$

Finally, in order for the function Ω to be determined completely, one must impose the value 0 at the origin. Our formulas will not suffer any specialization as a result of that, as was observed in § **5**. All of those conditions demand that one must have:

$$\alpha' = 0, \qquad \alpha'' = 0, \qquad \alpha_0 = 0, \qquad \beta = 0, \qquad \gamma = 0, \qquad \beta' = \left(\frac{\partial \Omega}{\partial x}\right)_0, \qquad \beta'' = \left(\frac{\partial \Omega}{\partial y}\right)_0,$$

so only the *six* arbitrary constants α, α_1, α_2, β_0, β_1, β_2 will remain in the expressions for the displacements, tensions, etc. The displacements are:

$$u = -\eta\left(\alpha x + \alpha_1 \frac{x^2 - y^2}{2} + \alpha_2\, xy\right) - \eta z\left(\beta_1 \frac{x^2 - y^2}{2} + \beta_2\, xy\right) + \beta_0\, yz$$
$$- \alpha_1 \frac{z^2}{2} - \beta_1 \frac{z^3}{6} + \left(\frac{\partial \Omega}{\partial x}\right)_0 z,$$

$$v = -\eta\left(\alpha y + \alpha_1 xy + \alpha_2 \frac{y^2 - x^2}{2}\right) - \eta z\left(\beta_1\, xy + \beta_2 \frac{y^2 - x^2}{2}\right) - \beta_0\, xz$$
$$- \alpha_2 \frac{z^2}{2} - \beta_2 \frac{z^3}{6} + \left(\frac{\partial \Omega}{\partial y}\right)_0 z,$$

$$w = -(k-\eta)(\beta_1\, xy^2 + \beta_2\, x^2 y) + (\alpha + \alpha_1 x + \alpha_2 y)\, z + (\beta_1 x + \beta_2 y)\,\frac{z^2}{2} + \Omega$$
$$- \left(\frac{\partial\Omega}{\partial x}\right)_0 x - \left(\frac{\partial\Omega}{\partial y}\right)_0 y\,.$$

8. – The function Ω, which depends upon the form of the section, is determined for any particular form by means of the indefinite equation $\Delta^2\Omega = 0$ and the boundary conditions – viz., $\Omega = 0$ for $x = 0$, $y = 0$, and:

$$\frac{d\Omega}{dn} = U\,\frac{dx}{dn} + V\,\frac{dy}{dn}$$

on the contour – in which U and V are two known functions of x and y, for which, it is useful to recall the expressions for them from what we saw in § **6**:

$$U = -\left(\frac{\partial u}{\partial z} + \frac{\partial\Phi}{\partial x} + \frac{\partial\Psi}{\partial x}\right), \qquad V = -\left(\frac{\partial v}{\partial z} + \frac{\partial\Phi}{\partial y} + \frac{\partial\Psi}{\partial y}\right).$$

Hence, when one substitutes the preceding expressions for u, v and the ones that were found to be pointed out in § **5**, one will get:

$$U = -\beta_0\, y + \frac{\beta_1}{2}[\eta\, x^2 + (2k - 3\eta)\, y^2] + (2k-\eta)\,\beta_2\, xy,$$

$$V = \beta_0\, x + \frac{\beta_2}{2}[\eta\, y^2 + (2k - 3\eta)\, x^2] + (2k-\eta)\,\beta_1\, xy.$$

9. – In order to determine functions that depend upon only the form of the section and not on the β coefficients, one sets:

$$\Omega = \beta_0\,\Omega_0 + \beta_1\,\Omega_1 + \beta_2\,\Omega_2\,.$$

The functions Ω_0, Ω_1, Ω_2 must be harmonic, zero at the origin, and satisfy the conditions:

$$\frac{d\Omega_0}{dn} = -y\,\frac{dx}{dn} + x\,\frac{dy}{dn}\,,$$

$$\frac{d\Omega_1}{dn} = \tfrac{1}{2}[\eta x^2 + (2k-3\eta)y^2]\frac{dx}{dn} + (2k-\eta)xy\,\frac{dy}{dn}\,,$$

$$\frac{d\Omega_2}{dn} = (2k - \eta)xy\frac{dx}{dn} + \tfrac{1}{2}[\eta y^2 + (2k - 3\eta)xy]\frac{dy}{dn}$$

on the contour.

It is important to observe that if the section is symmetric with respect to the x-axis then Ω_1 will be an *even* function of y. Indeed, the equation that relates to the boundary and the indefinite equations will not be altered when one changes y into $-y$, since the values of dy / dn at two points of the contour that are symmetric with respect to the x-axis will clearly differ only in sign. Since the function that satisfies those conditions will be unique, it will be Ω_1. However, that function will be *odd* in x when the section is symmetric with respect to the y-axis, since changing x into $-x$ will change only the sign of $d\Omega_1 / dn$, and the function $-\Omega_1$ will certainly satisfy the new conditions. Hence, one will have:

$$\Omega_1(x, -y) = \Omega_1(x, y), \qquad \Omega_1(-x, y) = -\Omega_1(x, y).$$

One shows analogously that Ω_2 will be odd in y and even in x under the stated hypothesis:

$$\Omega_2(x, -y) = -\Omega_2(x, y), \qquad \Omega_2(-x, y) = \Omega_2(x, y).$$

It will then follow that $\dfrac{\partial\Omega_1}{\partial y}$ and $\dfrac{\partial\Omega_2}{\partial x}$ are odd functions of y and x, respectively; therefore:

$$\left(\frac{\partial\Omega_1}{\partial y}\right)_0 = 0, \qquad \left(\frac{\partial\Omega_2}{\partial x}\right)_0 = 0.$$

Finally, if the section is symmetric with respect to the two axes then Ω_0 will be odd in x, as well as in y, and consequently, one will have:

$$\left(\frac{\partial\Omega_0}{\partial x}\right)_0 = 0, \quad \left(\frac{\partial\Omega_0}{\partial y}\right)_0 = 0, \quad \left(\frac{\partial^2\Omega_0}{\partial x^2}\right)_0 = 0, \quad \left(\frac{\partial^2\Omega_0}{\partial y^2}\right)_0 = 0,$$

while the mixed second derivative is even in y, as well as in x, and it will therefore not necessarily be zero at the center of the section, since it is not necessarily true that $\dfrac{\partial\Omega_1}{\partial x}$ and $\dfrac{\partial\Omega_2}{\partial y}$ will be zero.

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CHAPTER XVII

APPLICATIONS TO PRACTICAL PROBLEMS

1. – We shall now pass from the particular problems that we have treated so far to the practical problems that rest upon the hypothesis that two systems of forces that are statically equivalent will not produce different deformations. In reality, that is not the case, because the manner by which the deforming forces are distributed over the surface of the body will also influence the deformation. However, observations have shown that the difference that is due to that cause will become noticeable only in the vicinity of the points of applications when they occupy a small part of the surface. Meanwhile, we can recall that if we make an exception for two regions that are close to the bases then a cylindrical body whose length is much larger than the dimensions of the transversal sections will behave like a collection of fibers that are independent of each other. Having said that, we suppose that the forces that are applied to the free bases consist of the force (X, Y, Z) and the couple ($\mathcal{L}$, $\mathcal{M}$, $\mathcal{N}$) and examine whether we will find that the forces that act to produce the deformations that were studied up to now will consist of the same force and the same couple. In order to do that, we seek to express X, Y, Z, $\mathcal{L}$, $\mathcal{M}$, $\mathcal{N}$ in terms of the constants α, α_1, α_2, β_0, β_1, β_2 . We then suppose that the first six quantities are given, so the formulas that we obtain will serve to determine the latter constants inversely and will consequently characterize a deformation that is negligible in comparison to the one that is produced effectively in almost all of the cylinder.

2. – One then calculates the integrals:

$$\begin{aligned} X &= \int L\,ds\,, & \mathcal{L} &= \int [Ny - M(z-l)]\,ds = \int Ny\,ds\,, \\ Y &= \int M\,ds\,, & \mathcal{M} &= \int [L(z-l) - Nx]\,ds = -\int Nx\,ds\,, \\ Z &= \int N\,ds\,, & \mathcal{N} &= \int (Mx - Ly)\,ds\,, \end{aligned}$$

which extend over all of the free base. With that, one will have:

$$L = -p_{xz}\,, \qquad M = -p_{yz}\,, \qquad N = -p_{zz}\,,$$

and consequently:

$$X = G\left[-\frac{\beta_1\eta}{2}\int x^2 ds - \frac{\beta_1}{2}(2k-3\eta)\int y^2 ds - (2k-\eta)\beta_1\int xy\,ds + \int\frac{\partial\Omega}{\partial x}ds\right],$$

$$Y = G\left[-\frac{\beta_2\eta}{2}\int y^2 ds - \frac{\beta_2}{2}(2k-3\eta)\int x^2 ds - (2k-\eta)\beta_1\int xy\,ds + \int\frac{\partial\Omega}{\partial y}ds\right].$$

In order to simplify the calculations, refer to the figure whose axes are the principal axes of inertia and let λ and μ represent the radii of inertia, in such a way that:

$$\int x^2\,ds = \lambda^2\,s, \qquad \int y^2\,ds = \mu^2\,s, \qquad \int x^2\,ds = 0.$$

The preceding expressions will become:

$$X = -\frac{Gs}{2}\beta_1[\eta\lambda^2 + (2k-3\eta)\mu^2] + G\int\frac{\partial\Omega}{\partial x}ds\,,$$

$$Y = -\frac{Gs}{2}\beta_2[\eta\mu^2 + (2k-3\eta)\lambda^2] + G\int\frac{\partial\Omega}{\partial y}ds\,.$$

One directly finds $Z = Esa$ for the longitudinal component of the resultant. One also needs to calculate the two integrals that appear in X and Y, and it is noteworthy that their values can be obtained with knowing Ω. First of all, from Green's theorem, one has:

$$\int\left(x\frac{d\Omega}{dn} - \Omega\frac{dx}{dn}\right)d\sigma = 0.$$

Hence:

$$\int\frac{\partial\Omega}{\partial x}ds = -\int\Omega\frac{dx}{dn}d\sigma = -\int x\frac{d\Omega}{dn}d\sigma\,,$$

or

$$\int\frac{\partial\Omega}{\partial x}ds = -\int\left(U\frac{dx}{dn} + V\frac{dy}{dn}\right)d\sigma = \int\left(\frac{\partial Ux}{\partial x} + \frac{\partial Vx}{\partial y}\right)ds\,,$$

and finally:

$$\int\frac{\partial\Omega}{\partial x}ds = \int U\,ds + \int\left(\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y}\right)x\,ds\,,$$

$$\int\frac{\partial\Omega}{\partial y}ds = \int V\,ds + \int\left(\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y}\right)y\,ds\,.$$

On the other hand, if one adopts the expressions that were found in the preceding § **8** then one will have immediately that:

$$\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} = 2k\,(\beta_1\,x + \beta_2\,y),$$

and consequently:

$$\int\left(\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y}\right)x\,ds = 2k\,\beta_1\,\lambda^2\,s, \qquad \int\left(\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y}\right)y\,ds = 2k\,\beta_2\,\mu^2\,s.$$

In addition:

$$\int U\,ds = \frac{\beta_1 s}{2}[\eta\lambda^2 + (2k - 3\eta)\mu^2],$$

$$\int V\,ds = \frac{\beta_2 s}{2}[\eta\mu^2 + (2k - 3\eta)\lambda^2].$$

Hence:

$$\int \frac{\partial\Omega}{\partial x}\,ds = \frac{\beta_1 s}{2}[(4k + \eta)\lambda^2 + (2k - 3\eta)\mu^2],$$

$$\int \frac{\partial\Omega}{\partial y}\,ds = \frac{\beta_2 s}{2}[(4k + \eta)\mu^2 + (2k - 3\eta)\lambda^2],$$

and finally:

$$X = E\,\lambda^2 s\beta_1, \qquad Y = E\,\mu^2 s\beta_2.$$

Thus, we already know that if we are given the resultant then the constants:

$$\beta_1 = \frac{X}{E\lambda^2 s}, \qquad \beta_2 = \frac{Y}{E\mu^2 s}, \qquad \alpha = \frac{Z}{Es}$$

will be determined. In order to determine the other ones, observe that:

$$\int Nx\,ds = E\lambda^2 s\,(\alpha_1 + \beta_1 z), \qquad \int Ny\,ds = E\mu^2 s\,(\alpha_2 + \beta_2 z).$$

However, if one sets $\lambda^2 + \mu^2 = \rho^2$ then one will get:

$$\begin{aligned}\int (Mx - Ly)\,ds = -\,G\rho^2 s\,\beta_0 &+ G\int\left(x\frac{\partial\Omega}{\partial y} - y\frac{\partial\Omega}{\partial x}\right)ds \\ &+ \frac{G\beta_1}{2}\int[(2k - 3\eta)y^2 + (4k + \eta)x^2]\,y\,ds \\ &- \frac{G\beta_2}{2}\int[(2k - 3\eta)x^2 + (4k + \eta)y^2]\,x\,ds.\end{aligned}$$

If the figure is symmetric with respect to the axes, as is ordinarily the case, the integrals that are multiplied by β_1 and β_2 will obviously be zero, and the last formula will simplify. The observations that were made at the end of the preceding chapter in regard to the parity of the functions Ω permit us to say, in addition, that the integrals:

$$\int\left(x\frac{\partial\Omega_1}{\partial y} - y\frac{\partial\Omega_1}{\partial x}\right)ds, \qquad \int\left(x\frac{\partial\Omega_2}{\partial y} - y\frac{\partial\Omega_2}{\partial x}\right)ds$$

are zero, because their elements can be grouped into pairs with equal values and opposite signs. It follows that:

$$\int \left(x\frac{\partial \Omega}{\partial y} - y\frac{\partial \Omega}{\partial x} \right) ds = \beta_0 \int \left(x\frac{\partial \Omega_0}{\partial y} - y\frac{\partial \Omega_0}{\partial x} \right) ds .$$

Therefore, the formulas that serve to determine α_1, α_2, β_0 are:

$$\alpha_1 + \beta_1 \, l = -\frac{\mathcal{M}}{E\lambda^2 s}, \quad \alpha_2 + \beta_2 \, l = -\frac{\mathcal{L}}{E\mu^2 s}, \quad \beta_0 = \frac{-\mathcal{N}}{G\left[\rho^2 s - \int \left(x\frac{\partial \Omega_0}{\partial y} - y\frac{\partial \Omega_0}{\partial x} \right) ds \right]} .$$

3. – The formulas that were established in the preceding paragraph enable us to analyze an arbitrary deformation and show that it can always be made to result from four special deformations:

a) *Traction:* Of the six arbitrary constants, one supposes that only α is non-zero.

Since the β will always be zero then, one will have $\Omega = 0$. The final formulas in § **7** will then become:

$$u = -\eta \, \alpha x, \quad v = -\eta \, \alpha y, \quad w = \alpha z$$

and characterizes a *traction*, under which the cylinder will contract transversally and elongate by:

$$w = \alpha z = \frac{Zl}{Es} .$$

However, in the case where α is negative, one will have compression. The transversal sections will remain planar and the fibers will remain rectilinear under those deformations. They are produced by only the force Z, since the formulas in the preceding paragraph will show that X, Y, $\mathcal{L}$, $\mathcal{M}$, $\mathcal{N}$ are zero.

b) *Torsion:* One annuls all of the constants except β_0, so the function Ω will reduce to $\beta_0 \, \Omega_0$, and the usual formulas will give:

$$u = \beta_0 \, z \left[y + \left(\frac{\partial \Omega_0}{\partial x} \right)_0 \right], \quad v = -\beta_0 \, z \left[x - \left(\frac{\partial \Omega_0}{\partial y} \right)_0 \right], \quad w = \beta_0 \left[\Omega_0 - \left(\frac{\partial \Omega_0}{\partial x} \right)_0 x - \left(\frac{\partial \Omega_0}{\partial y} \right)_0 y \right].$$

Apart from w, one studies the projection of the motion onto a transverse section, and one transports the origin to the point O of that section that has the coordinates $\left(\frac{\partial \Omega_0}{\partial y} \right)_0$,

$-\left(\frac{\partial \Omega_0}{\partial x}\right)_0$ and coincides with the center when the section is symmetric with respect to both axes, in which case, one will also have $w = \beta_0\,\Omega_0$. One will then see directly that the displacement (u, v) consists of a very small rotation $-\beta_0\,z$ around O' in the opposite sense to the one in which the hand of a clock will move. Since the points O' are all along one fiber, one can say that all of the sections rotate around that fiber, and the angles will rotate such that they will vary from 0 at one extreme of the other cylinder to $-\beta_0\,l$ the other one. One will then have a *torsion* that is produced by only the couple $\mathcal{N}$, which acts in the plane of the extreme section and makes it rotate through an angle of:

$$\omega = -\beta_0\,l = \frac{\mathcal{N}\,l}{G\left[\rho^2 s - \int\left(x\frac{\partial \Omega_0}{\partial y} - y\frac{\partial \Omega_0}{\partial x}\right)ds\right]}.$$

The transverse sections do not remain planar. Indeed, in the vicinity of the central fiber, one will have:

$$w = \frac{\beta_0}{2}\left[\left(\frac{\partial^2 \Omega_0}{\partial x^2}\right)_0 x^2 + 2\left(\frac{\partial^2 \Omega_0}{\partial x\,\partial y}\right)_0 xy + \left(\frac{\partial^2 \Omega_0}{\partial y^2}\right)_0 y^2\right],$$

and since the coefficients of the extreme terms differ only in sign, one will see that the section is curved into the form of a hyperbolic paraboloid. However, from what was said at the end of the preceding chapter, if the section is symmetric with respect to the axes then one will have:

$$w = \beta_0\left(\frac{\partial^2 \Omega_0}{\partial x\,\partial y}\right)_0 xy,$$

and therefore those axes will divide the section into four regions, such that two opposite regions will be below the original plane, while the other two will be above it. Nonetheless, if Ω_0 is zero then the section will deform while remaining in its own plane. In that case, the preceding value of the angle of torsion will reduce to:

$$\omega = \frac{\mathcal{N}\,l}{G\rho^2 s},$$

and one will then get the formula that is adopted "in practice," and which is established precisely by freely making the hypothesis that the sections will remain planar.

c) *Simple flexure:* One keeps only the constant α_1, so one once more has $\Omega = 0$, and:

$$u = -\frac{\alpha_1}{2}\,[\eta\,(x^2 - y^2) + z^2], \qquad v = -\,\eta\,\alpha_1\,x\,y, \qquad w = \alpha_1\,x\,z.$$

In order to see how one deforms a fiber, one keeps x and y. v is then constant, and the elimination of z from u and w will show that all of the fibers curve parabolically in planes that are parallel to Oxy (viz., *the planes of flexure*). In particular, for the central fiber, one has $u = -(\alpha_1/2)\, z^2$, $v = 0$, $w = 0$. That fiber will then bend into a parabola in that plane of flexure, and the maximum deflection (*saetta di flessione*) of the old position will be:

$$u = -\frac{\alpha_1}{2} l^2 = \frac{M\, l^2}{2E\lambda^2 s}.$$

As one sees, that deformation is produced by just one couple that acts in the plane of flexure. It is known that the transverse sections will remain planar. If one keeps α_2, instead of α_1, then one will still have a flexure, but this time, it will be parallel to the plane Oyz.

d) *Complex flexure.* The deformations that are characterized by the constants β_1 and β_2 are more complicated and are consequently produced by a tangential force X or Y. Consider the one that corresponds to β_1, and in order for the force X to not be accompanied by the couple $\mathcal{M}$, set $\alpha_1 + \beta_1\, l = 0$, instead of $\alpha_1 = 0$. Recall the formulas of § **7** in the preceding chapter then, and suppose that only the constants β_1 and $\alpha_1 = -\beta_1\, l$ are non-zero. In the case of a symmetric section, that will give:

$$u = \beta_1 \left[\eta \frac{x^2 - y^2}{2}(l - z) + \frac{lz^2}{2} - \frac{z^2}{6} + \left(\frac{\partial \Omega_1}{\partial x}\right)_0 z \right],$$

$$v = \beta_1\, \eta\, xy\, (l - x),$$

$$w = \beta_1 \left[lxz + \frac{xz^2}{2} - (k - \eta)xy^2 + \Omega_1 - \left(\frac{\partial \Omega_1}{\partial x}\right)_0 x \right].$$

It is noteworthy that the hypothesis $z = l$ will make u and v independent of x and y. Hence, if one leaves aside the longitudinal displacements then one can say that the free base will displace laterally as if it were rigid. Not only will the transverse sections not remain planar then, but they will assume various forms (viz., third-order surfaces) according to their position in the cylinder. Indeed, as in the case of torsion, w will no longer be a function of only the variables x and y. In addition, the fibers will be bent only slightly, except the ones that are situated on the plane OXZ (viz., the plane of flexure). In particular, for $x = 0$, $y = 0$, one will have $v = 0$, $w = 0$. Therefore, the central fiber will bend in that plane into the form of a cubic parabola, since one also has:

$$u = \beta_1 \left[\frac{lz^2}{2} - \frac{z^2}{6} + \left(\frac{\partial \Omega_1}{\partial x}\right)_0 z \right].$$

For $z = l$, one will get the value of the maximum deflection, and one will recover the practical formula, which neglects the last term and which is the only one that depends upon the form of the section and which assumes values whose ratios to l^2 are negligible, like the ratio of the area of that section to l^2. The approximate value of the maximum will then be:

$$u = \frac{\beta_1}{3} l^3 = \frac{X\, l^3}{3E\lambda^2 s}.$$

4. – Let us apply the preceding results to the circular cylinder. First of all, we must determine the function Ω. The first of the boundary conditions that was written down in § **9** (Chap. XVI) will become:

$$x\frac{\partial \Omega_0}{\partial x} + y\frac{\partial \Omega_0}{\partial y} = 0.$$

That must be true for $x^2 + y^2 = R^2$. However, it and the Laplace equation can be satisfied for any pair of values of x and y by taking Ω_0 to be constant, and since one must have $\Omega_0 = 0$ at a point, one must have $\Omega_0 = 0$ over the entire section. Therefore, if one refers to what was said in the preceding paragraph then one can assert that under the torsion of cylinders with circular transverse sections, those sections will remain planar. It is precisely that special case that has induced the experimenters to assume hypothetically that the latter fact was true for all forms, while it will already cease to be true for the elliptic forms. Indeed, for an ellipse with semi-axes a and b, one will get:

$$\Omega_0 = \frac{a^2 - b^2}{a^2 + b^2}\, xy\,;$$

one will have $w = \beta_0\, \Omega_0$: All of the sections change into equal pieces of a hyperbolic paraboloid. Let us turn to the circular sections and determine Ω_0. For $x^2 + y^2 = R^2$, that function must satisfy the condition:

$$x\frac{\partial \Omega_1}{\partial x} + y\frac{\partial \Omega_1}{\partial y} = \tfrac{1}{2}\eta\, x^3 + \left(3k - \tfrac{5}{2}\eta\right) xy^2\,.$$

It is natural that one would try to verify this by taking Ω_1 to be a function of degree three in x and y. Meanwhile, one will see (Chap. XVI, § **9**) that Ω_1 is *even* in y and *odd* in x, and one will then need to have:

$$\Omega_1 = ax + \beta x^3 + \gamma y^2 + \delta xy^2.$$

In order to satisfy the Laplace equation, it will be necessary for one to have $6\beta x + 2\,(\gamma + \delta x) = 0$ identically; i.e., $\gamma = 0$, $\delta = -3\beta$. Consequently:

$$\Omega_1 = \alpha x + \beta\,(x^3 - 3xy^2).$$

Now, the boundary conditions will become:

$$\alpha x + \beta (x^3 - 3xy^2) = \tfrac{1}{2}\eta\, x^3 + \left(3k - \tfrac{5}{2}\eta\right) xy^2 ,$$

and it can be satisfied identically on the contour if, after multiplying $x^2 + y^2$ by the term αx and the other terms by R^2, one takes:

$$\alpha + 3\beta R^2 = \frac{\eta}{2} R^2 , \qquad \alpha - 9\beta R^2 = \left(3k - \tfrac{5}{2}\eta\right) R^2 ,$$

i.e.:

$$\alpha = (3k - \eta)\, R^2, \qquad \beta = -\tfrac{1}{4}(k - \eta),$$

and finally:

$$\Omega_1 = \tfrac{1}{4}[(3k - \eta)\, R^2\, x - (k - \eta)\, (x^3 - 3xy^2)].$$

One will then deduce that:

$$\left(\frac{\partial \Omega_1}{\partial x}\right)_0 = (3k - \eta)\frac{R^2}{4} , \qquad \left(\frac{\partial \Omega_1}{\partial y}\right)_0 = 0.$$

On the other hand:

$$\int x^2 ds = \int y^2 ds = \tfrac{1}{2}\int r^2 ds = \pi\int_0^R r^3 dr = \frac{\pi}{4} R^4 = \tfrac{1}{4} R^2 s ,$$

and consequently, $\lambda = \mu = \tfrac{1}{2} R$, $\rho = R / \sqrt{2}$. It follows from all of this that the angle of torsion, the maximum of simple flexure, that of complex flexure, etc., will have the values:

$$\frac{2\mathcal{N}\, l}{\pi G R^4} , \qquad \frac{2\mathcal{M}\, l^2}{\pi E R^4} , \qquad \frac{2X\, l^2}{3\pi E R^4} , \qquad \text{etc.}$$

The last maximum is, more precisely:

$$\frac{4X\, l^3}{3\pi E R^4}\left[1 + \tfrac{3}{4}(3k - \eta)\left(\frac{R}{l}\right)^2\right].$$

In the case of elliptic sections, the radii λ and μ have the values a / 2 and b / 2. The formula that is adopted in practice in order to express the angle of torsion is:

$$\omega = \frac{4\mathcal{N}\, l}{\pi G\, ab\, (a^2 + b^2)} .$$

When the section is strongly eccentric, that will induce serious errors. In order to correct them, one first needs to calculate the integral:

$$\int \left(x\frac{\partial \Omega_0}{\partial y} - y\frac{\partial \Omega_0}{\partial x} \right) ds = \frac{a^2 - b^2}{a^2 + b^2} \int (x^2 - y^2)\, ds = \frac{a^2 - b^2}{a^2 + b^2} (\lambda^2 - \mu^2)\, s\,.$$

The denominator in the exact formula then changes into the product of *Gs* with:

$$\lambda^2 + \mu^2 - \frac{a^2 - b^2}{a^2 + b^2} (\lambda^2 - \mu^2) = \frac{a^2 - b^2}{a^2 + b^2} (b^2\, \lambda^2 + a^2 \mu^2) = \frac{a^2 b^2}{a^2 + b^2}\,.$$

Hence, the true angle of torsion is:

$$\omega = \frac{(a^2 + b^2)\mathcal{N}\, l}{\pi G\, a^2 b^2}\,.$$

This value, which is always greater than the empirical one, has been confirmed sufficiently by all of the experiments (*). The mathematical theory of elasticity has been accused of not being in agreement with experiments, so one must then address the various theoretical claims that have been contrived in order to justify *a posteriori* and generalize beyond measure some empirical formulas by combining contradictory hypotheses that are unjustified and unjustifiable. As Clebsch said (**), the differences that one finds:

> "get one in the habit of attributing imperfections to the theory, rather than to the irregularities that are committed in the applications. Is that perhaps why, too often, the misconception is created that *the theory itself* has been greatly discredited in certain circles?"

(*) See a note in CLEBSCH's *Traité*, pp. 209, and to some other notes of SAINT-VENANT on pp. 210, *et seq.*

(**) One should read the interesting § 38 in *Traité*.

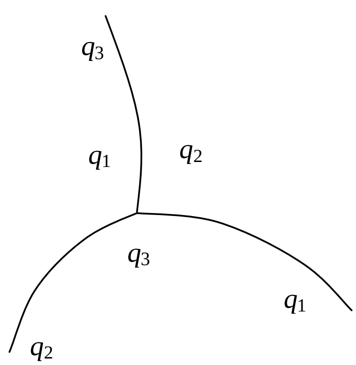
CHAPTER XVIII

SOME NOTIONS RELATING TO CURVILINEAR COORDINATES

1. – Let x_1, x_2, x_3 be the Cartesian coordinates of a point. The equation $f(x_1, x_2, x_3) = q$ represents a surface for each value of q. If one considers q to be a *parameter* that is capable of taking on all real values then that equation will represent a simple infinitude of surfaces. One now considers three families of surfaces:

$$f_1(x_1, x_2, x_3) = q_1\,, \qquad f_2(x_1, x_2, x_3) = q_2\,, \qquad f_3(x_1, x_2, x_3) = q_3\,,$$

such that three surfaces that are taken from the three families in any way will generally have just one common point. That point will then be individuated by the special values that the parameters q_1, q_2, q_3 have on the three surfaces that contain it. Therefore, q_1, q_2, q_3 can be assumed to be the *coordinates* of the point. The three surfaces and their lines of intersection will be called the *coordinate surfaces* and the *coordinate lines*, resp., of the point, and will take the name of *corresponding parameters.* Therefore, the *line* q_i is that coordinate line along which only the parameter q_i varies, while the *surface* q_i is the coordinate surface upon which the parameter q_i remains constant. That system of coordinates is called *orthogonal* if the coordinate surfaces are mutually-perpendicular at every point. If that were the case then it would be clear that the coordinate lines would also prove to be mutually-perpendicular.



2. – The derivatives of x_1, x_2, x_3 with respect to q_1, q_2, q_3 are obviously proportional to the *direction cosines of the tangent to the line* q_i at the point considered, since when one moves that point along that line, q_i will vary like a function of *only the arc* traversed. Its cosines will therefore be:

$$\frac{1}{Q_i}\frac{\partial x_1}{\partial q_i}, \qquad \frac{1}{Q_i}\frac{\partial x_2}{\partial q_i}, \qquad \frac{1}{Q_i}\frac{\partial x_3}{\partial q_i}, \tag{1}$$

if one takes:

$$Q_i^2 = \left(\frac{\partial x_1}{\partial q_i}\right)^2 + \left(\frac{\partial x_2}{\partial q_i}\right)^2 + \left(\frac{\partial x_3}{\partial q_i}\right)^2 . \tag{2}$$

If one then lets σ_i represent the contact arc length along the line q_i when one starts from an arbitrarily-fixed origin, and if one observes that the cosines (1) are also equal to the derivatives of x_1, x_2, x_3 with respect to σ_i then one will see immediately that $d\sigma_i = Q_i\, dq_i$. In other words, Q_i is the coefficient that one needs to give in order to obtain the line element along the line q_i. That observation serves precisely to expeditiously exhibit the functions Q_1, Q_2, Q_3, which have great significance in that theory, in various particular cases. As for the general expression for the line element, it is obviously given by the formula:

$$d\sigma^2 = Q_1^2\, dq_1^2 + Q_2^2\, dq_2^2 + Q_3^2\, dq_3^2$$

in orthogonal systems, since $d\sigma$ measures the diagonal of a rectangular parallelpiped whose sides are measured by $d\sigma_1$, $d\sigma_2$, $d\sigma_3$, up to higher infinitesimals.

3. – The first partial derivatives of q_i are proportional to the *direction cosines of the normal to the surface* q_i. Those cosines are therefore equal to those derivatives, divided by $\pm\sqrt{\Delta q_i}$. Since the normal to the surface q_i is not that of the tangent to the line q_i, one will have:

$$\frac{1}{\sqrt{\Delta q_i}}\frac{\partial q_i}{\partial x_1} = \frac{1}{Q_i}\frac{\partial x_1}{\partial q_i}, \quad \frac{1}{\sqrt{\Delta q_i}}\frac{\partial q_i}{\partial x_2} = \frac{1}{Q_i}\frac{\partial x_2}{\partial q_i}, \quad \frac{1}{\sqrt{\Delta q_i}}\frac{\partial q_i}{\partial x_3} = \frac{1}{Q_i}\frac{\partial x_3}{\partial q_i}$$

when one fixes the positive senses of the directions being varied conveniently. When those formulas are multiplied by $\frac{\partial q_i}{\partial x_1}$, $\frac{\partial q_i}{\partial x_2}$, $\frac{\partial q_i}{\partial x_3}$, respectively, and summed, that will give:

$$\sqrt{\Delta q_i} = \frac{1}{Q_i}. \tag{3}$$

Therefore:

$$Q_i \frac{\partial q_i}{\partial x_j} = \frac{1}{Q_i}\frac{\partial x_j}{\partial q_i}. \tag{4}$$

4. – Recall that the determinant of the cosines of the angles that one line of an orthogonal triad makes with those of another orthogonal triad is equal to ± 1, and that can always be done in such a way that it is equal to $+ 1$, in which case, one will merely have that any element of the determinant is equal to its own algebraic complement. Since the determinant:

$$\nabla = \begin{vmatrix} \dfrac{\partial x_1}{\partial q_1} & \dfrac{\partial x_1}{\partial q_2} & \dfrac{\partial x_1}{\partial q_3} \\ \dfrac{\partial x_2}{\partial q_1} & \dfrac{\partial x_2}{\partial q_2} & \dfrac{\partial x_2}{\partial q_3} \\ \dfrac{\partial x_3}{\partial q_1} & \dfrac{\partial x_3}{\partial q_2} & \dfrac{\partial x_3}{\partial q_3} \end{vmatrix}$$

is deduced from the determinant of the cosines, divided by the verticals to Q_1, Q_2, Q_3, respectively, one will see immediately that $\nabla = Q_1\, Q_2\, Q_3$. In addition, the elements of each vertical to ∇ are proportional to their own algebraic complements. On the other hand, let us take:

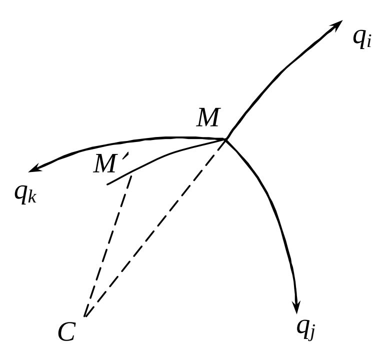
$$s_k = \frac{\partial x_1}{\partial q_k}\frac{\partial^2 x_1}{\partial q_i\,\partial q_j} + \frac{\partial x_2}{\partial q_k}\frac{\partial^2 x_2}{\partial q_i\,\partial q_j} + \frac{\partial x_3}{\partial q_k}\frac{\partial^2 x_3}{\partial q_i\,\partial q_j},$$

for the moment, in which i, j, k represent an arbitrary assignment of the indices 1, 2, 3. The orthogonality condition between the lines q_i and q_j amounts to saying that when:

$$\frac{\partial x_1}{\partial q_i}\frac{\partial x_1}{\partial q_j} + \frac{\partial x_2}{\partial q_i}\frac{\partial x_2}{\partial q_j} + \frac{\partial x_3}{\partial q_i}\frac{\partial x_3}{\partial q_j} = 0$$

is differentiated with respect to q_k , that will give $s_i + s_j = 0$. It will then follow that $s_1 = s_2 = s_3 = 0$, and by virtue of the aforementioned property, one can give the equality $s_k = 0$ the form:

$$\begin{vmatrix} \dfrac{\partial x_1}{\partial q_i} & \dfrac{\partial x_1}{\partial q_j} & \dfrac{\partial^2 x_1}{\partial q_i\,\partial q_j} \\ \dfrac{\partial x_2}{\partial q_i} & \dfrac{\partial x_2}{\partial q_j} & \dfrac{\partial^2 x_2}{\partial q_i\,\partial q_j} \\ \dfrac{\partial x_3}{\partial q_i} & \dfrac{\partial x_3}{\partial q_j} & \dfrac{\partial^2 x_3}{\partial q_i\,\partial q_j} \end{vmatrix} = 0. \qquad (5)$$



5. – Having assumed that, we seek the lines of curvature of the surface q_i . Displace the point on that surface M in such a way that the position M (q_1, q_2, q_3) will go to the position:

$$M'(q_1 + \delta q_1 , q_2 + \delta q_2 , q_3 + \delta q_3),$$

in which $\delta q_i = 0$. Since MM' is along a line of curvature, it will be necessary that the normals at M and M' to the surface q_i must meet. Let $x_1 - \lambda \frac{\partial x_1}{\partial q_i}$, $x_2 - \lambda \frac{\partial x_2}{\partial q_i}$, $x_3 - \lambda \frac{\partial x_3}{\partial q_i}$ be the Cartesian coordinates of a point on the first normal. Since it will belong to the second normal, one will need to have that the variations of the coordinates under the passage from M to M' must be zero; i.e., that one must have:

$$\left\{\begin{aligned} \delta x_1 - \lambda\,\delta \frac{\partial x_1}{\partial q_i} - \frac{\partial x_1}{\partial q_i}\,\delta\lambda = 0, \\ \delta x_2 - \lambda\,\delta \frac{\partial x_2}{\partial q_i} - \frac{\partial x_2}{\partial q_i}\,\delta\lambda = 0, \\ \delta x_3 - \lambda\,\delta \frac{\partial x_3}{\partial q_i} - \frac{\partial x_3}{\partial q_i}\,\delta\lambda = 0. \end{aligned}\right. \tag{6}$$

The elimination of λ and $\delta\lambda$ will lead immediately to the condition:

$$\begin{vmatrix} \frac{\partial x_1}{\partial q_i} & \delta x_1 & \delta \frac{\partial x_1}{\partial q_i} \\ \frac{\partial x_2}{\partial q_i} & \delta x_2 & \delta \frac{\partial x_2}{\partial q_i} \\ \frac{\partial x_3}{\partial q_i} & \delta x_3 & \delta \frac{\partial x_3}{\partial q_i} \end{vmatrix} = 0,$$

which is equivalent to:

$$\begin{vmatrix} \frac{\partial x_1}{\partial q_i} & \frac{\partial x_1}{\partial q_j}\delta q_j + \frac{\partial x_1}{\partial q_k}\delta q_k & \frac{\partial^2 x_1}{\partial q_i\,\partial q_j}\delta q_j + \frac{\partial^2 x_1}{\partial q_i\,\partial q_k}\delta q_k \\ \frac{\partial x_2}{\partial q_i} & \frac{\partial x_2}{\partial q_j}\delta q_j + \frac{\partial x_2}{\partial q_k}\delta q_k & \frac{\partial^2 x_2}{\partial q_i\,\partial q_j}\delta q_j + \frac{\partial^2 x_2}{\partial q_i\,\partial q_k}\delta q_k \\ \frac{\partial x_3}{\partial q_i} & \frac{\partial x_3}{\partial q_j}\delta q_j + \frac{\partial x_3}{\partial q_k}\delta q_k & \frac{\partial^2 x_3}{\partial q_i\,\partial q_j}\delta q_j + \frac{\partial^2 x_3}{\partial q_i\,\partial q_k}\delta q_k \end{vmatrix} = 0.$$

If one decomposes the columns then one will get a quadratic form in δq_i and δq_k in the left-hand side in which the coefficients of δq_j^2 and δq_k^2 will be:

$$\begin{vmatrix} \dfrac{\partial x_1}{\partial q_i} & \dfrac{\partial x_1}{\partial q_j} & \dfrac{\partial^2 x_1}{\partial q_i\,\partial q_j} \\ \dfrac{\partial x_2}{\partial q_i} & \dfrac{\partial x_2}{\partial q_j} & \dfrac{\partial^2 x_2}{\partial q_i\,\partial q_j} \\ \dfrac{\partial x_3}{\partial q_i} & \dfrac{\partial x_3}{\partial q_j} & \dfrac{\partial^2 x_3}{\partial q_i\,\partial q_j} \end{vmatrix}, \qquad \begin{vmatrix} \dfrac{\partial x_1}{\partial q_i} & \dfrac{\partial x_1}{\partial q_k} & \dfrac{\partial^2 x_1}{\partial q_i\,\partial q_k} \\ \dfrac{\partial x_2}{\partial q_i} & \dfrac{\partial x_2}{\partial q_k} & \dfrac{\partial^2 x_2}{\partial q_i\,\partial q_k} \\ \dfrac{\partial x_3}{\partial q_i} & \dfrac{\partial x_3}{\partial q_k} & \dfrac{\partial^2 x_3}{\partial q_i\,\partial q_k} \end{vmatrix},$$

resp., which will therefore be equal to zero, by virtue of (5). What will remain are the terms in $\delta q_i\ \delta q_k$, whose coefficients cannot be zero *identically*, since all of the lines that emanate from M would *always* be lines of curvature. Therefore, $\delta q_i = 0$ or $\delta q_k = 0$. One then has Dupin's theorem:

In any triply-orthogonal system, the surfaces of two families will trace out all of the lines of curvature on any surface of the third family (*).

6. – In order to find the principal radii of curvature of the surface q_i at the point M, one needs to displace that point along the coordinate lines q_j and q_k . If r_{ij} and r_{ik} are the radii that are defined in those two directions then their values will obviously be given by the expressions $Q_i\ \lambda$, in which λ is calculated by means of equations (6). Suppose that one would like to calculate r_{ij} , for example. In that case, $\delta q_i = 0$, $\delta q_k = 0$, and one will have $\delta = \dfrac{\partial}{\partial q_j} \cdot \delta q_j$, and therefore equations (6) will become:

$$\begin{cases} \dfrac{\partial x_1}{\partial q_j} - \lambda \dfrac{\partial^2 x_1}{\partial q_i\,\partial q_j} - \dfrac{\partial x_1}{\partial q_i}\dfrac{\partial \lambda}{\partial q_j} = 0, \\[2ex] \dfrac{\partial x_2}{\partial q_j} - \lambda \dfrac{\partial^2 x_2}{\partial q_i\,\partial q_j} - \dfrac{\partial x_2}{\partial q_i}\dfrac{\partial \lambda}{\partial q_j} = 0, \\[2ex] \dfrac{\partial x_3}{\partial q_j} - \lambda \dfrac{\partial^2 x_3}{\partial q_i\,\partial q_j} - \dfrac{\partial x_3}{\partial q_i}\dfrac{\partial \lambda}{\partial q_j} = 0. \end{cases}$$

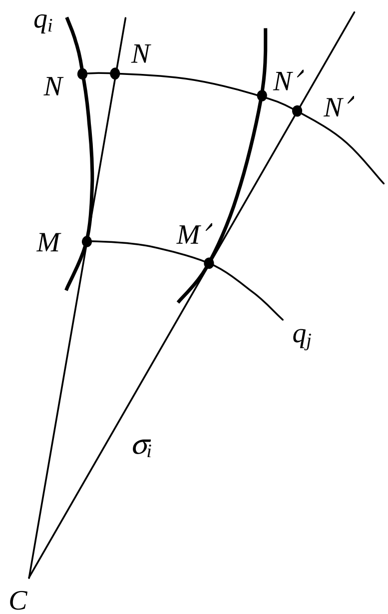
If one multiplies this by $\dfrac{\partial x_1}{\partial q_j}$, $\dfrac{\partial x_2}{\partial q_j}$, $\dfrac{\partial x_3}{\partial q_j}$, respectively, then sums, and takes into account the orthogonality of the lines q_i and q_j then one will get:

$$Q_j^2 = \frac{\lambda}{2}\frac{\partial}{\partial q_i} Q_j^2 = \lambda\, Q_j\, \frac{\partial Q_j}{\partial q_i}.$$

(*) See LAMÉ: *Leçons sur les coordonnées*, §§ XXVIII, XXIV; or BIANCHI: *Lezioni di Geometria differenziale* (Pisa, Nistri, 1885-86, § 122). Another interesting proof that is based upon kinematic considerations is due to Beltrami (Rendiconti dell'Istituto lombardo, 1872, pp. 483)

Therefore:

$$\frac{1}{r_{ij}} = \frac{1}{Q_i\,Q_j}\frac{\partial Q_j}{\partial q_i}. \tag{7}$$



7. – Formula (7) can be proved geometrically in a very simple way by taking an infinitesimal arc MM' along the line q_j and considering the arc NN' into which MM' will change when q_i becomes $q_i + dq_i$. One will then have:

$$MN = d\sigma_i\,, \qquad MM' = d\sigma_j\,, \qquad NN' = d\sigma_j + \frac{\partial Q_j}{\partial q_i}\,dq_i\,dq_j\,,$$

and the obvious relation:

$$\frac{NN' - MM'}{MM'} = \frac{MN}{r_{ij}}$$

will transform into formula (7) directly. That mainly serves to exhibit the idiosyncrasy that Lamé adopted in his classic *Leçons*. Constantly preoccupied with giving the geometric form of the results of his calculations, that illustrious inventor of curvilinear coordinates always introduced the curvatures of the coordinate arcs and the derivatives with respect to those arcs in the final formulas. That serves to exhibit the true significance of the formula "inasmuch as" (as Laplace said) it is interesting that the results of the analysis will apply to space, and conversely, one can read of the modifications to lines and surfaces and the variations of the motions of bodies in the equations that express them. That rapprochement of geometry and analysis sheds a new light upon the two sciences: The intellectual operations of the second one, which are made sensible by the images in the first one, are easier to comprehend and more interesting to follow, and when observations realize those images and transform the geometric results into natural laws, one's view of that sublime spectacle will embody the noblest of all the pleasures that are reserved for human nature.

CHAPTER XIX

DIGRESSION ON DIFFERENTIAL PARAMETERS

1. – If one is given a function V of the orthogonal Cartesian coordinates x_1, x_2, x_3 then the functions:

$$\Delta V = \left(\frac{\partial V}{\partial x_1}\right)^2 + \left(\frac{\partial V}{\partial x_2}\right)^2 + \left(\frac{\partial V}{\partial x_3}\right)^2, \qquad \Delta^2 V = \frac{\partial^2 V}{\partial x_1^2} + \frac{\partial^2 V}{\partial x_2^2} + \frac{\partial^2 V}{\partial x_3^2},$$

which one calls the first-order and second-order *differential parameters* (*) of V, have the property that they do not depend upon the system of axes with respect to which they are calculated. Indeed, it is easy to prove that if one rotates the orthogonal triad of axes arbitrarily then the values of the differential parameters at each point will remain invariant (**). That will give the geometric and mechanical significance of those parameters, which is *independent of the orientations of the axes.* For example, let V_1 be the mean of the values that the function V assumes over a spherical surfaces of infinitesimal radius r, and let V_0 be the value of V at the center of the sphere. If one ignores infinitesimals of order higher than the second then one will have:

$$V = V_0 + x_1 \frac{\partial V}{\partial x_1} + \cdots + \frac{1}{2}\left(x_1^2 \frac{\partial^2 V}{\partial x_1^2} + \cdots + 2 x_1 x_2 \frac{\partial^2 V}{\partial x_1 \partial x_2} + \cdots \right),$$

when one supposes that all of the derivatives are calculated at the center of the sphere. If one multiplies by ds and integrates over the entire spherical surface, while observing that:

$$\int x_i \, ds = 0, \qquad \int x_2 \, x_3 \, ds = \ldots = 0, \qquad \int x_1^2 \, ds = \ldots = \tfrac{1}{3}\int r^2 ds = \tfrac{1}{3} s r^2,$$

then upon dividing by sr^2, one will get:

(*) Considered for the first time by LAMÉ. See his *Leçons sur les coordonnées curvilignes*, § III. Whoever wants to learn about the general theory of differential parameters, when it is established upon a purely analytical basis, should read a paper by Prof. BELTRAMI that was published in the *Accademia di Bologna* (vol. VIII of series 2, pp. 549) and another one by Prof. RICCI in the *Annali di Matematica* (vol. XIV, series 2, pp. 1).

(**) For an infinitesimal rotation of the axes (cf., Chap. III, § 3), the first and second derivatives of V with respect to x_1 will change by:

$$a_2 \frac{\partial V}{\partial x_3} - a_3 \frac{\partial V}{\partial x_2}, \quad 2\left(a_2 \frac{\partial^2 V}{\partial x_1 \partial x_3} - a_3 \frac{\partial V}{\partial x_1 \partial x_2} \right),$$

resp., and therefore the means of the variations of ΔV and $\Delta^2 V$ will bee:

$$\Sigma\left(a_2 \frac{\partial V}{\partial x_3} - a_3 \frac{\partial V}{\partial x_2} \right)\frac{\partial V}{\partial x_1} = 0, \quad \Sigma\left(a_2 \frac{\partial^2 V}{\partial x_1 \partial x_3} - a_3 \frac{\partial^2 V}{\partial x_1 \partial x_2} \right) = 0,$$

resp.

$$\lim_{r=0} \frac{V_1 - V_0}{r^2} = \tfrac{1}{6} \Delta^2 V .$$

One can establish an analogous equality for ΔV, when one considers the mean of the values of V^2. That equality will make the invariance property of ΔV and $\Delta^2 V$ obvious.

2. – Let α_1, α_2, α_3 be the cosines that define an arbitrary direction. The integral of $\alpha_i \alpha_j\, ds$, when extended over a spherical surface of radius 1, will have the value $4\pi / 3$ or 0, according to whether $i = j$ or $i \neq j$, resp. Indeed, in the first case, one can write:

$$\int \alpha_i^2\, ds = \int_0^{\pi}\int_0^{2\pi} \cos^2\theta \sin\theta\, d\theta\, d\psi = 2\pi \int_0^{\pi} \cos^2\theta \sin\theta\, d\theta = \frac{4\pi}{3},$$

and in the second:

$$\int \alpha_i\, \alpha_j\, ds = \int_0^{\pi}\int_0^{2\pi} \sin^2\theta \cos\theta \cos\psi\, d\theta\, d\psi = 0.$$

Given the above, the derivative of V in the direction considered will be:

$$\frac{dV}{d\sigma} = \alpha_1 \frac{\partial V}{\partial x_1} + \alpha_2 \frac{\partial V}{\partial x_2} + \alpha_3 \frac{\partial V}{\partial x_3} . \tag{1}$$

If one squares this, multiplies by ds, and integrates over the whole sphere then it will follow that:

$$\int \left(\frac{dV}{d\sigma}\right)^2 ds = \frac{4\pi}{3} \Delta V.$$

Similarly, if one observes that:

$$\frac{d^2V}{d\sigma^2} = \alpha_1^2 \frac{\partial^2 V}{\partial x_1^2} + \alpha_2^2 \frac{\partial^2 V}{\partial x_2^2} + \cdots + 2\alpha_1\, \alpha_3 \frac{\partial^2 V}{\partial x_1\, \partial x_2} + \ldots \tag{2}$$

and multiplies by ds and integrates then one will get:

$$\int \frac{d^2V}{d\sigma^2} ds = \frac{4\pi}{3} \Delta^2 V.$$

Therefore, the values of $\Delta^2 V$ and ΔV are proportional to the mean values of the second derivatives and the squares of the first derivatives, resp., over all directions that can be considered around each point (*).

(*) BOUSSINESQ: *Cours d'Analyse infinitesimale*, t. 1, 2nd fasc., pp. 57, 71.

3. – The invariant significance of ΔV can also be deduced immediately from formula (1) when one considers the direction that is defined by the cosines to be proportional to the first derivative of V. Let θ be the angle that the direction makes with the other one (α_1, α_2, α_3). One can then give (1) the form:

$$\frac{dV}{d\sigma} = \sqrt{\Delta V} \cdot \cos \theta.$$

The *maximum* value of dV / ds is then attained for $\theta = 0$, and is precisely $\sqrt{\Delta V}$. In other words, $\sqrt{\Delta V}$ is the derivative of V in the direction along which the function varies *most rapidly*. Note that this direction is precisely that of the normal to the surface V = const. Similarly to formula (2), one deduces that the directions for which the second derivative is zero will constitute a quadric cone, and it is obvious that it cannot depend upon the axes. The discriminant of the quadratic form (2), which is the Hessian of the function V, is then *invariant*, and the sum of its principal minors of first and second order also enjoys the invariant property, as well as $\Delta^2 V$, in particular. Other interesting interpretations can be given for the differential parameters. They all serve to show that "the second differential parameter is, so to speak, the *derivative par excellence*, namely, a derivative that expresses how much more generality one has in the mode of variation of the function." (*). That is why that parameter has the highest importance in all branches of mathematical physics.

> "When a class of physical phenomena depends upon the variations of a certain function, almost always that will come about by means of its second differential parameters, as if it were a *natural derivative*, which is more essential, simpler, and at the same time, more complete in all of the partial derivatives (which are chosen more or less arbitrarily) than the derivatives that one is used to considering." (**).

4. – The differential parameters also present themselves quite naturally when one makes use of *quaternions* (***), or complex numbers that result from the combination (of the additive kind) of a *scalar* (viz., a real number, with no sense of direction) and a *vector*, which is a rectilinear segment in space whose magnitude and direction are defined by its projections onto three arbitrary orthogonal axes. Therefore, the displacement (u, v, w) of a point is a vector $\omega = iu + jv + kw$, and the double rotation of the medium is another vector $\mathcal{T} = i\,\mathcal{T}_1 + j\,\mathcal{T}_2 + k\,\mathcal{T}_3$, in which the *units* i, j, k, which are linearly-independent, are subject to only the conditions that:

$$i^2 = j^2 = k^2 = -1, \qquad jk = -kj = i, \qquad ki = -ik = j, \qquad ij = -ji = k.$$

The dilatation Θ is, however, scalar. When one applies the Hamilton operator (****):

(*) BOUSSINESQ, *loc. cit.*, pp. 72.
(**) LAMÉ, *Leçons sur les coordonnées curvilignes*, § XV.
(***) I highly recommend that the reader should study the chapters "Kinematics" and "Physical Applications" in TAIT's *Elementary treatise on quaternions.*
(****) TAIT, *loc. cit.*, part two, pp. 35.

$$\nabla = i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}$$

to the displacement, which is meant to preserve the properties of the usual algebraic calculations (except for the commutative property of multiplication, if necessary), one will get:

$$\nabla\omega = -\Theta + \mathcal{T},$$

which is to say: The *dilatation* and twice the *rotation* of the medium are the scalar and vectorial part of $\nabla\omega$, respectively. However, when the operator ∇ is applied to a scalar, one can only say that the square of the modulus of the result is equal to precisely the first-order differential parameter of that scalar, namely, $|\nabla|^2 = \Delta$. If one repeats the first result of the operation ∇ once more then one will get:

$$\nabla^2 = -\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right) = -\Delta^2,$$

and that will obviously remain true when one operates on a vector in such a way that the second differential parameter is not the result of two operations of ∇, applied in succession, up to sign. It is useful to point out the great simplicity that the problems of elasticity will assume when one makes use of the symbols that were just defined. The three indefinite equations of elastic equilibrium in isotropic media (Chap. IV, § 5) can be summarized in one $\mathcal{F} = \nabla\varphi$, in which $\mathcal{F}$ is the vector that represents the volume forces – i.e., $iX + jY + kZ$ – and φ represents the quaternion $-A\Theta + B\,\mathcal{T}$, which is a slight modification of $\nabla\omega$. Moreover, the introduction of the vector $\Omega = iU + jV + kW$, which was considered in Chap. XII (§§ 4, 5), will permit one to write, first of all, $\nabla\Omega = 4\pi\,\varphi$, and finally $\nabla^2\Omega = 4\pi\,\mathcal{F}$. That makes it obvious that it is possible to always reduce the questions of elasticity to the Dirichlet problem. Those considerable simplifications should not appear marvelous, when one reflects that:

> "In physics, in order to reason, and not calculate, it is often desirable to avoid the explicit introduction of Cartesian coordinates, and to advantageously fix one's attention upon a point in space, taken by itself, and not upon its three coordinates, such as when one fixes one's attention upon the magnitude and direction of a force, and not upon its three components. That way of considering geometric and physical questions is more natural then the other one, and it is the first to come to mind. Nevertheless, the ideas that result from it were not developed completely up to the time in which Hamilton made a second great leap in the study of space thanks to the invention of the calculus of quaternions." (*)

5. Expression for ΔV in curvilinear coordinates. – One has:

$$\frac{\partial V}{\partial x_i} = \frac{\partial V}{\partial q_1}\frac{\partial q_1}{\partial x_i} + \frac{\partial V}{\partial q_2}\frac{\partial q_2}{\partial x_i} + \frac{\partial V}{\partial q_3}\frac{\partial q_3}{\partial x_i} \qquad (i = 1, 2, 3)\,.$$

If one then squares this and sums, while taking into account the orthogonality of the coordinate surfaces and formula (3) of the preceding chapter, then one will get:

(*) MAXWELL expressed this in the preliminaries (§ 10) of his immortal *Treatise on Electricity and Magnetism*. Here, one should also be careful to consult the whole text of these precious *preliminaries*.

$$\Delta V = \frac{1}{Q_1^2}\left(\frac{\partial V}{\partial q_1}\right)^2 + \frac{1}{Q_2^2}\left(\frac{\partial V}{\partial q_2}\right)^2 + \frac{1}{Q_3^2}\left(\frac{\partial V}{\partial q_3}\right)^2 . \tag{3}$$

6. Expression for $\Delta^2 V$ in curvilinear coordinates. – We first treat the special case of $V = q_i$. One deduces from formula (4) of the previous chapter that:

$$\frac{\partial^2 q_i}{\partial x_j^2} = \frac{\partial x_j}{\partial q_i}\frac{\partial}{\partial x_j}\frac{1}{Q_i^2} + \frac{1}{Q_i^2}\frac{\partial}{\partial x_j}\frac{\partial x_j}{\partial q_i};$$

setting j = 1, 2, 3 and summing will then give:

$$\Delta^2 q_i = \frac{\partial}{\partial x_j}\frac{1}{Q_i^2} + \frac{1}{Q_i^2}\sum_j \frac{\partial}{\partial x_j}\frac{\partial x_j}{\partial q_i}.$$

Now, if one observes the predicted formula (4) and inverts the orders of certain differentiations then one will have:

$$\frac{\partial}{\partial x_j}\frac{\partial x_j}{\partial q_i} = \frac{\partial q_1}{\partial x_j}\frac{\partial}{\partial q_1}\frac{\partial x_j}{\partial q_i} + \frac{\partial q_2}{\partial x_j}\frac{\partial}{\partial q_2}\frac{\partial x_j}{\partial q_i} + \frac{\partial q_3}{\partial x_j}\frac{\partial}{\partial q_3}\frac{\partial x_j}{\partial q_i}$$
$$= \frac{1}{Q_1^2}\frac{\partial x_j}{\partial q_1}\frac{\partial}{\partial q_i}\frac{\partial x_j}{\partial q_1} + \frac{1}{Q_2^2}\frac{\partial x_j}{\partial q_2}\frac{\partial}{\partial q_i}\frac{\partial x_j}{\partial q_2} + \frac{1}{Q_3^2}\frac{\partial x_j}{\partial q_3}\frac{\partial}{\partial q_i}\frac{\partial x_j}{\partial q_3}.$$

Summing will give:

$$\sum_j \frac{\partial}{\partial x_j}\frac{\partial x_j}{\partial q_i} = \frac{1}{Q_1^2}\left(\frac{\partial x_1}{\partial q_1}\frac{\partial}{\partial q_i}\frac{\partial x_1}{\partial q_1} + \frac{\partial x_2}{\partial q_1}\frac{\partial}{\partial q_i}\frac{\partial x_2}{\partial q_1} + \frac{\partial x_3}{\partial q_1}\frac{\partial}{\partial q_i}\frac{\partial x_3}{\partial q_1}\right)$$
$$+ \frac{1}{Q_2^2}\left(\frac{\partial x_1}{\partial q_2}\frac{\partial}{\partial q_i}\frac{\partial x_1}{\partial q_2} + \frac{\partial x_2}{\partial q_2}\frac{\partial}{\partial q_i}\frac{\partial x_2}{\partial q_2} + \frac{\partial x_3}{\partial q_2}\frac{\partial}{\partial q_i}\frac{\partial x_3}{\partial q_2}\right)$$
$$+ \frac{1}{Q_3^2}\left(\frac{\partial x_1}{\partial q_3}\frac{\partial}{\partial q_i}\frac{\partial x_1}{\partial q_3} + \frac{\partial x_2}{\partial q_3}\frac{\partial}{\partial q_i}\frac{\partial x_2}{\partial q_3} + \frac{\partial x_3}{\partial q_3}\frac{\partial}{\partial q_i}\frac{\partial x_3}{\partial q_3}\right).$$

In the meantime, if one takes into account the definition of Q_k [form. (2) of the preceding chapter] then one will have:

$$\frac{\partial x_1}{\partial q_k}\frac{\partial}{\partial q_i}\frac{\partial x_1}{\partial q_k} + \frac{\partial x_2}{\partial q_k}\frac{\partial}{\partial q_i}\frac{\partial x_2}{\partial q_k} + \frac{\partial x_3}{\partial q_k}\frac{\partial}{\partial q_i}\frac{\partial x_3}{\partial q_k} = \frac{1}{2}\frac{\partial}{\partial q_i}Q_k^2 = Q_k\frac{\partial Q_k}{\partial q_i}.$$

Therefore:

$$\sum_j \frac{\partial}{\partial x_j}\frac{\partial x_j}{\partial q_i} = \frac{1}{Q_1}\frac{\partial Q_1}{\partial q_i} + \frac{1}{Q_2}\frac{\partial Q_2}{\partial q_i} + \frac{1}{Q_3}\frac{\partial Q_3}{\partial q_i} = \frac{\partial}{\partial q_i}\log Q_1\, Q_2\, Q_3$$

and consequently:

$$\Delta^2 q_i = \frac{\partial}{\partial q_i}\frac{1}{Q_i^2} + \frac{1}{Q_i^2}\frac{\partial \log \nabla}{\partial q_i} = \frac{1}{\nabla}\left(\nabla \frac{\partial}{\partial q_i}\frac{1}{Q_i^2} + \frac{1}{Q_i^2}\frac{\partial \nabla}{\partial q_i}\right) = \frac{1}{\nabla}\frac{\partial}{\partial q_i}\frac{\nabla}{Q_i^2}. \tag{4}$$

7. – We now pass on to the general case. One has:

$$\frac{\partial^2 V}{\partial x_j^2} = \sum_i \frac{\partial V}{\partial q_i}\frac{\partial^2 q_i}{\partial x_j^2} + \sum_i \frac{\partial q_i}{\partial x_j}\frac{\partial}{\partial x_j}\frac{\partial V}{\partial q_i}.$$

Setting $j = 1, 2, 3$ and summing will then give:

$$\Delta^2 V = \sum_i \frac{\partial V}{\partial q_i}\,\Delta^2 q_i + \sum_i \left(\frac{\partial q_i}{\partial x_1}\frac{\partial}{\partial x_1}\frac{\partial V}{\partial q_i} + \frac{\partial q_i}{\partial x_2}\frac{\partial}{\partial x_2}\frac{\partial V}{\partial q_i} + \frac{\partial q_i}{\partial x_3}\frac{\partial}{\partial x_3}\frac{\partial V}{\partial q_i}\right).$$

The expression that is found in the second summation sign can also be written as:

$$\frac{1}{Q_i^2}\left(\frac{\partial x_1}{\partial q_i}\frac{\partial}{\partial x_1}\frac{\partial V}{\partial q_i} + \frac{\partial x_2}{\partial q_i}\frac{\partial}{\partial x_2}\frac{\partial V}{\partial q_i} + \frac{\partial x_3}{\partial q_i}\frac{\partial}{\partial x_3}\frac{\partial V}{\partial q_i}\right) = \frac{1}{Q_i^2}\frac{\partial}{\partial q_i}\frac{\partial V}{\partial q_i}.$$

Therefore:

$$\Delta^2 V = \sum_i \left(\frac{\partial V}{\partial q_i}\,\Delta^2 q_i + \frac{1}{Q_i^2}\frac{\partial^2 V}{\partial q_i^2}\right); \tag{5}$$

hence, by virtue of (4):

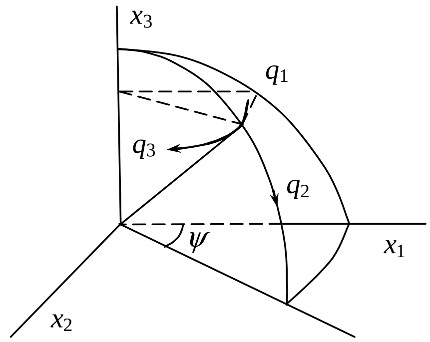
$$\Delta^2 V = \frac{1}{\nabla}\sum_i \left(\frac{\partial V}{\partial q_i}\frac{\partial}{\partial q_i}\frac{\nabla}{Q_i^2} + \frac{\nabla}{Q_i^2}\frac{\partial^2 V}{\partial q_i^2}\right) = \frac{1}{\nabla}\sum_i \frac{\partial}{\partial q_i}\left(\frac{\nabla}{Q_i^2}\frac{\partial V}{\partial q_i}\right),$$

i.e.:

$$\Delta^2 V = \frac{1}{Q_1\,Q_2\,Q_3}\left[\frac{\partial}{\partial q_1}\left(\frac{Q_2\,Q_3}{Q_1}\frac{\partial V}{\partial q_1}\right) + \frac{\partial}{\partial q_2}\left(\frac{Q_3\,Q_1}{Q_2}\frac{\partial V}{\partial q_2}\right) + \frac{\partial}{\partial q_3}\left(\frac{Q_1\,Q_2}{Q_3}\frac{\partial V}{\partial q_3}\right)\right].$$

That is the very important *Lamé formula* (*).

(*) *Leçons…*, § XIV. The proof that is given there does not differ, in substance, from the one that LAMÉ carried out in §§ XII, XIII, XIV of his *Leçons*. A more general formula that relates to the case of n variables can be found in Prof. BRIOSCHI's *Teorica dei determinanti* (pp. 93).

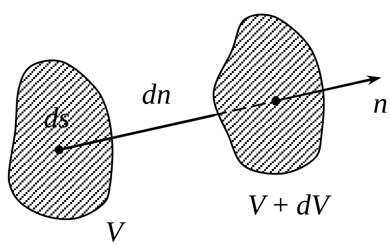


8. – Apply this to the system of *polar coordinates*. – In that system, one has a family of concentric spheres, a family of cones of rotations, with their vertices at the center of the sphere and a common axis, and finally, a pencil of planes with the same axis. The parameters are the radius r of each sphere, the angle θ that the generator of one of the cones makes with the axis of rotation, and the angle ψ that each plane makes with a fixed plane. At any point, a *radius*, a *meridian*, and a *parallel* cross at right angles. Along those lines, which are the *coordinate lines*, the line elements are dr, $r\,d\theta$, $r \sin\theta\, d\psi$. On the other hand, one knows that these elements are expressed by $Q_1\,dr$, $Q_2\,d\theta$, $Q_3\,d\psi$. Therefore:

$$Q_1 = 1, \qquad Q_2 = r, \qquad Q_3 = r \sin\theta, \qquad \nabla = r^2 \sin\theta,$$

and Lamé's formula will become:

$$\Delta^2 V = \frac{1}{r^2}\left[\frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial V}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 V}{\partial\psi^2}\right].$$



9. Second proof (*). – Lamé's formula presents itself in a rather simple form when one seeks to establish the general equation of the theory of heat in curvilinear coordinates. Let V represent the temperature at the various points of a homogeneous and isotropic medium, and seek to express the quantity of heat that crosses a planar element ds during the time dt. Take an infinitesimally-close element that is parallel to the first one, with the temperature $V + dV$, and imagine that they form a *wall* of infinitesimal thickness dn. Note (**) that the difference $V - (V + dV)$ of the temperatures on the two

(*) Also due to LAMÉ, *Leçons*…, § XVI.
(**) See, for example, JAMIN, *Cours de physique*, 2nd ed., t. II, pp. 335.

faces of the wall, divided by the thickness and multiplied by a coefficient c, which is called the coefficient of *calorific conductivity*, will give the quantity of heat that passes through a unit area on the surface per unit time. The heat flux that traverses ds during the time dt will be:

$$c\,ds\,dt\,\frac{V-(V+dV)}{dn} = -\,c\,\frac{dV}{dn}\,ds\,dt.$$

Given that, consider the parallelepiped that is constructed from the line elements $d\sigma_i = Q_i\,dq_i$ (i = 1, 2, 3). It will receive a quantity of heat that is expressed by $-\frac{c}{Q_i}\frac{\partial V}{\partial q_i}ds_i\,dt$ upon crossing the face ds_i, and will lose the quantity:

$$-\left[\frac{1}{Q_i}\frac{\partial V}{\partial q_i}ds_i + \frac{\partial}{\partial q_i}\left(\frac{1}{Q_i}\frac{\partial V}{\partial q_i}ds_i\right)dq_i\right]c\,dt,$$

in such a way that the quantity of heat:

$$c\frac{\partial}{\partial q_i}\left(\frac{1}{Q_i}\frac{\partial V}{\partial q_i}ds_i\right)dq_i\,dt = c\frac{\partial}{\partial q_i}\left(\frac{\nabla}{Q_i^2}\frac{\partial V}{\partial q_i}\right)dq_1\;dq_2\;dq_3\,dt$$

will remain in the spatial element. The total quantity of heat the element dS acquires during the time dt is therefore:

$$\frac{c\,dS\,dt}{\nabla}\sum_i\frac{\partial}{\partial q_i}\left(\frac{\nabla}{Q_i^2}\frac{\partial V}{\partial q_i}\right). \tag{6}$$

On the other hand, one should note that this quantity of heat must be proportional to the rise in temperature $\frac{\partial V}{\partial t}dt$, and to the mass $\rho\,dS$, and that the coefficient of proportionality must be the specific heat C. Hence, the expression (6) is equivalent to:

$$C\frac{\partial V}{\partial t}\rho\,dS\,dt.$$

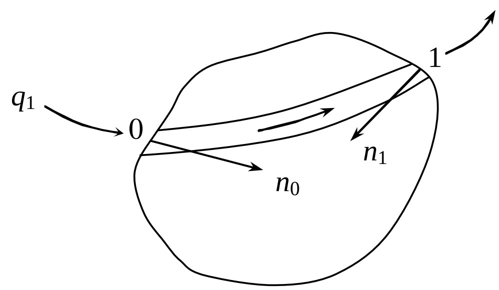
If one lets k denote the constant $C\rho / c$ then a comparison will give:

$$\frac{1}{\nabla}\sum_i\frac{\partial}{\partial q_i}\left(\frac{\nabla}{Q_i^2}\frac{\partial V}{\partial q_i}\right) = k\,\frac{\partial V}{\partial t}.$$

The right-hand side of this cannot depend upon the choice of system of coordinates, on the strength of its physical significance. Therefore, the left-hand side must preserve the proper heat unaltered when one specializes the coordinate system. In the case of Cartesian coordinates, it will become $\Delta^2 V$, since one has $q_i = x_i$, $Q_i = 1$, $\nabla = 1$. It will then remain to prove the Lamé formula, and at the same time, one will see that the propagation of heat in the homogeneous and isotropic medium is regulated by *Fourier's equation* $\Delta^2 V = k\,\dfrac{\partial V}{\partial t}$. In particular, one observes that if a body is in thermal equilibrium then it must satisfy the equation $\Delta^2 V = 0$ [viz., the *Laplace equation* (*)] at all points.

10. Transformation of integrals. – Let V be a finite function that is continuous and uniform, and consider the triple integral:

$$\int \frac{\partial V}{\partial q_1} \frac{dS}{\nabla}.$$



For a line q_1, take the surfaces q_2 and q_3 and consider the two infinitely-close surfaces $q_2 + dq_2$ and $q_3 + dq_3$. Those four surfaces cut out a *channel* from space. Decompose the body into an infinitude of similar channels, which one assumes are being traversed in the sense by which one computes the q_1, and let 0 and 1 distinguish everything that refers to the points of *entrance* and *exit* of the channels on the surface of the body, respectively. Given that, the integral considered can be written as:

$$\iiint \frac{\partial V}{\partial q_1} dq_1\, dq_2\, dq_3 = \iint dq_2\, dq_3 \int \frac{\partial V}{\partial q_1} dq_1 = \iint (V_1 - V_0)\, dq_2\, dq_3 .$$

Each channel cuts out an element ds_0 or ds_1 from the surface whose projection onto the tangent plane to the surface q_1 at the point considered will give a right section of the channel – i.e., a rectangle that the dimensions $Q_2\, dq_2$, $Q_3\, dq_3$. Since the angle between the line q_1 and the normal to the surface of the body is acute at the entrance and obtuse at the exit, one will have:

$$Q_2\, Q_3\, dq_2\, dq_3 = \quad ds_0 \cos(n_0, q_1) \qquad \text{at the entrance,}$$
$$Q_2\, Q_3\, dq_2\, dq_3 = - ds_0 \cos(n_0, q_1) \qquad \text{at the exit,}$$

and therefore:

$$\iint V_1\, dq_2\, dq_3 - \iint V_0\, dq_2\, dq_3$$

(*) *Mécanique céleste*, Liv. III, XI.

$$= -\int \frac{V_1 \cos(n_1\, q_1)}{Q_2\, Q_3} ds_1 - \int \frac{V_0 \cos(n_0\, q_1)}{Q_2\, Q_3} ds_0 = -\int \frac{V_1 \cos(n\, q_1)}{Q_2\, Q_3} ds\,.$$

Hence:

$$\int \frac{\partial V}{\partial q_1} \frac{dS}{\nabla} = -\int Q_1\, V \cos(n\, q_1) \frac{ds}{\nabla}\,. \tag{7}$$

(9) of Chap. I is included in that formula.

11. Third proof (*). – The transformation (7) puts us in a position to present another beautiful proof of Lamé's formula. Consider the integral:

$$J = \int \Delta V \cdot dS$$

and vary *V*. By virtue of (3), one can write:

$$J = \int \left[\frac{\nabla}{Q_1^2} \left(\frac{\partial V}{\partial q_1} \right)^2 + \frac{\nabla}{Q_2^2} \left(\frac{\partial V}{\partial q_2} \right)^2 + \frac{\nabla}{Q_3^2} \left(\frac{\partial V}{\partial q_3} \right)^2 \right] \frac{dS}{\nabla}\,.$$

If one takes the variation and integrates by parts then:

$$\tfrac{1}{2}\delta J = \int \left(\frac{\nabla}{Q_1^2} \frac{\partial V}{\partial q_1} \frac{\partial \delta V}{\partial q_1} + \frac{\nabla}{Q_2^2} \frac{\partial V}{\partial q_2} \frac{\partial \delta V}{\partial q_2} + \frac{\nabla}{Q_3^2} \frac{\partial V}{\partial q_3} \frac{\partial \delta V}{\partial q_3} \right) \frac{dS}{\nabla}$$

$$= \int \left[\frac{\partial}{\partial q_1} \left(\frac{\nabla}{Q_1^2} \frac{\partial V}{\partial q_1} \delta V \right) + \frac{\partial}{\partial q_2} \left(\frac{\nabla}{Q_2^2} \frac{\partial V}{\partial q_2} \delta V \right) + \frac{\partial}{\partial q_3} \left(\frac{\nabla}{Q_3^2} \frac{\partial V}{\partial q_3} \delta V \right) \right] \frac{dS}{\nabla}$$

$$- \int \left[\frac{\partial}{\partial q_1} \left(\frac{\nabla}{Q_1^2} \frac{\partial V}{\partial q_1} \right) + \frac{\partial}{\partial q_2} \left(\frac{\nabla}{Q_2^2} \frac{\partial V}{\partial q_2} \right) + \frac{\partial}{\partial q_3} \left(\frac{\nabla}{Q_3^2} \frac{\partial V}{\partial q_3} \right) \right] \delta V \cdot \frac{dS}{\nabla}\,.$$

If one adopts (7) then one will see that the first integral transforms into:

$$\int \left(\frac{1}{Q_1} \frac{\partial V}{\partial q_1} \cos(nq_1) + \frac{1}{Q_2} \frac{\partial V}{\partial q_2} \cos(nq_2) + \frac{1}{Q_3} \frac{\partial V}{\partial q_3} \cos(nq_3) \right) \delta V \frac{dS}{\nabla}\,;$$

i.e., into $-\int \delta V \cdot \frac{dV}{dn} ds$. Hence:

(*) Due to JACOBI (vol. 2 of his *Math. Werke*).

$$\tfrac{1}{2}\delta J = -\int \delta V \cdot \frac{dV}{dn}\, ds - \int \delta V \left\{ \frac{\partial}{\partial q_i} \left(\frac{\nabla}{Q_i^2} \frac{\partial V}{\partial q_i} \right) \right\} \frac{dS}{\nabla} .$$

In particular, if one takes Cartesian coordinates then:

$$\tfrac{1}{2}\delta J = -\int \delta V \cdot \frac{dV}{dn}\, ds - \int \delta V \cdot \Delta^2 V \cdot dS .$$

If one equates the two expressions for δJ and observes that the δV that relate to the various points of the body are completely independent of each other then one will see that Lamé's formula must be true at any point in space.

CHAPTER XX

ISOTHERMAL SYSTEMS

1. Level, equipotential, isothermal, isostatic surfaces. – Although not everything that we are going to present has direct importance in theory of elasticity, we nonetheless believe that is useful to discuss those things in order to either exhibit more evidence of the theory of curvilinear coordinates or to shed some light on the laws that exist between the various branches of mathematical physics. In all of the mathematical theories of natural phenomena, one is led to the notion of a certain function and the study of the surfaces upon which that function remains constant. They are the *level* surfaces of hydrostatics, the *equipotential* surfaces of the theory of universal attraction, and the *isothermal* surfaces in the theory of heat. From the geometric viewpoint, there is no essential difference between all of those families of surfaces. Therefore, it is enough to speak of isothermal surfaces.

2. – If a body is in thermal equilibrium then it can be considered to be the geometric locus of an infinitude of surfaces, upon each of which the temperature V is constant. If q is the parameter of that family of surfaces – which are called *isothermal* – then V cannot vary when q does not vary, and therefore V is a function of only q. One will then have, in succession:

$$\frac{\partial V}{\partial x_i} = \frac{dV}{dq}\frac{\partial q}{\partial x_i}, \qquad \frac{\partial^2 V}{\partial x_i^2} = \frac{dV}{dq}\frac{\partial^2 q}{\partial x_i^2} + \frac{d^2 V}{dq^2}\left(\frac{\partial q}{\partial x_i}\right)^2 \qquad (i = 1, 2, 3),$$

so when one sums, one will get:

$$\Delta^2 V = \frac{dV}{dq}\Delta^2 q + \frac{d^2 V}{dq^2}\Delta q .$$

We have seen that in order to have thermal equilibrium, we must have $\Delta^2 V = 0$. It then follows that:

$$\frac{\Delta^2 q}{\Delta q} = -\frac{d^2 V / dq^2}{dV / dq} . \tag{1}$$

If we observes that the right-hand side is a function of only q then we will reach the following conclusion:

In order for a family of surfaces with parameter q to be isothermal, it is necessary that the ratio of the differential parameters of q should be a function of only q (*). This condition is also *sufficient.*

(*) This important proposition is due to LAMÉ, *Leçons*, §§ XX, XXI.

In fact, suppose that the ratio of the differential parameters of q has been found to be expressed by $\varphi(q)$, and seek to determine V. Equation (1) will become:

$$\frac{d}{dq}\log\frac{dV}{dq} = -\varphi(q),$$

and with two successive integrations, one will deduce that $V = \lambda\tau + \mu$, with λ and μ arbitrary constants, and:

$$\tau = \int e^{-\int \varphi(q)\,dq}\,dq\,.$$

The function τ depends upon q alone. One can then assume that τ is the parameter of the family of surfaces, which is distinguished from q by the name of the *thermometric parameter*, since it verifies the stationary temperature equation $\Delta^2\tau = 0$. One observes that adopting the thermometric parameter will lead to noteworthy simplifications to the calculations in curvilinear coordinates. In particular, if q_1, q_2, q_3 are the thermometric parameters of a triple family of coordinate surfaces then Lamé's formula will become:

$$\Delta^2 V = \frac{1}{Q_1^2}\frac{\partial^2 V}{\partial q_1^2} + \frac{1}{Q_2^2}\frac{\partial^2 V}{\partial q_2^2} + \frac{1}{Q_3^2}\frac{\partial^2 V}{\partial q_3^2}\,.$$

In order to insure this, it is enough to suppose that $\Delta^2 q_i = 0$ in formula (5) of the preceding chapter.

3. – It is useful to know some families of isothermal surfaces. In the first place, observe that the families that constitute the system of polar coordinates are all isothermal. Indeed, for the concentric sphere, one will have $x_1^2 + x_2^2 + x_3^2 = r^2$; hence:

$$\frac{\partial r}{\partial x_i} = \frac{x_i}{r}, \qquad \frac{\partial^2 r}{\partial x_i^2} = \frac{1}{r} - \frac{x_i^2}{r^3}, \qquad \Delta r = 1, \qquad \Delta^2 r = \frac{2}{r}, \qquad \frac{\Delta^2 r}{\Delta r} = \frac{2}{r}\,.$$

If one wants to find the thermometric parameter then one will have:

$$\varphi(r) = \frac{2}{r}, \qquad \int \varphi(r)\,dr = 2\log r, \qquad \int e^{-\int \varphi(r)\,dr}\,dr = \int \frac{dr}{r^2} = -\frac{1}{r},$$

and one can take $\tau = 1 / r$. With that, one can know the distribution of the temperature in a spherical shell whose terminal surfaces are maintained at constant temperatures V_0 and V_1. Indeed, one has $V = \lambda / r + \mu$, and the constants λ and μ are determined by means of the equations $\frac{\lambda}{r_0} + \mu = V_0$, $\frac{\lambda}{r_1} + \mu = V_1$. One shows in an analogous manner that the families of cones and planes upon which the thermometric parameters are log tan $\theta / 2$ and ψ, respectively, are isothermal.

4. – Three interesting families of isothermal surfaces are then provided by the homofocal surfaces of order two. They are represented by the equation:

$$\frac{x_1^2}{q^2-\alpha_1^2}+\frac{x_2^2}{q^2-\alpha_2^2}+\frac{x_3^2}{q^2-\alpha_3^2}=1, \tag{2}$$

in which one must suppose that $\alpha_1 = 0$, $\alpha_2 = b$, $\alpha_3 = c$. One has *ellipsoids* when q is greater than b and c, *hyperboloids of one sheet* when q is between b and c, and *hyperboloids of two sheets* when q is less than b and c. One then has *three* families of surfaces, and it is known from analytic geometry that those families are pair-wise orthogonal to each other. We shall now prove, by means of Lamé's rule, that any of those families is isothermal. If we partially-differentiate (2) and set:

$$M=\frac{x_1^2}{(q^2-\alpha_1^2)^2}+\frac{x_2^2}{(q^2-\alpha_2^2)^2}+\frac{x_3^2}{(q^2-\alpha_3^2)^2}$$

then we will get:

$$\frac{x_i}{q^2-\alpha_i^2}=Mq\frac{\partial q}{\partial x_i} \qquad (i = 1, 2, 3), \tag{3}$$

and upon squaring and summing this, we will get:

$$\Delta q=\frac{1}{Mq^2}.$$

After a new differentiation, one will similarly get:

$$\frac{1}{q^2-\alpha_1^2}+\frac{1}{q^2-\alpha_2^2}+\frac{1}{q^2-\alpha_3^2}=M\,\Delta q+Mq\,\Delta^2 q+\sum_i\left(\frac{2x_i}{(q^2-\alpha_i^2)^2}+\frac{\partial M}{\partial x_i}\right)q\frac{\partial q}{\partial x_i}. \tag{4}$$

Now, if we set:

$$N=\frac{x_1^2}{(q^2-\alpha_1^2)^3}+\frac{x_2^2}{(q^2-\alpha_2^2)^3}+\frac{x_3^2}{(q^2-\alpha_3^2)^3}$$

then we will have:

$$\frac{\partial M}{\partial x_i}=\frac{2x_i}{(q^2-\alpha_1^2)^2}-4Nq\frac{\partial q}{\partial x_i},$$

and therefore:

$$\sum_i \left(\frac{2x_i}{(q^2-\alpha_1^2)^2} + \frac{\partial M}{\partial x_i} \right) q \frac{\partial q}{\partial x_i} = 4q \sum_i \frac{x_i \dfrac{\partial q}{\partial x_i}}{(q^2-\alpha_i^2)^2} - 4N^2\,\Delta q.$$

Ultimately, by virtue of (3):

$$\sum_i \frac{x_i \dfrac{\partial q}{\partial x_i}}{(q^2-\alpha_i^2)^2} = \frac{1}{Mq} \sum_i \frac{x_i^2}{(q^2-\alpha_i^2)^3} = \frac{N}{Mq}.$$

Consequently:

$$\sum_i \left(\frac{2x_i}{(q^2-\alpha_1^2)^2} + \frac{\partial M}{\partial x_i} \right) q \frac{\partial q}{\partial x_i} = \frac{4N}{M} - 4N\, q^2\,\Delta q = 0,$$

and the equality (4) will become:

$$\frac{1}{q^2-\alpha_1^2} + \frac{1}{q^2-\alpha_2^2} + \frac{1}{q^2-\alpha_3^2} = \frac{1}{q^2} + Mq\,\Delta^2 q.$$

Finally, one then has:

$$\frac{\Delta^2 q}{\Delta q} = \frac{q}{q^2-b^2} + \frac{q}{q^2-c^2} = \varphi(q), \qquad \text{etc.}$$

5. – What is quite interesting, from the viewpoint of pure analysis, as well, are the families of coordinate surfaces that are composed as follows: A family of parallel planes and two families of cylinders that are perpendicular to those planes and orthogonal to each other. We would like to show that if a family of cylinders is isothermal then the family that is orthogonal to it will also be isothermal. We will have $Q_3 = 1$ for it, while Q_1 and Q_2 will be functions of q_1 and q_2, but not $q_3 = z$. Note that [Chap. XVII, form. (3), Chap. XVIII, form. (4)]:

$$\Delta q_i = \frac{1}{Q_i^2}, \qquad \Delta^2 q_i = \frac{1}{\nabla} \frac{\partial}{\partial q_i} \frac{\nabla}{Q_i^2};$$

hence:

$$\frac{\Delta^2 q_i}{\Delta q_i} = \frac{\partial}{\partial q_i} \log \frac{\nabla}{Q_i^2},$$

i.e.:

$$\frac{\Delta^2 q_1}{\Delta q_1} = \frac{\partial}{\partial q_1} \log \frac{Q_2}{Q_1}, \qquad \frac{\Delta^2 q_2}{\Delta q_2} = \frac{\partial}{\partial q_2} \log \frac{Q_1}{Q_2}. \tag{5}$$

Consequently:

$$\frac{\partial}{\partial q_2} \frac{\Delta^2 q_1}{\Delta q_1} + \frac{\partial}{\partial q_1} \frac{\Delta^2 q_2}{\Delta q_2} = 0.$$

If the family q_1 is isothermal then Lamé's rule (§ 2) will say that $\frac{\Delta^2 q_1}{\Delta q_1}$ does not depend upon q_2, and the last equation shows that $\frac{\Delta^2 q_1}{\Delta q_1}$ will not depend upon q_1 in that case. Therefore, the family q_2 is isothermal. If one takes q_1 and q_2 to be the thermometric parameters then formulas (5) will show that Q_1 / Q_2 is constant. It is clear that a thermometric parameter can always be multiplied by a constant without losing its characteristic property. By virtue of a known formula [Chap. XVII, form. (3)], the corresponding function Q will then prove to have been multiplied by a constant. One can then do that in such a way that the ratio Q_1 / Q_2 is equal to unity. In that case, take $Q_1 = Q_2 = 1/h$, and Lamé's formula will assume the simple form:

$$\Delta^2 V = h^2 \left(\frac{\partial^2 V}{\partial q_1^2} + \frac{\partial^2 V}{\partial q_2^2}\right) + \frac{\partial^2 V}{\partial z^2}.$$

6. – One can construct an infinitude of pairs of families of orthogonal isothermal cylinders by taking the parameters q_1 and q_2 to be equal to the real part and the coefficient of $\sqrt{-1}$, respectively, of a function of the complex variable $x_1 + x_2\sqrt{-1}$. It is known that one has:

$$\frac{\partial q_1}{\partial x_1} = \frac{\partial q_2}{\partial x_2}, \qquad \frac{\partial q_1}{\partial x_2} = -\frac{\partial q_2}{\partial x_1},$$

and one will see immediately that $\Delta^2 q_1 = 0$, $\Delta^2 q_2 = 0$. That proves that the two families of cylinders are isothermal and, at the same time, shows that q_1 and q_2 are precisely the thermometric parameters of the two families. In addition, one will see that:

$$\frac{\partial q_1}{\partial x_1}\frac{\partial q_2}{\partial x_1} + \frac{\partial q_1}{\partial x_2}\frac{\partial q_2}{\partial x_2} = 0,$$

and that is precisely the orthogonality condition. Conversely, it is easy to see that the preceding construction will prove *all* of the possible families of orthogonal isothermal cylinders. Indeed, one will see that one can always suppose that $Q_1 = Q_2$; i.e.:

$$\left(\frac{\partial q_1}{\partial x_1}\right)^2 + \left(\frac{\partial q_1}{\partial x_2}\right)^2 = \left(\frac{\partial q_2}{\partial x_1}\right)^2 + \left(\frac{\partial q_2}{\partial x_2}\right)^2$$

or

$$\left(\frac{\partial q_1}{\partial x_1}\right)^2 - \left(\frac{\partial q_2}{\partial x_1}\right)^2 = \left(\frac{\partial q_2}{\partial x_2}\right)^2 - \left(\frac{\partial q_1}{\partial x_2}\right)^2.$$

On the other hand, in order to express the orthogonality, one will have the equivalence:

$$\frac{\partial q_1}{\partial x_1}\cdot\frac{\partial q_2}{\partial x_1} = \frac{\partial q_1}{\partial x_2}\cdot\frac{\partial q_2}{\partial x_2}.$$

If one multiplies this by $2\sqrt{-1}$, adds it to the preceding equivalence, and takes the square root then one will get:

$$\frac{\partial q_1}{\partial x_1}+\frac{\partial q_2}{\partial x_1}\sqrt{-1} = \pm\left(\frac{\partial q_2}{\partial x_2}-\frac{\partial q_1}{\partial x_2}\sqrt{-1}\right);$$

i.e., one will simultaneously have:

$$\frac{\partial q_1}{\partial x_1} = \pm\frac{\partial q_2}{\partial x_2}, \quad \frac{\partial q_1}{\partial x_2} = \mp\frac{\partial q_2}{\partial x_1}.$$

Therefore, $q_1 + q_2\sqrt{-1}$ will be is a function of $x_1 + x_2\sqrt{-1}$.

7. – "While hydrostatics and potential theory have introduced families of level or equipotential surfaces, resp., and the theory of heat has introduced families of isothermal surfaces, it is the mathematical theory of elastic equilibrium in solid bodies that has given rise to the consideration of three conjugate orthogonal families. Indeed, it results from that theory that there will always exist three orthogonal planar elements that are subjected to elastic forces normally at any point of a solid in elastic equilibrium. If one then considers, at the same time, that triad of elements to vary continuously with position at all points of the body then that will define (*) three families of orthogonal surfaces that one calls an *isostatic system*, and they are endowed with the fundamental property of being the only surfaces that are subjected to elastic forces normally… One knows that any orthogonal system can occasionally become isostatic when those of its surfaces that form the walls of the solid are subject to normal pressures: It is enough that the signs and the intensities of those pressures vary conveniently from one point of the surface to another. The property of being isostatic therefore has a very different nature from that of being isothermal, which belongs to only a certain family of surfaces. However, the true fundamental property of any isostatic system is the obligatory meeting of the three families of surfaces and their necessary orthogonality. It is that property, so neatly characterized, that gave birth to the idea of curvilinear coordinates… Its use is indispensible when one would like to treat bodies of a well-defined form in the various

(*) Unfortunately, that surface does not exist, in general, because a pair of orthogonal directions in a plane that are defined at any point can always be considered to be those of the tangents at that point to two lines of a doubly-orthogonal system in space, but the analogous property is not true for an orthogonal triad of planar elements that are also defined at any point. In a paper "Zur Theorie der isostatischen Flächen," [Crelle's Journal (1881), pp. 18], WEINGARTEN has carried out a search for the restrictive conditions under which one can verify the existence of an isostatic system. The question can also be treated by means of the "Formole par lo studio delle linee e delle superficie orthogonalii" that was developed by Prof. BELTRAMI in the Rendiconti dell'Istituto Lombardo (1872), pp. 474.

branches of mathematical physics, in which one indeed always deals with integrating, that is to say, to determine one or more functions that must verify one or more second-order partial differential equations that express the physical laws that those functions must obey. In addition, those functions or their general integrals must verify other first-order partial differential equations for all points of the surface of the body that is considered. Now, that problem of double integration would be completely inaccessible if one did not refer the points of the body to a coordinate system such that the surfaces are represented by equating one of the coordinates to a constant… The idea of curvilinear coordinates came out of the mathematical theory of elasticity, and it is also in that theory that the new tool leads to laws that are more complete and meet up with a larger number of applications. The equations of elasticity, when transformed by means of the various parameters of the orthogonal system, will be presented in the form that is best suited to the integrations… Coordinate systems characterize the phases or stages of science. Without the invention of rectilinear coordinates, algebra would perhaps still be at the point at which Diophantus and his commentators left it, and would not have arrived at either the infinitesimal calculus or analytical mechanics. Celestial mechanics would be absolutely impossible without the introduction of spherical coordinates. Without elliptical coordinates, famous geometers would not have been able to solve several important questions of that theory that were left hanging, and the full scope of those three types of special coordinates has merely been touched upon. However, whereas it will transform and complete all of the solutions of celestial mechanics, we need to address mathematical physics or terrestrial mechanics seriously. It will then necessarily be only in the realm of arbitrary curvilinear coordinates that we can address the new questions in full generality.” (*)

(*) LAMÉ, *Leçons sur les coord. curvilignes*, Discours préliminaire and §§ CXLVIII, CC.

CHAPTER XXI

GENERAL EQUATIONS OF ELASTICITY IN CURVILINEAR COORDINATES (*)

1. – We have seen that the line element is given by the formula:

$$d\sigma^2 = Q_1^2\,dq_1^2 + Q_2^2\,dq_2^2 + Q_3^2\,dq_3^2$$

in orthogonal curvilinear coordinates. If one varies the position at each point then it will follow that:

$$\begin{aligned} d\sigma\,\delta d\sigma = {} & Q_1^2\,dq_1\,\delta dq_1 + Q_2^2\,dq_2\,\delta dq_2 + Q_3^2\,dq_3\,\delta dq_3 \\ & + Q_1\,\delta Q_1 \cdot dq_1^2 + Q_2\,\delta Q_2 \cdot dq_2^2 + Q_3\,\delta Q_3 \cdot dq_3^2 . \end{aligned}$$

If α_1, α_2, α_3 are the direction cosines of an element then one will have:

$$\alpha_1 = Q_1 \frac{dq_1}{d\sigma}, \quad \alpha_2 = Q_2 \frac{dq_2}{d\sigma}, \quad \alpha_3 = Q_3 \frac{dq_3}{d\sigma},$$

and the preceding equality can then be written:

$$\frac{\delta d\sigma}{d\sigma} = \alpha_1\,Q_1 \frac{\delta dq_1}{d\sigma} + \alpha_2\,Q_2 \frac{\delta dq_2}{d\sigma} + \alpha_3\,Q_3 \frac{\delta dq_3}{d\sigma} + \alpha_1^2 \frac{\delta Q_1}{Q_1} + \alpha_2^2 \frac{\delta Q_2}{Q_2} + \alpha_3^2 \frac{\delta Q_3}{Q_3} .$$

One now observes that:

$$\delta dq_i = d\delta q_i = \frac{\delta dq_1}{d\sigma} dq_1 + \frac{\delta dq_2}{d\sigma} dq_2 + \frac{\delta dq_3}{d\sigma} dq_3 .$$

Therefore:

$$\begin{aligned} \frac{\delta d\sigma}{d\sigma} = {} & \alpha_1\,Q_1 \left(\frac{\alpha_1}{Q_1} \frac{\delta dq_1}{dq_1} + \frac{\alpha_2}{Q_2} \frac{\delta dq_1}{dq_2} + \frac{\alpha_3}{Q_3} \frac{\delta dq_1}{dq_3} \right) + \alpha_1^2 \frac{\delta Q_1}{Q_1} \\ & + \alpha_2\,Q_2 \left(\frac{\alpha_1}{Q_1} \frac{\delta dq_2}{dq_1} + \frac{\alpha_2}{Q_2} \frac{\delta dq_2}{dq_2} + \frac{\alpha_3}{Q_3} \frac{\delta dq_2}{dq_3} \right) + \alpha_2^2 \frac{\delta Q_2}{Q_2} \\ & + \alpha_3\,Q_3 \left(\frac{\alpha_1}{Q_1} \frac{\delta dq_3}{dq_1} + \frac{\alpha_2}{Q_2} \frac{\delta dq_3}{dq_2} + \frac{\alpha_3}{Q_3} \frac{\delta dq_3}{dq_3} \right) + \alpha_3^2 \frac{\delta Q_3}{Q_3} . \end{aligned}$$

As a consequence, if one sets:

(*) See the paper "Sulle equazioni generali dell'elasticità" by Prof. BELTRAMI in the *Annali di Matematica* (1881).

$$\left.\begin{aligned}
\delta\theta_1 &= \frac{\partial\delta q_1}{\partial q_1}+\frac{\delta Q_1}{Q_1}, & \delta\omega_1 &= \frac{Q_2}{Q_3}\frac{\partial\delta q_2}{\partial q_3}+\frac{Q_3}{Q_2}\frac{\partial\delta q_3}{\partial q_2},\\
\delta\theta_2 &= \frac{\partial\delta q_2}{\partial q_2}+\frac{\delta Q_2}{Q_2}, & \delta\omega_2 &= \frac{Q_3}{Q_1}\frac{\partial\delta q_3}{\partial q_1}+\frac{Q_3}{Q_2}\frac{\partial\delta q_3}{\partial q_2},\\
\delta\theta_3 &= \frac{\partial\delta q_3}{\partial q_3}+\frac{\delta Q_3}{Q_3}, & \delta\omega_3 &= \frac{Q_1}{Q_2}\frac{\partial\delta q_1}{\partial q_2}+\frac{Q_2}{Q_1}\frac{\partial\delta q_2}{\partial q_1},
\end{aligned}\right\} \tag{1}$$

one will get:

$$\frac{\delta d\sigma}{d\sigma} = \alpha_1^2\,\delta\theta_1+\alpha_2^2\,\delta\theta_2+\alpha_3^2\,\delta\theta_3 + \alpha_2\,\alpha_3\,\delta\omega_1 + \alpha_3\,\alpha_1\,\delta\omega_2 + \alpha_1\,\alpha_2\,\delta\omega_3\,. \tag{2}$$

2. General equations. – Take a body that *has already been deformed and equilibrated* under the action of the volume forces ($F_1\,dS$, $F_2\,dS$, $F_3\,dS$), surface pressures ($\varphi_1\,ds$, $\varphi_2\,ds$, $\varphi_3\,ds$), and internal forces. Imagine a virtual motion around that equilibrium position that takes each point (q_1, q_2, q_3) to the position ($q_1 + \delta q_1$, $q_2 + \delta q_2$, $q_3 + \delta q_3$), where the variations are δq_1 , δq_2 , δq_3 . The work done by the external forces per unit volume is:

$$Q_1\,F_1\,\delta q_1 + Q_2\,F_2\,\delta q_2 + Q_3\,F_3\,\delta q_3$$

for each point of the body, and the work done per unit area at each point of the surface is:

$$Q_1\,\varphi_1\,\delta q_1 + Q_2\,\varphi_2\,\delta q_2 + Q_3\,\varphi_3\,\delta q_3\,.$$

As for the work done by internal forces, it is solely due to the alteration of the relative distances between the points of the body, and by virtue of (2), it will therefore depend upon the variations $\delta\theta_1$, $\delta\theta_2$, $\delta\theta_3$, $\delta\omega_1$, $\delta\omega_2$, $\delta\omega_3$. Since they are very small, by hypothesis, the aforementioned work per unit volume at any point will be represented by an expression:

$$\Theta_1\,\delta\theta_1 + \Theta_2\,\delta\theta_2 + \Theta_3\,\delta\theta_3 + \Omega_1\,\delta\omega_1 + \Omega_2\,\delta\omega_2 + \Omega_3\,\delta\omega_3\,,$$

in which the Θ and Ω are certain functions of q_1, q_2, q_3 . An application of *Lagrange's principle* will then lead to the equality:

$$\begin{aligned}
&\int (Q_1\,F_1\,\delta q_1+Q_2\,F_2\,\delta q_2+Q_3\,F_3\,\delta q_3)\,dS+\int (Q_1\,\varphi_1\,\delta q_1+Q_2\,\varphi_2\,\delta q_2+Q_3\,\varphi_3\,\delta q_3)\,ds\\
&\qquad + \int (\Theta_1\,\delta\theta_1+\Theta_2\,\delta\theta_2+\cdots+\Omega_3\,\delta\omega_3)\,dS = 0.
\end{aligned} \tag{3}$$

We now seek to free the variations δq_1, δq_2, δq_3 from the third integral by the usual process in such a way that they will appear explicitly, as they do in the first two integrals. If one recalls (1) then:

$$\int \Theta_1\,\delta\theta\,dS = \int \nabla\Theta_1 \frac{\partial \delta q_1}{\partial q_1}\cdot\frac{dS}{\nabla} + \int \frac{\Theta_1}{Q_1}\,\delta Q_1 dS\,.$$

If one integrates by parts and uses a known transformation (Chap. XIX, § **10**) then one will see that the first integral is equal to:

$$\int \nabla\Theta_1 \frac{\partial \delta q_1}{\partial q_1}\cdot\frac{dS}{\nabla} + \int \delta q_1 \frac{\partial \delta\Theta_1}{\partial q_1}\cdot\frac{dS}{\nabla} = -\int Q_1\,\Theta_1 \cos(nq_1)\,\delta q_1\,ds - \int \delta q_1 \frac{\partial \delta\Theta_1}{\partial q_1}\frac{dS}{\nabla}\,.$$

Similarly:

$$\int \Omega_1\,\delta\omega_1\,dS = \int\left(\frac{Q_1\,\Omega_1}{Q_3}\frac{\partial \delta q_2}{\partial q_3} + \frac{Q_3\,\Omega_1}{Q_2}\frac{\partial \delta q_3}{\partial q_2}\right)dS = \int Q_2^2 \frac{\partial \delta q_2}{\partial q_3}\frac{dS}{\nabla} + \int Q_3^2\,Q_1 \frac{\partial \delta q_3}{\partial q_2}\frac{dS}{\nabla}\,.$$

One now has:

$$\begin{aligned}\int Q_2^2 \frac{\partial \delta q_2}{\partial q_3}\frac{dS}{\nabla} &= \int \frac{\partial}{\partial q_3}(Q_2^2\,Q_1\,\Omega_1\,\delta q_2)\frac{dS}{\nabla} - \int \delta q_2 \frac{\partial Q_2^2\,Q_1\,\Omega_1}{\partial q_3}\frac{dS}{\nabla} \\ &= -\int Q_1\,\Omega_1 \cos(nq_3)\,\delta q_1\,ds - \int \delta q_2 \frac{\partial Q_2^2\,Q_1\,\Omega_1}{\partial q_3}\frac{dS}{\nabla}\,.\end{aligned}$$

Consequently:

$$\begin{aligned}&\int \Omega_1\,\delta\omega_1\,dS \\ &= -\int [Q_2 \cos(nq_3)\,\delta q_2 + Q_3 \cos(nq_2)\,\delta q_3]\,\Omega_1\,ds - \int\left(\frac{\partial Q_2^2\,Q_1\,\Omega_1}{\partial q_3}\delta q_2 + \frac{\partial Q_3^2\,Q_1\,\Omega_1}{\partial q_2}\delta q_3\right)\frac{dS}{\nabla}\,.\end{aligned}$$

The work done by the internal forces is then composed of three parts that are analogous to the following one:

$$\begin{aligned}&-\int \delta q_1 \frac{\partial \nabla\,\Theta_1}{\partial q_1}\frac{dS}{\nabla} - \int\left(\frac{\partial Q_2^2\,Q_1\,\Omega_1}{\partial q_3}\delta q_2 + \frac{\partial Q_3^2\,Q_1\,\Omega_1}{\partial q_2}\delta q_3\right)\frac{dS}{\nabla} \\ &\quad + \int \frac{\Theta_1}{Q_1}\left(\frac{\partial Q_1}{\partial q_1}\delta q_1 + \frac{\partial Q_1}{\partial q_2}\delta q_2 + \frac{\partial Q_1}{\partial q_3}\delta q_3\right)\frac{dS}{\nabla} \\ &- \int Q_1\,\Theta_1 \cos(nq_1)\,\delta q_1\,ds - \int [Q_2 \cos(nq_1)\,\delta q_1 + Q_3 \cos(nq_2)\,\delta q_3\,]\,\Omega_1\,ds\,.\end{aligned}$$

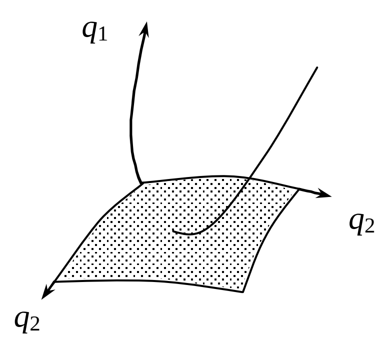
If one substitutes this in (3) and equates the multipliers of δq_1, δq_2, δq_3 to zero individually, first in the spatial integral and then in the surface integral, then one will get the *indefinite equations:*

$$
(4)\qquad \begin{cases}
Q_1\,F_1 = \dfrac{1}{\nabla}\left(\dfrac{\partial\nabla\Theta_1}{\partial q_1} + \dfrac{\partial Q_1^2\,Q_3\,\Omega_3}{\partial q_2} + \dfrac{\partial Q_1^2\,Q_2\,\Omega_2}{\partial q_3}\right) - \left(\dfrac{\Theta_1}{Q_1}\dfrac{\partial Q_1}{\partial q_1} + \dfrac{\Theta_2}{Q_2}\dfrac{\partial Q_2}{\partial q_1} + \dfrac{\Theta_3}{Q_3}\dfrac{\partial Q_3}{\partial q_1}\right), \\[2ex]
Q_2\,F_2 = \dfrac{1}{\nabla}\left(\dfrac{\partial Q_2^2\,Q_2\,\Omega_3}{\partial q_1} + \dfrac{\partial\nabla\Theta_2}{\partial q_2} + \dfrac{\partial Q_2^2\,Q_1\,\Omega_1}{\partial q_3}\right) - \left(\dfrac{\Theta_1}{Q_1}\dfrac{\partial Q_1}{\partial q_2} + \dfrac{\Theta_2}{Q_2}\dfrac{\partial Q_2}{\partial q_2} + \dfrac{\Theta_3}{Q_3}\dfrac{\partial Q_3}{\partial q_2}\right), \\[2ex]
Q_3\,F_3 = \dfrac{1}{\nabla}\left(\dfrac{\partial Q_2^2\,Q_2\,\Omega_2}{\partial q_1} + \dfrac{\partial Q_2^2\,Q_1\,\Omega_1}{\partial q_2} + \dfrac{\partial\nabla\Theta_3}{\partial q_3}\right) - \left(\dfrac{\Theta_1}{Q_1}\dfrac{\partial Q_1}{\partial q_3} + \dfrac{\Theta_2}{Q_2}\dfrac{\partial Q_2}{\partial q_3} + \dfrac{\Theta_3}{Q_3}\dfrac{\partial Q_3}{\partial q_3}\right),
\end{cases}
$$

and the *boundary equations:*

$$
(5)\qquad \begin{cases}
\varphi_1 = \Theta_1\cos(nq_1) + \Omega_3\cos(nq_2) + \Omega_2\cos(nq_3), \\
\varphi_2 = \Omega_3\cos(nq_1) + \Theta_2\cos(nq_2) + \Omega_1\cos(nq_3), \\
\varphi_3 = \Omega_2\cos(nq_1) + \Omega_1\cos(nq_2) + \Theta_3\cos(nq_3).
\end{cases}
$$

In reality, since these relations are the *equations of equilibrium*, which serve to determine the new form of the body and the new distribution of internal actions, it is necessary that q_1, q_2, q_3 should represent the coordinates of the points in their natural positions and not the *unknown* coordinates that the points take on as a result of the deformation. In order to have the equations of equilibrium, if q_1, q_2, q_3 denote the *initial* coordinates, and $q_1 + \kappa_1$, $q_2 + \kappa_2$, $q_3 + \kappa_3$ denote the coordinates *after the deformation*, in such a way that the displacements are $Q_1\,\kappa_1$, $Q_2\,\kappa_2$, $Q_3\,\kappa_3$ then one will need to substitute the $q + \kappa$ for the q in the Q, Θ, Ω in equations (4) and (5). That substitution will not yield anything but the additions of some terms that are negligible in comparison to the ones that were written already if, as one supposes, the κ are negligible with respect to the q. Therefore, equations (4) and (5) will be the equations of equilibrium, provided that one attributes the new significance to the q, which will be done from now on.



3. Observations. – The significance of the Θ and Ω results directly from equations (5). They obviously apply to any surface inside of the body, provided that one suppresses a portion of the body that is situated on one part of the surface, and replaces it with the system of pressures that the portion exerts upon the other portion. In particular, if p_1 represents the unit pressure at any point on a surface q_1, and one suppresses that part of the body in which q_1 increases (when it is removed from the surface) then one will have:

$$\cos(n\,q_1) = -1, \qquad \cos(n\,q_2) = \cos(n\,q_3) = 0,$$

and the formulas (5) will give $p_{11} = -\Theta_1$, $p_{12} = -\Omega_3$, $p_{13} = -\Omega_2$. If one repeats everything for the other surface coordinates then one will see that:

$$-p_{11} = \Theta_1,\ -p_{22} = \Theta_2,\ -p_{33} = \Theta_3\ ;$$

$$-p_{23} = -p_{32} = \Omega_1, \qquad -p_{31} = -p_{13} = \Omega_2, \qquad -p_{12} = -p_{21} = \Omega_3.$$

Therefore, the Θ represent the unit *stresses* that are developed *normally* to the coordinate surfaces, and the Ω are the ones that are developed *tangentially* to that surface. The Θ and the Ω are then the unknowns of the problem. One addresses the integration of equations (4) in such a way that (5) is satisfied on the surface of the body. One will then have three equations to determine *six* functions. However, one must observe that the concept of *elasticity* has not been introduced yet.

4. – Before going any further, it is interesting to profit from the last observations in order to see the formal elegance that the relations (7) of Chap. XVIII confer upon certain results of analysis. In order to prove the *law* that was stated by Lamé for *isostatic systems*, imagine that there exists an isostatic system in a body that is subject to only external pressures, and its parameters can be taken to be coordinates. In that case (Chap. XX, § **7**), the tangential pressures Ω_1, Ω_2, Ω_3 will be zero at any point of the body, and the formulas (4) will become:

$$\frac{1}{\nabla}\frac{\partial \nabla\Theta_i}{\partial q_i} = \frac{\Theta_1}{Q_1}\frac{\partial\Theta_1}{\partial q_i} + \frac{\Theta_2}{Q_2}\frac{\partial\Theta_2}{\partial q_i} + \frac{\Theta_3}{Q_3}\frac{\partial\Theta_3}{\partial q_i} \qquad (i = 1, 2, 3).$$

The left-hand side is equal to:

$$\frac{\partial\Theta_i}{\partial q_i} + \frac{\Theta_i}{\nabla}\frac{\partial\nabla}{\partial q_i} = \frac{\partial\Theta_i}{\partial q_i} + \Theta_i\left(\frac{1}{Q_1}\frac{\partial Q_1}{\partial q_i} + \frac{1}{Q_2}\frac{\partial Q_2}{\partial q_i} + \frac{1}{Q_3}\frac{\partial Q_3}{\partial q_i}\right).$$

Consequently, if one adopts the stated relations (7) then:

$$\frac{\partial\Theta_i}{\partial\sigma_i} = \frac{\Theta_1-\Theta_i}{r_{i1}} + \frac{\Theta_2-\Theta_i}{r_{i2}} + \frac{\Theta_3-\Theta_i}{r_{i3}}.$$

Hence, when one sets i = 1, 2, 3, one will get the following relations between the principal elastic forces, which were pointed out by Lamé (*):

$$\begin{cases} \dfrac{\partial\Theta_1}{\partial\sigma_1} = \dfrac{\Theta_2-\Theta_1}{r_{12}} + \dfrac{\Theta_3-\Theta_1}{r_{13}}, \\[2ex] \dfrac{\partial\Theta_2}{\partial\sigma_2} = \dfrac{\Theta_3-\Theta_2}{r_{23}} + \dfrac{\Theta_1-\Theta_2}{r_{21}}, \\[2ex] \dfrac{\partial\Theta_3}{\partial\sigma_3} = \dfrac{\Theta_1-\Theta_3}{r_{31}} + \dfrac{\Theta_2-\Theta_3}{r_{32}}. \end{cases}$$

(*) *Leçons*…, § CXLIX.

These relations constitute a necessary complement to the law that is represented by the elasticity ellipsoid, since it says how the axes of elasticity will vary when one passes from any point to an infinitely-close one.

5. – Return to the questions of § 2 and assume, once and for all, that the q are the initial coordinates and the $q + \kappa$ are the final coordinates. The virtual motion consists of passing from the position $q + \kappa$ to the position $q + \kappa + \delta(q + \kappa) = q + \kappa + \delta\kappa$, instead of passing from the positions q to the positions $q + \delta q$. Therefore, the δq from the beginning of this chapter are now representable by the $\delta\kappa$. It will then follow that if one sets:

$$\theta_1 = \frac{\partial \kappa_1}{\partial q_1} + \frac{1}{Q_1}\left(\frac{\partial Q_1}{\partial q_1}\kappa_1 + \frac{\partial Q_1}{\partial q_2}\kappa_2 + \frac{\partial Q_1}{\partial q_3}\kappa_3\right), \quad \omega_1 = \frac{Q_2}{Q_3}\frac{\partial \kappa_2}{\partial q_3} + \frac{Q_3}{Q_2}\frac{\partial \kappa_2}{\partial q_2}, \ldots \tag{6}$$

then those θ and ω will be precisely the θ and ω in formulas (1), since one will recover formulas (1) with the $\delta\kappa$ changed into δq when one varies the κ in (6), while naturally the q and the functions that depend upon them will remain invariant. In order to get the coefficient of elongation in the direction $(\alpha_1, \alpha_3, \alpha_3)$, it is enough to imagine that one has repeated the calculation that one did in order to obtain formula (2), except that one needs to intend that the κ in formula (1) have been substituted for the δq. By virtue of (6), that is equivalent to substituting the θ and ω for the $\delta\theta$ and $\delta\omega$, resp. Consequently, the requisite coefficient will be:

$$\varepsilon = \theta_1 \alpha_1^2 + \theta_2 \alpha_2^2 + \theta_3 \alpha_3^2 + \omega_1 \alpha_2 \alpha_3 + \omega_2 \alpha_3 \alpha_1 + \omega_3 \alpha_1 \alpha_2 . \tag{7}$$

What is the significance of the θ and ω? Let $d\sigma_1, d\sigma_2, d\sigma_3$ be the projections of $d\sigma$ onto the coordinate lines in such a way that $d\sigma_i = \alpha_i\, d\sigma$. If $\varepsilon_1, \varepsilon_2, \varepsilon_3$ are the values of ε along the tangents to the coordinates lines, and one supposes that the angles of the parallelepiped that is constructed from the edges $d\sigma_1, d\sigma_2, d\sigma_3$ are diminished by η_1, η_2, η_3 then one will obtain an oblique parallelepiped after the deformation, whose diagonals, edges, and angles will be $(1 + \varepsilon)\, d\sigma$, $(1 + \varepsilon_1)\, d\sigma$, …, $\pi / 2 - \eta_1$, … Therefore, one has:

$$(1 + \varepsilon)^2\, d\sigma^2 = (1 + \varepsilon_1)^2\, d\sigma_1^2 + (1 + \varepsilon_2)^2\, d\sigma_2^2 + (1 + \varepsilon_3)^2\, d\sigma_3^2$$

$$+ 2\,(1 + \varepsilon_2)(1 + \varepsilon_3)\, d\sigma_2\, d\sigma_3 \cdot \cos\left(\frac{\pi}{2} - \eta_1\right) + \ldots,$$

and if one divides this by $d\sigma^2$ and drops infinitesimals of order higher than one then one will deduce that:

$$\varepsilon = \varepsilon_1 \alpha_1^2 + \varepsilon_2 \alpha_2^2 + \varepsilon_3 \alpha_3^2 + \eta_1 \alpha_2 \alpha_3 + \eta_2 \alpha_3 \alpha_1 + \eta_3 \alpha_1 \alpha_2 .$$

When one compares this with (7), one will see that $\theta_i = \varepsilon_i$, $\omega_i = \eta_i$; i.e., the θ and ω are the unit elongations of the edges and the decrements in the angles, resp., of a parallelepiped element that is bounded by its coordinate surfaces.

6. – We shall now introduce the concept of elasticity into the equations of equilibrium. As one knows, it is expressed by writing down that the elementary unit work done by internal forces – i.e.:

$$\Theta_1\,\delta\theta_1 + \Theta_2\,\delta\theta_2 + \ldots + \Omega_3\,\delta\omega_3\,,$$

is an exact variation with respect to the quantity that deformation that has already been made. If one sets, for brevity:

$$\kappa_{ij} = \frac{\partial\kappa_i}{\partial q_j}$$

then formulas (6) can be written as:

$$\left.\begin{aligned}
\theta_1 &= \kappa_{11} + \frac{1}{Q_1}\left(\frac{\partial Q_1}{\partial q_1}\kappa_1 + \frac{\partial Q_1}{\partial q_2}\kappa_2 + \frac{\partial Q_1}{\partial q_3}\kappa_3\right), \quad & \omega_1 &= \frac{Q_2}{Q_3}\kappa_{23} + \frac{Q_3}{Q_2}\kappa_{32}, \\
\theta_2 &= \kappa_{22} + \frac{1}{Q_3}\left(\frac{\partial Q_2}{\partial q_1}\kappa_1 + \frac{\partial Q_2}{\partial q_2}\kappa_2 + \frac{\partial Q_2}{\partial q_3}\kappa_3\right), \quad & \omega_2 &= \frac{Q_3}{Q_1}\kappa_{31} + \frac{Q_1}{Q_3}\kappa_{13}, \\
\theta_3 &= \kappa_{33} + \frac{1}{Q_3}\left(\frac{\partial Q_3}{\partial q_1}\kappa_1 + \frac{\partial Q_3}{\partial q_2}\kappa_2 + \frac{\partial Q_3}{\partial q_3}\kappa_3\right), \quad & \omega_3 &= \frac{Q_1}{Q_2}\kappa_{12} + \frac{Q_2}{Q_1}\kappa_{21}.
\end{aligned}\right\} \tag{8}$$

If one substitutes this in the expression for the work then one will get:

$$\delta\Pi = \Theta_1\left[\delta\kappa_{11} + \frac{1}{Q_1}\left(\frac{\partial Q_1}{\partial q_1}\delta\kappa_1 + \frac{\partial Q_1}{\partial q_2}\delta\kappa_2 + \frac{\partial Q_1}{\partial q_3}\delta\kappa_3\right)\right] + \ldots + \Omega_3\left(\frac{Q_1}{Q_2}\delta\kappa_{12} + \frac{Q_2}{Q_1}\delta\kappa_{21}\right),$$

or

$$\begin{aligned}
\delta\Pi = {} & \sum_i\left[\frac{\Theta_1}{Q_1}\frac{\partial Q_1}{\partial q_i} + \frac{\Theta_2}{Q_2}\frac{\partial Q_2}{\partial q_i} + \frac{\Theta_3}{Q_3}\frac{\partial Q_3}{\partial q_i}\right]\delta\kappa_i \\
& + \Theta_1\,\delta\kappa_{11} + \frac{Q_1\,\Omega_3}{Q_2}\,\delta\kappa_{12} + \frac{Q_1\,\Omega_2}{Q_3}\,\delta\kappa_{13} \\
& + \frac{Q_2\,\Omega_3}{Q_1}\,\delta\kappa_{21} + \Theta_2\,\delta\kappa_{22} + \frac{Q_2\,\Omega_1}{Q_3}\,\delta\kappa_{23} \\
& + \frac{Q_3\,\Omega_2}{Q_1}\,\delta\kappa_{31} + \frac{Q_3\,\Omega_1}{Q_2}\,\delta\kappa_{23} + \Theta_3\,\delta\kappa_{33}\,.
\end{aligned}$$

One sees that Π is necessarily a function of the κ_i and κ_{ij}, and one must have:

$$\left.\begin{aligned}\frac{\partial\Pi}{\partial\kappa_i}&=\frac{\Theta_1}{Q_1}\frac{\partial Q_1}{\partial q_i}+\frac{\Theta_2}{Q_2}\frac{\partial Q_2}{\partial q_i}+\frac{\Theta_3}{Q_3}\frac{\partial Q_3}{\partial q_i},\\ \frac{\partial\Pi}{\partial\kappa_i}=\Theta_i,\quad \frac{\partial\Pi}{\partial\kappa_{23}}&=\frac{Q_2\,\Omega_1}{Q_3},\quad \frac{\partial\Pi}{\partial\kappa_{32}}=\frac{Q_3\,\Omega_1}{Q_2},\quad\cdots\end{aligned}\right\}\tag{9}$$

If one compares the values for Ω_1 then one will get:

$$\Omega_1=\frac{Q_3}{Q_2}\frac{\partial\Pi}{\partial\kappa_{23}}=\frac{Q_2}{Q_3}\frac{\partial\Pi}{\partial\kappa_{32}}=\frac{\partial\Pi}{\partial\left(\dfrac{Q_2}{Q_3}\kappa_{23}+\dfrac{Q_3}{Q_2}\kappa_{32}\right)}=\frac{\partial\Pi}{\partial\omega_1},\qquad\cdots$$

Therefore, the κ_{ij} ($i \neq j$) will not enter into Π, except in combinations of the ω. Furthermore, if one observes the first of (8) then one can write:

$$\Theta_1=\frac{\partial\Pi}{\partial\kappa_{11}}=\frac{\partial\Pi}{\partial\theta_1},\qquad\qquad \frac{\partial\theta_1}{\partial\kappa_i}=\frac{1}{Q_1}\frac{\partial Q_1}{\partial q_i},\qquad\qquad\cdots$$

As a result, one can give the following form to the first of (9):

$$\frac{\partial\Pi}{\partial\kappa_i}=\frac{\partial\Pi}{\partial\theta_1}\frac{\partial\theta_1}{\partial\kappa_i}+\frac{\partial\Pi}{\partial\theta_2}\frac{\partial\theta_2}{\partial\kappa_i}+\frac{\partial\Pi}{\partial\theta_3}\frac{\partial\theta_3}{\partial\kappa_i}.$$

Since the ω do not contain the κ_i, one will see that they cannot enter into the expression for Π except in combinations of the $\theta_1, \theta_2, \theta_3$. In summary, the *twelve* variables:

$$\kappa_1,\ \kappa_2,\ \kappa_3,\ \kappa_{11},\ \kappa_{22},\ \kappa_{33},\ \kappa_{23},\ \kappa_{32},\ \kappa_{31},\ \kappa_{13},\ \kappa_{12},\ \kappa_{21}$$

upon which the expression for Π depends, can be grouped with each other in that expression in such a way that Π will behave like a function of only the *six* quantities θ_1, θ_2, θ_3, ω_1, ω_2, ω_3.

7. – If one substitutes the last results in the equations of equilibrium then they will not depend upon more than *three* functions, and that is befitting of only *elastic* bodies. In the first indefinite equation, the last parenthesis will reduce immediately to $\partial\Pi / \partial\kappa_1$, while in the first parenthesis, the functions $\nabla\Theta_1$, $Q_3\,Q_1^2\,\Omega_3$, $Q_2\,Q_1^2\,\Omega_2$ will become:

$$\nabla\frac{\partial\Pi}{\partial\kappa_{11}},\qquad Q_3\,Q_1^2\cdot\frac{Q_2}{Q_1}\frac{\partial\Pi}{\partial\kappa_{12}}=\nabla\frac{\partial\Pi}{\partial\kappa_{12}},\qquad Q_2\,Q_1^2\cdot\frac{Q_3}{Q_1}\frac{\partial\Pi}{\partial\kappa_{13}}=\nabla\frac{\partial\Pi}{\partial\kappa_{13}}.$$

One has analogous reductions in the other two equations. The boundary equations suffer slight changes in form, and one finally obtains the equations of elastic equilibrium:

$$\begin{cases} Q_1 F_1 = \dfrac{1}{\nabla}\left[\dfrac{\partial}{\partial q_1}\left(\nabla \dfrac{\partial \Pi}{\partial \kappa_{11}}\right) + \dfrac{\partial}{\partial q_2}\left(\nabla \dfrac{\partial \Pi}{\partial \kappa_{12}}\right) + \dfrac{\partial}{\partial q_3}\left(\nabla \dfrac{\partial \Pi}{\partial \kappa_{13}}\right)\right] - \dfrac{\partial \Pi}{\partial \kappa_1}, \\ Q_2 F_2 = \dfrac{1}{\nabla}\left[\dfrac{\partial}{\partial q_1}\left(\nabla \dfrac{\partial \Pi}{\partial \kappa_{21}}\right) + \dfrac{\partial}{\partial q_2}\left(\nabla \dfrac{\partial \Pi}{\partial \kappa_{22}}\right) + \dfrac{\partial}{\partial q_3}\left(\nabla \dfrac{\partial \Pi}{\partial \kappa_{23}}\right)\right] - \dfrac{\partial \Pi}{\partial \kappa_2}, \\ Q_3 F_3 = \dfrac{1}{\nabla}\left[\dfrac{\partial}{\partial q_1}\left(\nabla \dfrac{\partial \Pi}{\partial \kappa_{31}}\right) + \dfrac{\partial}{\partial q_2}\left(\nabla \dfrac{\partial \Pi}{\partial \kappa_{32}}\right) + \dfrac{\partial}{\partial q_3}\left(\nabla \dfrac{\partial \Pi}{\partial \kappa_{33}}\right)\right] - \dfrac{\partial \Pi}{\partial \kappa_3}, \end{cases}$$

$$\begin{cases} Q_1 \varphi_1 = Q_1 \dfrac{\partial \Pi}{\partial \kappa_{11}} \cos(nq_1) + Q_2 \dfrac{\partial \Pi}{\partial \kappa_{12}} \cos(nq_2) + Q_3 \dfrac{\partial \Pi}{\partial \kappa_{13}} \cos(nq_3), \\ Q_2 \varphi_2 = Q_1 \dfrac{\partial \Pi}{\partial \kappa_{31}} \cos(nq_1) + Q_2 \dfrac{\partial \Pi}{\partial \kappa_{22}} \cos(nq_2) + Q_3 \dfrac{\partial \Pi}{\partial \kappa_{23}} \cos(nq_3), \\ Q_3 \varphi_3 = Q_1 \dfrac{\partial \Pi}{\partial \kappa_{31}} \cos(nq_1) + Q_2 \dfrac{\partial \Pi}{\partial \kappa_{32}} \cos(nq_2) + Q_3 \dfrac{\partial \Pi}{\partial \kappa_{33}} \cos(nq_3). \end{cases}$$

These equations can be written succinctly as follows (*):

$$Q_i F_i = \frac{1}{\nabla} \sum_j \frac{\partial}{\partial q_j}\left(\nabla \frac{\partial \Pi}{\partial \kappa_{ij}}\right) - \frac{\partial \Pi}{\partial \kappa_i}, \qquad Q_i \varphi_i = \sum_j Q_j \frac{\partial \Pi}{\partial \kappa_{ij}} \cos(nq_j) \quad (i = 1, 2, 3). \tag{10}$$

8. Expression for Θ in curvilinear coordinates. – We know that Θ is the sum of the coefficients of elongation relative to three arbitrary orthogonal axes. If one adopts formulas (6) then it will follow that:

$$\Theta = \theta_1 + \theta_2 + \theta_3 = \sum_i \left[\frac{\partial \kappa_i}{\partial q_i} + \left(\frac{1}{Q_1}\frac{\partial Q_1}{\partial q_i} + \frac{1}{Q_2}\frac{\partial Q_2}{\partial q_i} + \frac{1}{Q_3}\frac{\partial Q_3}{\partial q_i}\right) \kappa_i\right].$$

The expression that is subordinate to the summation sign is equivalent to:

$$\frac{\partial \kappa_i}{\partial q_i} + \kappa_i \frac{\partial}{\partial q_i} \log Q_1 Q_2 Q_3 = \frac{1}{\nabla}\left(\nabla \frac{\partial \kappa_i}{\partial q_i} + \kappa_i \frac{\partial \nabla}{\partial q_i}\right).$$

Hence:

$$\Theta = \frac{1}{\nabla}\left(\frac{\partial \nabla \kappa_1}{\partial q_1} + \frac{\partial \nabla \kappa_2}{\partial q_2} + \frac{\partial \nabla \kappa_3}{\partial q_3}\right). \tag{11}$$

(*) However, if one has exhibited the displacements $Q\kappa$ of the κ then one will get the equations of equilibrium in the form that was given to them by C. NEUMANN in the paper "Zur Theorie der Elastistizität." The first to translate the general equations of elasticity into curvilinear coordinates was LAMÉ. See *Leçons sur les coord. curvilignes*, § CXLIV and § CVLVII.

9. Expressions for $\mathcal{T}_1$, $\mathcal{T}_2$, $\mathcal{T}_3$ in curvilinear coordinates. – For the moment, take the Cartesian axes to be the tangents to the coordinate lines and trace out a closed curve around the origin in the *xy*-plane. It is known that when the integral $\int_\sigma (u\,dx + v\,dy)$ is extended over that curve, it will transform into the integral $\int \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) ds$, when it is extended over the area that is enclosed by that curve, and it is clear that when one shrinks the curve around the origin indefinitely, one will have:

$$\lim \frac{1}{s}\int_\sigma (u\,dx + v\,dy) = \lim \frac{\int \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)}{\int ds} = \mathcal{T}_3 ,$$

since $\mathcal{T}_3$ is the value of $\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$ at the point considered. Given that, evaluate the first integral in curvilinear coordinates. One has $ds = Q_1\, Q_2\, dq_1\, dq_2$, and the displacements u, v are expressed in terns of $Q_1\ \kappa_1$, $Q_2\ \kappa_2$, while $dx = Q_1\, dq_1$, $dy = Q_2\, dq_1$. Consequently:

$$\begin{aligned} \mathcal{T}_3 &= \lim \frac{1}{s}\int_\sigma (Q_1^2\, \kappa_1\, dq_1 + Q_2^2\, \kappa_2\, dq_2) \\ &= \lim \frac{1}{s}\int_\sigma \left(\frac{\partial Q_2^2\, \kappa_2}{\partial q_1} - \frac{\partial Q_1^2\, \kappa_1}{\partial q_2}\right) dq_1\, dq_2 , \end{aligned}$$

or

$$\mathcal{T}_3 = \lim \frac{\int \left(\frac{\partial Q_2^2\, \kappa_2}{\partial q_1} - \frac{\partial Q_1^2\, \kappa_1}{\partial q_2}\right)\frac{ds}{Q_1\, Q_2}}{\int ds} = \frac{1}{Q_1\, Q_2}\left(\frac{\partial Q_2^2\, \kappa_2}{\partial q_1} - \frac{\partial Q_1^2\, \kappa_1}{\partial q_2}\right).$$

Therefore (*):

$$\left.\begin{aligned} \mathcal{T}_1 &= \frac{1}{Q_1\, Q_2}\left(\frac{\partial Q_3^2\, \kappa_3}{\partial q_2} - \frac{\partial Q_2^2\, \kappa_2}{\partial q_3}\right), \\ \mathcal{T}_2 &= \frac{1}{Q_3\, Q_1}\left(\frac{\partial Q_1^2\, \kappa_1}{\partial q_3} - \frac{\partial Q_3^2\, \kappa_3}{\partial q_1}\right), \\ \mathcal{T}_3 &= \frac{1}{Q_1\, Q_3}\left(\frac{\partial Q_2^2\, \kappa_2}{\partial q_1} - \frac{\partial Q_1^2\, \kappa_1}{\partial q_2}\right). \end{aligned}\right\} \tag{12}$$

10. – We shall now seek to find the special form that the equations of equilibrium take in the case of isotropy. If we were to proceed with the direct substitution of the particular form that Π has in isotropic bodies then the calculations would become very

(*) See the first paper on *rational hydrodynamics* that was published by Prof. BELTRAMI in the Memorie dell'Accademie di Bologna (1871), pp. 463, 471.

complicated, and as Beltrami has observed, those complications "are not merely a defect of the algorithm, but have their roots in the *nature* of space itself (*)." However, we can make noteworthy simplifications when we recall a known (Chap. V, § **3**) decomposition of the potential Π into two parts, one of which has no influence upon the indefinite equations. We intend to address only those equations, since the substitution of Θ and Ω in the boundary equations presents no difficulty. Therefore, substitute:

$$\Pi_0 = -\frac{1}{2}\left[A\Theta^2 + B(\mathcal{T}_1^2 + \mathcal{T}_2^2 + \mathcal{T}_3^2)\right]$$

for Π in the indefinite equations, while setting the Θ and $\mathcal{T}$ equal to the expressions (11) and (12), resp. Instead of performing the direct substitution, it is better to repeat the calculations that led to the general equations. To that end, consider:

$$\int \delta\Pi_0\, dS = -\int [A\Theta\,\delta\Theta + B(\mathcal{T}_1\delta\mathcal{T}_1 + \mathcal{T}_2\delta\mathcal{T}_2 + \mathcal{T}_3\delta\mathcal{T}_3)]\, dS\,.$$

First of all, one has:

$$\int \Theta\,\delta\Theta\, dS = \sum_i \int \left[\frac{\partial\delta\kappa_i}{\partial q_i} + \left(\frac{1}{Q_1}\frac{\partial Q_1}{\partial q_i} + \frac{1}{Q_2}\frac{\partial Q_2}{\partial q_i} + \frac{1}{Q_3}\frac{\partial Q_3}{\partial q_i}\right)\right]\Theta\, dS\,.$$

The integral inside the summation sign is equivalent to:

$$\int \nabla\Theta \frac{\partial\kappa_i}{\partial q_i}\frac{dS}{\nabla} + \int \Theta\,\delta\kappa_i \frac{\partial\nabla}{\partial q_i}\frac{dS}{\nabla} = \int \frac{\partial}{\partial q_i}(\nabla\Theta\,\kappa_i)\frac{dS}{\nabla} - \int \delta\kappa_i \frac{\partial\nabla\Theta}{\partial q_i}\frac{dS}{\nabla} + \int \Theta\,\delta\kappa_i \frac{\partial\nabla}{\partial q_i}\frac{dS}{\nabla}\,.$$

The first integral transforms into a surface integral and in the other two, one finds – $\delta\kappa_i$ dS / ∇, multiplied by:

$$\frac{\partial\nabla\Theta}{\partial q_i} - \Theta\frac{\partial\nabla}{\partial q_i} = \nabla\frac{\partial\Theta}{\partial q_i}\,.$$

Consequently:

$$\int \Theta\,\delta\Theta\, dS = -\int \left(\frac{\partial\Theta}{\partial q_1}\delta\kappa_1 + \frac{\partial\Theta}{\partial q_2}\delta\kappa_2 + \frac{\partial\Theta}{\partial q_3}\delta\kappa_3\right) dS + \text{surf. int.}$$

Similarly:

$$\int \mathcal{T}_1\,\delta\mathcal{T}_1\, dS = \int Q_1\,\mathcal{T}_1\left[\frac{\partial}{\partial q_2}(Q_3^2\delta\kappa_3) - \frac{\partial}{\partial q_3}(Q_2^2\delta\kappa_2)\right]\frac{dS}{\nabla}$$

$$= \int \left[\frac{\partial}{\partial q_2}(Q_3^2\, Q_1\,\mathcal{T}_1\,\delta\kappa_3) - \frac{\partial}{\partial q_3}(Q_2^2\, Q_1\,\mathcal{T}_1\,\delta\kappa_2)\right]\frac{dS}{\nabla} - \int \left(Q_3^2\,\delta\kappa_3\frac{\partial Q_1\mathcal{T}_1}{\partial q_2} - Q_2^2\delta\kappa_2\frac{\partial Q_1\mathcal{T}_1}{\partial q_3}\right)\frac{dS}{\nabla}\,,$$

(*) See the following chapter.

etc. Therefore the indefinite equations for elastic equilibrium are:

$$\left.\begin{aligned}
F_1 + \frac{A}{Q_1}\frac{\partial\Theta}{\partial q_1} + \frac{B}{Q_2\,Q_3}\left(\frac{\partial Q_2\,\mathcal{T}_2}{\partial q_3} - \frac{\partial Q_3\,\mathcal{T}_3}{\partial q_2}\right) &= 0,\\
F_2 + \frac{A}{Q_2}\frac{\partial\Theta}{\partial q_2} + \frac{B}{Q_3\,Q_1}\left(\frac{\partial Q_3\,\mathcal{T}_3}{\partial q_1} - \frac{\partial Q_1\,\mathcal{T}_1}{\partial q_3}\right) &= 0,\\
F_3 + \frac{A}{Q_3}\frac{\partial\Theta}{\partial q_3} + \frac{B}{Q_1\,Q_2}\left(\frac{\partial Q_1\,\mathcal{T}_1}{\partial q_2} - \frac{\partial Q_2\,\mathcal{T}_2}{\partial q_1}\right) &= 0.
\end{aligned}\right\} \tag{13}$$

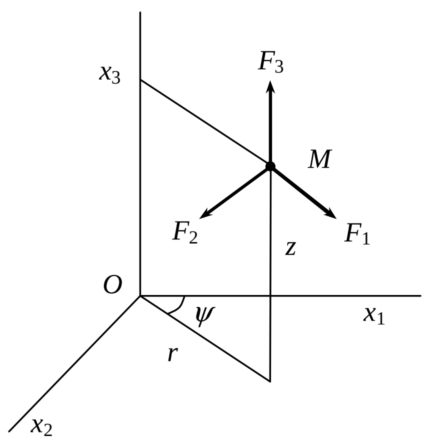
Suppose that they have been integrated. One will then know the κ; and one can therefore calculate the θ and the ω by means of the defining formulas. The Θ and Ω will then be given by the formula:

$$\left.\begin{aligned}
\Theta_1 &= -A\Theta + 2B(\theta_2 + \theta_3), \quad \Omega_1 = -B\omega_1,\\
\Theta_2 &= -A\Theta + 2B(\theta_3 + \theta_1), \quad \Omega_2 = -B\omega_2,\\
\Theta_3 &= -A\Theta + 2B(\theta_1 + \theta_2), \quad \Omega_3 = -B\omega_3,
\end{aligned}\right\} \tag{14}$$

which are obtained by deriving the unitary potential:

$$\Pi = -\tfrac{1}{2}\,[A\,(\theta_1 + \theta_2 + \theta_3)^2 + B\,(\omega_1^2 + \omega_2^2 + \omega_3^2 - 4\theta_2\,\theta_3 - 4\theta_3\,\theta_1 - 4\theta_1\,\theta_2)].$$

Finally, if one substitutes the Θ and the Ω in the boundary equations then one can determine the arbitrary constants.



11. Example. – Assume cylindrical coordinates; i.e., consider the triply-orthogonal system that is composed as follows: Cylinders of rotation with the common axis Ox_3, planes that pass through Qx_3, planes perpendicular to Ox_3. Parameters: $q_1 = r$, $q_2 = \psi$, $q_3 = z$. Coordinate line elements:

$$Q_1\,dq_1 = dr, \qquad Q_3\,dq_3 = r\,d\psi, \qquad Q_3\,dq_3 = dz.$$

Therefore, $Q_1 = 1$, $Q_2 = r$, $Q_3 = 1$, $\nabla = r$. Finally, set $\kappa_1 = u$, $\kappa_2 = v$, $\kappa_3 = w$, while being aware that the displacements are u, rv, w. Formulas (11) and (12) then give:

$$\Theta = \frac{1}{r}\frac{\partial(ur)}{\partial r} + \frac{\partial v}{\partial \psi} + \frac{\partial w}{\partial z}, \tag{15}$$

$$\mathcal{T}_1 = \frac{1}{r}\frac{\partial w}{\partial \psi} - r\frac{\partial v}{\partial z}, \qquad \mathcal{T}_2 = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial r}, \ \mathcal{T}_3 = \frac{1}{r}\left(\frac{\partial(vr^2)}{\partial r} - \frac{\partial u}{\partial \psi}\right), \tag{16}$$

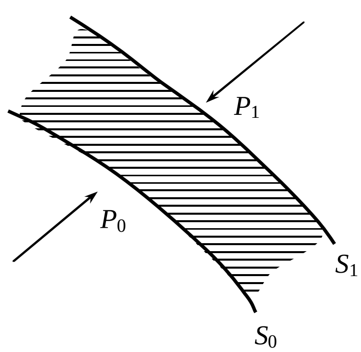
and equations (13) will become:

$$\left.\begin{aligned} &F_1 + A\frac{\partial\Theta}{\partial r} + B\left(\frac{\partial\mathcal{T}_2}{\partial z} - \frac{1}{r}\frac{\partial\mathcal{T}_3}{\partial\psi}\right) = 0, \\ &F_2 + \frac{A}{r}\frac{\partial\Theta}{\partial r} + B\left(\frac{\partial\mathcal{T}_3}{\partial r} - \quad \frac{\partial\mathcal{T}_1}{\partial z}\right) = 0, \\ &F_3 + A\frac{\partial\Theta}{\partial r} + B\left(\frac{\partial\mathcal{T}_1}{\partial\psi} - \frac{\partial(r\mathcal{T}_3)}{\partial r}\right) = 0. \end{aligned}\right\} \tag{17}$$

The θ and ω are given by the formulas:

$$\left.\begin{aligned} &\theta_1 = \frac{\partial u}{\partial r}, && \omega_1 = r\frac{\partial v}{\partial z} + \frac{1}{r}\frac{\partial w}{\partial\psi}, \\ &\theta_2 = \frac{\partial v}{\partial\psi} + \frac{u}{r}, && \omega_2 = \frac{\partial w}{\partial r} + \frac{\partial u}{\partial z}, \\ &\theta_3 = \frac{\partial w}{\partial z}, && \omega_3 = \frac{1}{r}\frac{\partial u}{\partial\psi} + r\frac{\partial v}{\partial z}. \end{aligned}\right\} \tag{18}$$

Finally, one can get the Θ and Ω from formulas (14).



12. Application to the cylindrical casing. – Let r_0, p_0 be the radius and unit pressure, resp., on an internal surface, and let r_1, p_1 be the radius and unit pressure, resp., on an external surface. If the pressure is distributed uniformly over the surface then one will know that the deformation will be independent of ψ and z by reason of symmetry; i.e., one will have that u, Θ, $\mathcal{T}_1$, …, are functions of only r, while v and w will be equal to zero. With that, equations (17) will become:

$$\frac{d\Theta}{dr} = 0, \qquad \frac{d\mathcal{T}_3}{dr} = 0, \qquad \frac{d(\mathcal{T}_2\, r)}{dr} = 0.$$

On the other hand, from (16), one has $\mathcal{T}_3 = 0$, $\mathcal{T}_2 = 0$. Therefore, the last two equations will be satisfied, while the first one will become:

$$\frac{d}{dr}\left(\frac{1}{r}\frac{dur}{dr}\right) = 0,$$

by virtue of (15). One will then deduce that:

$$u = \lambda\, r + \frac{\mu}{r}$$

by integration. In order to determine the λ and μ, we will need to appeal to the boundary equations. However, we must first calculate the Θ and Ω, which are given by formulas (14). In the present case, formulas (18) then give:

$$\theta_1 = \frac{du}{dr}, \qquad \theta_2 = \frac{u}{r}, \qquad \theta_3 = \omega_1 = \omega_2 = \omega_3 = 0,$$

and one will then have, from (14):

$$\Omega_1 = \Omega_2 = \Omega_3 = 0,$$

$$\left.\begin{aligned} \Theta_1 &= -A\Theta + 2B\frac{u}{r} = -2\lambda A + 2B\left(\lambda + \frac{\mu}{r^2}\right) = -2\lambda(A-B) + \frac{2\mu B}{r^2}, \\ \Theta_2 &= -A\Theta + 2B\frac{du}{dr} = -2\lambda A + 2B\left(\lambda - \frac{\mu}{r^2}\right) = -2\lambda(A-B) - \frac{2\mu B}{r^2}, \\ \Theta_3 &= -A\Theta + 2B\left(\frac{u}{r} + \frac{du}{dr}\right) = -2\lambda A + 4\lambda B = -2\lambda(A-2B). \end{aligned}\right\} \tag{19}$$

Given that, the boundary equations will become $\varphi_1 = \Theta_1 \cos(n\, r)$, $\varphi_2 = \varphi_3 = 0$. On the internal surface, $\varphi_1 = p_0$, $\cos(n\, r) = 1$. On the external surface, $\varphi_1 = -p_1$, $\cos(n\, r) = -1$. Therefore:

$$p_0 = -2\lambda\,(A - B) + \frac{2\mu B}{r_0^2}, \qquad p_1 = -2\lambda\,(A - B) + \frac{2\mu B}{r_1^2}.$$

One will then deduce that:

$$\lambda = -\frac{1}{2(A-B)} \cdot \frac{p_1\, r_1^2 - p_0\, r_0^2}{r_1^2 - r_0^2}, \qquad \mu = -\frac{1}{2B} \cdot \frac{(p_1 - p_0)\, r_1^2\, r_0^2}{r_1^2 - r_0^2}.$$

The displacements, unit dilatations, etc., will then remain completely determined. Finally, the formulas (19) will tell one about the distribution of internal actions at each point of the body. For example, in the case of an indefinite cylindrical cavity, one has $\lambda = 0$, $2\mu B = p_0\, r_0^2$, and formulas (19) give:

$$\Theta_1 = -\Theta_2 = p_0\,\frac{r_0^2}{r^2}, \qquad \Theta_3 = 0.$$

Considering the infinitude of planar sections that are made in the body perpendicular to the axis, it will follow that they will behave as if they were mutually-independent, and radial pressures and lateral tensions will develop in any section that are equal in absolute value for every point and vary from one point to the other in inverse proportion to the square of the radius.

13. Betti's theorem. – Betti's theorem can be presented in orthogonal curvilinear coordinates. That is not obvious *a priori*, since the orientation of the triad of axes with respect to which one computes the displacements $Q\kappa$ will vary from point to point in the case of general curvilinear coordinates. Let $Q\kappa'$ represent some other displacements that are due to the forces (F_1', F_2', F_3'), $(\varphi_1', \varphi_2', \varphi_3')$. Multiplying equations (10) by $\kappa_i'\, dS$ and integrating, one will obtain:

$$\begin{aligned}&\int Q_i\, F_i\, \kappa_i'\, dS \\ &= \int \kappa_i' \left[\frac{\partial}{\partial q_1}\left(\nabla \frac{\partial \Pi}{\partial \kappa_{i1}}\right) + \frac{\partial}{\partial q_2}\left(\nabla \frac{\partial \Pi}{\partial \kappa_{i2}}\right) + \frac{\partial}{\partial q_3}\left(\nabla \frac{\partial \Pi}{\partial \kappa_{i3}}\right)\right] \frac{dS}{\nabla} - \int \kappa_i' \frac{\partial \Pi}{\partial \kappa_i}\, dS.\end{aligned}$$

The first integral is equivalent to:

$$\begin{aligned}&\int \left[\frac{\partial}{\partial q_1}\left(\kappa_i' \nabla \frac{\partial \Pi}{\partial \kappa_{i1}}\right) + \frac{\partial}{\partial q_2}\left(\kappa_i' \nabla \frac{\partial \Pi}{\partial \kappa_{i2}}\right) + \frac{\partial}{\partial q_3}\left(\kappa_i' \nabla \frac{\partial \Pi}{\partial \kappa_{i3}}\right)\right] \frac{dS}{\nabla} \\ &\qquad - \int \left(\kappa_i' \frac{\partial \Pi}{\partial \kappa_{i1}} + \kappa_i' \frac{\partial \Pi}{\partial \kappa_{i2}} + \kappa_i' \frac{\partial \Pi}{\partial \kappa_{i3}}\right) dS.\end{aligned}$$

The first of these integrals, in turn, transforms into:

$$-\int\left[Q_1\frac{\partial\Pi}{\partial\kappa_{i1}}\cos(nq_1)+Q_2\frac{\partial\Pi}{\partial\kappa_{i2}}\cos(nq_2)+Q_3\frac{\partial\Pi}{\partial\kappa_{i3}}\cos(nq_3)\right]\kappa_i'\,ds=-\int Q_i\,\varphi_i\,\kappa_i'\,ds\,.$$

Therefore, if one sets i = 1, 2, 3 and sums then:

$$\sum_i\left\{\int Q_i\,F_i\,\kappa_i'\,dS+\int Q_i\,\theta_i\,\kappa_i'\,ds\right\}$$

$$=-\int\left[\kappa_1'\frac{\partial\Pi}{\partial\kappa_1}+\kappa_2'\frac{\partial\Pi}{\partial\kappa_2}+\cdots+\kappa_{11}'\frac{\partial\Pi}{\partial\kappa_{11}}+\kappa_{12}'\frac{\partial\Pi}{\partial\kappa_{12}}+\cdots+\kappa_{33}'\frac{\partial\Pi}{\partial\kappa_{33}}\right]dS\,.$$

Therefore, Π will be the quadratic form in κ_1, κ_2, κ_3, κ_{11}, κ_{12}, κ_{13}, …, κ_{23}, κ_{33} . The right-hand side will not change when one exchanges the κ with the κ'. Thus:

$$\begin{aligned}&\int(Q_1\,F_1\,\kappa_1'+Q_2\,F_2\,\kappa_2'+Q_3\,F_3\,\kappa_3')\,dS+\int(Q_1\,\varphi_1\,\kappa_1'+Q_2\,\varphi_2\,\kappa_2'+Q_3\,\varphi_3\,\kappa_3')\,ds\\&=\int(Q_1\,F_1'\kappa_1'+Q_2\,F_2'\,\kappa_2'+Q_3\,F_3'\,\kappa_3')\,dS+\int(Q_1\,\varphi_1'\,\kappa_1'+Q_2\,\varphi_2'\,\kappa_2'+Q_3\,\varphi_3'\,\kappa_3')\,ds\,.\end{aligned}\tag{20}$$

14. – One can make some applications of the preceding theorem, as was done in Chap. VI. However, everything depends upon the effective integration of equations (8), in which one can suppose that the θ and the ω are *constants.* One will then get expressions for the κ' that contain six arbitrary constants a_1, a_2, …, a_6 linearly, along with the θ and ω. When the θ and the ω are set equal to zero, the κ' that relate to them will correspond to the rigidity hypothesis, and therefore the F' and φ' will be zero. Hence, formula (20) will reduce the left-hand side to zero. The equation thus-obtained will then split into *six* distinct equations for any domain of the a's, and those are the equations of rigid equilibrium. However, if one sets the a's equal to zero and determines the θ and ω by means of the six first-degree equations:

$$\Theta_1=\Theta_2=\Theta_3=1,\qquad\Omega_1=\Omega_2=\Omega_3=0$$

then equations (4) and (5) will give:

$$F_i'=0,\qquad\varphi_i'=\cos(n\,q_i)\qquad(i=1,2,3).$$

Thus, the right-hand side of (20) will become:

$$\int[Q_1\,\kappa_1\cos(n\,q_1)+Q_2\,\kappa_2\cos(n\,q_2)+Q_3\,\kappa_3\cos(n\,q_3)]\,ds$$

$$=-\int\left(\frac{\partial\nabla\kappa_1}{\partial q_1}+\frac{\partial\nabla\kappa_2}{\partial q_2}+\frac{\partial\nabla\kappa_3}{\partial q_3}\right)\frac{dS}{\nabla}\,.$$

If one substitutes in (20) and observes (11) then it will follow that:

$$-\int \Theta\, dS = \int (Q_1 F_1 \kappa_1' + Q_2 F_2 \kappa_2' + Q_3 F_3 \kappa_3')\, dS + \int (Q_1 \varphi_1 \kappa_1' + Q_2 \varphi_2 \kappa_2' + Q_3 \varphi_3 \kappa_3')\, ds\,.$$

It remains for one to determine the total dilatations for any deformation.

15. Example. – One must integrate equations (18), while supposing that the θ and the ω are constants in the case of cylindrical coordinates. One can distinguish those equations by means of their left-hand sides. If one differentiates (ω_2) with respect to z and r in succession then one will get:

$$\frac{\partial^2 u}{\partial z^2} = 0, \qquad \frac{\partial^2 w}{\partial r^2} = 0.$$

One then sees that $\partial u / \partial z$ is independent of z, while $\partial u / \partial r = \theta_1$. Similarly, $\partial w / \partial r$ is independent of r, while $\partial w / \partial z = \theta_3$. Therefore:

$$u = \theta_1 r + z\, F(\psi) + G(\psi), \qquad w = \theta_3 z + r\, U(\psi) + V(\psi). \tag{21}$$

If one substitutes this in (ω_2) then one will see that:

$$F(\psi) + U(\psi) = \omega_2\,. \tag{22}$$

If one differentiates (ω_2) and (ω_1) with respect to ψ and takes (ω_2) into account then one will get:

$$\frac{\partial^2 u}{\partial \psi^2} = r^2 \frac{\partial}{\partial r}\frac{u}{r}\,, \qquad \frac{\partial^2 w}{\partial \psi^2} = r\frac{\partial u}{\partial z}\,.$$

One then deduces, by virtue of (21), that:

$$F + F'' = 0, \qquad G + G'' = 0, \qquad V'' = 0,$$

so

$$F(\psi) = a_1 \sin\psi + a_2 \cos\psi, \quad G(\psi) = a_2 \sin\psi + a_4 \cos\psi, \quad V = m\,\psi + a_5\,.$$

If one takes (22) into account then (21) will become:

$$\begin{cases} u = \theta_1 r + (a_1 \sin\psi + a_2 \cos\psi)\, z + a_3 \sin\psi + a_4 \cos\psi, \\ w = \theta_3 z + (\omega_2 - a_1) \sin\psi - a_2 \cos\psi)\, z + m\psi + a_5. \end{cases}$$

If one now integrates (θ_2) then one will get:

$$v = (\theta_2 - \theta_1)\,\psi + \frac{z}{r}(a_1 \cos\psi - a_2 \sin\psi) + \frac{1}{r}(a_3 \cos\psi - a_4 \sin\psi) + H(r, s).$$

The substitution of this result in (ω_3) and (ω_1) will show that the function H must satisfy the equations:

$$\frac{\partial H}{\partial r} = \frac{\omega_3}{r}, \qquad \frac{\partial H}{\partial z} = \frac{\omega_1}{r} - \frac{m}{r^2}.$$

Therefore, $\omega_1 = 0$ (*), $m = 0$, $H = \omega_3 \log r + a_6$. Finally:

$$\begin{cases} u = \theta_1 r + (a_1 \sin\psi + a_2 \cos\psi)\, z + a_3 \sin\psi + a_4 \cos\psi, \\ w = \theta_3 r + (\omega_3 - a_1 \sin\psi - a_2 \cos\psi)\, r + a_5, \\ v = (\theta_2 - \theta_1)\,\psi + \dfrac{z}{r}(a_1 \cos\psi - a_2 \sin\psi) + \dfrac{1}{r}(a_3 \cos\psi - a_4 \sin\psi) + \omega_3 \log r + a_6. \end{cases}$$

It is now easy to find the six conditions for rigid equilibrium, the total dilatations, etc., from the process that was described. The results that one will get in this special case can be reached more rapidly thanks to the direct transformation of the analogous results that are obtained in Cartesian coordinates. However, in order to perform that transformation, it is necessary to know the relations that exist between the q and the x, and after all, the general process that was presented has the advantage of remaining applicable when one rejects the validity of Euclid's postulate in the space considered.

(*) What one can call a *homogeneous* deformation relative to the particular cylindrical representation that one considers is therefore not possible, unless one supposes that two arbitrary surface elements that are situated in a plane perpendicular to the axis and a plane that contains the axis will remain orthogonal under deformation. With that, one sees that one cannot attribute constant values to the θ and ω arbitrarily, as one can in the Cartesian representation. The conditions that those quantities must satisfy, whether they are constant or variable, have been pointed out by Prof. E. PADOVA in vol. III of the *Studii offerti dall'Università Padovana alla Bolognese nell'VIII centenario, ecc.*

CHAPTER XXII

ELASTICITY IN CURVED SPACES

1. – Before we enter into the field of research that was initiated by Prof. Beltrami for the study of elasticity in *spaces of constant curvature* α, it is good to remember that they are characterized by the property that any rigid figure in them possesses the property that it can always be superimposed with itself after any motion. That property is assumed dogmatically in ordinary geometry, which is based upon two known postulates, in addition, one of which characterizes *Euclidean space* ($\alpha = 0$) among all of the spaces of constant curvature – i.e., *Euclid's postulate* – and the other of which is the postulate of the *infinitude* of space (*). It is intuitive that in spaces of constant curvature, the concept of isotropy will persist as it is from the homogeneous geometric constitution that such spaces admit around each point by virtue of the aforementioned characteristic property. However, when one visualizes that concept in an arbitrary space, the coefficient A and B will need to be considered to be variable from point to point along with the curvature (**). Finally, observe that the *Cartesian representation* supposes that space is infinite and *includes the Euclidian hypothesis*, in such a way that all of the results that are obtained in Cartesian coordinates are applicable to Euclidian spaces exclusively. It will then follow that in order to study elasticity in non-Euclidian spaces, one will need to make use of general curvilinear coordinates that assume nothing about the nature of space. However, the results that one obtains can then be applied solely to spaces of constant curvature when one considers the coefficients A and B to be constants, in addition.

2. – The preceding considerations might perhaps become more precise when one recalls the analytical conceptualization of space – i.e., when one would like to give the name of (*three*-dimensional) space to the set of all triples of values for the parameters q_1, q_2, q_3 . Any *arbitrary* triple of functions Q_1, Q_2, Q_3 of q_1, q_2, q_3 corresponds to a particular space in which the square of the line element can be expressed as:

$$d\sigma^2 = Q_1^2\, dq_1^2 + Q_2^2\, dq_2^2 + Q_3^2\, dq_3^2 .$$

In order for such a space to be Euclidian space (which is representable in Cartesian coordinates, as we have said), it is a necessary and sufficient condition that one can find three functions x_1, x_2, x_3 of q_1, q_2, q_3 such that:

$$dx_1^2 + dx_2^2 + dx_3^2 = Q_1^2\, dq_1^2 + Q_2^2\, dq_2^2 + Q_3^2\, dq_3^2 .$$

(*) See, e.g., the translator's note in § 6 of the interesting book by CLIFFORD *Il senso commune nelle scienze esatta*, Milan, Dumolard, 1886. The younger generation of our school of media can draw much knowledge of general mathematical culture from this well.

(**) In order to gain a precise notion of the curvature of spaces, one can study Prof. BELTRAMI's "Teoria degli spazii di curvature costante," Annali di Matematica, v. II of the second series, pp. 232. See also CLIFFORD *loc. cit.*, pp. 255.

When one performs the effective integration (*), one will get six relations between the Q and their derivatives that are due to Lamé, and one can show that they are *necessary* and *sufficient* conditions for the integration, and consequently for the *Euclidian-ness* of space. Those conditions result spontaneously from the following analysis, which will provide the analogous conditions for the space considered to have constant curvature, more generally. Here it is convenient to recall that according to Riemann (**), the square of the line element in a space of constant curvature can always be given the form:

$$d\sigma^2 = Q^2\,(dq_1^2 + dq_2^2 + dq_3^2)\,,$$

in which Q_1, Q_2, Q_3 are equal to:

$$Q = \frac{1}{1+\dfrac{\alpha}{4}(q_1^2+q_2^2+q_3^2)}\,. \tag{1}$$

The coordinates q_1, q_2, q_3 that appear here are the ones that Prof. Beltrami (***) called *stereographic*, and the constant α measures the *curvature of space.*

3. – In order to account for the ultimate form of $d\sigma$, it is perhaps useful to seek to establish it by means of elementary considerations, while being guided by the analogy with two-dimensional spaces. It is known that the stereographic representation of a spherical surface is accomplished by projecting it from one of its points P onto the tangent place at the diametrically-opposite point O. The stereographic coordinates of a point on the surface are the Cartesian coordinates of the stereographic image of that point. If N and N' are the images of the points M and M' then one will have:

$$OP^2 = PM \cdot PN = PM' \cdot PN'.$$

The triangles PMM' and $PN'N$ are similar then, and therefore $MM' : NN' = PM : PN'$. Thus, if M' is infinitely close to M then:

$$\frac{d\sigma}{NN'} = \frac{PM}{PN} = \frac{OP^2}{PN^2} = \frac{1}{1+\left(\dfrac{ON}{OP}\right)^2}\,,$$

and one will then get Riemann's formula by observing that:

(*) See BIANCHI, *Geometria differenziale*, § 125.

(**) Read the celebrated paper on the fundamental hypotheses of geometry in *B. Riemann's math. Werke*, pp. 264.

(***) *Loc. cit.*, pp. 242.

$$NN'^2 = dq_1^2 + dq_2^2, \qquad ON^2 = q_1^2 + q_2^2, \qquad OP^2 = \frac{4}{\alpha}.$$

4. – From what was said before, the general equations of equilibrium [Chap. XX, forms. (4) and (10)] are applicable to all spaces, while equations (13) of the previous chapter are suitable to only Euclidian spaces, since the artifice that was resorted to in order to abbreviate the calculations was to adopt Cartesian coordinates (Chap. V, § **3**). It will then follow that if one would like to obtain the equations of equilibrium of isotropic bodies in an arbitrary space then one would need to start with the general equations (4) and introduce the isotropy hypothesis into them directly; i.e., to suppose that:

$$\Theta_i = -(A - 2B)\,\Theta - 2B\,\theta_i, \qquad \Omega_i = -B\,\omega_i \qquad (i = 1, 2, 3),$$

with A and B variable, like α. We limit ourselves to the case of constant α and first consider the terms that are multiplied by A. That will then give rise to the expression:

$$-A\left(\frac{1}{\nabla}\frac{\partial\nabla\Theta}{\partial q_1} - \frac{\Theta}{\nabla}\frac{\partial\nabla}{\partial q_1}\right) = -A\,\frac{\partial\Theta}{\partial q_1}$$

in the first equation (4). The terms that are multiplied by B easily reduce to:

$$-\frac{2B}{\nabla}\left(Q_1\,\theta_1\frac{\partial Q_2\,Q_3}{\partial q_1} - Q_3\,Q_1\frac{\partial Q_2\,\theta_1}{\partial q_1} + \frac{1}{2}\frac{\partial Q_1^2\,Q_2\omega_3}{\partial q_2} + \frac{1}{2}\frac{\partial Q_1^2\,Q_2\omega_2}{\partial q_3}\right). \tag{2}$$

One can predict that this expression will be included in what one obtains for Euclidian space, i.e.:

$$-\frac{BQ_1}{Q_2\,Q_3}\left(\frac{\partial Q_2\mathcal{T}_2}{\partial q_3} - \frac{\partial Q_3\mathcal{T}_3}{\partial q_2}\right), \tag{3}$$

in which:

$$\mathcal{T}_2 = \frac{1}{Q_3\,Q_1}\left(\frac{\partial Q_2^2\kappa_1}{\partial q_3} - \frac{\partial Q_3^2\kappa_2}{\partial q_1}\right), \qquad \mathcal{T}_3 = \frac{1}{Q_1\,Q_2}\left(\frac{\partial Q_2^2\kappa_2}{\partial q_1} - \frac{\partial Q_1^2\kappa_1}{\partial q_2}\right). \tag{4}$$

If one substitutes (4) in (3) and then subtracts the result from (2) then what will remain (*) is a homogeneous linear function of κ, which one can give the following form:

$$\frac{2B}{Q_2\,Q_3}[(H_{22} + H_{33})\,Q_1\,\kappa_1 - H_{12}\,Q_2\,\kappa_2 - H_{13}\,Q_3\,\kappa_3],$$

after having set:

(*) See the following paragraph for the details of the calculation.

$$\left.\begin{aligned}
H_{11} &= Q_1\left[\frac{\partial}{\partial q_2}\left(\frac{1}{Q_1}\frac{\partial Q_3}{\partial q_2}\right)+\frac{\partial}{\partial q_3}\left(\frac{1}{Q_3}\frac{\partial Q_2}{\partial q_3}\right)\right]+\frac{1}{Q_1}\frac{\partial Q_2}{\partial q_1}\frac{\partial Q_3}{\partial q_1},\\
H_{22} &= Q_2\left[\frac{\partial}{\partial q_3}\left(\frac{1}{Q_3}\frac{\partial Q_1}{\partial q_3}\right)+\frac{\partial}{\partial q_1}\left(\frac{1}{Q_1}\frac{\partial Q_3}{\partial q_1}\right)\right]+\frac{1}{Q_2}\frac{\partial Q_3}{\partial q_2}\frac{\partial Q_1}{\partial q_2},\\
H_{33} &= Q_3\left[\frac{\partial}{\partial q_1}\left(\frac{1}{Q_1}\frac{\partial Q_1}{\partial q_1}\right)+\frac{\partial}{\partial q_2}\left(\frac{1}{Q_2}\frac{\partial Q_1}{\partial q_2}\right)\right]+\frac{1}{Q_3}\frac{\partial Q_1}{\partial q_3}\frac{\partial Q_2}{\partial q_3},\\
H_{23} &= H_{32} = \frac{1}{Q_2}\frac{\partial Q_1}{\partial q_2}\frac{\partial Q_2}{\partial q_3}+\frac{1}{Q_3}\frac{\partial Q_1}{\partial q_3}\frac{\partial Q_3}{\partial q_2}-\frac{\partial^2 Q_1}{\partial q_2\,\partial q_3},\\
H_{31} &= H_{13} = \frac{1}{Q_3}\frac{\partial Q_2}{\partial q_3}\frac{\partial Q_3}{\partial q_1}+\frac{1}{Q_1}\frac{\partial Q_2}{\partial q_1}\frac{\partial Q_1}{\partial q_3}-\frac{\partial^2 Q_2}{\partial q_3\,\partial q_1},\\
H_{12} &= H_{21} = \frac{1}{Q_1}\frac{\partial Q_3}{\partial q_3}\frac{\partial Q_1}{\partial q_2}+\frac{1}{Q_2}\frac{\partial Q_3}{\partial q_2}\frac{\partial Q_2}{\partial q_1}-\frac{\partial^2 Q_3}{\partial q_1\,\partial q_2}.
\end{aligned}\right\}\qquad(5)$$

If one sets, in addition:

$$\Phi = \sum_{i,j} H_{ij}\,Q_i\,Q_j\,\kappa_i\,\kappa_j-(H_{11}+H_{22}+H_{33})\sum_i Q_i^2\,\kappa_i^2 \qquad (6)$$

then one will finally see that the expression (2) is equivalent to:

$$-\frac{BQ_1}{Q_2\,Q_3}\left(\frac{\partial Q_3\mathcal{T}_2}{\partial q_3}-\frac{\partial Q_2\mathcal{T}_3}{\partial q_2}\right)-\frac{B}{Q_2\,Q_3}\frac{\partial\Phi}{\partial Q_i\,\kappa_1}.$$

The indefinite equations of equilibrium will then reduce to:

$$\left.\begin{aligned}
F_1+\frac{A}{Q_1}\frac{\partial\Theta}{\partial q_1}+\frac{B}{Q_2\,Q_3}\left(\frac{\partial Q_2\mathcal{T}_2}{\partial q_3}-\frac{\partial Q_3\mathcal{T}_3}{\partial q_2}\right)+\frac{B}{Q_1\,Q_2\,Q_3}\frac{\partial\Phi}{\partial Q_1\,\kappa_1}&=0,\\
F_2+\frac{A}{Q_2}\frac{\partial\Theta}{\partial q_2}+\frac{B}{Q_3\,Q_1}\left(\frac{\partial Q_3\mathcal{T}_3}{\partial q_1}-\frac{\partial Q_1\mathcal{T}_1}{\partial q_3}\right)+\frac{B}{Q_1\,Q_2\,Q_3}\frac{\partial\Phi}{\partial Q_2\,\kappa_2}&=0,\\
F_3+\frac{A}{Q_3}\frac{\partial\Theta}{\partial q_3}+\frac{B}{Q_1\,Q_2}\left(\frac{\partial Q_1\mathcal{T}_1}{\partial q_2}-\frac{\partial Q_2\mathcal{T}_2}{\partial q_1}\right)+\frac{B}{Q_1\,Q_2\,Q_3}\frac{\partial\Phi}{\partial Q_3\,\kappa_3}&=0.
\end{aligned}\right\}\qquad(7)$$

5. – In order to perform in detail the calculation that was pointed out in the preceding paragraph, recall expression (2), and after changing the sign, write:

$$\frac{2B}{Q_2\,Q_3}\left(\theta_1\frac{\partial Q_3\,Q_3}{\partial q_1}-Q_3\frac{\partial Q_2\,\theta_2}{\partial q_1}-Q_2\frac{\partial Q_3\,\theta_3}{\partial q_1}+\frac{1}{2Q_1}\frac{\partial Q_1^2\,Q_3\,\omega_3}{\partial q_2}+\frac{1}{2Q_1}\frac{\partial Q_1^2\,Q_2\,\omega_2}{\partial q_3}\right).$$

Subtracting:

$$\frac{BQ_1}{Q_2\,Q_3}\left(\frac{\partial Q_2\,\mathcal{T}_2}{\partial q_3}-\frac{\partial Q_3\,\mathcal{T}_3}{\partial q_2}\right)$$

and dropping the factor $2B\,/\,(Q_2\,Q_3)$ will give:

$$\begin{aligned}&\theta_1\frac{\partial Q_2\,Q_3}{\partial q_1}-Q_3\frac{\partial Q_2\,Q_2}{\partial q_1}-Q_2\frac{\partial Q_3\,Q_3}{\partial q_1}\\&+\frac{1}{2Q_1}\frac{\partial Q_1^2\,Q_3\,\omega_3}{\partial q_2}+\frac{1}{2Q_1}\frac{\partial Q_1^2\,Q_2\,\omega_2}{\partial q_3}-\frac{Q_1}{2}\frac{\partial Q_2\,\mathcal{T}_2}{\partial q_3}+\frac{Q_1}{2}\frac{\partial Q_3\,\mathcal{T}_3}{\partial q_2}\,.\end{aligned}\tag{8}$$

One now notes that:

$$\begin{aligned}\frac{1}{2Q_1}\frac{\partial Q_1^2\,Q_3\,\omega_3}{\partial q_2}+\frac{Q_1}{2}\frac{\partial Q_3\,\mathcal{T}_3}{\partial q_2}&=\frac{1}{2}\frac{\partial Q_1\,Q_3\,\omega_3}{\partial q_2}+\frac{Q_3\omega_3}{2}\frac{\partial Q_1}{\partial q_2}+\frac{1}{2}\frac{\partial Q_1\,Q_3\,\mathcal{T}_3}{\partial q_2}-\frac{Q_3\mathcal{T}_3}{2}\frac{\partial Q_1}{\partial q_2}\\&=\frac{1}{2}\frac{\partial}{\partial q_2}Q_1\,Q_3(\omega_3+\mathcal{T}_3)+\frac{1}{2}Q_3(\omega_3-\mathcal{T}_3)\frac{\partial Q_1}{\partial q_2}\,.\end{aligned}$$

On the other hand, one has:

$$\omega_3=\frac{Q_1}{Q_2}\frac{\partial\kappa_1}{\partial q_2}+\frac{Q_2}{Q_1}\frac{\partial\kappa_2}{\partial q_1}=\frac{1}{Q_1}\frac{\partial Q_2\,\kappa_2}{\partial q_1}+\frac{1}{Q_2}\frac{\partial Q_1\,\kappa_1}{\partial q_2}-\frac{\kappa_1}{Q_2}\frac{\partial Q_1}{\partial q_2}-\frac{\kappa_2}{Q_1}\frac{\partial Q_2}{\partial q_1}\,,$$

$$\mathcal{T}_3=\frac{1}{Q_1\,Q_2}\left(\frac{\partial Q_2^2\,\kappa_2}{\partial q_1}-\frac{\partial Q_1^2\,\kappa_1}{\partial q_2}\right)=\frac{1}{Q_1}\frac{\partial Q_2\,\kappa_2}{\partial q_1}-\frac{1}{Q_2}\frac{\partial Q_1\,\kappa_1}{\partial q_2}-\frac{\kappa_1}{Q_2}\frac{\partial Q_1}{\partial q_2}+\frac{\kappa_2}{Q_1}\frac{\partial Q_2}{\partial q_1}\,,$$

in such a way that:

$$\tfrac{1}{2}(\omega_3+\mathcal{T}_3)=\frac{1}{Q_1}\frac{\partial Q_2\,\kappa_2}{\partial q_1}-\frac{\kappa_1}{Q_2}\frac{\partial Q_1}{\partial q_2}\,,\quad \tfrac{1}{2}(\omega_3-\mathcal{T}_3)=\frac{1}{Q_2}\frac{\partial Q_1\,\kappa_1}{\partial q_2}-\frac{\kappa_2}{Q_1}\frac{\partial Q_2}{\partial q_1}\,.$$

Therefore:

$$\begin{aligned}&\frac{1}{2Q_1}\frac{\partial Q_1^2\,Q_3\,\omega_3}{\partial q_2}+\frac{Q_1}{2}\frac{\partial Q_3\,\mathcal{T}_3}{\partial q_2}\\&=\frac{\partial}{\partial q_2}\left(Q_3\frac{\partial Q_2\,\kappa_2}{\partial q_1}-\frac{Q_3}{Q_2}\frac{\partial Q_1}{\partial q_2}Q_1\,\kappa_1\right)+\frac{\partial Q_1}{\partial q_2}\left(\frac{Q_3}{Q_2}\frac{\partial Q_1\,\kappa_1}{\partial q_2}-\frac{Q_3}{Q_1}\kappa_2\frac{\partial Q_2}{\partial q_1}\right)\end{aligned}$$

$$= Q_3 \frac{\partial^2 Q_2 \kappa_2}{\partial q_1 \partial q_2} + \frac{\partial Q_3}{\partial q_2}\frac{\partial Q_2 \kappa_2}{\partial q_1} - Q_1 \kappa_1 \frac{\partial}{\partial q_2}\left(\frac{Q_3}{Q_2}\frac{\partial Q_1}{\partial q_2}\right) - \frac{Q_3}{Q_1}\frac{\partial Q_1}{\partial q_2}\frac{\partial Q_2}{\partial q_1}\kappa_2 .$$

Analogously, one gets:

$$\frac{1}{2Q_1}\frac{\partial Q_1^2 Q_2 \omega_2}{\partial q_3} - \frac{Q_1}{2}\frac{\partial Q_2 \mathcal{T}_2}{\partial q_3}$$

$$= Q_2 \frac{\partial^2 Q_3 \kappa_3}{\partial q_1 \partial q_3} + \frac{\partial Q_2}{\partial q_3}\frac{\partial Q_3 \kappa_3}{\partial q_1} - Q_1 \kappa_1 \frac{\partial}{\partial q_3}\left(\frac{Q_2}{Q_3}\frac{\partial Q_1}{\partial q_3}\right) - \frac{Q_2}{Q_1}\frac{\partial Q_1}{\partial q_3}\frac{\partial Q_3}{\partial q_1}\kappa_3 .$$

In addition, one has:

$$Q_1\ \theta_1 = Q_1 \frac{\partial \kappa_1}{\partial q_1} + \frac{\partial Q_1}{\partial q_1}\kappa_1 + \frac{\partial Q_1}{\partial q_2}\kappa_2 + \frac{\partial Q_1}{\partial q_3}\kappa_3 = \frac{\partial \kappa_1 Q_1}{\partial q_1} + \frac{\partial Q_1}{\partial q_2}\kappa_2 + \frac{\partial Q_1}{\partial q_3}\kappa_3 ,$$

and therefore the expression (8) will become:

$$\begin{aligned}
&\frac{1}{Q_1}\frac{\partial Q_2 Q_3}{\partial q_1}\left(\frac{\partial Q_1 \kappa_1}{\partial q_1} + \frac{\partial Q_1}{\partial q_2}\kappa_2 + \frac{\partial Q_1}{\partial q_3}\kappa_3\right) \\
&- Q_3 \frac{\partial}{\partial q_1}\left(\frac{\partial Q_2 \kappa_2}{\partial q_2} + \frac{\partial Q_2}{\partial q_1}\kappa_1 + \frac{\partial Q_2}{\partial q_3}\kappa_3\right) - Q_2 \frac{\partial}{\partial q_1}\left(\frac{\partial Q_3 \kappa_3}{\partial q_3} + \frac{\partial Q_3}{\partial q_1}\kappa_1 + \frac{\partial Q_3}{\partial q_2}\kappa_2\right) \\
&+ Q_3 \frac{\partial^2 Q_2 \kappa_2}{\partial q_1 \partial q_2} + \frac{\partial Q_3}{\partial q_2}\frac{\partial Q_2 \kappa_2}{\partial q_1} - Q_1 \kappa_1 \frac{\partial}{\partial q_2}\left(\frac{Q_3}{Q_2}\frac{\partial Q_1}{\partial q_2}\right) + \frac{Q_3}{Q_1}\frac{\partial Q_1}{\partial q_2}\frac{\partial Q_2}{\partial q_1}\kappa_2 \\
&+ Q_2 \frac{\partial^2 Q_3 \kappa_3}{\partial q_1 \partial q_3} + \frac{\partial Q_2}{\partial q_3}\frac{\partial Q_3 \kappa_3}{\partial q_1} - Q_1 \kappa_1 \frac{\partial}{\partial q_3}\left(\frac{Q_2}{Q_3}\frac{\partial Q_1}{\partial q_3}\right) - \frac{Q_2}{Q_1}\frac{\partial Q_1}{\partial q_3}\frac{\partial Q_3}{\partial q_1}\kappa_3 .
\end{aligned}$$

The terms:

$$\frac{1}{Q_1}\frac{\partial Q_2 Q_3}{\partial q_1}\frac{\partial Q_1 \kappa_1}{\partial q_1}, \qquad \frac{\partial Q_3}{\partial q_2}\frac{\partial Q_2 \kappa_2}{\partial q_1}, \qquad \frac{\partial Q_2}{\partial q_3}\frac{\partial Q_3 \kappa_3}{\partial q_1}$$

cancel with:

$$-\frac{Q_3}{Q_1}\frac{\partial Q_2}{\partial q_1}\frac{\partial Q_1 \kappa_1}{\partial q_1} - \frac{Q_2}{Q_1}\frac{\partial Q_3}{\partial q_1}\frac{\partial Q_1 \kappa_1}{\partial q_1}, \qquad - Q_2 \cdot \frac{1}{Q_2}\frac{\partial Q_3}{\partial q_1}\frac{\partial Q_2 \kappa_2}{\partial q_1}, \qquad - Q_3 \cdot \frac{1}{Q_3}\frac{\partial Q_2}{\partial q_3}\frac{\partial Q_3 \kappa_3}{\partial q_1},$$

respectively, and the expression under consideration will become:

$$\begin{aligned}
&\frac{1}{Q_1}\frac{\partial Q_2 Q_3}{\partial q_1}\left(\frac{\partial Q_1}{\partial q_2}\kappa_2 + \frac{\partial Q_1}{\partial q_3}\kappa_3\right) - Q_3\left[Q_1 \kappa_1 \frac{\partial}{\partial q_1}\left(\frac{1}{Q_1}\frac{\partial Q_2}{\partial q_1}\right) + Q_3 \kappa_3 \frac{\partial}{\partial q_1}\left(\frac{1}{Q_3}\frac{\partial Q_2}{\partial q_3}\right)\right] \\
&\qquad - Q_2\left[Q_1 \kappa_1 \frac{\partial}{\partial q_1}\left(\frac{1}{Q_1}\frac{\partial Q_3}{\partial q_1}\right) + Q_2 \kappa_2 \frac{\partial}{\partial q_1}\left(\frac{1}{Q_2}\frac{\partial Q_3}{\partial q_2}\right)\right]
\end{aligned}$$

$$- Q_1 \kappa_1 \frac{\partial}{\partial q_2}\left(\frac{Q_3}{Q_2}\frac{\partial Q_1}{\partial q_2}\right) - \frac{Q_3}{Q_1}\frac{\partial Q_1}{\partial q_2}\frac{\partial Q_2}{\partial q_1}\kappa_1 - Q_1 \kappa_1 \frac{\partial}{\partial q_3}\left(\frac{Q_2}{Q_3}\frac{\partial Q_1}{\partial q_3}\right) - \frac{Q_2}{Q_1}\frac{\partial Q_1}{\partial q_3}\frac{\partial Q_3}{\partial q_1}\kappa_3 .$$

This is linear in the κ's. The coefficient of $Q_2\ \kappa_2$ is:

$$\kappa_1 \frac{1}{Q_1 Q_2}\frac{\partial Q_2}{\partial q_1}\frac{Q_3}{}\frac{\partial Q_1}{\partial q_2} - Q_2 \frac{\partial}{\partial q_1}\left(\frac{1}{Q_2}\frac{\partial Q_3}{\partial q_2}\right) - \frac{Q_3}{Q_1 Q_2}\frac{\partial Q_1}{\partial q_2}\frac{\partial Q_2}{\partial q_1}$$

$$= \frac{1}{Q_1}\frac{\partial Q_3}{\partial q_1}\frac{\partial Q_1}{\partial q_3} - \frac{1}{Q_2}\frac{\partial Q_2}{\partial q_1}\frac{\partial Q_3}{\partial q_2} - \frac{\partial^2 Q_3}{\partial q_1 \partial q_2} = H_{12} .$$

Similarly, the coefficient of $Q_3\ \kappa_3$ is H_{13} and that of $Q_1\ \kappa_1$ is:

$$- Q_3 \frac{\partial}{\partial q_1}\left(\frac{1}{Q_1}\frac{\partial Q_2}{\partial q_1}\right) - Q_2 \frac{\partial}{\partial q_1}\left(\frac{1}{Q_1}\frac{\partial Q_3}{\partial q_1}\right) - \frac{\partial}{\partial q_2}\left(\frac{Q_3}{Q_2}\frac{\partial Q_1}{\partial q_2}\right) - \frac{\partial}{\partial q_3}\left(\frac{Q_2}{Q_3}\frac{\partial Q_1}{\partial q_3}\right)$$

$$= - Q_3 \left[\frac{\partial}{\partial q_1}\left(\frac{1}{Q_1}\frac{\partial Q_2}{\partial q_1}\right) + \frac{\partial}{\partial q_2}\left(\frac{1}{Q_2}\frac{\partial Q_1}{\partial q_2}\right)\right] - \frac{1}{Q_3}\frac{\partial Q_2}{\partial q_3}\frac{\partial Q_1}{\partial q_3}$$
$$- Q_2 \left[\frac{\partial}{\partial q_1}\left(\frac{1}{Q_1}\frac{\partial Q_3}{\partial q_1}\right) + \frac{\partial}{\partial q_3}\left(\frac{1}{Q_3}\frac{\partial Q_1}{\partial q_3}\right)\right] - \frac{1}{Q_2}\frac{\partial Q_3}{\partial q_2}\frac{\partial Q_1}{\partial q_2};$$

i.e., $-(H_{22} + H_{33})$. Therefore, the first indefinite equation of equilibrium is:

$$Q_1 F_1 + A\frac{\partial \Theta}{\partial q_1} + \frac{B Q_1}{Q_2 Q_3}\left(\frac{\partial Q_2 \mathcal{T}_2}{\partial q_3} - \frac{\partial Q_3 \mathcal{T}_3}{\partial q_2}\right)$$
$$+ \frac{B}{Q_2 Q_3}[-(H_{22} + H_{33})\, Q_1\, \kappa_1 + H_{12}\, Q_2\, \kappa_2 + H_{13}\, Q_3\, \kappa_3] = 0,$$

which is to say:

$$F_1 + \frac{A}{Q_1}\frac{\partial \Theta}{\partial q_1} + \frac{B}{Q_2 Q_3}\left(\frac{\partial Q_2 \mathcal{T}_2}{\partial q_3} - \frac{\partial Q_3 \mathcal{T}_3}{\partial q_2}\right) + \frac{B}{Q_1 Q_2 Q_3}\frac{\partial \Phi}{\partial Q_1 \kappa_1} = 0, \qquad \ldots$$

6. – Equations (7) have still not found their definitive form, in which the hypothesis of *constant curvature*, which has been taken into account only by supposing that A and B are constants, has not yet been introduced into the coordinate system. Observe, first of all, that equations (7) can be likewise arrived at by the general process when one takes into consideration, not the potential Π, but one of its components Π_0, which is expressed as follows:

$$\Pi_0 = -\tfrac{1}{2}\left[A\Theta^2 + B\left(\mathcal{T}_1^2 + \mathcal{T}_2^2 + \mathcal{T}_3^2\right)\right] + \frac{B\Phi}{\nabla}.$$

That exhibits the invariant character of the expression Φ / ∇, and that authorizes us to specialize the coordinate system in order to find its significance. Assume stereographic coordinates then, for which Q_1, Q_2, Q_3 have the expression (1), as we have said. Formulas (5) and (6) then give:

$$H_{ij} = \begin{cases} -Q^2\alpha & \text{if} \quad i = j, \\ 0 & \text{if} \quad i \neq j, \end{cases} \qquad \frac{\Phi}{\nabla} = 2\alpha Q^2\,(\kappa_1^2 + \kappa_2^2 + \kappa_3^2),$$

in succession. Therefore, Φ / ∇ *is the product of* 2α *with the square of the displacement.* It will then follow that in spaces of constant curvature α, one will have:

$$\frac{\Phi}{\nabla} = 2\alpha\,(Q_1^2\,\kappa_1^2 + Q_2^2\kappa_2^2 + Q_3^2\kappa_3^2) \tag{9}$$

for any orthogonal coordinate system, and consequently:

$$\frac{1}{\nabla}\frac{\partial\Phi}{\partial Q_i\,\kappa_i} = 4\alpha\,Q_i\,\kappa_i \quad (i = 1, 2, 3). \tag{10}$$

Finally, equations (7) become:

$$\left.\begin{aligned} F_1 + \frac{A}{Q_1}\frac{\partial\Theta}{\partial q_1} + \frac{B}{Q_2\,Q_3}\left(\frac{\partial Q_2\mathcal{T}_2}{\partial q_3} - \frac{\partial Q_3\mathcal{T}_3}{\partial q_2}\right) + 4\alpha B\,Q_1\kappa_1 = 0, \\ F_2 + \frac{A}{Q_2}\frac{\partial\Theta}{\partial q_2} + \frac{B}{Q_3\,Q_1}\left(\frac{\partial Q_3\mathcal{T}_3}{\partial q_1} - \frac{\partial Q_1\mathcal{T}_1}{\partial q_3}\right) + 4\alpha B\,Q_2\kappa_2 = 0, \\ F_3 + \frac{A}{Q_3}\frac{\partial\Theta}{\partial q_3} + \frac{B}{Q_1\,Q_2}\left(\frac{\partial Q_1\mathcal{T}_1}{\partial q_2} - \frac{\partial Q_2\mathcal{T}_2}{\partial q_1}\right) + 4\alpha B\,Q_3\kappa_3 = 0. \end{aligned}\right\} \tag{11}$$

These are the *equations of elastic equilibrium of isotropic bodies for spaces of constant curvature* α in arbitrary orthogonal coordinates.

7. Observations:

a) In order for equations (7) to coincide with (11), it is necessary and sufficient that the relations (10) should be verified; i.e., that Φ must have the form (9), and one will then have:

$$H_{11} = H_{22} = H_{33} = -\,\alpha\nabla, \qquad H_{11} = H_{22} = H_{33} = 0.$$

These are therefore the necessary and sufficient conditions for the invariability of the curvature of the space. For $\alpha = 0$, one will recover the conditions that were pointed out by Lamé as necessary and sufficient for the *Euclidian-ness* of the space, i.e.:

$$H_{11} = H_{22} = H_{33} = H_{23} = H_{21} = H_{12} = 0.$$

By virtue of a known relation [Chap. XVII, form. (7)], one can give them the following form (*):

$$\begin{cases} \dfrac{\partial \frac{1}{r_{23}}}{\partial \sigma_2} + \dfrac{\partial \frac{1}{r_{32}}}{\partial \sigma_3} + \dfrac{1}{r_{23}^2} + \dfrac{1}{r_{32}^2} + \dfrac{1}{r_{12}\, r_{13}} = 0, \\[2ex] \dfrac{\partial \frac{1}{r_{31}}}{\partial \sigma_3} + \dfrac{\partial \frac{1}{r_{13}}}{\partial \sigma_1} + \dfrac{1}{r_{31}^2} + \dfrac{1}{r_{13}^2} + \dfrac{1}{r_{23}\, r_{21}} = 0, \\[2ex] \dfrac{\partial \frac{1}{r_{22}}}{\partial \sigma_1} + \dfrac{\partial \frac{1}{r_{21}}}{\partial \sigma_2} + \dfrac{1}{r_{12}^2} + \dfrac{1}{r_{21}^2} + \dfrac{1}{r_{31}\, r_{32}} = 0, \end{cases}$$

$$\begin{cases} \dfrac{\partial \frac{1}{r_{21}}}{\partial \sigma_3} = \dfrac{1}{r_{31}}\left(\dfrac{1}{r_{23}} - \dfrac{1}{r_{21}}\right), \quad \text{or} \quad \dfrac{\partial \frac{1}{r_{31}}}{\partial \sigma_2} = \dfrac{1}{r_{21}}\left(\dfrac{1}{r_{32}} - \dfrac{1}{r_{31}}\right), \\[2ex] \dfrac{\partial \frac{1}{r_{32}}}{\partial \sigma_1} = \dfrac{1}{r_{12}}\left(\dfrac{1}{r_{31}} - \dfrac{1}{r_{32}}\right), \quad \text{"} \quad \dfrac{\partial \frac{1}{r_{22}}}{\partial \sigma_3} = \dfrac{1}{r_{32}}\left(\dfrac{1}{r_{13}} - \dfrac{1}{r_{12}}\right), \\[2ex] \dfrac{\partial \frac{1}{r_{23}}}{\partial \sigma_2} = \dfrac{1}{r_{23}}\left(\dfrac{1}{r_{12}} - \dfrac{1}{r_{13}}\right), \quad \text{"} \quad \dfrac{\partial \frac{1}{r_{23}}}{\partial \sigma_1} = \dfrac{1}{r_{13}}\left(\dfrac{1}{r_{21}} - \dfrac{1}{r_{23}}\right). \end{cases}$$

b) The Lamé relations are ordinarily obtained by trying to integrate the equations:

$$\frac{\partial x_1}{\partial q_i}\frac{\partial x_1}{\partial q_j} + \frac{\partial x_2}{\partial q_i}\frac{\partial x_2}{\partial q_j} + \frac{\partial x_3}{\partial q_i}\frac{\partial x_3}{\partial q_j} = \begin{cases} Q_i^2 & \text{if} \quad i = j \\ 0 & \text{if} \quad i \neq j, \end{cases}$$

which serve to determine the Cartesian coordinates x_1, x_2, x_3 as functions of q_1, q_2, q_3 . For a two-dimensional space, the analogous equations are obtained by supposing that x_1 and x_2 are functions of Q_1 and Q_2 , while allowing that x_3 is an arbitrary function of only

(*) LAMÉ, *Leçons*…, § XLVI.

q_3, and therefore Q_3 will be a function of only q_3, while Q_1 and Q_2 are independent of q_3. Five Lamé relations will then be satisfied, and the single relation $H_{33} = 0$ will reduce to:

$$\frac{\partial}{\partial q_1}\left(\frac{1}{Q_1}\frac{\partial Q_2}{\partial q_1}\right)+\frac{\partial}{\partial q_2}\left(\frac{1}{Q_2}\frac{\partial Q_1}{\partial q_2}\right)=0.$$

That is therefore the single condition that must be satisfied by the functions Q_1 and Q_2 in order for it to be possible to pass from the form $Q_1^2\,dq_1^2 + Q_2^2\,dq_2^2$ for the square of line element in a two-dimensional space to the Cartesian form $dx_1^2 + dx_2^2$. If r_1 and r_2 are the radii of geodetic curvature of the lines q_1 and q_2 on the surface considered then one can give that condition the form:

$$\frac{\partial \frac{1}{r_2}}{\partial \sigma_1}+\frac{\partial \frac{1}{r_1}}{\partial \sigma_2}+\frac{1}{r_1^2}+\frac{1}{r_2^2}=0.$$

It is easy to verify that this condition is satisfied by the ordinary systems of orthogonal curves that one adopts in the plane.

c) If α_1 is the curvature of the surface q_1 at a given point then we need to substitute the form that the condition that was just obtained will take for three-dimensional spaces:

$$\frac{\partial}{\partial q_1}\left(\frac{1}{Q_1}\frac{\partial Q_2}{\partial q_1}\right)+\frac{\partial}{\partial q_2}\left(\frac{1}{Q_2}\frac{\partial Q_1}{\partial q_2}\right)+\alpha_1\,Q_2\,Q_3=0.$$

One then has:

$$H_{11} = -\,\alpha_1\,Q_1\,Q_2\,Q_3+\frac{1}{Q_1}\frac{\partial Q_2}{\partial q_1}\frac{\partial Q_3}{\partial q_1}=\left(\frac{1}{r_{12}\,r_{13}}-\alpha_1\right)\nabla,$$

and the conditions $H_{ii} + \alpha\nabla = 0$ will become:

$$\alpha_1=\alpha+\frac{1}{r_{12}\,r_{13}},\qquad \alpha_2=\alpha+\frac{1}{r_{23}\,r_{21}},\qquad \alpha_3=\alpha+\frac{1}{r_{31}\,r_{32}}.$$

For $\alpha = 0$, that will give the measure of the curvature (viz., the product of the principal curvatures) of the three surface coordinates according to GAUSS's theorem. When α is non-zero, one will see that the geodetic curvature of each surface in a curved space must be augmented with the curvature of that space.

8. – Returning to (11), observe, with Prof. Beltrami, that "one can predict that the curvature of the space must not be devoid of influence on the equations of elasticity. However, it is, with a doubt, supremely noteworthy that this influence manifests itself in such a simple form." Nonetheless, we can add that "despite that simplicity, the theory of elastic media in spaces of constant curvature presents very relevant differences when compared to the ordinary ones." As a first example, he has written equations (11), in the absence of volume forces, in the following way:

$$A\frac{\nabla}{Q_1^2}\frac{\partial\Theta}{\partial q_1}+B\left(\frac{\partial Q_2\mathcal{T}_2}{\partial q_3}-\frac{\partial Q_3\mathcal{T}_3}{\partial q_2}\right)+4\alpha B\,\nabla\kappa_1=0, \qquad \ldots$$

If one differentiates this with respect to q_1, q_2, q_3 and then sums then one will get:

$$A\,\Delta^2\,\Theta+4\alpha B\,\Theta=0.$$

Therefore the cubic dilatation in a curved space cannot satisfy the Laplace equation, as it does in Euclidian space. For example, it is not possible for Θ to have a constant, non-zero value in the entire body.

9. – What is more noteworthy is the result that one obtains by considering certain potential deformations. Since twice the components $\mathcal{T}_1$, $\mathcal{T}_2$, $\mathcal{T}_3$ of the rotation are zero for those deformations, the formulas (12) of the preceding chapter will imply that $Q_i^2\,\kappa_i$ are the first derivatives of a function U, and the total dilatation will be expressed by:

$$\Theta=\frac{1}{\nabla}\sum_i\frac{\partial\nabla\kappa_i}{\partial q_i}=\frac{1}{\nabla}\sum_i\frac{\partial}{\partial q_i}\left(\frac{\nabla}{Q_i^2}\frac{\partial U}{\partial q_i}\right)=\Delta^2 U,$$

in such a way that equations (11) will become:

$$F_i+(A\,\Delta^2\,U+4\alpha B\,U)=0 \qquad (i=1,2,3),$$

and that will show that the force F also admits a potential function. If V is that function then the three indefinite equations will reduce to the single one:

$$A\,\Delta^2 U+4\alpha BU+V=0,$$

in which one intends that the constant of integration is included in U. If one takes $V = -\alpha BU$ and $\Delta^2 U = 0$ then one will see that one has $F_i = -4\alpha\,B\,U_i\,\kappa_i$. One will then obtain a deformation that is devoid of either dilatations or rotations in which the force and displacement have a constant ratio at any point and have the same direction (if $\alpha < 0$: Gaussian space or *pseudo-sphere*) or opposite directions (if $\alpha > 0$: Riemannian space or *sphere*). That result, as Prof. Beltrami said, "is not encountered in Euclidian space, and presents a singular analogy with certain modern concepts in regard to the action of dielectric media."

10. – The last observation brings to mind the ingenious hypotheses that have been proposed in order to explain light, heat, magnetism, etc., when one considers those phenomena to have been produced by a reaction under which space opposes the variation of its curvature on time. It is important to observe that the additional term $B\Phi / \nabla$ in the effective part (for the formation of the indefinite equations) of the elastic potential can be considered to be precisely the expression for the energy of the reactions that space, whose geometric constitution is rigid, opposes the elastic matter that fills it, which one supposes to be *inert*, in the sense that when it is required to deform in that space, it will tend to behave as if the space were Euclidian. Recent progress in the theory of elastic media in curved space might allow one to respond to Clifford's demand to know (*):

> "Whether we cannot say that we can consider certain real effects that are due to changes in the curvature of our space to be physical variations; in other words, whether any of the causes that we call physical (and probably all of them) are not, perchance, due to the geometric constitution of the space in which they live."

(*) *Loc. cit.*, pp. 267.