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Infinite continuous pseudogroups

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1. Introduction and examples. The groups of local transformations that are defined by the general integrals of a system of partial differential equations were studied for the first time by Sophus Lie ([1]). Later, Élie Cartan gave a more satisfactory treatment of the subject and solved some of the fundamental problems ([2]). However, their viewpoints were purely local. In the present article, we shall attempt to pose the foundations of that theory and show, in particular, that it is a natural generalization and has great importance in the theory of Lie groups and complex manifolds.

A situation that frequently presents itself in differential geometry is the following one: Let M be a real analytic manifold of dimension n with coordinate neighborhoods U_α, U_β, etc., that form a covering of M. Suppose that for each U_α one is given n independent linear Pfaffian forms on U_α that will be denoted by θ_α^i ($i = 1, \ldots, n$) ([3]). We then have, for the case where $U_\alpha \cap U_\beta \neq \varnothing$:

(1) $$\theta_\alpha^i = \sum_j g_{\alpha\beta,j}^i(x)\,\theta_\beta^j \qquad x \in U_\alpha \cap U_\beta.$$

The matrix:

(2) $$g_{\alpha\beta}(x) = \left[g_{\alpha\beta,j}^i(x)\right]$$

may be considered as an element of the general linear group $GL(n, \mathbb{R})$ in n real variables, and we suppose that the association $x \to g_{\alpha\beta}(x)$ ($x \in U_\alpha \cap U_\beta$) defines an analytic map of $U_\alpha \cap U_\beta$ into $GL(n, \mathbb{R})$.

Now, let G be a subgroup of $GL(n, \mathbb{R})$. A fundamental problem in the theory of fiber bundles is the following one: May one choose the forms θ_α^i in such a fashion that $g_{\alpha\beta}(x)$ belongs to G for all pairs α, β of indices and all $x \in U_\alpha \cap U_\beta$? If this is possible then we say, in the terminology of fiber bundles, that the structure group of the principal fiber bundle of the tangent fiber bundle to M may be *reduced* to G. More simply, we say that

([1]) The numbers in brackets [.] refer to the bibliography at the end of the article.
([2]) [**1**], [**2**], [**3**], [**4**], [**5**].
([3]) Unless stated to the contrary, in all of the present article, we shall use the following system of indices: $1 \leq i, j, k \leq n$; $1 \leq \rho, \sigma, \tau \leq r$.

M has a *G-structure*, and that the set of Pfaff forms θ_α^i defines a *G-structure* on M. The following examples illustrate the significance of that notion:

1. G is composed of just the identity element. M is then parallelizable.

2. G is the orthogonal group $O(n, \mathbb{R})$ in n real variables. Then:

(3) $$(\theta_\alpha^1)^2 + \cdots + (\theta_\alpha^n)^2 = (\theta_\beta^1)^2 + \cdots + (\theta_\beta^n)^2$$

is a positive-definite, quadratic, differential form on M and defines a Riemannian metric. Conversely, a Riemannian metric on M gives rise to a G-structure on M with $G = O(n, \mathbb{R})$ in an obvious way.

3. $n = 2m$ is even, and G is the general linear group $GL(m, \mathbb{C})$ in m complex variables, which is regarded as a subgroup of $GL(n, \mathbb{R})$. The corresponding structure group G will generally be called an *almost-complex* structure group, while M will be called an *almost-complex manifold.*

While the theory of fiber bundles is occupied with the existence or non-existence of G-structures on M for a given G, one of the goals of differential geometry consists precisely in the study of differential invariants of a particular G-structure and their global implications. For example, the existence of a Riemannian metric (or, at least, a non-analytic metric) on M is a simple theorem in the theory of fiber bundles, but Riemannian geometry is precisely the study of properties of a Riemannian metric whose most interesting aspects are the global, or topological, implications of these properties.

2. First invariants of a *G*-structure. Integrable structure. – We shall show how one may define the first differential invariants of a G-structure. Let θ_α refer to the matrix with one row whose elements are $\theta_\alpha^1, \ldots, \theta_\alpha^n$. Equations (1) may then be written in the matrix form:

(4) $$\theta_\alpha = \theta_\beta\, g_{\alpha\beta}(x), \qquad x \in U_\alpha \cap U_\beta.$$

We identify $x \in U_\alpha \cap U_\beta$ with the elements $(x, y_\alpha) \in U_\alpha \times G$ and $(x, y_\beta) \in U_\beta \times G$, with $y_\alpha, y_\beta \in G$ by the condition that $y_\beta = g_{\alpha\beta}(x)\, y_\alpha$. The space B_G thus obtained is called the *principal fiber bundle* with the *structure group G*, and it is locally homeomorphic to a Cartesian product $U_\alpha \times G$. Since $\theta_\alpha\, y_\alpha = \theta_\beta\, y_\beta$, their common expression ω is a matrix with just one row of Pfaff forms that are defined globally on B_G. As a result, instead of remaining on M, we pass to B_G, and we shall study the properties of these Pfaff forms in B_G.

The first step is the calculation of the exterior derivative $d\omega$. In U_α, we utilize the representation $\omega = \theta_\alpha\, y_\alpha$, and we find:

(5) $$d\omega = -\ \omega \wedge\ y_\alpha^{-1}\, dy_\alpha + d\theta_\alpha\, y_\alpha ,$$

where y_α^{-1} is a matrix of left-invariant Pfaff forms on G. Since the elements ω^i $(1 \le i \le n)$ of ω are linearly independent, we may write that equation more explicitly as follows:

(6) $$d\omega^i = -\sum_{\rho,k} a^i_{\rho k}\,\omega^k \wedge \pi^\rho + \tfrac{1}{2}\sum_{j,k} c^i_{jk}(b)\,\omega^\alpha \wedge \omega^k ,$$

where r is the dimension of G, $a^i_{\rho k}$ are constants, and $c^i_{jk}(b)$ are functions on $U_\alpha \times G$ that are subject to the condition of being anti-symmetric in j, k, in order to be determined completely. The π^p are left-invariant Pfaff forms on G. The formula (6) is local, being valid only on $p^{-1}(U_\alpha)$, where p is the projection of B_G onto M. We now apply the transformation:

(7) $$\pi^p \to \pi'^p = \pi^p + \sum_k b^\rho_k\,\omega^k$$

to the forms π^p.

Equations (6) preserve the same form under this transformation, where the new coefficients c'^i_{jk} are given by the equations:

(8) $$c'^i_{jk} = c^i_{jk} + \sum_\rho (-a^i_{\rho k} b^\rho_j + a^i_{\rho j} b^\rho_k) .$$

The $n^2(n-1)/2$ expressions $\sum(-a^i_{\rho k} b^\rho_j + a^i_{\rho j} b^\rho_k)$ are linear in b^ρ_k and have constant coefficients. Suppose that they are mutually linearly independent. If $n^2(n-1)/2 = s$ then we may choose b^ρ_k in such a way that all the c'^i_{jk} disappear.

If $n^2(n-1)/2 > s$ then we choose b^ρ_k in such a way that s of the c'^i_{jk} are annulled, when chosen conveniently. We now suppose that such a transformation (7) has been performed, and we suppose that there are now primes (′) on the π'^p and c'^i_{jk}. In order that our conditions should remain unchanged, the π^p must be determined up to the transformation (7) that satisfies:

(9) $$\sum_\rho (-a^i_{\rho k} b^\rho_j + a^i_{\rho j} b^\rho_k) = 0.$$

It results from (8) that the new coefficients c^i_{jk} are functions on B_G that are defined globally. They are thus the first invariants of a G-structure. We say that the G-structure is *integrable* if all of these invariants are constants.

We mention the following examples of integrable G-structures:

1. M is a connected Lie group or an open subset of that group, and ω^i are the left-invariant Pfaff forms. In this case, G consists of just the identity, and the c^i_{jk} are the structure constants of the group.

2. Consider an almost-complex structure, as in example 3 of paragraph 1. In order to study its local invariants, it is convenient to use Pfaff forms with complex values – i.e., Pfaff forms $\omega = \varphi + i\psi$, where φ, ψ are real analytic; we also write $\bar{\omega}$ for $\varphi - i\psi$. With these conventions, suppose that $n = 2m$, and suppose that a $GL(m, \mathbb{C})$-structure – i.e., an almost-complex structure – is given on M. It is defined by m Pfaff forms with complex values ([4]) θ_α^t $(1 \leq t \leq m)$ in each U_α, such that θ_α^t, $\bar{\theta}_\alpha^t$ are linearly independent and:

$$\theta_\alpha^t = \sum_u g_{\alpha\beta,u}^t(x)\,\theta_\beta^u, \quad x \in U_\alpha \cap U_{\alpha\beta}, \tag{10}$$

where $\left[g_{\alpha\beta,u}^t(x)\right] \in GL(m, \mathbb{C})$. By following the general method above, we obtain m Pfaff forms with complex values in B_G ($G = GL(m, \mathbb{C})$) that may be represented locally by:

$$\omega^t = \sum_u y_{\alpha u}^t \theta_\alpha^u, \tag{11}$$

where $(y_{\alpha u}^t) \in GL(m, \mathbb{C})$. In $p^{-1}(U_\alpha) \subset B_G$, their exterior derivatives may be written in the form:

$$d\omega^t = \sum_u \pi_u^t \wedge \omega^u + \tfrac{1}{2}\sum_{u,v} c_{uv}^t \bar{\omega}^u \wedge \bar{\omega}^v, \quad \left(c_{uv}^t + c_{vu}^t\right) = 0, \tag{12}$$

where the π_u^t are determined up to the transformation:

$$\pi_u^t \to \pi_u'^t = \pi_u^t + \sum_v b_{uv}^t \omega^v, \qquad \left(b_{uv}^t + b_{vu}^t\right) = 0. \tag{13}$$

It then results that c_{uv}^t, as functions on B_G, are first invariants of an almost-complex structure. The latter is integrable if the c_{uv}^t are constants. One may verify, by a brief calculation, that this happens only when $c_{uv}^t = 0$. From a theorem of Frobenius, this is a necessary and sufficient condition for the system $\omega^t = 0$ to be completely integrable. When these conditions are satisfied, we may take a system of m functionally independent first integrals $z^1, \ldots, z^n$ to be local coordinates on M. Any other system of first integrals is attached to them by a complex analytic transformation with a non-zero Jacobian. This shows that an integrable almost-complex structure on M defines a complex structure on M.

The invariants c_{uv}^t were given for the first time by Ehresmann-Libermann ([5]), and are essentially equivalent to a tensor that was defined by Eckmann-Fröhlicher ([6]).

([4]) In all of this example, and on occasion later, whenever we are dealing with an almost-complex structure, we use the following series of indices: $1 \leq t, u, v \leq m$.

([5]) [**11**].

([6]) [**9**].

If one reverts to the case of a general G-structure then we desire to remark that the constants $a^i_{\rho k}$ in (6) have a simple geometric interpretation. In fact, if one considers G to be a linear group that acts on an n-dimensional vector space with coordinates ξ^i that conform to the equations:

(14) $$\xi^i \to \xi'^i = \sum_j y^i_j \xi^j \qquad \left[y^i_j\right] \in G$$

then:

(15) $$X_\rho = \sum_{i,k} a^i_{\rho k} \xi^k \frac{\partial}{\partial \xi^i}$$

are r infinitesimal transformations that generates G.

3. Notion of general integral of a differential system. Suppose that there exist two coordinate neighborhoods U and U', with local coordinates x and x', respectively, and in each of them one is given an integrable G-structure whose first invariants have the same constant values. Let θ^i, θ'^i be Pfaff forms that define these G-structures; they are determined up to a transformation of G. A homeomorphism of U into U' that is defined by the equations:

(16) $$x'^i = x'^i(x^1, \ldots, x^n)$$

is called *admissible* if it satisfies the equations:

(17) $$\theta'^i\,[x'^1(x^k), \ldots, x'^n(x^k)] = \sum_j \theta^j g^i_j(x) \qquad (g^i_j(x) \in G).$$

This condition remains unchanged if we replace θ^i or θ'^i by forms that differ by a transformation of G; it is therefore a condition on the G-structures. Equations (17) may be regarded as a system of partial differential equations on $U \times U'$. We would like to study the question of whether it admits a "general integral," upon shrinking the neighborhoods U and U', if necessary. Consequently, a precise definition of the notion of a general integral, and other things, that refers to the theory of differential systems on a manifold will be given in the present section ([7]).

Let X be a real analytic manifold. At each point $x \in X$, we let $V(x)$ denote the space of (contravariant) vectors at x and let $V^*(x)$ denote the space of covectors; denote their exterior algebras by $\Lambda(V(x))$ and $\Lambda(V^*(x))$, respectively. By the term *differential system* Σ on X, we intend this to mean a submanifold $Y \subset X$, along with the association of an ideal $O(x) \in \Lambda(V^*(x))$ at each point $x \in Y$, in such a way that the following condition is satisfied: For each homogeneous element $\alpha \in O(x)$, there exists a neighborhood U of x on Y and a real analytic differential form ω on U that belongs to $O(y)$ for each point $y \in U$ and reduces to α at x; one says that such a form *belongs to* Σ. Since the vector spaces $\Lambda(V(x))$ and $\Lambda(V^*(x))$ are dual, a homogeneous element $l(x) \in \Lambda(V(x))$, $x \in Y$ is called an

([7]) For a detailed treatise on the subject, we refer the reader to [**6**] or [**13**].

integral element if it annuls all of the elements of $O(x)$. A submanifold $Q \subset Y$ is called an *integral manifold* of Σ if its tangent space at any point is an integral element. We say that Σ is *closed* if exterior differential of every differential form in Σ belongs to Σ. In order to study integral manifolds, we may suppose that our differential system is closed.

We give ourselves a decomposable, analytic, exterior differential form W of degree p on Y that is nowhere zero. A fundamental problem in the local theory of differential systems is the formulation of theorems on p-dimensional integral manifolds whose tangent spaces nowhere annul W. One may formulate a sufficient condition as follows: Since W is decomposable, we locally write $W = \varphi^1 \wedge \ldots \wedge \varphi^p$. If n denotes the dimension of Y then one chooses chooses $n - p$ other Pfaff forms $\varphi^{p+1}, \ldots, \varphi^n$, in such a way that $\varphi^1 \wedge \ldots \wedge \varphi^n \neq 0$ in the neighborhood of Y in question. Let $\mathbf{e}_1, \ldots, \mathbf{e}_n$ be the dual basis to $\varphi^1, \ldots, \varphi^n$, and consider the vectors:

(18) $$V_i = \mathbf{e}_i + \sum_{\sigma=1}^{n} l_i^\sigma \mathbf{e}_\sigma \qquad (i = 1, \ldots, p).$$

Let $l_k(x)$, $k = 1, \ldots, p$ denote the space generated by $v_1, \ldots, v_k$. Let:

(19) $$O_k(x, l_1^\sigma, \ldots, l_k^\sigma) = 0$$

be the conditions for $l_k(x)$ to be a k-dimensional integral element at x; it is easy to see that these equations are linear in l_k^σ. We say that Σ is *involutive* (with respect to the decomposable form W) if, for a certain choice of φ, equations (19) are linearly independent in the l_k^σ, after taking into account the equations:

(20) $$O_k(x, l_1^\sigma) = 0, \ldots, O_{k-1}(x, l_1^\sigma, \ldots, l_{k-1}^\sigma) = 0.$$

A fundamental theorem in the local theory of differential systems confirms the existence of a p-dimensional integral manifold that satisfies appropriate initial conditions and the condition that $W \neq 0$ when the system is involutive. One proves this by successive applications of the Cauchy-Kowalewsky theorem. The integral manifolds whose existence is thus confirmed by this existence theorem are called "general," in contrast to the integrals of partial differential equations that are called "singular." It is important to remark that the condition that a differential system is involutive is obviously independent of the choice of local coordinates.

A notion that is closely attached to the notion of involution of a differential system is that of prolongation. In order to define that notion, let $\tilde{M}$ be the multiplicity of all p-dimensional integral elements that are not annulled by W. Upon associating any integral element with its origin x, we obtain a representation $\bar{n}: \tilde{M} \to Y$. In the dual representation of p, W becomes a form W', and the differential forms that belong to Σ become forms on $\tilde{M}$. The latter assign a set of elements of $\Lambda(V^*(m))$ that define an ideal to each point $m \in M$. It is not difficult to see that these ideals define a differential system Σ' on M; in general, Σ' is not closed. We let Σ denote the corresponding closed differential system that is obtained by starting from Σ' and adding the exterior derivatives of the differential forms in Σ'. The system Σ on $\tilde{M}$, combined with the form W', constitutes the *prolongation* of the given system Σ. It results easily from our definitions that the prolongation of an involutive system is involutive.

There exists a particularly important case in which the condition of involution may be formulated more explicitly. It is the case in which the ideals $O(x)$, for x belonging to a neighborhood U of Y, can be generated by:

1. h analytic Pfaffian forms $\theta^1, \ldots, \theta^h$

and

2. m analytic, exterior, quadratic, differential forms:

$$\Phi^i = \sum_{\rho=1}^{v}\sum_{k=1}^{p} a^i_{\rho k}\pi^\rho \wedge \varphi^k + \sum_{j,k=1}^{p} c^i_{jk}\varphi^j \wedge \varphi^k \qquad (1 \le i \le m) \tag{21}$$

such that:

$$W \wedge \theta^1 \wedge \ldots \wedge \theta^h \wedge \pi^1 \wedge \ldots \wedge \pi^n \neq 0. \tag{22}$$

In this case, let $t^k_1, \ldots, t^k_{p-1}$ $(1 \le k \le p)$ be $p(p-1)$ generic constants. Let σ_1 be the number of linearly independent forms among $\sum_\rho\sum_k a^i_{\rho k} t^k_1 \pi^\rho$, let $\sigma_1 \ldots \sigma_2$ be the number of linearly independent forms among:

$$\sum_\rho\sum_k a^i_{\rho k} t^k_1 \pi^\rho, \qquad \sum_\rho\sum_k a^i_{\rho k} t^k_2 \pi^\rho,$$

and finally let $\sigma_1 + \sigma_2 + \ldots + \sigma_{p-1}$ be the number of linearly independent forms among:

$$\sum_\rho\sum_k a^i_{\rho k} t^k_1 \pi^\rho, \qquad \sum_\rho\sum_k a^i_{\rho k} t^k_{p-1} \pi^\rho.$$

Let q be the dimension of the manifold of p-dimensional integral elements that passes through a point of Y. We then have the inequality:

$$q \le pv - (p-1)\, \sigma_1 \ldots \sigma_{p-1}. \tag{23}$$

Our criterion of involution confirms that the "equal" sign is valid if and only if the given differential system is involutive. (It is implicit that we assume that our system is closed.) This criterion may be applied to our differential system (17). In order to do this, we transform the system into a new form. Let $V = U \times G$, $V' = U' \times G$, and we introduce the Pfaff forms:

$$\begin{aligned} \omega^i &= \sum_j y^i_j \theta^j, \\ \omega'^i &= \sum_j y'^i_j \theta'^j, \end{aligned} \tag{24}$$

on V and V', respectively, where $\left[y^i_j\right]$, $\left[y'^i_j\right]$ are generic elements of G. The differential system (17) is then equivalent to the following one:

$$\omega'^i - \omega^i = 0. \tag{25}$$

In our terminology, the ideal on $V' \times V$ that is *associated* with the differential system at a point of $V' \times V$ is generated by the left-hand sides of (25). In order to make that system closed, we add the generators $d(\omega'^i - \omega^i)$ to the ideal at any point. Since the two G-structures are integrable with the same first invariants, one may choose Pfaff forms π^ρ, in such a way that:

$$d(\omega'^i - \omega^i) = \sum_{\rho,k} a^i_{\rho k}(\pi'^\rho - \pi^\rho) \wedge \omega^k .$$

Our differential system is then of the particular type that was discussed above, with $W = \omega^1 \wedge \ldots \wedge \omega^n$. In order to find the condition of involution, we introduce the integers q, $\sigma_1, \ldots, \sigma_{n-1}$, as before, with p placed by n. According to the criterion above, a necessary and sufficient condition for the differential system to be involutive is:

$$q = nv - (n-1)\ \sigma_1, \ldots, \sigma_{n-1} . \tag{26}$$

In particular, we see that this condition depends only upon the group G. When it is satisfied, we say that the linear group G is involutive. One will say that a G-structure is involutive if G is involutive.

4. Pseudogroups defined by an involutive integrable structure. – In the present section, we shall show how an involutive integrable structure on a manifold defines a pseudogroup of transformations, and a corresponding family of admissible coordinate systems. We first recall the definition of a pseudogroup.

Let E be a topological space, and let F be a family of open subsets such that the union of an arbitrary number of subsets of F and the intersection of a finite number of subsets of F belong to F. A family H of homeomorphisms, each of which represents a subset of F on a subset of F, is said to constitute a *pseudogroup* if the following conditions are satisfied:

1. If $U \in F$ then the identity representation belongs to H. If $h \in H$ represents U in V then the inverse representation belongs to H. If $h, h' \in H$ and their product hh' is well-defined then $hh' \in H$.

2. Let V be a union of subsets V_α of F. In order for a homeomorphism h that is defined on V to belong to H, it is necessary and sufficient that its restriction $h \mid V_\alpha$ belong to H.

Now, let X be a topological space, and let $\{U_\alpha\}$ be a covering of X with open subsets. Suppose that for each α there is a homeomorphisms f_α of a subset V_α of F onto U_α, in such a fashion that the following condition is satisfied: For two arbitrary indices α, β with $U_\alpha \cap U_\beta \neq \varnothing$, there is a representation $h_{\alpha\beta} \in H$ of V_α onto V_β with the property that the representation $g_{\alpha\beta} : f_\alpha^{-1}(U_\alpha \cap U_\beta) \to f^{-1}(U_\alpha \cap U_\beta)$ that is defined by $f_\alpha(x) = f_\beta(x)[g_{\alpha\beta}(x)]$, $x \in f_\alpha^{-1}(U_\alpha \cap U_\beta)$ is the restriction of $h_{\alpha\beta}$ to $f_\alpha^{-1}(U_\alpha \cap U_\beta)$. If this is the case then we say that X has a family of local coordinate systems that are *compatible* with the pseudogroup H.

Suppose that we have an involutive, integrable G-structure on M. Using the notations of paragraph 2, we have, in particular, equations (6), in which the c^i_{jk} are constants, while the $\pi^{\rho'}$ are determined up to the transformation (7), always assuming the condition (9). In addition, the group G is now assumed to be involutive. The coefficients $a^i_{\rho k}$, c^i_{jk} are subject to certain relations. In order to establish them, we remark that $d\pi^{\rho'}$ are exterior, quadratic, differential forms, in such a way that we may locally write:

$$d\pi^{\rho'} = \tfrac{1}{2}\sum_{\sigma,\tau} \gamma^\rho_{\sigma\tau}\pi^\sigma \wedge \pi^\tau + \sum_{\sigma,i} u^\rho_{\sigma i}\pi^\sigma \wedge \omega^i + \tfrac{1}{2}\sum_{i,j} v^\rho_{ij}\omega^i \wedge \omega^j , \tag{27}$$

where:

$$\gamma^\rho_{\sigma\tau} + \gamma^\rho_{\tau\sigma} = 0, \qquad v^\rho_{ij} + v^\rho_{ji} = 0. \tag{28}$$

Upon using (27), we find that the conditions $d(d\omega^i) = 0$ may be written explicitly:

$$\sum_i - a^k_{\rho i} a^i_{\sigma j} + a^k_{\sigma i} a^i_{\sigma j} = \sum_i a^k_{\bar{i}j}\, a^{\bar{i}}_{\sigma\rho} , \tag{29}$$

$$\sum_i \left(-c^k_{im}\, a^i_{\rho l} + c^k_{il}\, a^i_{\rho m} + c^i_{lm}\, a^k_{\rho i}\right) + \sum_\sigma \left(a^k_{\sigma l}\, u^\sigma_{\rho m} - a^k_{\sigma m}\, u^\sigma_{\rho l}\right) = 0, \tag{30}$$

$$\sum_i \left(c^k_{ij}\, c^i_{lm} + c^k_{il}\, c^i_{mj} + c^k_{im}\, a^i_{jk}\right) + \sum_\rho \left(a^k_{\rho j}\, v^\rho_{lm} + a^k_{\rho l}\, v^\rho_{mj} + a^k_{\rho m}\, v^\rho_{jl}\right) = 0. \tag{31}$$

The compatibility of these equations is therefore a necessary condition on $a^i_{\rho k}$, c^i_{jk}. We remark, in passing, that if G is composed of only the identity then (29), (30) are satisfied identically, and (31) reduces to the well-known Jacobi identities.

Equations (29) have a simple interpretation. Indeed, in terms of the infinitesimal transformations X_ρ in (15), they are equivalent to the equations:

$$(X_\rho, X_\sigma) = \sum_\tau \gamma^\tau_{\rho\sigma} X_\tau . \tag{32}$$

It then results that $\gamma^\tau_{\rho\sigma}$ are constants – and, in fact, they are the structure constants of G.

A fundamental theorem states that these results have a converse. More precisely, we have the following theorem: Suppose that the constants $a^i_{\rho k}$, c^i_{jk} are such that equations

(29), (30), (31) are compatible. Let (y^i_j) be a generic element of G, and let π'^{ρ} be r linearly independent, left-invariant Pfaff forms on G. We may find a connected open subset E in the n-dimensional Euclidian space (which may be any space), and n linearly independent, analytic Pfaff forms φ^i in E with the property: In $E \times G$, there are Pfaff forms:

$$\pi^{\rho} = \pi'^{\rho} + \sum_k d^{\rho}_k \varphi^k \,, \tag{33}$$

which, when combined with $\omega^i = \sum_k y^{\rho}_k \varphi^k$, satisfy equations (6).

This theorem is a generalization of the converse of Lie's third fundamental theorem for Lie groups. One proves it by applying the existence theorem that was given in paragraph 3. Neither the set E, nor the forms φ^k, are determined in a unique fashion. However, in the discussion that we made we directed our attention to a particular choice of these elements.

When a choice of this type has been made, we say that an analytic representation f of an open subset $U \subset E$ into an open subset $V \subset E$ is admissible if the restrictions φ^i_v of the forms φ^k to V are represented by the dual representation of f by the forms:

$$\sum_k y^i_k(x)\varphi^k_u \,, \qquad y^i_k(x) \in G, \quad x \in U. \tag{34}$$

If we take the sets of F to be open subsets of E and take the representations of H to be admissible homeomorphisms then all of these homeomorphisms define a pseudogroup, in the sense that was defined at the beginning of the present section.

Suppose that our manifold M has a covering $\{U_\alpha\}$ by coordinate neighborhoods, with respect to which, the given involutive, integrable G-structure is defined in each U_α by the Pfaff forms θ^i_α. Let $x \in U_\alpha$. It results from the fundamental theorem above that we may find a neighborhood $V \subset E$, and a homeomorphism f of V onto a neighborhood U, $x \in U \subset U_\alpha$, on which the θ^i_α are represented by the forms:

$$\sum_j y^i_j(v)\varphi^k_v \,, \qquad \left(y^i_j(v)\right) \in G, \quad v \in V.$$

Among these neighborhoods U, we may choose a covering of M that, when combined with the corresponding homeomorphisms f, obviously defines local coordinate systems that are compatible with the pseudogroup H above. Such systems of local coordinates will be called *admissible*.

5. Other examples. – In order to make our concepts clearer, we shall give some examples. All of the structures of the present section will be involutive and integrable.

1. G consists of just the identity. As in example 1, paragraph 2, such a structure exists on the manifold of a Lie group or on one of its open submanifolds. In a certain

sense, these are the only possible ones. Indeed, define the structure by the linearly independent Pfaff forms ω^i. Consider a parameterized curve $x(t)$, $0 \leq t \leq 1$ in M, and let $\omega^i = p^i(t)$ along the curve. The structure is called *complete* if $\lim x(t)$ exists whenever:

$$\int_0^1 \sqrt{(p^1)^2 + \cdots + (p^n)^2}\, dt$$

converges. We then have the following theorem: Let M be a connected and simply connected manifold with an integrable G-structure for which G consists of just the identity. Suppose the structure is complete. M is then a Lie group, with ω^i for the left-invariant Pfaff forms.

The condition of being complete is a property of the structure. However, it results from the definition that the structure is complete if M is compact. As a consequence, we see that the seven-dimensional sphere S^7 gives an example of a parallelizable manifold on which no integrable structure exists with G = identity.

2. $G = GL(n, \mathbb{R})$. Such a structure is always integrable, and equations (6) may be written in the form:

(35) $$d\omega^i = \sum_k \pi_k^i \wedge \omega^k .$$

In this case, we take E to be n-dimensional Euclidian space and let:

(36) $$\varphi^i = dx^i,$$

where x^i are coordinates on E. The systems of admissible local coordinates are nothing but the original systems that correspond to the given analytic structure on M.

3. Almost-complex structure on an even-dimensional manifold. This was studied in example 2, paragraph 2, and we adopt the notations that were employed there. We have shown that if the almost-complex structure is integrable then systems of local complex coordinates may be introduced on M.

It is well-known that the existence of an almost-complex structure on M is topologically equivalent to that of an exterior, quadratic differential form Φ that is everywhere of rank equal to the highest n. Such a manifold has been called *symplectic* by Ehresmann ([8]) if Φ is closed ($d\Phi = 0$). Among the symplectic manifolds, one finds the ones for which Φ is locally reducible to the form:

(37) $$\Phi = dx^1 \wedge dx^{m+1} \wedge \ldots + dx^m \wedge dx^{2m}.$$

Such a manifold then has an integrable, involutive structure. Its systems of admissible local coordinates are the ones for which Φ takes the form (37).

([8]) [**11**].

4. M has odd dimension $n = 2m + 1$, and G is the group of all non-singular matrices of the form:

(38)
$$\begin{bmatrix} g_{00} & g_{01} & \cdots & g_{02m} \\ 0 & & & \\ \vdots & & C & \\ 0 & & & \end{bmatrix}, \quad g_{00} > 0,$$

where C is the most general $2m \times 2m$ matrix that satisfies the relation $CJ\,C^{-1} = J$, where J is the matrix:

(39)
$$J = \begin{bmatrix} 0 & I_m \\ -I_m & 0 \end{bmatrix}, \qquad I_m = \left.\begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix}\quad\right\} m\,.$$

One may write equations (6) in the form:

(40)
$$\begin{cases} d\omega_0 = \pi_0 \wedge \omega_0 - \sum_k \omega_k \wedge \omega_{k'}, \\ d\omega_i = \tfrac{1}{2}\pi_0 \wedge \omega_i + \pi_i \wedge \omega_0 + \sum_k \pi_{ik} \wedge \omega_k + \sum_{k'} \pi_{ik'} \wedge \omega_{k'}, \\ d\omega_{i'} = \tfrac{1}{2}\pi_0 \wedge \omega_{i'} + \pi_{i'} \wedge \omega_0 + \sum_k \pi_{i'k} \wedge \omega_k + \sum_{k'} \pi_{i'k'} \wedge \omega_{k'}, \end{cases}$$

where $i' = m + i$, $k' = m + k$, and:

(41)
$$\pi_{ik'} + \pi_{ki'} = 0, \quad \pi_{i'k} + \pi_{k'i} = 0, \quad \pi_{ik} + \pi_{i'k'} = 0, \qquad 1 \le i, k \le m.$$

For the space E, we may take the Euclidian space of dimension $2m + 1$, with the coordinates $x_0, \ldots, x_{2m}$, and we may take:

(42)
$$\varphi_0 = dx_0 + \sum_u x_{u'} dx_u,$$
$$\varphi_u = dx_u, \ \varphi_{u'} = dx_{u'}.$$

The pseudogroup is essentially that of the contact transformations on a space of dimension $m - 1$. The structure on M is defined by a Pfaff form θ_0 that is defined up to a positive factor, and is locally reducible to the form:

(43)
$$\theta_0 = dx_0 + x_{m+1}\, dx_1 + \ldots + x_{2m}\, dx_m\,.$$

The admissible local coordinate systems are the ones for which that Pfaff form may be written in the form (43).

5. M is three-dimensional, and G is the group of all matrices of the form:

$$(44) \qquad \begin{bmatrix} u & v & w \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad u > 0.$$

Equations (6) become:

$$(45) \qquad \begin{cases} d\omega_1 = \pi_1 \wedge \omega_1, \\ d\omega_2 = \pi_2 \wedge \omega_1, \\ d\omega_3 = \pi_3 \wedge \omega_1 - \pi_2 \wedge \omega_3. \end{cases}$$

In the present case, we take the space E to be the half-space $y > 0$ of three-dimensional Euclidian space (x, y, z). Equations (45) then have the solutions:

$$(46) \qquad \omega_1 = u\,dx, \qquad \omega_2 = v\,dx + \frac{1}{y}dy, \qquad \omega_3 = w\,dx + \frac{1}{y}dz.$$

An admissible representation of a neighborhood with the coordinates x, y, z in a neighborhood that has the coordinates X, Y, Z satisfies the differential equations:

$$(47) \qquad \begin{aligned} dX &= u\,dx, \\ \frac{1}{y}dY &= v\,dx + \frac{1}{y}dy, \\ \frac{1}{y}dZ &= w\,dx + \frac{1}{y}dz. \end{aligned}$$

The finite equations are:

$$(48) \qquad X = f(x), \qquad Y = y\,g(x), \qquad Z = z\,g(x) + h(x),$$

where f, g, h are arbitrary analytic functions of x.

6. Involutive and semi-involutive linear groups. Connections. – It results from the discussions above that the study of involutive linear groups plays an important role in the theory of infinite continuous groups. It is a purely algebraic problem. Meanwhile, we shall give some geometric considerations that will help to clarify the situation.

To continue the general discussions of paragraph 2, we see that the forms π^ρ on B are determined up to the transformation (7), with the restriction of conditions (9). The latter are homogeneous, linear equations in b_k^ρ with constant coefficients. Suppose that b_{ks}^ρ, $1 \le s \le r_1$ are a fundamental system of solutions of this linear system, in such a way that any other solution may be written:

$$(49) \qquad b_k^\rho = \sum_{s=1}^{r_1} b_{ks}^\rho\, t^s,$$

where the t^s are arbitrary parameters. It follows that in B_G we have $n + r$ Pfaff forms ω^j – ρ, the first of which are completely determined, while the latter are determined up to the transformation:

(50) $$\pi^\rho \to \pi'^\rho = \pi^\rho + \sum_{s=1}^{r_1} b^\rho_{ks}\, t^s \omega^k .$$

If G_1 denotes the group of all matrices of the form:

(51) $$\begin{bmatrix} I_n & \mathcal{C} \\ 0 & I_r \end{bmatrix}, \qquad \mathcal{C} = \left[\sum_s b^\rho_{js}\, t^s\right]$$

then this must say precisely that B_G has a G_1-structure, where G_1 is an Abelian group of dimension r that acts on a vector space of dimension $n + r$. It is determined completely by G, and one may call it the *derived group* of G.

Upon repeating this process, we obtain a sequence of manifolds B_G, B_{G_1}, B_{G_2}, …, where G_k is the derived group of G_{k-1}. Naturally, these structures are not necessarily integrable, even if the original G-structure on M is integrable. We say that G is *semi-involutive* if G_k is involutive and different from the identity for a certain $k \geq 1$. Since, according to our existence theorem on differential equations, the derived group of an involutive group is involutive, an involutive group that is different from the identity is semi-involutive. A necessary condition for G to be semi-involutive is that no G_k is the identity for $k \geq 1$.

This situation is in direct contrast with the case where a connection may be defined by starting with a G-structure. Indeed, the condition G = identity signifies precisely that one may define linearly independent Pfaff forms on $B_{G_{k-1}}$ whose number is equal to the dimension of $B_{G_{k-1}}$. These Pfaff forms define a connection on the fiber bundle $B_{G_{k-1}} \to B_{G_{k-2}}$ (with the convention that $B_{G_0} = B_G$, $B_{G_{-1}} = M$) whose structure group is G_{k-1}. As a particular example, we may take $G = O(n, \mathbb{R})$. Then, as one easily shows, G_1 is composed of just the identity, and we define a connection in the sheaf $B_G \to M$ that is the well-known parallelism of Levi-Cività. This proves, in particular, that the orthogonal group $O(n, \mathbb{R})$ is not involutive.

By a broad use of the theory of representations of semi-simple Lie algebras, E. Cartan has determined all of the semi-simple complex linear groups that are irreducible, and whose derived group is not the identity. On the basis of that result, he has determined all of the linear groups that are irreducible and semi-involutive. In particular, one shows that the only complex, semi-simple, irreducible, semi-involutive, linear groups are the special linear group and the symplectic group. The proof of the theorem is long and demands many calculations. A simplified proof and an extension to the real case would be very desirable.

Cartan appealed to this theorem to determine what he called the *simple* pseudogroups. In the ultimate developments of the theory of infinite pseudogroups, these simple pseudogroups will probably be the primary objects of the study. Consequently, we shall summarize his results here. The following concepts are entirely local.

As is the custom, a pseudogroup is called *simple* if it contains no sub-pseudogroup, properly speaking, that is distinct from the identity. One calls it *imprimitive* if it transforms a family of disjoint k-dimensional submanifolds ($0 < k < n$) that the manifold is subdivided into amongst themselves. If it is not imprimitive then one says that it is *primitive*. A simple pseudogroup is necessarily primitive. By determining all of the primitive infinite pseudogroups, Cartan has shown that in the complex domain the simple infinite pseudogroups are of four types:

A. The pseudogroup of all complex analytic transformations of n complex variables with non-zero Jacobian.

B. The pseudogroup of all complex analytic transformations of n complex variables with Jacobian equal to 1.

C. The pseudogroup of all complex analytic transformations of $n = 2m$ complex variables $z^1, \ldots, z^{2m}$ that leave invariant the exterior quadratic differential form:

$$dz^1 \wedge dz^{m+1} + \ldots + dz^m \wedge dz^{2m}. \tag{52}$$

D. The pseudogroup of all complex analytic transformations of $n = 2m + 1$ complex variables $z^0, \ldots, z^{2m}$ that reproduce the Pfaff form:

$$dz^0 + z^{m+1}\, dz^1 + \ldots + dz^{2m}\, dz^m, \tag{53}$$

up to a non-zero factor.

These pseudogroups have their analogues in the real variables. We say that a manifold has an *A*-structure if it has a covering by systems of local coordinate systems that are compatible with the pseudogroup *A*, and similarly for *B*, *C*, and *D*-structures. In the real case, we intend the term "pseudogroup *D*" to mean all real transformations that reproduce the form (53), up to a positive factor. This pseudogroup has been considered in example 4, paragraph 5.

7. Global problems. – Let G be an involutive linear group in n variables. The global problems that we immediately encounter are those of knowing whether a manifold M has a G-structure, and whether it has an integrable G-structure. For $n = 2m$ and $G = G(m, \mathbb{C})$, this amounts to the problems of the existence of almost-complex and complex structures on M, about which much work has been done ([9]). In the present section, we shall give a sampling of these results for the case of a *D*-structure on a multiplicity of odd dimension.

We recall that the only local invariant of a Pfaffian equation $\omega = 0$ is its class, which is an odd integer $c = 2r + 1$ such that $2r$ is the rank of $d\omega$ (mod ω). On a multiplicity of odd dimension $n = 2m + 1$, there always exists a non-zero Pfaffian form (i.e., a Pfaffian

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([9]) See, for example, [**8**], [**11**], [**12**].

form that is never annulled). The existence of a D-structure is equivalent to the existence of a Pfaffian form that is everywhere of class $2m + 1$. Such a property has topological implications on the manifold. It is easy to that if M has a D-structure then it must be orientable.

Other necessary conditions for the existence of a D-structure may be deduced as follows: Let E be the real Euclidian space of dimension $2m + 2N + 1$, and let O be a fixed point of E. Let E' be a hyperplane that passes through O, and let v be a unit vector that is perpendicular to E'. E' may be considered as a complex Euclidian space of complex dimension $m + N$. A complex vector space ξ of complex dimension m that issues from O in E' determines a real vector space of dimension $2m$, which, along with v, determines a real vector space of dimension $2m + 1$. We let $G(m, N; \mathbb{C})$ denote the complex Grassmann manifold of all complex vector spaces of dimension m that issue from O and are found E', and let $G(2m + 1, 2N; \mathbb{R})$ denote the real Grassmann manifold of all real vector spaces of dimension $2m + 1$ in E. The construction above then defines a representation:

$$h: G(m, N; \mathbb{C}) \to G(2m + 1, 2N; \mathbb{R}). \tag{54}$$

For N sufficiently large, let $\tau: M \to G(2m + 1, 2N; \mathbb{R})$ be the tangential representation of M; i.e., the representation that induces the tangent fiber bundle of M. It results from standard considerations regarding fiber bundles that M has a D-structure only if there exists a representation $\sigma: M \to G(m, N; \mathbb{C})$ such that τ and σ are homotopic. Since $G(m, N; \mathbb{C})$, as a complex Grassmann manifold, has no non-zero cohomology group of odd dimension, we have the following theorem: If M has a D-structure then all of the Stiefel-Whitney characteristic classes of odd dimension must be zero.

This necessary condition easily leads to the multiplicities of odd dimension that have no D-structures. In particular, we may consider the five-dimensional manifold of Wu Wen-Tsun that is defined as follows: Let P_2 be the complex projective plane, and let l be the unit interval $0 \leq t \leq 1$. At a point $z \in P_2$, let z be the point whose homogeneous coordinates are the complex conjugates of z. We take the Cartesian product $P_2 \times l$ and identify the points $z \times 0$ and $\bar{z} \times 1$. The resulting space is an orientable five-dimensional manifold whose three-dimensional Stiefel-Whitney class is not annulled ([10]). As a result, it has no D-structure.

On the other hand, there do exist manifolds with a D-structure. Let μ be an analytic manifold of dimension $m - 1$. By the term *co-ray* of μ, we intend this to mean the class of all covectors that differ from one of the others by a positive factor. The manifold of all co-rays of μ has dimension $2m - 1$, and obviously has a D-structure. Instead of considering the manifold of all co-rays, we may also take the manifold of all rays, which is generally called the *tangent manifold* to μ. If we introduce a Riemannian metric on μ then we see that the tangent manifold is differentiably homeomorphic to the manifold of all co-rays, and, in turn, to a D-structure.

([10]) [**15**].

Another example of a multiplicity with a D-structure is given by a sphere S^{2m+1} of dimension 2^{m-1}. Indeed, consider a complex Euclidian space of dimension $m - 1$ with the coordinates $z^0, \ldots, z^m$, and define S^{2m+1} to be the locus of the equation:

(55) $$z^0\bar{z}^0 + z^1\bar{z}^1 + \cdots + z^m\bar{z}^m = 0.$$

The Pfaff form:

(56) $$\frac{1}{i}\sum_{\alpha=0}^{m}\left(z^\alpha d\bar{z}^\alpha - \bar{z}^\alpha dz^\alpha\right)$$

is then real and everywhere of class $2m - 1$.

We need to remark that the second example is not included in the first one. The following theorem was communicated to me by Professor Spanier: Any sphere S of dimension $2m - 1$ is the tangent manifold of a multiplicity of dimension $m - 1$.

This theorem is proved as follows: Suppose that S is the tangent manifold of a manifold μ of dimension $m - 1$. S is then a bundle over M that is fibered into spheres of dimension m. If μ is non-orientable then the tangent manifold of its two-sheeted orientable covering covers S twice. Since this is not possible, μ must be orientable. As the case $m = 1$ easily leads to a contradiction, we suppose that μ is greater than 1. Consider the Gysin sequence of the fiber bundle $S \to \mu$ ([11]):

(57) $$\ldots \to H^{z-1}(S) \xrightarrow{j} H^{r-m-1}(\mu) \xrightarrow{h} H^z(\mu) \xrightarrow{p^*} H^r(S) \xrightarrow{j} H^{r-m}(\mu) \xrightarrow{h}$$
$$\xrightarrow{h} H^{r+1}(\mu) \to \ldots,$$

where the groups are the cohomology groups with integer coefficients. p^* is the dual homomorphism that is induced by p, j is the integration on the fiber, and h is multiplication by the characteristic class W^{m+1}. We may infer the following conclusions from this exact sequence:

1. For $r = m - 1$, $H^0(\mu) \cong H^{m+1}(\mu)$, and W^{m+1} generates $H^{m+1}(\mu)$, which implies that the Euler-Poincaré characteristic $\chi(\mu)$ is equal to ± 1.

2. For $r < m$, $H^m(\mu) \sim H^r(S) = 0$.

3. For $r > 1$, $H^r(\mu) \cong H^{r+m}(S)$, in such a way that $H^r(\mu) = 0$.

From the last two results, we infer that $H^r(\mu) = 0$ $(1 \leq r \leq m)$. This latter condition implies that $\chi(\mu)$ is equal to 0 or 2, which contradicts 1). This completes the proof of the theorem.

8. Observations. – The infinite continuous groups that were considered in the present article constitute only a particular case of the ones that were studied by E. Cartan. One may obtain other generalities in two directions: First, the transformations of

([11]) [**7**].

pseudogroups may be defined as being the general integrals of a system of partial differential equations of higher order, rather than being of first order, as with equations (17) in our case. For this generalization, Ehresmann's theory of jets or N. Bourbaki's theory of infinitely close points reveals its utility. Secondly, Cartan has taken under consideration what one calls the *intransitive* groups. They are important for the infinite continuous groups pseudogroups, because, with a notion of local isomorphism that we have not discussed here, contrary to the case of Lie groups, there exist intransitive pseudogroups that are not isomorphic to any transitive pseudogroup.

Despite the restricted nature of our class of pseudogroups, it seems that one has sufficient grounds for reflection. Other than the type of problems that were considered in paragraph 7, there is the question of the study of particular families of functions on M. If the structure group G has an invariant subspace then a natural family of functions of this type is composed of the ones whose gradient vectors everywhere belong to the corresponding invariant subspaces. This notion contains that of a complex analytic function on an almost-complex manifold. If we use the notation of paragraph 2 then a function f with complex values on an almost-complex manifold is complex analytic if df is everywhere a linear combination of the ω^j. Given the extreme richness of the theory of functions of one or more complex variables, we are of the opinion that it might perhaps be worthwhile to consider this generalization a little more closely.

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