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On a quantity that is analogous to potential and a theorem that relates to it

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One knows that a large part of mathematical physics is based upon the potential theory. Since knowledge of that theory is not as widespread as one might desire, I believed that it would be useful to publish a small treatise in which I presented the most essential properties of the potential function and the potential.

Folie, although occupied with some very important geometric research in his own right, and who has already published some interesting results, has nonetheless been kind enough to take the trouble to produce a French translation of that treatise, and it is not necessary to add that it was accomplished with the greatest care and propriety. Permit me to pay homage to the name of that translator before this academy.

I shall take this occasion to communicate a theorem to the Academy that I discovered in my research into the mechanical theory of heat, and which pertains to the subject that I treated in that book.

In a paper that I had the honor of communicating to the Academy, I stated the following theorem: *The force that acts upon heat is proportional to the absolute temperature* (1). Since, according to my way of thinking, heat is nothing but a motion, I have no doubt that this theorem corresponds to a general theorem in mechanics that allows one to derive the equations of motion in the same way that the principle of the equivalence of heat is only a special case of the principle of the equivalence of an mechanical work. Here is that theorem, which refers to stationary motion of an arbitrary system of material points – i.e., to a motion in which the positions and velocities of the points do not always change in the same sense, but remain within certain limits.

Let a system of material points m, m', m'', \ldots be given with coordinates $x, y, z; x', y', z'; x'', y'', z''; \ldots$, which are subject to forces whose components are $X, Y, Z; X', Y', Z'; X'', Y'', Z''; \ldots$ Form the sum:

$$\sum \frac{m}{2} \left[\left(\frac{dx}{dt} \right)^2 + \left(\frac{dy}{dt} \right)^2 + \left(\frac{dz}{dt} \right)^2 \right],$$

or the sum (which is known by the name of the *vis viva* of the system):

^{(&}lt;sup>1</sup>) Poggendorff Annalen, v. CXCI. – Journal de Liouville (2), v. VIII. – *Théorie mécanique de la Chaleur*, translated by F. Folie, v. I, pps. 257, 261, and 324.

$$\sum \frac{m}{2}v^2$$
,

when one denotes the velocities of the points by v, v', v'', ..., and further form the sum:

$$\sum -\frac{1}{2}(Xx+Yy+Zz),$$

to whose mean values I propose to give the name of the *virial* of the system (in German *virial*, from the Latin word *vis* for force). We will then have the theorem:

The mean vis viva of the system is equal to its virial.

If we distinguish the mean value of a quantity from its variable value by putting an overbar on the formula that represents the variable quantity then our theorem can be expressed by the following equation:

$$\sum \frac{m}{2} \overline{v^2} = -\frac{1}{2} \sum \left(\overline{Xx + Yy + Zz} \right).$$

As for the value of the virial, it will take some very simply forms in the most important cases in nature.

When there are two points *m* and *m*' that are separated by a distance of *r* and which exert an attractive or repulsive force upon each other that is represented by the function $\varphi(r)$, and which we will suppose to be positive or negative according to whether the force is attractive or repulsive, respectively, we will have:

$$X x + X' x' = \varphi(r) \frac{x' - x}{r} x + \varphi(r) \frac{x - x'}{r} x' = -\varphi(r) \frac{(x' - x)^2}{r},$$

and as a result:

$$-\frac{1}{2}(Xx + Yy + Zz + X'x' + Y'y' + Z'z') = \frac{1}{2}r \varphi(r).$$

When that result is extended to an arbitrary number of points that are subject to only attractive or repulsive forces that they exert upon each other, one will have:

$$-\frac{1}{2}\sum (X x + Y y + Z z) = \frac{1}{2}\sum r \varphi(r),$$

in which the sum on the right relates to all pair-wise combinations of given points. The virial of the system of points will then have the expression:

$$\frac{1}{2}\sum \overline{r\,\varphi(r)}$$

in this case.

One easily recognizes the analogy between that quantity and another known quantity. If we introduce the function $\Phi(r)$ by setting:

$$\Phi(r)=\int \varphi(r)\,dr\,,$$

then we will have:

$$-\sum (X x + Y y + Z z) = d\sum \Phi(r).$$

In the special case where the attractive or repulsive forces are inversely proportional to the squares of the distances, the sum $\sum \Phi(r)$, up to sign, is called the *potential* of the system. Since that quantity has not further been given a name in the general case, I propose the name of *ergon* for it (from the Greek word $\xi \rho \gamma ov$, for work), whose German form is *ergal*, but it might be pronounced *ergiel* in French. The known theorem of the equivalence of *vis viva* and mechanical work is then expressed very simply, and in order to show more clearly the analogy between that theorem and the one that concerns the virial, I will juxtapose two theorems:

1. The sum of the vis viva and the ergon is constant.

2. The mean vis viva is equal to the virial.

In order to apply our theorem to heat, consider a body to be a system of material points in motion. Those points act upon each other, and in addition, they are subject to external forces. We can then separate the virial into two parts, one of which refers to the internal forces and the other, to the external forces, which we will call the *internal virial* and the *external virial*, respectively. The internal virial is represented by the formula that was cited before:

$$\frac{1}{2}\sum r \varphi(r),$$

in which the overbar is no longer necessary because due to the large number of atoms that move irregularly, the value that the sum possesses at a certain time will be equal to its mean value. As for the external virial, in the most common case, in which the only external force that acts is a uniform pressure that is normal to the surface, one can express it by the following formula, in which p represents the pressure, and v represents the volume:

$$\frac{3}{2}pv$$

If we further denote that *vis viva* of the motion that we call heat by *h* then we will have:

$$h = \frac{1}{2} \sum r \varphi(r) + \frac{3}{2} pv.$$

It remains for us to prove the stated theorem about the virial. That proof is very easy. The equations of motion of a material point *m* are:

$$m\frac{d^2x}{dt^2} = X, \qquad m\frac{d^2y}{dt^2} = Y, \qquad m\frac{d^2z}{dt^2} = Z.$$

Now one has:

$$\frac{d^2(x^2)}{dt^2} = 2\frac{d}{dt}\left(x\frac{dx}{dt}\right) = 2\left(\frac{dx}{dt}\right)^2 + 2x\frac{d^2x}{dt^2}.$$

Upon multiplying that equation by m / 4 and putting X in place of $m \frac{d^2 x}{dt^2}$, one will obtain:

$$\frac{m}{2} \left(\frac{dx}{dt}\right)^2 = -\frac{1}{2} X x + \frac{m}{4} \frac{d^2(x^2)}{dt^2},$$

so upon integrating this and dividing by *t*, one will infer that:

$$\frac{m}{2t} \int_{0}^{t} \left(\frac{dx}{dt}\right)^{2} dt = -\frac{1}{2t} \int_{0}^{t} X x dt + \frac{m}{4t} \left[\frac{d(x^{2})}{dt} - \left(\frac{d(x^{2})}{dt}\right)_{0}\right]$$

in which $\left(\frac{d(x^2)}{dt}\right)_0$ represents the initial value of $\frac{d(x^2)}{dt}$.

The formulas:

$$\frac{1}{t} \int_{0}^{t} \left(\frac{dx}{dt}\right)^2 dt$$
 and $\frac{1}{t} \int_{0}^{t} X x dt$,

when taken over a large value of time t, represent the mean values of $\left(\frac{dx}{dt}\right)^2$ and Xx,

which we have denoted by $\left(\frac{dx}{dt}\right)^2$ and \overline{Xx} , respectively. For a periodic motion, the last term in the equation will be equal to zero at the end of each period, because $d(x^2) / dt$ will take on its initial value $\left(\frac{d(x^2)}{dt}\right)_0$. If the motion is not regularly periodic, but irregular, like the motion of the atoms in the interior of a body, then the difference $\frac{d(x^2)}{dt} - \left(\frac{d(x^2)}{dt}\right)_0$ will not regularly represent the value zero, but nonetheless that value will present itself from time to time, and other than that, the divisor *t* will make the last term vanish when time *t* becomes very large.

Hence, upon suppressing that term, we can write:

$$\frac{m}{2} \overline{\left(\frac{dx}{dt}\right)^2} = -\frac{1}{2} \overline{X x} \,.$$

Since the same equation will be true for the other coordinates, we will get:

$$\frac{m}{2}\left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2\right] = -\frac{1}{2}\left(\overline{X\ x + Y\ y + Z\ z}\right),$$

or, more briefly:

$$\frac{m}{2}\overline{v^2} = -\frac{1}{2}\overline{(X \ x + Y \ y + Z \ z)},$$

and for a system with an arbitrary number of points:

$$\sum \frac{m}{2} \overline{v^2} = -\frac{1}{2} \overline{(X x + Y y + Z z)}.$$

Our theorem has then been proved, and we will likewise see that it is not only true for the entire system of points and the three coordinates, when taken together, but also for each point and each coordinate, when taken separately.