“Ueber die Gleichgewichtsfigur eines biegsamen Fadens,” J. f. reine u. angew. Math. **57** (1860), 93-110.

On the equilibrium figure of a flexible string

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§ 1. – General equations

The general principles that made it possible for **Jacobi** to reduce the equation of motion to the solution of one partial differential equation (as along as a force function exists) allow an application to the ascertainment of the equilibrium figure of a string in the same case, and it likewise comes down to a problem in the calculus of variations. Moreover, it is not without interest to carry out the corresponding considerations in that problem, especially because they also seem to suggest some simplifications in the calculations.

If we denote the arc-length of the string by ds, the force function by $U\,ds$, and finally the stress that the string experiences at the element ds by T then the equations upon which the form of the string depends are known to be:

$$(1)\qquad \left\{\begin{aligned} &\frac{d}{ds}\left(T\frac{dx}{ds}\right)+\frac{\partial U}{\partial x}=0,\\ &\frac{d}{ds}\left(T\frac{dy}{ds}\right)+\frac{\partial U}{\partial y}=0,\\ &\frac{d}{ds}\left(T\frac{dz}{ds}\right)+\frac{\partial U}{\partial z}=0.\end{aligned}\right.$$

One will also get them from the problem of *finding a minimum for the integral:*

$$(2)\qquad \int_{s_0}^{s_1} U\,ds\,,$$

while the equation exists:

$$(3)\qquad \left(\frac{dx}{ds}\right)^2+\left(\frac{dy}{ds}\right)^2+\left(\frac{dz}{ds}\right)^2=1.$$

However, at the same time, this can be converted into the problem of integrating a partial differential equation when one imagines that a function V of x, y, z has been defined such that:

(4)
$$\begin{cases} T\dfrac{dx}{ds} = \dfrac{\partial V}{\partial x}, \quad T\dfrac{dy}{ds} = \dfrac{\partial V}{\partial y}, \quad T\dfrac{dz}{ds} = \dfrac{\partial V}{\partial z}, \\ -U = \dfrac{\partial V}{\partial s} + \dfrac{\partial V}{\partial x}\dfrac{dx}{ds} + \dfrac{\partial V}{\partial y}\dfrac{dy}{ds} + \dfrac{\partial V}{\partial z}\dfrac{dz}{ds}. \end{cases}$$

When one eliminates the differential from these equations, one will get the two equations:

(5)
$$T = -\frac{\partial V}{\partial s} - U,$$

(6)
$$\left(\frac{\partial V}{\partial x}\right)^2 + \left(\frac{\partial V}{\partial y}\right)^2 + \left(\frac{\partial V}{\partial z}\right)^2 = \left(\frac{\partial V}{\partial s} + U\right)^2.$$

The latter is a partial differential equation that has V for a general solution. If it includes three arbitrary constants h, a, b then one will finally have:

(7)
$$\kappa = \frac{\partial V}{\partial h}, \qquad \alpha = \frac{\partial V}{\partial a}, \qquad \beta = \frac{\partial V}{\partial b}$$

for the equations of the string, while κ; α, β denote new arbitrary constants.

The force function U of the arc-length in this can be obtained in an arbitrary way when one treats it as a constant in the definition of the differential quotients $\frac{\partial U}{\partial x}$, $\frac{\partial U}{\partial y}$, $\frac{\partial U}{\partial z}$. However, if U is free of s, in particular, then one might set:

(8)
$$\frac{\partial V}{\partial s} = h$$

in equation (6). The complete solution of that equation will then take the form:

(9)
$$V = hs + f(x, y, z),$$

and the first of equations (7) will give the arc-length as a function of the coordinates with no further analysis:

(10)
$$\kappa = s + \frac{\partial f}{\partial h}.$$

Since that must be true in most cases, one must usually replace equation (6) with the simpler one:

(11)
$$\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + \left(\frac{\partial f}{\partial z}\right)^2 = (h + U)^2,$$

and the stress will take the simpler expression:

$$T = -h - U. \tag{12}$$

This is an expression that must be essentially positive, and therefore imply restrictions on the values of the integration constant, in general.

§ 2. – Catenaries

The simplest application of these equations is defined by the catenary. If G is the weight per unit length of the string, and the Z-axis points vertically, moreover, then one will have $U = Gz$, and one will have to set:

$$\frac{\partial f}{\partial x} = a, \qquad \frac{\partial f}{\partial y} = b, \qquad \frac{\partial f}{\partial z} = \sqrt{(h+Gz)^2 - a^2 - b^2}$$

in equation (11), or also $b = 0$, when one chooses the XY-plane to be the plane of the string; that will then imply that:

$$V = hs + ax + \int dz \sqrt{(h+Gz)^2 - a^2}\ ,$$

and therefore, from (7), the equations of the catenary would be:

$$\alpha = x - \int \frac{a\,dz}{\sqrt{(h+Gz)^2 - a^2}} = x - \frac{a}{G} \log\,[(h + Gz) + \sqrt{(h+Gz)^2 - a^2}\ ],$$

$$\kappa = s + \int \frac{(h-Gz)\,dz}{\sqrt{(h-Gz)^2 - a^2}} = s - \frac{1}{G}\sqrt{(h+Gz)^2 - a^2}\ .$$

However, from (5), the stress is:

$$T = -h - Gz.$$

It would be superfluous to pursue the treatment of these equations any further.

§ 3. – String under the influence of centrifugal force

I would now like to consider a string whose endpoints are rigidly fixed along a uniformly-rotating axis. Abstracting from the force of gravity, only the centrifugal force will affect it. If the Z-axis is the axis of rotation, while ω is the constant angular velocity, and G is, in turn, the weight per unit length then:

$$U = \frac{G\omega^2}{g}\int (x\,dx + y\,dy) = \frac{G\omega^2}{2g} r^2,$$

where r is the distance from the element ds to the axis of rotation. The integral (2) will then assumes the form:

$$\int r^2\, ds\,,$$

and one will then get the theorem:

The form of the string will be one for which its moment of inertia is a maximum.

Now, if one introduces the polar coordinates r, φ into equation (11), instead of x, y, then one will get the differential equation:

$$\left(\frac{\partial f}{\partial r}\right)^2 + \left(\frac{\partial f}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial f}{\partial \varphi}\right)^2 = \mu^2\,(h - r^2)^2 \tag{13}$$

from a known transformation formula, in which $\mu = G\omega^2 / 2g$, and h has been replaced with $-\mu\, h^2$. The latter is justified by the fact that the stress assumes the expression:

$$T = \mu\,(h - r^2)^2, \tag{14}$$

which actually requires that $\mu\, h^2$ must have a positive value, since T must be positive.

However, one must set:

$$\frac{\partial f}{\partial z} = \mu \cdot a, \qquad \frac{\partial f}{\partial \varphi} = \mu \cdot b,$$

$$\frac{\partial f}{\partial r} = \frac{\mu}{r}\sqrt{r^2(h^2 - r^2)^2 - a^2 r^2 - b^2}\,,$$

and one will get the expression:

$$V = -\,h^2\, s + a\, z + b\, \varphi + \int \frac{dr}{r}\sqrt{r^2(h^2 - r^2)^2 - a^2 r^2 - b^2} \tag{15}$$

for the function V when one drops the common factor of μ. One then sees *that the magnitude of the angular velocity has no influence upon the form of the string.*

However, from (7), the integral of the problem will be the following:

$$(16)\qquad \begin{cases} \kappa = -s + \displaystyle\int \frac{(h^2 - r^2)\, r\, dr}{\sqrt{r^2(h^2-r^2)^2 - a^2 r^2 - b^2}}, \\ \alpha = \quad z - a \displaystyle\int \frac{(h^2 - r^2)\, r\, dr}{\sqrt{r^2(h^2-r^2)^2 - a^2 r^2 - b^2}}, \\ \beta = \quad \varphi - b \displaystyle\int \frac{(h^2 - r^2)\, r\, dr}{\sqrt{r^2(h^2-r^2)^2 - a^2 r^2 - b^2}}. \end{cases}$$

These equations show that the z-coordinate can be expressed by an elliptic integral of the first kind, the arc-length, by one of the second kind, and the angle φ, by one of the third kind. It can also be expressed rationally in terms of sin am ($p\ z$), while s, φ can be represented in terms of the functions Z and Π with the same arguments.

In order to develop this manner of representation, it is next necessary to consider the quantity under the square root sign:

For	$r = \infty$,	$r = h$,	$r = 0$,
it will assume the value	$+\infty$,	$-(a^2 h^2 + b^2)$,	$-b^2$, resp.

The cubic equation:

$$(17)\qquad R = r^2(h^2 - r^2) - a^2 r^2 - b^2 = 0$$

will then have a real root between $r^2 = h^2$ and $r^2 = \infty$, in any event. However, since one must always have $r^2 < h^2$, from (14), it is necessary that the roots should also assume a real value between $r^2 = 0$ and $r^2 = h^2$, so, since R is negative at both limits, the sign must alternate twice within them. The equation $R = 0$ will then have three positive real roots, two of which are smaller than h^2, and one of which is larger.

We then set:

$$(18)\qquad R = (r^2 - \rho^2)(r^2 - \sigma^2)(r^2 - \tau^2),$$

where:

$$\rho > \sigma > \tau.$$

A comparison of (17), (18) will then give:

$$(18a)\qquad \begin{cases} h^2 = \dfrac{\rho^2 + \sigma^2 + \tau^2}{2}, \quad b^2 = \rho^2\sigma^2\tau^2, \\ h^4 - a^2 = \rho^2\sigma^2 + \sigma^2\tau^2 + \tau^2\rho^2, \quad \text{or} \\ a^2 = \dfrac{(\rho^2 + \sigma^2 + \tau^2)^2}{2} - (\rho^2\sigma^2 + \sigma^2\tau^2 + \tau^2\rho^2) \\ \quad = \tfrac{1}{4}(\rho + \sigma + \tau)(\rho + \sigma - \tau)(\rho - \sigma + \tau)(\rho - \sigma - \tau). \end{cases}$$

Therefore, in order for a, b, h to be real, ρ, σ, τ will only have to satisfy the condition that:

(19) $$\rho \geq \sigma + \tau.$$

Now, since r must also lie between σ and τ, one can set:

(19)[*sic*] $$\begin{cases} r = \sigma^2 \sin^2 \operatorname{am} u + \tau^2 \cos^2 \operatorname{am} u \\ \quad = \tau^2 + (\sigma^2 - \tau^2) \sin^2 \operatorname{am} u, \end{cases}$$

and when one introduces this into (18), that will give:

(20) $$\begin{cases} R = (\rho^2 - \tau^2)(\sigma^2 - \tau^2)\, \Delta^2 \operatorname{am} u \cos^2 \operatorname{am} u \sin^2 \operatorname{am} u \\ \dfrac{r\,dr}{\sqrt{R}} = \dfrac{du}{\sqrt{\rho^2 - \tau^2}}, \quad k^2 = \dfrac{\sigma^2 - \tau^2}{\rho^2 - \tau^2}, \end{cases}$$

from which, equations (16) will assume the form:

(21) $$\begin{cases} \sqrt{\rho^2 - \tau^2}\,(s + \kappa) = \int_0^u [(h^2 - \tau^2) - (\sigma^2 - \tau^2) \sin^2 \operatorname{am} u]\, du, \\ \sqrt{\rho^2 - \tau^2}\, \dfrac{z - \alpha}{a} = u, \\ \sqrt{\rho^2 - \tau^2}\, \dfrac{\varphi - \beta}{b} = \int_0^u \dfrac{du}{\tau^2 + (\sigma^2 - \tau^2) \sin^2 \operatorname{am} u}. \end{cases}$$

Furthermore, in order to bring the last integral into the usual form of the third type, set:

(22) $$\sigma^2 - \tau^2 = -\,\tau^2 k^2 \sin^2 \operatorname{am}(i\,c),$$

in which c always represents a positive real argument. When one brings in the value of k^2 from (20), one will get:

(23) $$\sigma = \tau\, \Delta \operatorname{am}(i\,c), \qquad \rho = \tau \cos \operatorname{am}(i\,c).$$

Finally, if one sets:

(24) $$z - \alpha = nu, \qquad n = \frac{a}{\sqrt{\rho^2 - \tau^2}} = \frac{\tau i}{2} \cdot \frac{\sqrt{k'^4 \sin^4 \operatorname{am}(ic) + 2(1 + k^2) \sin^2 \operatorname{am}(ic) - 3}}{\sin \operatorname{am}(ic)}$$

then one can now represent the roots ρ, σ, τ in terms of the three quantities n, k, c, and when one introduces this into equations (21), one will get from the first and third ones:

$$(25)\qquad \begin{cases} \dfrac{s+\kappa}{n} = \dfrac{[1-(1+k^2)\sin^2\operatorname{am}(ic)]u + 2k^2\sin^2\operatorname{am}(ic)\int_0^u \sin^2\operatorname{am} u\,du}{\sqrt{k'^4\sin^4\operatorname{am}(ic) + 2(1+k^2)\sin^2\operatorname{am}(ic) - 3}}, \\ \varphi - \beta = i\dfrac{\cos\operatorname{am}(ic)\,\Delta\operatorname{am}(ic)}{\sin\operatorname{am}(ic)}\displaystyle\int_0^u \frac{du}{1-k^2\sin^2\operatorname{am}(ic)\sin^2\operatorname{am} u}. \end{cases}$$

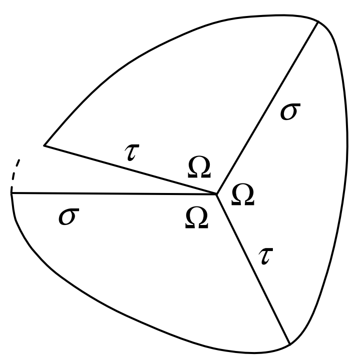
Now, if K, E denote the entire integrals of the first and second kind, resp.:

$$\int_0^u \sin^2\operatorname{am} u\,du = \left(1-\frac{K}{E}\right) u - Z(u),$$

$$\int_0^u \frac{du}{1-k^2\sin^2\operatorname{am}(ic)\sin^2\operatorname{am} u} = u + \frac{\sin\operatorname{am}(ic)}{\cos\operatorname{am}(ic)\Delta\sin\operatorname{am}(ic)}\Pi(u, ic)$$

then one will ultimately get the following formulas, which express r, s, and φ in terms of z and the constants n, k, c:

$$(26)\quad \begin{cases} \dfrac{r^2}{n^2} = \dfrac{-4\sin^2\operatorname{am}(ic)}{k'^4\sin^4\operatorname{am}(ic) + 2(1+k^2)\sin^2\operatorname{am}(ic) - 3}\left(1-k^2\sin^2\operatorname{am}(ic)\sin^2\operatorname{am}\dfrac{z-\alpha}{n}\right), \\ \varphi - \beta = i\dfrac{\cos\operatorname{am}(ic)\,\Delta\operatorname{am}(ic)}{\sin\operatorname{am}(ic)}\dfrac{z-\alpha}{n} + i\,\Pi\left(\dfrac{z-\alpha}{n}, ic\right), \\ \dfrac{s+\kappa}{n} = \dfrac{[1-(1+k^2)\sin^2\operatorname{am}(ic)]\dfrac{z-\alpha}{n} + 2\sin^2\operatorname{am}(ic)\left[\left(1-\dfrac{K}{E}\right)\dfrac{z-\alpha}{n} - Z\left(\dfrac{z-\alpha}{n}\right)\right]}{\sqrt{k'^4\sin^4\operatorname{am}(ic) + 2(1+k^2)\sin^2\operatorname{am}(ic) - 3}}. \end{cases}$$



If one measures the z-coordinate from a height at which the radius r attains a maximum and then sets $\alpha = 0$ then that radius will itself be a periodic function of z that will not change when z goes to $-z$, while s and φ will, at the same time, change signs with z, as well as including a periodic part with terms that are proportional to z. The form of the curve is then that of a spiral that approaches the Z-axis more or less closely at times. The projection onto the XY-plane exhibits congruent and alternately symmetric parts whose limiting radii alternately have the values σ and τ and always subtend the same central angle. In order to find it, one must only determine the growth that φ experiences when $\frac{z-\alpha}{n}$ increases by K. For that, it is not inconvenient to first appeal to a manner of representation that **Jacobi** had applied to the rotation of a body. One has:

$$\Pi\,(u, ic) = u\,Z\,(ic) + \tfrac{1}{2}\log\frac{\Theta(u-ic)}{\Theta(u+ic)}.$$

Thus:

$$\frac{\Theta(u-ic)}{\Theta(u+ic)} = e^{-2i\,\Phi} = \cos 2\Phi - i\sin 2\Phi,$$

in which one sets:

$$\Phi = \varphi - \beta - i\left(\frac{\cos\operatorname{am}(ic)\Delta\operatorname{am}(ic)}{\sin\operatorname{am}(ic)} + Z(ic)\right)\frac{z-\alpha}{n},$$

for brevity, so one will have:

$$\tan\,2\left[\varphi-\beta-i\left(\frac{\cos\operatorname{am}(ic)\Delta\operatorname{am}(ic)}{\sin\operatorname{am}(ic)}+Z(ic)\right)\right]\frac{z-\alpha}{n} \tag{27}$$

$$= \frac{1}{i}\cdot\frac{\Theta\left(\frac{z-\alpha}{n}+ic\right)-\Theta\left(\frac{z-\alpha}{n}-ic\right)}{\Theta\left(\frac{z-\alpha}{n}+ic\right)+\Theta\left(\frac{z-\alpha}{n}-ic\right)}.$$

Since the expression on the right does not change when one lets the argument increase by $2K$, φ must, in the meantime, increase through the angle:

$$2\Omega = i\left[\frac{\cos\operatorname{am}(ic)\Delta\operatorname{am}(ic)}{\sin\operatorname{am}(ic)}+Z(ic)\right]2K, \tag{28}$$

and since that angle belongs to a curve length at which the radius vector once more assumes precisely the previous values, it will yield one-half (Ω) of the angle that corresponds to a single curve segment.

In the special case that will be of interest here, the case $k = 0$ must be singled out to begin with. In that case, r will equal to a constant in (26), but $\varphi - \beta$ will be proportional to $z - \alpha$, so Π will vanish. One will then be dealing with an ordinary helix of the most general form.

If the curve lies entirely on a cylindrical surface in that case then it will be, by contrast, contained in a plane whenever b or a vanishes. For $b = 0$, φ is constant, so the curve will be contained in a plane that goes through the rotational axis. As a result of equations (18a), it will then be necessary that τ must also be zero, and if σ, ρ are not also supposed to vanish then it would be necessary for Δ am (ic) and cos am (ic) to become infinitely large; i.e.:

$$c = K'.$$

Since, at the same time, sin am (ic) = ∞, equations (26) will go to:

$$\frac{r^2}{n^2} = \frac{4k^2}{k'^4}\sin^2 \operatorname{am}\frac{z-\alpha}{n},$$

$$\frac{s+\kappa}{n} = \frac{(1+k^2)\dfrac{z-\alpha}{n} - 2\left[\left((1-\dfrac{K}{E}\right)\dfrac{z-\alpha}{n} - Z\left(\dfrac{z-\alpha}{n}\right)\right]}{k'^2}.$$

By contrast, if $a = 0$ then the curve will be contained completely within the plane that is perpendicular to the rotational axis. As a consequence of equation (24), one will then have the condition:

$$n = 0 = k'^4 \sin^4 \operatorname{am}(ic) + 2\,(1 + k^2) \sin^2 \operatorname{am}(ic) - 3,$$

which shows how to express the constant c in terms of the modulus k. One will get:

$$\sin^2 \operatorname{am}(ic) = \frac{-(1+k^2) - 2\sqrt{1+k^2+k^4}}{k'^4},$$

since the other sign must be discarded, because $\sin^2$ am (ic) is essentially negative. It should be remarked that n must again be expressed in terms of τ [using (24)] in equations (26). The argument $\dfrac{z-\alpha}{n}$ will assume an indeterminate value, and one will ultimately have:

$$r^2 = \tau^2\,(1 - k^2 \sin^2 \operatorname{am}(ic) \sin^2 \operatorname{am} u),$$

$$\varphi - \beta = \frac{\cos \operatorname{am}(ic)\,\Delta \operatorname{am}(ic)}{i \sin \operatorname{am}(ic)}u + \frac{\Pi(u, ic)}{i},$$

$$s + \kappa = \frac{\tau}{2i \sin \operatorname{am}(ic)}\left\{[1-(1+k^2)\sin \operatorname{am}(ic)]u + 2\sin^2 \operatorname{am}(ic)\left[\left(1-\frac{K}{E}\right)u - Zu\right]\right\}.$$

Finally, a combination of the last two cases will give the simplest case for $a = b = 0$, for which the string defines a straight line that is perpendicular to the rotational axis.

§ 4. – Equations for a string that is not free

If the string is not free, but required to remain on a certain surface, then one will define the problem of finding a maximum for $\int U\,ds$, while x, y, z, s are coupled by the equation of surface and the equation:

$$1 = \left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2 + \left(\frac{dz}{ds}\right)^2. \tag{29}$$

The equation of the surface might be represented by the three equations:

$$x = \varphi(p, q), \quad y = \psi(p, q), \quad z = \chi(p, q), \tag{30}$$

in which p, q denote arbitrary variables. Thus, the condition equation (29) will assume the form:

$$1 = P\left(\frac{dp}{ds}\right)^2 + Q\left(\frac{dq}{ds}\right)^2 + 2R\frac{dp}{ds}\frac{dq}{ds}. \tag{31}$$

In one then sets:

$$\left\{\begin{aligned} \frac{\partial V}{\partial p} &= T\left(P\frac{dp}{ds} + Q\frac{dq}{ds}\right), \\ \frac{\partial V}{\partial q} &= T\left(R\frac{dp}{ds} + Q\frac{dq}{ds}\right), \\ -U &= \frac{\partial V}{\partial s} + \frac{\partial V}{\partial p}\frac{dp}{ds} + \frac{\partial V}{\partial q}\frac{dq}{ds}, \end{aligned}\right. \tag{32}$$

analogous to equations (1), then upon eliminating the differentials using (31), (32), one will get:

$$T = -U - \frac{\partial V}{\partial s} \tag{33}$$

and

$$\left(U + \frac{\partial V}{\partial s}\right)^2 (QP - R^2) = Q\left(\frac{\partial V}{\partial p}\right)^2 + P\left(\frac{\partial V}{\partial q}\right)^2 - 2R\frac{\partial V}{\partial p}\frac{\partial V}{\partial q} \tag{34}$$

will be the partial differential equation whose complete solution is V, and if h, a denote the arbitrary constants in it then that will give the equation of the desired curve and its arc-length by:

$$\kappa = \frac{\partial V}{\partial h}, \qquad \alpha = \frac{\partial V}{\partial a}. \tag{35}$$

In particular, if the system of curves:

$$p = \text{const.}, \qquad q = \text{const.}$$

intersect at right angles on the given surface then $R = 0$, and equation (34) will assume the simpler form:

$$\left(U + \frac{\partial V}{\partial s}\right)^2 = \frac{1}{P}\left(\frac{\partial V}{\partial p}\right)^2 + \frac{1}{Q}\left(\frac{\partial V}{\partial q}\right)^2. \tag{36}$$

If the force function U is also independent of s then one can set:

$$V = hs + f(p, q), \tag{37}$$

and equation (36) will go to:

$$(U + h)^2 = \frac{1}{P}\left(\frac{\partial f}{\partial p}\right)^2 + \frac{1}{Q}\left(\frac{\partial f}{\partial q}\right)^2. \tag{38}$$

§ 5. – String on a surface of revolution

One can adapt the two problems that were treated above, and even a generalization of them, to an arbitrary surface of revolution. We denote the distance from a point to the rotational axis by r and let that axis be the Z-axis, but let φ mean the angle that is described by the projection of the radius vector onto a plane that is perpendicular to Z. As is known, one will then have:

$$ds^2 = dr^2 + dz^2 + r^2\, d\varphi^2,$$

or when the equation of the surface of revolution is $r = Fz$:

$$ds^2 = dz^2\, [1 + (F'z)^2] + (Fz)^2\, d\varphi^2.$$

Therefore, in the equations of the previous paragraphs:

$$P = 1 + (F'z)^2, \qquad Q = (F\,z)^2, \qquad R = 0;$$

hence, from (36):

$$\left(U + \frac{\partial V}{\partial s}\right)^2 = \frac{1}{1+(F'z)^2}\left(\frac{\partial V}{\partial z}\right)^2 + \frac{1}{(F\,z)^2}\left(\frac{\partial V}{\partial \varphi}\right)^2. \tag{39}$$

Now, if U depends upon only r and z, or what amounts to the same thing, upon only z, then one might set:

$$\frac{\partial V}{\partial s} = h, \qquad \frac{\partial V}{\partial \varphi} = a, \qquad \frac{\partial V}{\partial z} = \sqrt{\left[(U+h)^2 - \frac{a^2}{(F\,z)^2}\right][1+(F'z)^2]};$$

so

(40) $$V = hs + a\varphi + \int dz \sqrt{\left[(U+h)^2 - \frac{a^2}{(F\,z)^2}\right][1+(F'\,z)^2]}\,,$$

and therefore the equations of the curve of the string will be represented by:

(41) $$\begin{cases} \kappa = s + \displaystyle\int \frac{(U+h)\,dz\sqrt{1+(F'\,z)^2}}{\sqrt{(U+h)^2 - \dfrac{a^2}{(F\,z)^2}}}, \\ \alpha = \varphi - \displaystyle\int \frac{a\,dz\sqrt{1+(F'\,z)^2}}{(F\,z)^2\sqrt{(U+h)^2 - \dfrac{a^2}{(F\,z)^2}}}, \end{cases}$$

to which, one must add the equation for the stress:

(42) $$T = -\,U - h.$$

In particular, let us consider the case that was mentioned at the beginning of this paragraph. If the Z-axis points downwards then the force function of gravity will be Gz, where G means the weight of a piece of the string of unit length; furthermore, the force function of the centrifugal force (cf., § **3**):

$$\frac{G\,\omega^2}{2g}r^2 = \frac{G\,\omega^2}{2g}(F\,z)^2 = \mu\,(F\,z)^2\,.$$

(41), (42) will then imply the equations:

(43) $$\begin{cases} \kappa = s + \displaystyle\int \frac{[h+Gz+\mu\,(F\,z)^2]\,dz\sqrt{1+(F'\,z)^2}}{\sqrt{[h+Gz+\mu\,(F\,z)^2]^2 - \dfrac{a^2}{(F\,z)^2}}}, \\ \alpha = \varphi - \displaystyle\int \frac{a\,dz\sqrt{1+(F'\,z)^2}}{(F\,z)^2\sqrt{[h+Gz+\mu\,(F\,z)^2]^2 - \dfrac{a^2}{(F\,z)^2}}}, \\ T = -h - Gz - \mu\,(F\,z)^2, \end{cases}$$

which give the equilibrium figure of a string on a surface of revolution that is subject to centrifugal force.

If one lets G and μ vanish then those equations will become the equations of the shortest line on such a surface.

§ 6. – Catenary on the sphere

Equations (43) will then give one the form that a massive string assumes on the surface of a ball. If one lets μ vanish and sets $Fz = \sqrt{l^2 - z^2}$, where l denotes the radius of the ball, then the equations above will become:

$$(44)\qquad \begin{cases} \kappa = s + \int \dfrac{(h+Gz)\,l\,dz}{\sqrt{(h+Gz)^2\,(l^2-z^2)-a^2}}, \\ \alpha = \varphi - \int \dfrac{a\,l\,dz}{(l^2-z^2)\sqrt{(h+Gz)^2\,(l^2-z^2)-a^2}}, \\ T = -h - Gz. \end{cases}$$

One sees that the angle φ is represented by an integral of the third kind, while the arc-length is represented by one of the second kind.

Without going into the series development of those integrals, I would at least like to give a different form to the roots that will lead to the normal form of the elliptic integral with no further analysis. Whereas one will often be led to discuss equations of third and fourth order whose coefficients depend upon arbitrary constants in their integration, it is still quite often possible to replace them with other ones that already indicate how to solve the equation, and the reduction to normal form will often point to such a replacement; an example of this was given before in § **3**.

The series development of the expressions (44) depends upon solving the biquadratic equation:

$$Z = (h + Gz)^2\,(l^2 - z^2) - a^2 = 0,$$

into which the arbitrary constants a, h enter. I next set $h = -Glm$, $a = Gbl^2$, $z = l\cos\vartheta$. One must then consider the expression:

$$\frac{Z}{G^2 l^4} = \Theta = (m - \cos\vartheta)^2 \sin^2\vartheta - b^2.$$

However, one can next give this expression the form:

$$\Theta = [(m - \cos\vartheta)\sin\vartheta - b]\,[(m - \cos\vartheta)\sin\vartheta + b],$$

and when one decomposes each of these factors, in turn, one can set:

$$\Theta = [\sin(\vartheta + \varepsilon) + \gamma]\,[\cos(\vartheta - \varepsilon) + \gamma'][\sin(\vartheta - \varepsilon) - \gamma]\,[\cos(\vartheta + \varepsilon) + \gamma'] .$$

If one then considers the equation:

$$(\cos\vartheta - m)\sin\vartheta + b = [\sin(\vartheta+\varepsilon) + \gamma][\cos(\vartheta-\varepsilon) + \gamma'],$$

from which the equation that corresponds to the other factor will emerge upon switching ϑ with $-\vartheta$, then one will get just the conditions:

$$\begin{aligned} \gamma\cos 2\varepsilon &= m\sin\varepsilon, \\ \gamma'\cos 2\varepsilon &= -m\cos\varepsilon, \\ \sin\varepsilon\cos\varepsilon + \gamma\gamma' &= b. \end{aligned}$$

One can then consider the quantities ε and $\rho = m / \cos 2\varepsilon$ to be new arbitrary constants, in place of m and b, as soon as one only shows that each value of m, b always actually correspond to real values of ρ, ε. In the case of ρ, that is obvious, but when one eliminates $\gamma\gamma'$ from the equations above, one will further get:

$$X = (x - 2b)(1 - x^2) - m^2 x = 0,$$

in which one sets $x = \sin 2\varepsilon$. This equation is nothing but the resolvent of the biquadratic equation. However, one has:

$$\begin{aligned} &\text{for} \quad x = +1, \quad X = -m^2, \\ &\text{for} \quad x = -1, \quad X = +m^2, \end{aligned}$$

so a real root must necessarily exist that might be set equal to a sine.

Now, when one multiplies the first factor in the expression for Θ by the third one and the second one with the fourth, it will take the following form:

$$\Theta = -[\sin^2\vartheta\cos^2\varepsilon - \sin^2\varepsilon(\cos\vartheta + \rho)^2][\sin^2\vartheta\sin^2\varepsilon - \cos^2\varepsilon(\cos\vartheta - \rho)^2],$$

or, when one replaces $\cos\vartheta$ with z again, it will be:

$$\begin{aligned} Z &= -G^2(z^2 + 2l\rho z\sin\varepsilon + l^2 r^2 \sin\varepsilon - l^2\cos^2\varepsilon)(z^2 + 2l\rho z\sin\varepsilon + l^2 r^2\sin\varepsilon - l^2\cos^2\varepsilon) \\ &= -G^2(z + l\rho\sin^2\varepsilon + l\cos\varepsilon\sqrt{1-\rho^2\sin^2\varepsilon})(z + l\rho\sin^2\varepsilon - l\cos\varepsilon\sqrt{1-\rho^2\sin^2\varepsilon}) \\ &\qquad \cdot (z - l\rho\cos^2\varepsilon + l\sin\varepsilon\sqrt{1-\rho^2\cos^2\varepsilon})(z - l\rho\cos^2\varepsilon - l\sin\varepsilon\sqrt{1-\rho^2\cos^2\varepsilon}) \\ &= -G^2\zeta. \end{aligned}$$

The equation $Z = 0$ will then have either four real roots, and indeed when $\rho^2 < 1/\sin^2\varepsilon$ and also $\rho^2 < 1/\cos^2\varepsilon$, or two of them when ρ^2 lies between those two quantities. The third case is excluded, since the existence of four imaginary roots would make $\sqrt{Z}$ continually imaginary. Finally, if one introduces the new constant ρ, ε into equations (44) then one will get:

(44*a*)
$$\begin{cases} \kappa = s - \int \dfrac{(l\rho\cos 2\varepsilon - z)\, l\, dz}{\sqrt{-\zeta}}, \\[2ex] \alpha = \varphi - \dfrac{l^2 \sin 2\varepsilon}{2}(1-\rho^2) \int \dfrac{dz}{(l^2 - z^2)\sqrt{-\zeta}}, \\[2ex] T = G\,(l\rho\cos 2\varepsilon - z), \end{cases}$$

which are equations from which one can easily derive the series development now.

§ 7. – String on a sphere that is subjected to centrifugal force

I would now like to consider the case in which a string that is found on a sphere is subject to only the centrifugal force, while the axis of rotation should be a diameter of the sphere. If one lets G vanish in (43) and likewise introduces the radius r in place of z, such that $z^2 = l^2 - r^2$ then one will get (cf., § **3**):

(45)
$$\begin{cases} \kappa = s - \int \dfrac{l\,(h^2 - r^2)\, r\, dr}{\sqrt{(h^2 - r^2)\, r^2 - a^2}\,\sqrt{l^2 - r^2}}, \\[2ex] \alpha = \varphi - \int \dfrac{l\, a\, r\, dr}{r^2\sqrt{(h^2 - r^2)\, r^2 - a^2}\,\sqrt{l^2 - r^2}}, \\[2ex] T = \mu(h^2 - r^2). \end{cases}$$

As before, one then replaces $-\mu\, h^2$ with h and $\mu^2\, a^2$ with a^2, in turn, and sets $\mu = G\omega^2 / 2g$, such that now, in order for the stress to be positive, one must have only $r < h$, and at the same, μ will vanish completely, from the equations of the curve.

The treatment of the integral depends upon the cubic equation:

(46)
$$R = (h^2 - r^2)^2\, r^2 - a^2 = 0.$$

However, for:

$$r^2 = \infty \qquad h^2 \qquad 0$$

the expression R will assume the values:

$$+\infty \quad -a^2 \quad -a^2, \quad \text{resp.}$$

Now, since the expression R must necessarily be positive for certain values of r that are smaller than h in order for a real curve to appear at all, one will see that the equation $R = 0$ must have three positive roots, two of which are smaller than h, and the third of which is greater than h. Therefore, let:

(47) $$R = (r^2 - \rho^2)\,(r^2 - \sigma^2)\,(r^2 - \tau^2),$$

so a comparison of the coefficients of (46), (47) will give:

(48) $$\begin{cases} 2h^2 = \rho^2 + \sigma^2 + \tau^2, \\ h^4 = \rho^2\sigma^2 + \sigma^2\tau^2 + \tau^2\rho^2, \\ a^2 = \rho^2\sigma^2\tau^2. \end{cases}$$

It follows from this that ρ, σ, τ will assume entirely arbitrary values within the limits that were envisioned above, and they will satisfy the single condition:

$$0 = (\rho^2 + \sigma^2 + \tau^2)^2 - 4\,(\rho^2\,\sigma^2 + \sigma^2\,\tau^2 + \tau^2\,\rho^2)^2$$

$$= (\rho + \sigma + \tau)(\rho + \sigma - \tau)(\rho - \sigma + \tau)(\rho - \sigma - \tau).$$

Therefore, if:

$$\rho > \sigma > \tau,$$

as one can assume, then it will follow that:

$$\rho = \sigma + \tau,$$

and the expression R can then be replaced with the following one:

(49) $$R = [r^2 - (\sigma + \tau)^2]\,(r^2 - \sigma^2)\,(r^2 - \tau^2).$$

Now, one can then replace equations (45) with the following ones:

(50) $$\begin{cases} \kappa = s - \displaystyle\int \frac{l\,(\sigma^2 + \tau^2 + \sigma\tau - r^2)\,r\,dr}{\sqrt{(l^2 - r^2)\,(r^2 - \sigma^2)(r^2 - \tau^2)[r^2 - (\sigma + \tau)^2]}}, \\ \alpha = \varphi - \displaystyle\int \frac{l\,\sigma\tau\,(\sigma + \tau)\,dr}{r^2\sqrt{(l^2 - r^2)\,(r^2 - \sigma^2)(r^2 - \tau^2)[r^2 - (\sigma + \tau)^2]}}, \\ T = \mu\,(\sigma^2 + \tau^2 + \sigma\tau - r^2), \end{cases}$$

in which σ, τ denote arbitrary constants and $\sigma > \tau$. One of those constants must also be smaller than l, and only τ will fulfill that condition, so r must lie between τ and l. By contrast, if σ is also smaller than l then r must lie between σ and τ. It is easy to go from the expressions above to the series developments.

§ 8. – Equilibrium of thin elastic strings

The problem of an elastic string is also susceptible to a similar treatment when it is subject to external force that admits a force function and when, at the same time, its cross-section is so narrow that one might neglect the resistance to bending.

Let $d\sigma$ be the original length of an element and let ds be the same thing after extension. The stress in the string in that element is then:

$$T = m^2 \frac{ds - d\sigma}{d\sigma} = m^2\,\lambda, \tag{51}$$

in which m denotes a constant and λ is the extension of the unit length. Hence, if $U\,ds$ is likewise the force function then, from (1), one will have the equations:

$$\left\{\begin{aligned} m^2 \frac{d}{d\sigma}\left(\lambda \frac{dx}{ds}\right) + \frac{\partial U}{\partial x} &= 0, \\ m^2 \frac{d}{d\sigma}\left(\lambda \frac{dy}{ds}\right) + \frac{\partial U}{\partial y} &= 0, \\ m^2 \frac{d}{d\sigma}\left(\lambda \frac{dz}{ds}\right) + \frac{\partial U}{\partial z} &= 0. \end{aligned}\right. \tag{52}$$

However, since:

$$\lambda = \sqrt{\left(\frac{dx}{d\sigma}\right)^2 + \left(\frac{dy}{d\sigma}\right)^2 + \left(\frac{dz}{d\sigma}\right)^2} - 1, \qquad \frac{ds}{d\sigma} = 1 + \lambda, \tag{53}$$

one can also replace equations (52) with the following ones:

$$\left\{\begin{aligned} \frac{m^2}{2} \frac{d}{d\sigma}\left(\frac{\partial \lambda^2}{\partial \frac{dx}{ds}}\right) + \frac{\partial U}{\partial x} &= 0, \\ \frac{m^2}{2} \frac{d}{d\sigma}\left(\frac{\partial \lambda^2}{\partial \frac{dy}{ds}}\right) + \frac{\partial U}{\partial y} &= 0, \\ \frac{m^2}{2} \frac{d}{d\sigma}\left(\frac{\partial \lambda^2}{\partial \frac{dz}{ds}}\right) + \frac{\partial U}{\partial z} &= 0, \end{aligned}\right. \tag{54}$$

which can be derived from the starting assumption that *the integral:*

$$\int\left(\frac{m^2\lambda^2}{2}-U\right)d\sigma$$

must take on a minimum.

If one sets:

$$(55)\qquad \begin{cases} \dfrac{\partial V}{\partial x}=m^2\lambda\dfrac{dx}{ds}=\dfrac{m^2}{2}\dfrac{\partial\lambda^2}{\partial\dfrac{dx}{ds}}, \\ \dfrac{\partial V}{\partial y}=m^2\lambda\dfrac{dy}{ds}=\dfrac{m^2}{2}\dfrac{\partial\lambda^2}{\partial\dfrac{dy}{ds}}, \\ \dfrac{\partial V}{\partial z}=m^2\lambda\dfrac{dz}{ds}=\dfrac{m^2}{2}\dfrac{\partial\lambda^2}{\partial\dfrac{dz}{ds}}, \end{cases}$$

$$(56)\qquad \frac{\partial V}{\partial\sigma}+\frac{\partial V}{\partial x}\frac{dx}{d\sigma}+\frac{\partial V}{\partial y}\frac{dy}{d\sigma}+\frac{\partial V}{\partial z}\frac{dz}{d\sigma}=\frac{m^2\lambda^2}{2}-U,$$

in order to convert that given assumption into a partial differential equation, and likewise considers the expression (53) for λ then one will obtain:

$$\left(\frac{\partial V}{\partial x}\right)^2+\left(\frac{\partial V}{\partial y}\right)^2+\left(\frac{\partial V}{\partial z}\right)^2=m^4\lambda^2,$$

but also:

$$\frac{\partial V}{\partial\sigma}+U=\frac{m^2\lambda^2}{2}-m^2\lambda\frac{ds}{d\sigma}=-m^2\left(\frac{\lambda^2}{2}+\lambda\right),$$

and upon eliminating λ from both equations:

$$(57)\qquad \left\{\frac{\partial V}{\partial\sigma}+U+\frac{1}{2m^2}\left[\left(\frac{\partial V}{\partial x}\right)^2+\left(\frac{\partial V}{\partial y}\right)^2+\left(\frac{\partial V}{\partial z}\right)^2\right]\right\}^2=\left(\frac{\partial V}{\partial x}\right)^2+\left(\frac{\partial V}{\partial y}\right)^2+\left(\frac{\partial V}{\partial z}\right)^2$$

or also

$$(58)\qquad \left(\frac{\partial V}{\partial x}\right)^2+\left(\frac{\partial V}{\partial y}\right)^2+\left(\frac{\partial V}{\partial z}\right)^2=m^4\left\{\sqrt{1-2\frac{U+\dfrac{\partial V}{\partial\sigma}}{m^2}}-1\right\}^2.$$

That is the partial differential equation whose complete solution must be sought. One can always work with it in the irrational form (58), as long as U is independent of σ, in which

$\partial V / \partial \sigma$ must then be set equal to a constant, and will have precisely the character of equation (6). *It will then admit the complete solution to any problem that was posed for an inelastic string above.*

It might be remarked that equation (58) will go to equation (6) as along as m is made infinitely large, which would correspond to the inelastic string. The previous problems can then be truly regarded as special cases of the present one.

§ 9. – Equilibrium of a thin elastic string under the influence of gravity

Let gravity be the only force that is active, let the Z-axis point upwards, and let G be the weight per unit length. One will then have $U = - Gz$, and if one then sets:

$$\frac{\partial V}{\partial x} = m^2\, a, \qquad \frac{\partial V}{\partial \sigma} = Gh$$

in (58) then:

$$\frac{\partial V}{\partial z} = m^2 \sqrt{\left\{\sqrt{1+2G\frac{z-h}{m^2}} - 1\right\}^2 - a^2}\ ,$$

so

$$V = m^2\, ax + Gh\sigma + m^2 \int dz \sqrt{\left\{\sqrt{1+2G\frac{z-h}{m^2}} - 1\right\}^2 - a^2}\ .$$

Now, the equation of the curve of the string is:

(59) $$\alpha = \frac{\partial V}{\partial a} = x - a \int \frac{dz}{\sqrt{\left\{\sqrt{1+2G\frac{z-h}{m^2}} - 1\right\}^2 - a^2}}\ .$$

However, when one differentiates this with respect to h and remarks that one can differentiate with respect to $- z$, instead of z, under the integral sign instead, the original arc-length will be expressed by:

(60) $$\kappa = \sigma - m^2 \sqrt{\left\{\sqrt{1+2G\frac{z-h}{m^2}} - 1\right\}^2 - a^2}\ .$$

The integral (59) is easy to perform. If one sets:

$$\sqrt{1+2G\frac{z-h}{m^2}} - 1 = u$$

then:

$$dz = \frac{m^2}{G}(1 + u)\,du,$$

so (59) will go to:

$$x - \alpha = \frac{am^2}{G}\int \frac{(1+u)\,du}{\sqrt{u^2 - a^2}}$$

$$= \frac{am^2}{G}\left[\log(u + \sqrt{u^2 - a^2}) + \sqrt{u^2 - a^2}\right].$$

If one again replaces *u* with its value in *z* then one will get:

(61) $x - \alpha =$

$$\frac{am^2}{G}\left\{\log\left[\sqrt{1 + 2G\frac{z-h}{m^2}} - 1 + \sqrt{\left(\sqrt{1 + 2G\frac{z-h}{m^2}} - 1\right)^2 - a^2}\right] + \sqrt{\left(\sqrt{1 + 2G\frac{z-h}{m^2}} - 1\right)^2 - a^2}\right\},$$

which is the equation of the desired curve. If one makes *m* infinitely large in it and replaces *a* with a / m^2 then it will go over to the equation of the ordinary catenary.

Karlsruhe, 26 May 1859.