“Über eine allgemeine Transformation der hydrodynamischen Gleichungen,” J. reine angew. Math. **54** (1857), 293-312.

On a general transformation of the hydrodynamical equations

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§ 1.

The equations upon which the theory of the motion of a fluid depends will be represented in two different ways, in general. On the one hand, one considers the unknown quantities in the problem to be the velocities that exist at a certain term at a location in the fluid, so one treats them as functions of position and time, and in that way, arrives at a system of four first-order partial differential equations that bears the name of **Euler**, and must then solve a system of three ordinary differential equations in order to solve the latter system if one is to determine the motion of an individual fluid particle. On the other hand, one introduces the coordinates of a fluid particle as dependent variables and once again arrives at a system of four partial differential equations by means of which one determines those coordinates as functions of time and the initial state. That is the path that **Lagrange** went down in his analytical mechanics. One gets around the problem of solving an additional system of ordinary differential equations, but the equations of the partial differential system will be second-order, except for one of them. It is also regrettable that the characteristic properties of the important *stationary* motion will emerge less clearly in that form.

Meanwhile, there is a third way of treating the problem that presents some singular advantages in precisely the aforementioned case. Namely, for the stationary motion, one can replace the differential equations with the equations of the following problem:

Minimize a triple integral that is extended over space in which the function to be integrated is the *vis viva* of a particle, plus an arbitrary quantity that remains the same for only those particles that traverse the same path. In that way, the aforementioned function is expressed in terms of those functions that will give the curves of motion of the particles when they are set equal to constants and the first partial derivatives of those functions.

With that theorem, which shall be derived in what follows along with some other theorems, and which exhibits a remarkable analogy with the principle of least action, one will then obtain a system of second-order partial differential equations for the stationary motion, and one will find the curves of motion immediately upon integrating them. One will then get the velocities by differentiating them. The corresponding result for the non-stationary state will be more complicated, in general. However, I would not like to fail to present the general development that also subsumes that case. I will even tentatively consider a general system of partial differential equations and exhibit the results of a transformation that corresponds to the one that was suggested.

Herr Dr. **Meissel** has already attempted to give a transformation of the greatest-possible generality for the stationary motion in the *plane* in an article that appeared in Bd. 95 of **Poggendorf**'s Annalen, and in which he excluded the usual assumption that the velocities of a molecule are set equal to the partial derivatives of a *single* function. However, one easily notes that **Meissel**, *loc. cit.*, was misled to some false conclusions that would, in turn, compromise the generality of the result almost completely. In fact, the equation that **Meissel** gave [pp. 278, (5)] must be replaced with another one that I have proposed in what follows [**§ 7**, (59)].

The source of the present investigations is defined by the theory of functional determinants that allow one to immediately recognize the true form in which the velocities are represented in general.

I shall first turn to the following general system of equations.

§ 2.

Let $u_1, u_2, \ldots, u_n$, and V be functions of the variables $x_1, x_2, \ldots, x_n$, and t, and as such, they are determined by the equations:

$$(1)\qquad \begin{cases} \dfrac{\partial V}{\partial x_1} = \dfrac{\partial u_1}{\partial t} + u_1 \dfrac{\partial u_1}{\partial x_1} + u_2 \dfrac{\partial u_1}{\partial x_2} + \cdots + u_n \dfrac{\partial u_1}{\partial x_n}, \\ \dfrac{\partial V}{\partial x_2} = \dfrac{\partial u_2}{\partial t} + u_1 \dfrac{\partial u_2}{\partial x_1} + u_2 \dfrac{\partial u_2}{\partial x_2} + \cdots + u_n \dfrac{\partial u_2}{\partial x_n}, \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots, \\ \dfrac{\partial V}{\partial x_n} = \dfrac{\partial u_n}{\partial t} + u_1 \dfrac{\partial u_n}{\partial x_1} + u_2 \dfrac{\partial u_n}{\partial x_2} + \cdots + u_n \dfrac{\partial u_n}{\partial x_n}, \end{cases}$$

$$(2)\qquad 0 = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \cdots + \frac{\partial u_n}{\partial x_n}.$$

The analytical meaning of the last equation can be deduced immediately. If one forms the functional determinant of the n functions of the variables $x_1, x_2, \ldots, x_n$:

$$a, \quad a', \quad a^{(2)}, \ldots, a^{(n-1)}$$

and arranges it in terms of the derivatives of a, such that it will assume the form:

$$(3)\qquad R = \Delta_1 \frac{\partial a}{\partial x_1} + \Delta_2 \frac{\partial a}{\partial x_2} + \cdots + \Delta_n \frac{\partial a}{\partial x_n},$$

then the Δ will be, in turn, functional determinants from which *a* and *one* of the variables *x* will always be excluded. As is known, one will then have the identity equation (cf., **Jacobi**, "Theoria novi multiplicatoris," this journal, Bd. 27, pp. 203, or *Mathem Werke*, I, pp. 51):

(4) $$\frac{\partial \Delta_1}{\partial x_1} + \frac{\partial \Delta_2}{\partial x_2} + \cdots + \frac{\partial \Delta_n}{\partial x_n} = 0 ,$$

and at the same time, the expressions $\Delta_1, \Delta_2, \ldots, \Delta_n$, which include $n - 1$ arbitrary functions $(a', a^{(2)}, \ldots, a^{(n-1)})$, are the most-general ones that satisfy equation (4). One can examine what equations (1) will imply when one introduces the functions a', $a^{(2)}$, …, $a^{(n-1)}$ in them as dependent variables and sets:

(5) $$u_1 = \Delta_1,\ u_2 = \Delta_2,\ \ldots,\ u_n = \Delta_n$$

in them.

Since we have fulfilled equation (2) identically, only equations (1) will remain, which can be summarized in the following symbolic equation:

(6) $$\delta V = \sum_{k=1}^{n} \frac{\partial \Delta_k}{\partial t} \delta x_k + \sum_{i=1}^{n} \sum_{k=1}^{n} \Delta_i \frac{\partial \Delta_k}{\partial x_i} \delta x_k ,$$

in which only $x_1, x_2, \ldots, x_n$, but not t, are considered to be variable under the variation.

If one lets $2T$ denote the expression:

(7) $$2T = \Delta_1^2 + \Delta_2^2 + \cdots + \Delta_n^2$$

then one will get this expression:

(8) $$\delta T = \sum_{i=1}^{n} \sum_{k=1}^{n} \Delta_i \frac{\partial \Delta_k}{\partial x_i} \delta x_k$$

upon variation, and in place of equation (6), one can consider the following one:

(9) $$\delta (T - V) = \sum_{k=1}^{n} \frac{\partial \Delta_k}{\partial t} \delta x_k + \sum_{i=1}^{n} \sum_{k=1}^{n} \Delta_i \left(\frac{\partial \Delta_k}{\partial x_i} - \frac{\partial \Delta_i}{\partial x_k} \right) \delta x_k .$$

If one sets:

(10) $$M_k = \sum_{i=1}^{n} \Delta_i \left(\frac{\partial \Delta_k}{\partial x_i} - \frac{\partial \Delta_i}{\partial x_k} \right) \delta x_k$$

for brevity, then equation (9) will go to:

(10.a) $$\delta(T-V) = \sum_{k=1}^{n}\left(\frac{\partial \Delta_k}{\partial t} + M_k\right)\delta x_k .$$

That form can be recognized already from the properties of the transformation. Namely, if one defines the sum:

$$M_1\,\Delta_1 + M_2\,\Delta_2 + \ldots + M_n\,\Delta_n = \sum_{i=1}^{n}\sum_{k=1}^{n}\Delta_i\,\Delta_k\left(\frac{\partial \Delta_k}{\partial x_i} - \frac{\partial \Delta_i}{\partial x_k}\right)$$

then one will see that this must therefore vanish identically when one switches i and k and changes its sign. The known property of determinants will emerge from that, namely, that the expression M must assume the form:

(11) $$\begin{cases} M_1 = A^{(1)}\dfrac{\partial a'}{\partial x_1} + A^{(2)}\dfrac{\partial a^{(2)}}{\partial x_1} + \cdots + A^{(n-1)}\dfrac{\partial a^{(n-1)}}{\partial x_1}, \\ M_2 = A^{(1)}\dfrac{\partial a'}{\partial x_2} + A^{(2)}\dfrac{\partial a^{(2)}}{\partial x_2} + \cdots + A^{(n-1)}\dfrac{\partial a^{(n-1)}}{\partial x_2}, \\ \dotfill \\ M_n = A^{(1)}\dfrac{\partial a'}{\partial x_n} + A^{(2)}\dfrac{\partial a^{(2)}}{\partial x_n} + \cdots + A^{(n-1)}\dfrac{\partial a^{(n-1)}}{\partial x_n}, \end{cases}$$

in which the A include the first and second derivatives of a in a known way. The form of the expressions A shall be given below.

However, equation (10.a) will go to the following one immediately:

(12) $$\delta(T-V) = \sum_{k=1}^{n}\frac{\partial \Delta_k}{\partial t}\delta x_k + A^{(1)}\,\delta a' + A^{(2)}\,\delta a^{(2)} + \cdots + A^{(n-1)}\,\delta a^{(n-1)} .$$

If we initially consider the case in which the quantities Δ are thought of as independent of t (which corresponds to the case of stationary motion) then we can integrate equation (12) because the equation:

(13) $$\delta(T-V) = A^{(1)}\,\delta a' + A^{(2)}\,\delta a^{(2)} + \cdots + A^{(n-1)}\,\delta a^{(n-1)} ,$$

in which only $n - 1$ variations occur on the right-hand side and n on the left, will imply that:

(14) $$\begin{cases} A^{(1)} = \Pi'(a'), \\ A^{(2)} = \Pi'(a^{(2)}), \\ \dotfill \\ A^{(n-1)} = \Pi'(a^{(n-1)}), \end{cases}$$

(14.a) $$V - T = \Pi(a', a^{(2)}, \ldots, a^{(n-1)}) \;,$$

with no further analysis. In those equations, Π is an arbitrary function of a, and $\Pi'(a^{(i)})$ refers to the partial derivative of that function with respect to $a^{(i)}$. In this case, one can then *replace equations* (1), (2) *with the* $n - 1$ *equations* (14), *which are second order.*

If the Δ are not independent of t then one can likewise exhibit a system of $n - 1$ differential equations, but they have order three and are much more complicated. Namely, equation (12) will represent the system:

(15) $$\begin{cases} \dfrac{\partial(V-T)}{\partial x_1} = A^{(1)}\dfrac{\partial a'}{\partial x_1} + A^{(2)}\dfrac{\partial a^{(2)}}{\partial x_1} + \cdots + A^{(n-1)}\dfrac{\partial a^{(n-1)}}{\partial x_1} + \dfrac{\partial \Delta_1}{\partial t}, \\ \dfrac{\partial(V-T)}{\partial x_2} = A^{(1)}\dfrac{\partial a'}{\partial x_2} + A^{(2)}\dfrac{\partial a^{(2)}}{\partial x_2} + \cdots + A^{(n-1)}\dfrac{\partial a^{(n-1)}}{\partial x_2} + \dfrac{\partial \Delta_2}{\partial t}, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \dfrac{\partial(V-T)}{\partial x_n} = A^{(1)}\dfrac{\partial a'}{\partial x_n} + A^{(2)}\dfrac{\partial a^{(2)}}{\partial x_n} + \cdots + A^{(n-1)}\dfrac{\partial a^{(n-1)}}{\partial x_n} + \dfrac{\partial \Delta_n}{\partial t}. \end{cases}$$

If one forms the functional determinants of order $n - 2$:

$$\frac{\partial \Delta_h}{\partial \dfrac{\partial a^{(m)}}{\partial x_k}}$$

and the sum:

$$S_h^{(m)} = \frac{\partial(V-T)}{\partial x_1}\cdot\frac{\partial \Delta_h}{\partial \dfrac{\partial a^{(m)}}{\partial x_1}} + \frac{\partial(V-T)}{\partial x_2}\cdot\frac{\partial \Delta_h}{\partial \dfrac{\partial a^{(m)}}{\partial x_2}} + \cdots + \frac{\partial(V-T)}{\partial x_n}\cdot\frac{\partial \Delta_h}{\partial \dfrac{\partial a^{(m)}}{\partial x_n}}$$

then $S_h^{(m)}$ will also be a functional determinant, and indeed I will obtain it when I replace the $a^{(m)}$ in Δ_h with the function of $V - T$. I can also consider $S_h^{(m)}$ to be the coefficients of $\dfrac{\partial a^{(m)}}{\partial x_h}$ in the n^{th}-order functional determinant of the functions a', $a^{(2)}$, …, $a^{(n-1)}$, $(V - T)$. From the aforementioned theorem, I will then have:

$$\frac{\partial S_1^{(m)}}{\partial x_1} + \frac{\partial S_2^{(m)}}{\partial x_2} + \cdots + \frac{\partial S_n^{(m)}}{\partial x_n} = 0$$

for every value of m from 1 to $n - 1$, which is an equation that will go to:

$$0 = \sum_{h=1}^{n}\sum_{k=1}^{n} \frac{\partial}{\partial x_h}\left\{ \frac{\partial(V-T)}{\partial x_k} \cdot \frac{\partial \Delta_h}{\partial \frac{\partial a^{(m)}}{\partial x_k}} \right\}$$

with the help of the expression for $S_h^{(m)}$.

If one also replaces $\frac{\partial(V-T)}{\partial x_k}$ with its value in (15) then the equation will cease to be an identity, and one will then get equations for determinant the *a*. Namely, one will then have:

$$(16) \qquad 0 = \sum_{h=1}^{n}\sum_{k=1}^{n} \frac{\partial}{\partial x_h}\left\{ \frac{\partial \Delta_h}{\partial \frac{\partial a^{(m)}}{\partial x_k}} \left(A^{(1)} \frac{\partial a'}{\partial x_k} + A^{(2)} \frac{\partial a^{(2)}}{\partial x_k} + \cdots + A^{(n-1)} \frac{\partial a^{(n-1)}}{\partial x_k} + \frac{\partial \Delta_k}{\partial t} \right) \right\} .$$

Meanwhile, that equation can be simplified considerably. The expression after the differentiation sign $\frac{\partial}{\partial x_h}$ is $A^{(1)}$ times:

$$\frac{\partial \Delta_h}{\partial \frac{\partial a^{(m)}}{\partial x_1}} \cdot \frac{\partial a'}{\partial x_1} + \frac{\partial \Delta_h}{\partial \frac{\partial a^{(m)}}{\partial x_2}} \cdot \frac{\partial a'}{\partial x_2} + \cdots + \frac{\partial \Delta_h}{\partial \frac{\partial a^{(m)}}{\partial x_n}} \cdot \frac{\partial a'}{\partial x_n} ,$$

and that is nothing but the functional determinant that will arise from Δ_h when one replaces $a^{(m)}$ in it with a'. It will then include two equal functions and must therefore vanish one does not have $m = 1$, in which case, it will coincide with Δ_h. Hence, the coefficient of $A^{(1)}$ will be zero, and likewise for $A^{(2)}$, etc., up to $A^{(m)}$, which will be Δ_h, and equation (16) will then be converted into:

$$(17) \qquad 0 = \sum_{h=1}^{n} \frac{\partial}{\partial x_h}(\Delta_h \cdot A^{(m)}) + \sum_{h=1}^{n}\sum_{k=1}^{n} \frac{\partial}{\partial x_h}\left(\frac{\partial \Delta_h}{\partial \frac{\partial a^{(m)}}{\partial x_k}} \cdot \frac{\partial \Delta_k}{\partial t} \right) .$$

If one then adds that from the repeatedly-applied theorem, one will have:

$$\sum_{h=1}^{n} \frac{\partial \Delta_h}{\partial x_h} = 0 , \qquad \sum_{h=1}^{n} \frac{\partial}{\partial x_h}\left(\frac{\partial \Delta_h}{\partial \frac{\partial a^{(m)}}{\partial x_h}} \right) = 0$$

then equation (17) will finally assume the following form:

(18) $$0 = \Delta_1 \frac{\partial A^{(m)}}{\partial x_1} + \Delta_2 \frac{\partial A^{(m)}}{\partial x_2} + \cdots + \Delta_n \frac{\partial A^{(m)}}{\partial x_n} + Q^{(m)},$$

in which one sets:

(18.a) $$Q^{(m)} = \sum_{h=1}^{n} \sum_{k=1}^{n} \frac{\partial^2 \Delta_k}{\partial t\, \partial x_h} \cdot \frac{\partial \Delta_h}{\partial \dfrac{\partial a^{(m)}}{\partial x_k}},$$

for brevity.

Equations (18) serve to determine the a. If one imagines that those equations have been solved, and the variables $x_1, x_2, \ldots, x_n$ in $V - T$ are then replaced with a', $a^{(2)}, \ldots, a^{(n-1)}$, and any other function a then according to equation (15), one will have:

$$\Delta_1 \frac{\partial (V-T)}{\partial x_1} + \Delta_2 \frac{\partial (V-T)}{\partial x_2} + \cdots + \Delta_n \frac{\partial (V-T)}{\partial x_n} = \frac{\partial T}{\partial t},$$

and likewise, the left-hand side is nothing but $R\left[\frac{\partial (V-T)}{\partial a}\right]$, namely, when we let R denote the determinant of all a with respect to x and let the square bracket suggest that the x in it are thought of as being expressed in terms of the a. We will then have:

$$R\left[\frac{\partial (V-T)}{\partial a}\right] = \frac{\partial T}{\partial t},$$

so, upon integrating that, we will have:

(19) $$V - T = \int \left[\frac{1}{R} \cdot \frac{\partial T}{\partial t}\right] da + \Pi(a', a'', \ldots, a^{(n-1)}, t),$$

in which Π is an arbitrary function. Equations (18), (19) will then give the functions a, V. Differentiating equation (19) will then lead to condition equations for the arbitrary functions which will give the complete integration of equations (18).

I further remark that as soon as *any* of the expressions $Q^{(m)}$ vanish, the corresponding equation (18) will give the integral:

(20) $$A^{(m)} = \Omega(a', a^{(2)}, \ldots, a^{(n-1)}, t),$$

in which Ω is an arbitrary function. Those expressions will *all* vanish in the case that was already considered above.

§ 3.

Before I continue, it is necessary for me to actually develop the expressions for the A. They are given by equations (9), (10), (11). Namely, one must have:

$$A^{(1)}\frac{\partial a'}{\partial x_k}+A^{(2)}\frac{\partial a^{(2)}}{\partial x_k}+\cdots+A^{(n-1)}\frac{\partial a^{(n-1)}}{\partial x_k}=\sum_{i=1}^{n}\Delta_i\left(\frac{\partial \Delta_k}{\partial x_i}-\frac{\partial \Delta_i}{\partial x_k}\right). \tag{21}$$

The right-hand side must be converted into the left-hand side. To that end, I shall consider the triple sum:

$$S_k=\sum_{i=1}^{n}\sum_{h=1}^{n}\sum_{m=1}^{n-1}\frac{\partial \Delta_h}{\partial x_i}\cdot\frac{\partial a^{(m)}}{\partial x_k}\cdot\frac{\partial \Delta_h}{\partial \dfrac{\partial a^{(m)}}{\partial x_i}}. \tag{22}$$

The sum:

$$\sum_{m=1}^{n-1}\frac{\partial a^{(m)}}{\partial x_k}\cdot\frac{\partial \Delta_h}{\partial \dfrac{\partial a^{(m)}}{\partial x_i}},$$

which is multiplied by $\frac{\partial \Delta_h}{\partial x_i}$ in S_k, obviously represents a functional determinant, and indeed the one that arises from Δ_h when one replaces derivatives that are performed with respect to x_i in with derivatives with respect to x_k. However, that determinant generally includes two series in which one differentiates with respect to x_k, and it will then vanish. It is only when the indices k and i are equal that it will remain unchanged under that transposition, namely, it will be Δ_h, and when $k = h$, Δ_h itself will no longer contain any derivatives with respect to x_k, and the value of the resulting determinant will be equal to $-\Delta_i$, since from a known theorem, one will have:

$$\frac{\partial \Delta_k}{\partial \dfrac{\partial a^{(m)}}{\partial x_i}}=-\frac{\partial \Delta_i}{\partial \dfrac{\partial a^{(m)}}{\partial x_k}}.$$

Thus, the sum (22) will reduce to:

$$S_k=\sum_{h=1}^{n}\Delta_h\frac{\partial \Delta_h}{\partial x_k}-\sum_{i=1}^{n}\frac{\partial \Delta_k}{\partial x_i}\Delta_i,$$

and one will then see that S_k differs from the right-hand side of equation (21) only in sign.

However, equation (22) will also immediately assume the form:

$$S_k = \sum_{m=1}^{n-1} \frac{\partial a^{(m)}}{\partial x_k} \left\{ \sum_{i=1}^{n} \sum_{h=1}^{n} \frac{\partial \Delta_h}{\partial x_i} \cdot \frac{\partial \Delta_h}{\partial \frac{\partial a^{(m)}}{\partial x_i}} \right\} .$$

That is already the form that is required in (21) since the index *k* no longer occurs in the bracket. One can then set:

$$A^{(m)} = - \sum_{i=1}^{n} \sum_{h=1}^{n} \frac{\partial \Delta_h}{\partial x_i} \cdot \frac{\partial \Delta_h}{\partial \frac{\partial a^{(m)}}{\partial x_i}} . \tag{23}$$

One can give that expression an even more suitable form when one remarks that:

$$0 = \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left(\frac{\partial \Delta_h}{\partial \frac{\partial a^{(m)}}{\partial x_i}} \right)$$

because one can then write:

$$A^{(m)} = - \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left\{ \sum_{h=1}^{n} \Delta_h \frac{\partial \Delta_h}{\partial \frac{\partial a^{(m)}}{\partial x_i}} \right\} .$$

However, the relation above can now be introduced:

$$2\,T = \Delta_1^2 + \Delta_2^2 + \cdots + \Delta_n^2 .$$

If we apply that then we will finally get $A^{(m)}$ in its simplest form:

$$- A^{(m)} = \frac{\partial}{\partial x_1} \frac{\partial T}{\partial \frac{\partial a^{(m)}}{\partial x_1}} + \frac{\partial}{\partial x_2} \frac{\partial T}{\partial \frac{\partial a^{(m)}}{\partial x_2}} + \cdots + \frac{\partial}{\partial x_n} \frac{\partial T}{\partial \frac{\partial a^{(m)}}{\partial x_n}} . \tag{24}$$

That is therefore the expression for $A^{(m)}$, into which second derivatives enter, as we see. Equation (18) will then lead to third derivatives. I further remark that equation (23) allows us to give equations (18) the form:

$$\Delta_1 \frac{\partial A^{(m)}}{\partial x_1} + \Delta_2 \frac{\partial A^{(m)}}{\partial x_2} + \cdots + \Delta_n \frac{\partial A^{(m)}}{\partial x_n} + \frac{\partial A^{(m)}}{\partial t} = R^{(m)} , \tag{25}$$

in which we set:

$$R^{(m)} = \sum_{i=1}^{n}\sum_{h=1}^{n} \frac{\partial \Delta_h}{\partial x_i} \cdot \frac{\partial}{\partial t} \frac{\partial \Delta_h}{\partial \frac{\partial a^{(m)}}{\partial x_i}}, \tag{25.a}$$

for brevity. This is a form that will be made use of in what follows.

However, for the simpler case in which the Δ (or the u) are independent of t, equation (14), along with (14), will give the following:

$$\frac{\partial}{\partial x_1} \frac{\partial T}{\partial \frac{\partial a^{(m)}}{\partial x_1}} + \frac{\partial}{\partial x_2} \frac{\partial T}{\partial \frac{\partial a^{(m)}}{\partial x_2}} + \cdots + \frac{\partial}{\partial x_n} \frac{\partial T}{\partial \frac{\partial a^{(m)}}{\partial x_n}} + \Pi'(a^m) = 0\,. \tag{26}$$

The system of equations that is presented in that way is nothing but the one that one will obtain from the problem of *minimizing the integral:*

$$\int^{(n)} (T - \Pi)\, dx_1\, dx_2 \cdots dx_n\,, \tag{27}$$

in which:

$$2T = \Delta_1^2 + \Delta_2^2 + \cdots + \Delta_n^2\,. \tag{28}$$

That theorem is important for the transformation of the differential equations in question and interesting due to the fact that it allows one to recognize the identity of the problems that are given by equations (1), (2), and (27).

§ 4.

We shall now connect the system of differential equations that was posed with the system of complete differential equations:

$$\frac{dx_1}{dt} = u_1\,, \qquad \frac{dx_2}{dt} = u_2\,, \qquad \ldots, \qquad \frac{dx_n}{dt} = u_n\,, \tag{29}$$

for which we can also set:

$$\frac{dx_1}{dt} = \Delta_1\,, \qquad \frac{dx_2}{dt} = \Delta_2\,, \qquad \ldots, \qquad \frac{dx_n}{dt} = \Delta_n \tag{29}$$

now.

The identity equation:

$$\frac{\partial \Delta_1}{\partial x_1} + \frac{\partial \Delta_2}{\partial x_2} + \cdots + \frac{\partial \Delta_n}{\partial x_n} = 0$$

initially shows that *a multiplier for the equations is equal to unity*, and one can then find the last integral when one knows the first $n - 1$.

One further sees that as long as the Δ are free of t, *the integral of the equations will immediately be the following one:*

(31) $$a' = \text{const.}, \qquad a^{(2)} = \text{const.}, \quad \ldots, \qquad a^{(n-1)} = \text{const.}$$

because it will follow from equations (30) that:

$$\frac{da^{(m)}}{dt} = \frac{\partial a^{(m)}}{\partial x_1} \cdot \frac{dx_1}{dt} + \frac{\partial a^{(m)}}{\partial x_2} \cdot \frac{dx_2}{dt} + \cdots = \Delta_1 \frac{\partial a^{(m)}}{\partial x_1} + \Delta_2 \frac{\partial a^{(m)}}{\partial x_2} + \cdots,$$

which is identically zero. *When the A are set equal to constants, they will also be integrals of the present equations* since they are functions of the a, according to (14).

One can add the following theorem in the general case:

1. *As long as one of the a is independent of t, a = const. will be an integral of equations* (30).

2. *As long as the expression* (25.a):

$$R^{(m)} = \sum_{i=1}^{n} \sum_{h=1}^{n} \frac{\partial \Delta_h}{\partial x_i} \cdot \frac{\partial}{\partial t} \frac{\partial \Delta_h}{\partial \frac{\partial a^{(m)}}{\partial x_i}}$$

vanishes:

$$A^{(m)} = \text{const.}$$

will be an integral of the present equations.

That is because, from (30), (25):

$$\frac{dA^{(m)}}{dt} = \Delta_1 \frac{\partial A^{(m)}}{\partial x_1} + \Delta_2 \frac{\partial A^{(m)}}{\partial x_2} + \cdots + \Delta_n \frac{\partial A^{(m)}}{\partial x_n} + \frac{\partial A^{(m)}}{\partial t} = R^{(m)} = 0 .$$

§ 5.

If we set $n = 3$ from now on then equations (1), (2) will go to **Euler**'s equations, namely:

$$(32)\quad \begin{cases} \dfrac{\partial V}{\partial x_1} = \dfrac{\partial u_1}{\partial t} + u_1 \dfrac{\partial u_1}{\partial x_1} + u_2 \dfrac{\partial u_1}{\partial x_2} + u_3 \dfrac{\partial u_1}{\partial x_3}, \\ \dfrac{\partial V}{\partial x_2} = \dfrac{\partial u_2}{\partial t} + u_1 \dfrac{\partial u_2}{\partial x_1} + u_2 \dfrac{\partial u_2}{\partial x_2} + u_3 \dfrac{\partial u_2}{\partial x_3}, \\ \dfrac{\partial V}{\partial x_3} = \dfrac{\partial u_3}{\partial t} + u_1 \dfrac{\partial u_3}{\partial x_1} + u_2 \dfrac{\partial u_3}{\partial x_2} + u_3 \dfrac{\partial u_3}{\partial x_3}, \end{cases}$$

$$0 = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3},$$

in which u_1, u_2, u_3 are the velocities that exist at the location (x_1, x_2, x_3) at time t, and $V = U - p / q$, when U denotes the force function, p is the pressure, and q is the constant density. As a result of equations (5), I will now set:

$$(33)\quad \begin{cases} u_1 = \Delta_1 = \dfrac{\partial a'}{\partial x_2} \cdot \dfrac{\partial a^{(2)}}{\partial x_3} - \dfrac{\partial a^{(2)}}{\partial x_2} \cdot \dfrac{\partial a'}{\partial x_3}, \\ u_2 = \Delta_2 = \dfrac{\partial a'}{\partial x_3} \cdot \dfrac{\partial a^{(2)}}{\partial x_1} - \dfrac{\partial a^{(2)}}{\partial x_3} \cdot \dfrac{\partial a'}{\partial x_1}, \\ u_3 = \Delta_3 = \dfrac{\partial a'}{\partial x_1} \cdot \dfrac{\partial a^{(2)}}{\partial x_2} - \dfrac{\partial a^{(2)}}{\partial x_1} \cdot \dfrac{\partial a'}{\partial x_2}. \end{cases}$$

After a known transformation, one will then have:

$$(34)\quad 2\,T = \Delta_1^2 + \Delta_2^2 + \Delta_3^2 = P' P^{(2)} - P\,P,$$

in which one sets:

$$(35)\quad \begin{cases} P' = \left(\dfrac{\partial a'}{\partial x_1}\right)^2 + \left(\dfrac{\partial a'}{\partial x_2}\right)^2 + \left(\dfrac{\partial a'}{\partial x_3}\right)^2, \\ P^{(2)} = \left(\dfrac{\partial a^{(2)}}{\partial x_1}\right)^2 + \left(\dfrac{\partial a^{(2)}}{\partial x_2}\right)^2 + \left(\dfrac{\partial a^{(2)}}{\partial x_3}\right)^2, \\ P = \dfrac{\partial a'}{\partial x_1} \cdot \dfrac{\partial a^{(2)}}{\partial x_1} + \dfrac{\partial a'}{\partial x_2} \cdot \dfrac{\partial a^{(2)}}{\partial x_2} + \dfrac{\partial a'}{\partial x_3} \cdot \dfrac{\partial a^{(2)}}{\partial x_3}, \end{cases}$$

for brevity, and the expressions for the A will become:

$$
(36) \qquad \begin{cases} -A^{(1)} = \sum_{i=1}^{3} \frac{\partial}{\partial x_i} \left\{ P^{(2)} \frac{\partial a'}{\partial x_i} - P \frac{\partial a^{(2)}}{\partial x_i} \right\}, \\ -A^{(2)} = \sum_{i=1}^{3} \frac{\partial}{\partial x_i} \left\{ P' \frac{\partial a^{(2)}}{\partial x_i} - P \frac{\partial a'}{\partial x_i} \right\}. \end{cases}
$$

The differential equations upon which the problem depends in general are:

$$
(37) \qquad \begin{cases} 0 = \Delta_1 \frac{\partial A^{(1)}}{\partial x_1} + \Delta_2 \frac{\partial A^{(1)}}{\partial x_2} + \Delta_3 \frac{\partial A^{(1)}}{\partial x_3} + \frac{\partial A^{(1)}}{\partial t} - R^{(1)}, \\ 0 = \Delta_1 \frac{\partial A^{(2)}}{\partial x_1} + \Delta_2 \frac{\partial A^{(2)}}{\partial x_2} + \Delta_3 \frac{\partial A^{(2)}}{\partial x_3} + \frac{\partial A^{(2)}}{\partial t} - R^{(2)}, \end{cases}
$$

in which:

$$
(37.a) \qquad \begin{cases} R^{(1)} = \frac{\partial^2 a^{(2)}}{\partial x_1 \, \partial t} \left(\frac{\partial \Delta_3}{\partial x_2} - \frac{\partial \Delta_2}{\partial x_3} \right) + \frac{\partial^2 a^{(2)}}{\partial x_2 \, \partial t} \left(\frac{\partial \Delta_1}{\partial x_3} - \frac{\partial \Delta_3}{\partial x_1} \right) + \frac{\partial^2 a^{(2)}}{\partial x_3 \, \partial t} \left(\frac{\partial \Delta_2}{\partial x_1} - \frac{\partial \Delta_1}{\partial x_2} \right), \\ -R^{(2)} = \frac{\partial^2 a'}{\partial x_1 \, \partial t} \left(\frac{\partial \Delta_3}{\partial x_2} - \frac{\partial \Delta_2}{\partial x_3} \right) + \frac{\partial^2 a'}{\partial x_2 \, \partial t} \left(\frac{\partial \Delta_1}{\partial x_3} - \frac{\partial \Delta_3}{\partial x_1} \right) + \frac{\partial^2 a'}{\partial x_3 \, \partial t} \left(\frac{\partial \Delta_2}{\partial x_1} - \frac{\partial \Delta_1}{\partial x_2} \right). \end{cases}
$$

The integrals of the equations:

$$
(38) \qquad \frac{dx_1}{dt} = \Delta_1 , \qquad \frac{dx_2}{dt} = \Delta_2 , \qquad \frac{dx_3}{dt} = \Delta_3
$$

represent a variable system of curves along which the particles move. From the theorems that were discussed in **§ 4**, *one will now have the integral:*

$$
A^{(m)} = \text{const.}
$$

for those equations when the term $R^{(m)}$ *vanishes in one of equations* (37). That is the case, e.g., $a^{(2)}$ when is independent of time. One will then have:

$$
A^{(1)} = \text{const.}
$$

as an integral. However:

$$
a^{(2)} = \text{const.}
$$

will also be one. If we couple that with the principle of the last multiplier then *we can integrate equations* (38) *as long as one of the systems of surfaces on which the motion takes place is independent of time*. Let the last integral be φ = const. φ must then satisfy the equation:

$$W = \frac{\partial\psi}{\partial t} + \Delta_1\frac{\partial\psi}{\partial x_1} + \Delta_2\frac{\partial\psi}{\partial x_2} + \Delta_3\frac{\partial\psi}{\partial x_3} = 0\,,$$

and since the multiplier is 1, that expression will also be equal to the determinant:

$$W \equiv \begin{vmatrix} \frac{\partial\psi}{\partial x_1} & \frac{\partial\varphi}{\partial x_1} & \frac{\partial a^{(2)}}{\partial x_1} & \frac{\partial A^{(1)}}{\partial x_1} \\ \frac{\partial\psi}{\partial x_2} & \frac{\partial\varphi}{\partial x_2} & \frac{\partial a^{(2)}}{\partial x_2} & \frac{\partial A^{(1)}}{\partial x_2} \\ \frac{\partial\psi}{\partial x_3} & \frac{\partial\varphi}{\partial x_3} & \frac{\partial a^{(2)}}{\partial x_3} & \frac{\partial A^{(1)}}{\partial x_3} \\ \frac{\partial\psi}{\partial t} & \frac{\partial\varphi}{\partial t} & \frac{\partial a^{(2)}}{\partial t} & \frac{\partial A^{(1)}}{\partial t} \end{vmatrix}.$$

If one now introduces $a^{(2)}$, $A^{(1)}$, and any new variable v in place of x_1, x_2, x_3, and one denotes the new derivatives by a parenthesis then that identity equation will go to:

$$\left(\frac{\partial\psi}{\partial t}\right) + \left(\frac{\partial\psi}{\partial v}\right)\left\{\frac{\partial v}{\partial t} + \Delta_1\frac{\partial v}{\partial x_1} + \Delta_2\frac{\partial v}{\partial x_2} + \Delta_3\frac{\partial v}{\partial x_3}\right\} \equiv D\cdot\begin{vmatrix} \left(\frac{\partial\psi}{\partial v}\right) & \left(\frac{\partial\varphi}{\partial v}\right) \\ \left(\frac{\partial\psi}{\partial t}\right) & \left(\frac{\partial\varphi}{\partial t}\right) \end{vmatrix},$$

in which D is the determinant of v, $a^{(2)}$, $A^{(1)}$ with respect to x_1, x_2, x_3. If one then sets:

(39) $$w = \frac{\partial v}{\partial t} + \Delta_1\frac{\partial v}{\partial x_1} + \Delta_2\frac{\partial v}{\partial x_2} + \Delta_3\frac{\partial v}{\partial x_3},$$

for brevity, then it will follow from the identity equation that:

$$-1 = D\left(\frac{\partial\varphi}{\partial v}\right), \qquad w = D\left(\frac{\partial\varphi}{\partial t}\right),$$

and the desired last integral will then be:

(40) $$\varphi = \int\frac{w\,dt - dv}{D}.$$

That case will come about *when the motion is the same in all directions around a vertical*, so one can give all integrals in that case since *one* integral (viz., the plane of motion that is laid through that vertical) is independent of t.

If the motion is stationary then according to (26), etc., one will have to *solve the equations that minimize the integral:*

$$\iiint \left(\frac{P' P^{(2)} - P P}{2} - \Pi(a', a^{(2)}) \right) dx_1\, dx_2\, dx_3 , \tag{41}$$

namely, the equations:

$$A^{(1)} = \Pi'(a') , \qquad A^{(2)} = \Pi'(a^{(2)}) . \tag{42}$$

The functions a', $a^{(2)}$ then give the system of surfaces along whose intersection curves the motion proceeds:

$$a' = \text{const.}, \qquad a^{(2)} = \text{const.}$$

That is the theorem that was stated in the introduction. The pressure will become finite as a result of equations (14.a) and (19), and will generally be given by the formula:

$$U - \frac{p}{q} - T = \int \left[\frac{1}{R} \frac{\partial T}{\partial t} \right] da + \Pi(a', a^{(2)}, t) ,$$

and for the stationary motion, in particular, it will be given by:

$$U - \frac{p}{q} - T = \Pi(a', a^{(2)}) .$$

That is, at the same time, the true form that the equation for vis viva assumes.

§ 6.

The introduction of new variables into the present equations raises no significant issues. For the stationary state, one has nothing to do besides transform the expression T, which obviously assumes only the knowledge of the form that the square of the line element assumes. If it is:

$$ds^2 = u_{11}\, dy_1^2 + u_{22}\, dy_2^2 + 2u_{12}\, dy_1\, dy_2 \cdots , \tag{43}$$

in which y_1, y_2, y_3 are the new variables (which are thought of as independent of t), then the determinant of the transformation will be:

(44) $$D = \sqrt{\begin{vmatrix} u_{11} & u_{12} & u_{13} \\ u_{21} & u_{22} & u_{23} \\ u_{31} & u_{32} & u_{33} \end{vmatrix}},$$

and furthermore:

$$2T = P' P^{(2)} - P P,$$

in which:

(45) $$P = -\begin{vmatrix} u_{11} & u_{12} & u_{13} & \dfrac{\partial a^{(2)}}{\partial y_1} \\ u_{21} & u_{22} & u_{23} & \dfrac{\partial a^{(2)}}{\partial y_2} \\ u_{31} & u_{32} & u_{33} & \dfrac{\partial a^{(2)}}{\partial y_3} \\ \dfrac{\partial a'}{\partial y_1} & \dfrac{\partial a'}{\partial y_2} & \dfrac{\partial a'}{\partial y_3} & 0 \end{vmatrix} \cdot \frac{1}{D^2},$$

and in which P' is obtained from P when one replaces $a^{(2)}$ with a', and $P^{(2)}$ will be obtained when one replaces with a' with $a^{(2)}$. The theory of determinants further implies that $2T$ also assumes the form:

(45.a) $$2T = \frac{1}{D^2} \cdot \begin{vmatrix} u_{11} & u_{12} & u_{13} & \dfrac{\partial a'}{\partial y_1} & \dfrac{\partial a^{(2)}}{\partial y_1} \\ u_{21} & u_{22} & u_{23} & \dfrac{\partial a'}{\partial y_2} & \dfrac{\partial a^{(2)}}{\partial y_2} \\ u_{31} & u_{32} & u_{33} & \dfrac{\partial a'}{\partial y_3} & \dfrac{\partial a^{(2)}}{\partial y_3} \\ \dfrac{\partial a'}{\partial y_1} & \dfrac{\partial a'}{\partial y_2} & \dfrac{\partial a'}{\partial y_3} & 0 & 0 \\ \dfrac{\partial a^{(2)}}{\partial y_1} & \dfrac{\partial a^{(2)}}{\partial y_2} & \dfrac{\partial a^{(2)}}{\partial y_3} & 0 & 0 \end{vmatrix}.$$

[Cf., **Hesse**, "Über Determinanten in der Geometrie," thus journal, Bd. 49, pp. 248, formulas (6), (7).]

That integral that must be minimized will then be:

$$\iiint (T - \Pi) \cdot D \, dy_1 \, dy_2 \, dy_3,$$

and the equations will follow from a known method:

$$(46)\qquad \begin{cases} 0 = D \cdot \Pi'(a') + \sum_{i=1}^{3} \dfrac{\partial}{\partial y_i}\left(D \cdot \dfrac{\partial T}{\partial \dfrac{\partial a'}{\partial y_i}} \right), \\[2ex] 0 = D \cdot \Pi'(a^{(2)}) + \sum_{i=1}^{3} \dfrac{\partial}{\partial y_i}\left(D \cdot \dfrac{\partial T}{\partial \dfrac{\partial a^{(2)}}{\partial y_i}} \right). \end{cases}$$

That equation also gives the transformation formula for the A for the more general case:

$$(47)\qquad -A^{(m)} = \frac{1}{D} \sum_{i=1}^{3} \frac{\partial}{\partial y_i}\left(D \cdot \frac{\partial T}{\partial \dfrac{\partial a^{(m)}}{\partial y_i}} \right).$$

We require this formula in order to transform equation (37). We further remark that the first part of that equation, namely:

$$U = \Delta_1 \frac{\partial A^{(m)}}{\partial x_1} + \Delta_2 \frac{\partial A^{(m)}}{\partial x_2} + \Delta_3 \frac{\partial A^{(m)}}{\partial x_3},$$

is nothing but the functional determinant of a', $a^{(2)}$, $A^{(m)}$ with respect to the x, so it will follow immediately that $D \cdot U$ is the functional determinant of a', $a^{(2)}$, $A^{(m)}$ with respect to the y. Thus, when one sets:

$$(48)\qquad \begin{cases} \nabla_1 = \dfrac{\partial a'}{\partial y_2} \cdot \dfrac{\partial a^{(2)}}{\partial y_3} - \dfrac{\partial a'}{\partial y_3} \cdot \dfrac{\partial a^{(2)}}{\partial y_2}, \\[2ex] \nabla_2 = \dfrac{\partial a'}{\partial y_3} \cdot \dfrac{\partial a^{(2)}}{\partial y_1} - \dfrac{\partial a'}{\partial y_1} \cdot \dfrac{\partial a^{(2)}}{\partial y_3}, \\[2ex] \nabla_3 = \dfrac{\partial a'}{\partial y_1} \cdot \dfrac{\partial a^{(2)}}{\partial y_2} - \dfrac{\partial a'}{\partial y_2} \cdot \dfrac{\partial a^{(2)}}{\partial y_1}, \end{cases}$$

the identity equation will come about:

$$(49)\qquad \sum_{i=1}^{3} \Delta_i \frac{\partial A^{(m)}}{\partial x_i} = \frac{1}{D} \sum_{k=1}^{3} \nabla_k \frac{\partial A^{(m)}}{\partial y_k}.$$

If one takes the coefficients of $\frac{\partial A^{(m)}}{\partial x_i}$ on both sides of that then one will get the transformation of Δ_i, namely:

(50) $$\Delta_i = \sum_{k=1}^{3} \frac{\nabla_k}{D} \cdot \frac{\partial x_i}{\partial y_k}.$$

That equation, in whose derivation the nature of a', $a^{(2)}$, $A^{(m)}$ remains completely irrelevant, involves the more-general equation:

$$\frac{\partial \varphi}{\partial x_2}\frac{\partial \psi}{\partial x_3} - \frac{\partial \varphi}{\partial x_3}\frac{\partial \psi}{\partial x_2} = \frac{1}{D}\begin{vmatrix} \frac{\partial \varphi}{\partial y_1} & \frac{\partial \psi}{\partial y_1} & \frac{\partial x_1}{\partial y_1} \\ \frac{\partial \varphi}{\partial y_2} & \frac{\partial \psi}{\partial y_2} & \frac{\partial x_1}{\partial y_2} \\ \frac{\partial \varphi}{\partial y_3} & \frac{\partial \psi}{\partial y_3} & \frac{\partial x_1}{\partial y_3} \end{vmatrix},$$

and when one applies that equation to (37.a), one will get:

(51) $$D\,R^{(1)} = \sum_{i=1}^{3} \begin{vmatrix} \frac{\partial^2 a^{(2)}}{\partial t\,\partial y_1} & \frac{\partial}{\partial y_1}\left(\frac{\nabla_1}{D}\frac{\partial x_i}{\partial y_1} + \frac{\nabla_2}{D}\frac{\partial x_i}{\partial y_2} + \frac{\nabla_3}{D}\frac{\partial x_i}{\partial y_3}\right) & \frac{\partial x_i}{\partial y_1} \\ \frac{\partial^2 a^{(2)}}{\partial t\,\partial y_2} & \frac{\partial}{\partial y_2}\left(\frac{\nabla_1}{D}\frac{\partial x_i}{\partial y_1} + \frac{\nabla_2}{D}\frac{\partial x_i}{\partial y_2} + \frac{\nabla_3}{D}\frac{\partial x_i}{\partial y_3}\right) & \frac{\partial x_i}{\partial y_2} \\ \frac{\partial^2 a^{(2)}}{\partial t\,\partial y_3} & \frac{\partial}{\partial y_3}\left(\frac{\nabla_1}{D}\frac{\partial x_i}{\partial y_1} + \frac{\nabla_2}{D}\frac{\partial x_i}{\partial y_2} + \frac{\nabla_3}{D}\frac{\partial x_i}{\partial y_3}\right) & \frac{\partial x_i}{\partial y_3} \end{vmatrix}.$$

The $\frac{\partial^2 a^{(2)}}{\partial t\,\partial y_1}$ in this will multiply the expression:

$$\sum_{i=1}^{3}\left[\frac{\partial x_i}{\partial y_3}\frac{\partial}{\partial y_2}\left(\frac{\nabla_1}{D}\frac{\partial x_i}{\partial y_1} + \frac{\nabla_2}{D}\frac{\partial x_i}{\partial y_2} + \frac{\nabla_3}{D}\frac{\partial x_i}{\partial y_3}\right) - \frac{\partial x_i}{\partial y_2}\frac{\partial}{\partial y_3}\left(\frac{\nabla_1}{D}\frac{\partial x_i}{\partial y_1} + \frac{\nabla_2}{D}\frac{\partial x_i}{\partial y_2} + \frac{\nabla_3}{D}\frac{\partial x_i}{\partial y_3}\right)\right],$$

or

$$\sum_{i=1}^{3}\sum_{k=1}^{3}\left\{\frac{\partial x_i}{\partial y_3}\frac{\partial}{\partial y_2}\left(\frac{\nabla_k}{D}\frac{\partial x_i}{\partial y_k}\right) - \frac{\partial x_i}{\partial y_2}\frac{\partial}{\partial y_3}\left(\frac{\nabla_k}{D}\frac{\partial x_i}{\partial y_k}\right)\right\},$$

or also when one performs the differentiation:

$$\sum_{k=1}^{3}\left\{\frac{\partial}{\partial y_2}\left(\frac{\nabla_k}{D}\right)\cdot\sum_{i=1}^{3}\frac{\partial x_i}{\partial y_3}\frac{\partial x_i}{\partial y_k} - \frac{\partial}{\partial y_3}\left(\frac{\nabla_k}{D}\right)\cdot\sum_{i=1}^{3}\frac{\partial x_i}{\partial y_2}\frac{\partial x_i}{\partial y_k}\right\} + \sum_{k=1}^{3}\frac{\nabla_3}{D}\left\{\sum_{i=1}^{3}\frac{\partial x_i}{\partial y_3}\frac{\partial^2 x_i}{\partial y_2\,\partial y_k} - \sum_{i=1}^{3}\frac{\partial x_i}{\partial y_2}\frac{\partial^2 x_i}{\partial y_3\,\partial y_k}\right\},$$

and when one notes that according to (43):

$$\text{(52)} \qquad \sum_{i=1}^{3} \frac{\partial x_i}{\partial y_k}\frac{\partial x_i}{\partial y_k} = u_{kh} \,,$$

the coefficient in question will go to:

$$\sum_{k=1}^{3} \left\{ \frac{\partial}{\partial y_2}\left(\frac{\nabla_k}{D}\right) \cdot u_{3k} - \frac{\partial}{\partial y_3}\left(\frac{\nabla_k}{D}\right) \cdot u_{2k} + \frac{\nabla_k}{D}\left(\frac{\partial u_{3k}}{\partial y_2} - \frac{\partial u_{2k}}{\partial y_3}\right) \right\} ,$$

and one will finally have the desired form for $R^{(1)}$:

$$\text{(53)} \qquad D\,R^{(1)} = \sum_{k=1}^{3} \frac{\nabla_k}{D} \left\{ \frac{\partial^2 a^{(2)}}{\partial t\,\partial y_1}\left(\frac{\partial u_{3k}}{\partial y_2} - \frac{\partial u_{2k}}{\partial y_3}\right) + \frac{\partial^2 a^{(2)}}{\partial t\,\partial y_2}\left(\frac{\partial u_{1k}}{\partial y_3} - \frac{\partial u_{3k}}{\partial y_1}\right) + \frac{\partial^2 a^{(2)}}{\partial t\,\partial y_3}\left(\frac{\partial u_{2k}}{\partial y_1} - \frac{\partial u_{1k}}{\partial y_2}\right) \right\}$$

$$+ \sum_{k=1}^{3} \begin{vmatrix} \dfrac{\partial^2 a^{(2)}}{\partial t\,\partial y_1} & \dfrac{\partial}{\partial y_1}\left(\dfrac{\nabla_k}{D}\right) & u_{1k} \\ \dfrac{\partial^2 a^{(2)}}{\partial t\,\partial y_2} & \dfrac{\partial}{\partial y_2}\left(\dfrac{\nabla_k}{D}\right) & u_{2k} \\ \dfrac{\partial^2 a^{(2)}}{\partial t\,\partial y_3} & \dfrac{\partial}{\partial y_3}\left(\dfrac{\nabla_k}{D}\right) & u_{3k} \end{vmatrix} .$$

That equation and equation (49) collectively complete the transformation of equations (37), which likewise comes down to nothing but the transformation of line elements.

In particular, if y are three systems of surfaces that intersect at right angles then u_{12}, u_{23}, u_{31} will vanish, and one will get the expression for $R^{(1)}$:

$$\text{(54)} \qquad R^{(1)} = \left\{ \frac{\partial}{\partial y_2}\left(u_{33}\frac{\nabla_3}{D}\right) - \frac{\partial}{\partial y_3}\left(u_{22}\frac{\nabla_2}{D}\right) \right\} \frac{\partial^2 a^{(2)}}{\partial t\,\partial y_1} + \left\{ \frac{\partial}{\partial y_3}\left(u_{11}\frac{\nabla_1}{D}\right) - \frac{\partial}{\partial y_1}\left(u_{33}\frac{\nabla_3}{D}\right) \right\} \frac{\partial^2 a^{(2)}}{\partial t\,\partial y_2}$$

$$+ \left\{ \frac{\partial}{\partial y_1}\left(u_{22}\frac{\nabla_2}{D}\right) - \frac{\partial}{\partial y_2}\left(u_{11}\frac{\nabla_1}{D}\right) \right\} \frac{\partial^2 a^{(2)}}{\partial t\,\partial y_3} ,$$

and the corresponding one for $R^{(2)}$ that one gets by switching a' and $a^{(2)}$. Finally, the transformation of equations (38) is included in equation (50) because when one multiplies it by $\frac{\partial y_h}{\partial x_i}$ and sums over i, one will get:

(55) $$\frac{dy_h}{dt} = \frac{\nabla_h}{D}.$$

§ 7.

In the individual cases, one will be in a position to determine *one* of the functions a', $a^{(2)}$, from the outset from the nature of the mechanical problem. One will then get a differential equation for the remaining function *a* that generally has order three, but for stationary motion, it will have order two.

Let the motion be such that all particles are required to move in parallel planes whose equations might be represented by x_3 = const. One can then set:

(56) $$a^{(2)} = x_3 , \qquad a' = f(x_1, x_2, t) .$$

Equations (35), (36) will now give:

(57) $$\begin{cases} P' = \left(\dfrac{\partial a'}{\partial x_1}\right)^2 + \left(\dfrac{\partial a'}{\partial x_2}\right)^2, \quad P^{(2)} = 1, \quad P = 0, \\ -A^{(1)} = \dfrac{\partial^2 a'}{\partial x_1^2} + \dfrac{\partial^2 a'}{\partial x_2^2}, \qquad A^{(2)} = 0, \end{cases}$$

and from (33), the expressions for the velocities will be:

(58) $$u_1 = \Delta_1 = \frac{\partial a'}{\partial x_2}, \qquad u_2 = \Delta_2 = -\frac{\partial a'}{\partial x_1}, \qquad u_3 = 0 .$$

Therefore, from equations (37), the second one vanishes identically, and one will have the single equation:

(59) $$\frac{\partial a'}{\partial x_2}\cdot\frac{\partial A^{(1)}}{\partial x_1} - \frac{\partial a'}{\partial x_1}\cdot\frac{\partial A^{(1)}}{\partial x_2} + \frac{\partial A^{(1)}}{\partial t} = 0 ,$$

in which *A* is defined by (57). In the case of stationary motion, equations (42), (57) will give:

(60) $$\frac{\partial^2 a'}{\partial x_1^2} + \frac{\partial^2 a'}{\partial x_2^2} + \Pi'(a') = 0 .$$

The integrals of the differential equations that belong to (59) are:

(61) $$A^{(1)} = \text{const.},$$

and a second one that one gets from the principle of the last multiplier, namely, (40), gives:

$$\text{const.} = \int \frac{\left(\frac{\partial v}{\partial t} + \frac{\partial a'}{\partial x_2}\cdot\frac{\partial v}{\partial x_1} - \frac{\partial a'}{\partial x_1}\cdot\frac{\partial v}{\partial x_2}\right)dt - dv}{\frac{\partial v}{\partial x_2}\cdot\frac{\partial A^{(1)}}{\partial x_1} - \frac{\partial v}{\partial x_1}\cdot\frac{\partial A^{(1)}}{\partial x_2}}, \tag{62}$$

in which $A^{(1)}$ and the arbitrary function v were introduced as variables under the integral sign in place of x_1, x_2 .

A second case in which the differential equations likewise reduce to *one* is given by the motion that is the same in all direction around an axis. Let that axis be the x_3. One can then introduce the coordinates:

$$x_1 = r\cos\varphi, \quad x_2 = r\sin\varphi, \quad x_3 = z\,. \tag{63}$$

The square of the line element is then known to be:

$$ds^2 = dr^2 + r^2 d\varphi^2 + dz^2\,. \tag{64}$$

At the same time, one will get:

$$\begin{cases} P' = \left(\frac{\partial a'}{\partial r}\right)^2 + \left(\frac{\partial a'}{\partial z}\right)^2 + \frac{1}{r^2}\left(\frac{\partial a'}{\partial \varphi}\right)^2, \\ P^{(2)} = \left(\frac{\partial a^{(2)}}{\partial r}\right)^2 + \left(\frac{\partial a^{(2)}}{\partial z}\right)^2 + \frac{1}{r^2}\left(\frac{\partial a^{(2)}}{\partial \varphi}\right)^2, \\ P = \frac{\partial a'}{\partial r}\frac{\partial a^{(2)}}{\partial r} + \frac{\partial a'}{\partial z}\frac{\partial a^{(2)}}{\partial z} + \frac{1}{r^2}\frac{\partial a'}{\partial \varphi}\frac{\partial a^{(2)}}{\partial \varphi}. \end{cases} \tag{65}$$

The determinant of the transformation is r. Now, since $2T = P'P^{(2)} - PP$, equations (47) give:

$$\begin{cases} -A^{(1)} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\left| P^{(2)}\frac{\partial a'}{\partial r} - P\frac{\partial a^{(2)}}{\partial r}\right|\right) + \frac{\partial}{\partial z}\left(P^{(2)}\frac{\partial a'}{\partial z} - P\frac{\partial a^{(2)}}{\partial z}\right) + \frac{1}{r^2}\frac{\partial}{\partial \varphi}\left(P^{(2)}\frac{\partial a'}{\partial \varphi} - P\frac{\partial a^{(2)}}{\partial \varphi}\right), \\ -A^{(2)} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\left| P'\frac{\partial a^{(2)}}{\partial r} - P\frac{\partial a'}{\partial r}\right|\right) + \frac{\partial}{\partial z}\left(P'\frac{\partial a^{(2)}}{\partial z} - P\frac{\partial a'}{\partial z}\right) + \frac{1}{r^2}\frac{\partial}{\partial \varphi}\left(P'\frac{\partial a^{(2)}}{\partial \varphi} - P\frac{\partial a'}{\partial \varphi}\right). \end{cases} \tag{66}$$

In order to consider motion that is symmetric around x_3, one can now try to set:

$$a^{(2)} = \varphi, \qquad a' = f(r, z, t)\,.$$

Equations (65) will then go to:

$$P' = \left(\frac{\partial a'}{\partial r}\right)^2 + \left(\frac{\partial a'}{\partial z}\right)^2, \qquad P^{(2)} = \frac{1}{r^2}, \qquad P = 0,$$

and equations (66) will give:

(67) $$-A^{(1)} = \frac{1}{r}\frac{\partial}{\partial r}\left(\frac{\partial a'}{\partial r}\right) + \frac{1}{r}\frac{\partial}{\partial z}\left(\frac{1}{r}\frac{\partial a'}{\partial z}\right), \quad A^{(2)} = 0.$$

Equations (48) will now go to:

(68) $$\nabla_1 = -\frac{\partial a'}{\partial z}, \qquad \nabla_1 = 0, \qquad \nabla_3 = \frac{\partial a'}{\partial r}.$$

When one adds that $D = r$, one will see, first of all, that the expressions for R both vanish, from (54), and equation (49) will then give:

(69) $$\frac{\partial a'}{\partial r}\cdot\frac{\partial A^{(1)}}{\partial z} - \frac{\partial a'}{\partial z}\cdot\frac{\partial A^{(1)}}{\partial z} + r\frac{\partial A^{(t)}}{\partial t} = 0,$$

in which $A^{(1)}$ is defined by (67), and:

(70) $$\frac{1}{r}\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial a'}{\partial r}\right) + \frac{1}{r}\frac{\partial}{\partial z}\left(\frac{1}{r}\frac{\partial a'}{\partial z}\right) + \Pi'(a') = 0$$

for the stationary motion.

The differential equations that must ultimately be integrated will be:

(71) $$\frac{dr}{dt} = -\frac{1}{r}\frac{\partial a'}{\partial z}, \qquad \frac{dz}{dt} = \frac{1}{r}\frac{\partial a'}{\partial r}.$$

One integral is, in turn, $A^{(1)}$ = const. The other one is obtained from the theory of multipliers, namely:

(72) $$\text{const.} = \int \frac{\left(r\frac{\partial v}{\partial t} - \frac{\partial a'}{\partial z}\cdot\frac{\partial v}{\partial r} + \frac{\partial a'}{\partial r}\cdot\frac{\partial v}{\partial z}\right)dt - r\,dv}{\frac{\partial v}{\partial r}\cdot\frac{\partial A^{(1)}}{\partial z} - \frac{\partial v}{\partial z}\cdot\frac{\partial A^{(1)}}{\partial r}},$$

in which $A^{(1)}$, which remains constant during the integration, and v, which is an arbitrary function of r, z, are introduced under the integral sign in place of r, z.

Berlin, 26 May 1857.