"Intorno alle superficie le quali deformandosi ritegno le stesse linee di curvatura," Ann. di sci. mat. fis. **7** (1856), 410-416.

On the surfaces whose deformations preserve the lines of curvature

By **Delfino CODAZZI**

Professor at the R. GINNASIO LICEAL in Pavia

Translated by D. H. Delphenich

---

1. – Let x, y, z be the rectangular coordinates of surface, which are functions of two variables μ, ν such that the equations μ = const., ν = const. express two systems of lines of curvature. Let m, r denote the arc length and radius of curvature that correspond to the line ν = const., and let n, s denote the analogous quantity for the line μ = const. Let a, b, c be the cosines of the angles that the normal to the surface at the point whose coordinates are μ, ν make with the axes. Finally, we shall denote the derivatives with respect to μ and ν by adding subscripts (†). It is known that the following equations will exist:

$$a_\mu = -\frac{x_\mu}{r}, \qquad b_\mu = -\frac{y_\mu}{r}, \qquad c_\mu = -\frac{z_\mu}{r},$$

$$a_\nu = -\frac{x_\nu}{s}, \qquad b_\nu = -\frac{y_\nu}{s}, \qquad c_\nu = -\frac{z_\nu}{s}.$$

If one eliminates the cosines a, b, c from them then one will deduce these three:

$$(1) \qquad \begin{cases} \left(\frac{1}{s} - \frac{1}{r}\right) x_{\mu\nu} = x_\mu \left(\frac{1}{r}\right)_\nu - x_\nu \left(\frac{1}{s}\right)_\mu , \\ \left(\frac{1}{s} - \frac{1}{r}\right) y_{\mu\nu} = y_\mu \left(\frac{1}{r}\right)_\nu - y_\nu \left(\frac{1}{s}\right)_\mu , \\ \left(\frac{1}{s} - \frac{1}{r}\right) z_{\mu\nu} = z_\mu \left(\frac{1}{r}\right)_\nu - z_\nu \left(\frac{1}{s}\right)_\mu . \end{cases}$$

Since:

$$(2) \qquad x_\mu^2 + y_\mu^2 + z_\mu^2 = m_\mu^2, \qquad x_\nu^2 + y_\nu^2 + z_\nu^2 = n_\nu^2,$$

(†) Translator: In the original, the notation was a prime for derivatives with respect to μ and a lower prime for derivatives with respect to ν, but the latter notation seemed difficult to typeset effectively.

which implies that:

$$x_\mu\, x_{\mu\nu} + y_\mu\, x_{\mu\nu} + z_\mu\, z_{\mu\nu} = m_\mu\, m_{\mu\nu}, \qquad x_\nu\, x_{\mu\nu} + y_\nu\, x_{\mu\nu} + z_\nu\, z_{\mu\nu} = n_\nu\, n_{\mu\nu}.$$

In addition:

$$x_\mu\, x_\nu + y_\mu\, x_\nu + z_\mu\, z_\nu = 0.$$

Hence, if one multiplies (1), first by x_μ , y_μ , z_μ , resp., and then by x_ν , y_ν , z_ν , resp., and then adds the results together, one will get the two equations:

$$\text{(3)} \qquad \left(\frac{1}{r}\right)_\nu = (\log m_\mu)_\nu \left(\frac{1}{s} - \frac{1}{r}\right), \qquad \left(\frac{1}{s}\right)_\mu = (\log n_\nu)_\mu \left(\frac{1}{r} - \frac{1}{s}\right).$$

If one differentiates the product $1 / rs$ and observes (3) then one will find that:

$$\frac{1}{s}\left(\frac{1}{r}\right)_\mu = \left(\frac{1}{rs}\right)_\mu - (\log n_\mu)_\mu \left(\frac{1}{r^2} - \frac{1}{rs}\right),$$

$$\frac{1}{r}\left(\frac{1}{s}\right)_\nu = \left(\frac{1}{rs}\right)_\nu - (\log n_\mu)_\nu \left(\frac{1}{r^2} - \frac{1}{rs}\right).$$

Therefore, if one sets $r / s = \rho$, one will have:

$$\rho_\mu = (\log n_\nu^2)_\mu - \rho\left(\log \frac{n_\nu^2}{rs}\right)_\mu, \qquad \rho_\nu = \rho\left(\log \frac{m_\mu^2}{rs}\right)_\mu - \rho^2 (\log m_\mu^2)_\nu .$$

If one calculates the two values of $\rho_{\mu\nu}$ and equates them to each other and replaces the respective values of ρ_μ, ρ_ν in the resulting equation then that will bring about the reduction:

$$\left[(\log m_\mu)_{\mu\nu} - (\log m_\mu)_\nu \left(\log \frac{n_\nu^2}{rs}\right)_\mu\right]\rho^2$$

$$\text{(4)} \qquad - \left[\left(\log \frac{m_\mu\, n}{rs}\right)_{\mu\nu} - (\log m_\mu^2)_\nu \left(\log n_\nu^2\right)_\mu\right]\rho$$

$$+ (\log n_\nu)_{\mu\nu} - (\log n_\nu)_\mu \left(\log \frac{m_\mu^2}{rs}\right)_\nu = 0.$$

That equation serves to express the ratio of the two radii of curvature of a surface in terms of the arc lengths of the lines of curvature when one knows the value of $1 / rs$ that is defined by those quantities. Now, Gauss's formula, from which one will know the value of $1 / rs$ as a function of the arc lengths of any two system of lines, will be true for the surface, and in the case where those lines are orthogonal, it will become:

$$(5) \qquad \frac{1}{rs} = -\frac{\left(\dfrac{m_{\mu\nu}}{n_\nu}\right)_\nu - \left(\dfrac{m_{\mu\nu}}{m_\mu}\right)_\mu}{m_\mu\, n_\nu} .$$

2. – For those surfaces that keep the same lines of curvature under deformation, equation (4) will be satisfied just the same independently of the value of ρ. Otherwise, the ratio of the radii of curvature would remain constant while the surface deformed. When one equates the first and third coefficients to zero and integrates the resulting equations, one will get:

$$(a) \qquad (\log m_\mu)_\nu = k(\nu)\frac{n_\nu^2}{rs}, \qquad (\log n_\nu)_\mu = h(\mu)\frac{m_\mu^2}{rs},$$

or

$$\frac{m_{\mu\nu}}{n_\nu} = k\frac{m_\mu\, n_\nu}{rs}, \qquad \frac{n_{\mu\nu}}{m_\mu} = h\frac{m_\mu\, n_\nu}{rs},$$

in which h, k are two arbitrary functions, one of which is a function of μ and the other of which is a function of ν. Set:

$$\log\frac{m_\mu\, n_\nu}{rs} = t,$$

and (5) will become:

$$(b) \qquad h\, t_\mu + k\, t_\nu + h_\mu + k_\nu + 1 = 0,$$

and if one equates the second coefficient in (4) to zero then one will get:

$$(c) \qquad t_{\mu\nu} - 4hk\, e^{2t} = 0.$$

Equations (b), (c) cannot be true simultaneously. In fact, if one differentiates the first one twice, once with respect to μ and then with respect to ν and eliminates the quantity $t_{\mu\nu}$ by hand by means of the second one then it will result that:

$$t_{\mu\mu\nu} = -4k\, e^{2t}\,(h_\mu + 2k_\nu + 2k\, t_\mu) .$$

If one differentiates (c) with respect to μ and equates the two values of $t_{\mu\mu\nu}$ then one will find that:

$$h\, t_\mu + k\, t_\nu + h_\mu + k_\mu = 0.$$

However, that equation is absurd, as one learns from (*b*). Therefore, the problem will not admit a solution whenever h, k both have non-zero values.

If one sets $k = 0$ then the first of (*a*) will become $m_{\mu\nu} = 0$, which will show that the lines ν = const. are geodetic, and therefore planar. In fact, m_μ will be a function of only μ in that case, such that if λ denotes the arc length of any line that is traced on the surface, one will have:

$$\left(\frac{dm}{d\lambda}\right)^2 + n_\nu^2\left(\frac{d\nu}{d\lambda}\right)^2 = 1,$$

and as is known, that will show precisely that ν = const. are geodetics. When the first of (*a*) and (*c*) are integrated, that will yield:

$$(d) \qquad m_\mu = A\,(\mu), \qquad \frac{m_\mu n_\nu}{rs} = B\,(\mu)\,C\,(\nu),$$

in which A, B, C are arbitrary functions of the variables that appear in the parentheses. As a result, the second of (*a*) and (*b*) will give:

$$(e) \qquad n_\nu = C\int h\,AB\,d\mu + D\,(\nu), \qquad (h\,B)_\mu + B = 0,$$

in which D denotes a new arbitrary function. One writes:

$$h = -\frac{E(\mu)}{E_\mu(\mu)},$$

in which E denotes a function of μ, and one sets:

$$C = 1, \quad h\,AB = 1,$$

which is equivalent to substituting the variables $\int C\,d\nu$, $\int h\,AB\,d\mu$ for μ, ν, resp. One will deduce from the second of (*e*) that:

$$B = E_\mu\,,$$

in which one writes E instead of the product of E with an arbitrary constant. One will then have:

$$(f) \qquad m_\mu = -\frac{1}{E(\mu)}, \qquad n_\mu = \mu + D\,(\nu), \qquad \frac{m_\mu\,n_\nu}{rs} = E_\mu\,(\mu)\,.$$

Those equations, along with (3), will reduce (1) to the following ones:

$$\frac{x_{\mu\nu}}{x_\nu} = \frac{1}{\mu + D}, \qquad \frac{y_{\mu\nu}}{y_\nu} = \frac{1}{\mu + D}, \qquad \frac{z_{\mu\nu}}{z_\nu} = \frac{1}{\mu + D},$$

from which, one will get:

$$x_\nu = \xi_\nu (\mu + D), \qquad y_\nu = \eta_\nu (\mu + D), \qquad z_\nu = \zeta_\nu (\mu + D),$$

in which the ξ, η, ζ are arbitrary functions of ν that satisfy the equation:

(*g*) $$\xi_\nu^2 + \eta_\nu^2 + \zeta_\nu^2 = 1,$$

as one would learn from the second of (2). Finally:

(*h*) $$x = \int D\,\xi_\nu \, d\nu + \xi\,\mu + \alpha(\mu), \quad y = \int D\,\eta_\nu \, d\nu + \eta\,\mu + \beta(\mu), \quad z = \int D\,\zeta_\nu \, d\nu + \zeta\,\mu + \gamma(\mu),$$

in which α, β, γ are arbitrary functions of μ that satisfy the equation:

(*k*) $$\xi^2 + \eta^2 + \zeta^2 + 2\,(\xi\,\alpha_\mu + \eta\,\beta_\mu + \zeta\,\gamma_\mu) + \alpha_\mu^2 + \beta_\mu^2 + \gamma_\mu^2 = \frac{1}{E^2},$$

as one will learn from the first of (2). In order to determine those functions, one sets:

$$\frac{1}{E^2} - \alpha_\mu^2 - \beta_\mu^2 - \gamma_\mu^2 = \delta^2,$$

and one will deduce the following equations from (*k*):

$$\xi\,\alpha_{\mu\mu} + \eta\,\beta_{\mu\mu} + \zeta\,\gamma_{\mu\mu} = \delta\,\delta_\mu, \qquad \xi\,\alpha_{\mu\mu\mu} + \eta\,\beta_{\mu\mu\mu} + \zeta\,\gamma_{\mu\mu\mu} = (\delta\,\delta_\mu)_\mu.$$

One must then have:

$$\alpha_{\mu\mu\mu}\,\gamma_{\mu\mu} - \alpha_{\mu\mu}\,\gamma_{\mu\mu\mu} = 0, \qquad \beta_{\mu\mu\mu}\,\gamma_{\mu\mu} - \beta_{\mu\mu}\,\gamma_{\mu\mu\mu} = 0, \qquad (\delta\,\delta_\mu)_\mu\,\gamma_{\mu\mu} - \delta\,\delta_\mu\,\gamma_{\mu\mu\mu} = 0,$$

from which, one will get:

(*l*) $$\alpha = G\,\gamma + H\,\mu + K, \qquad \beta = L\,\gamma + M\,\mu + N, \qquad \delta^2 = 2P\gamma_\mu + Q^2,$$

in which G, H, … are arbitrary constants. Thus, (*k*) will decompose into two equations:

(*m*) $$(\xi + H)^2 + (\eta + M)^2 + \zeta^2 = Q^2, \qquad G\,(\xi + H) + L\,(\eta + M) + \zeta = P.$$

Now, the functions ξ, η, ζ are determined by means of (*g*), (*m*), the functions α, β, by means of the first two in (*l*), and the functions $\gamma(\mu)$, $D(\nu)$ will remain arbitrary in equations (*h*).

If one multiplies (h), in turn, by G, L, 1, resp., and adds the results together then one will find that:

$$Gx + Ly + z = P\mu + (G^2 + L^2 + 1)\,\gamma + GK + LN,$$

which will show that the lines of curvature μ = const. are situated in mutually-parallel planes. Therefore, the surfaces that are expressed by equations (h) are the same ones that Monge considered in § XVII of his *Application de l'Analyse à la Géométrie*. (h) also contains two arbitrary functions that can be general integrals of the second-order partial differential equation that expresses the characteristic property of that surface. When (h) is multiplied by ξ_ν, η_ν, ζ_ν, respectively, and the results are summed, that will give:

$$\xi_\nu x + \eta_\nu y + \zeta_\nu z = \xi_\nu \int D\xi_\nu\, d\nu + \eta_\nu \int D\eta_\nu\, d\nu + \zeta_\nu \int D\zeta_\nu\, d\nu + K\,\xi_\nu + N\,\eta_\nu,$$

which exhibits the known property of those surfaces, namely, that the lines of curvature ν = const. are situated in planes that are perpendicular to the plane of the first line of curvature.

If one sets $k = 0$, $h = 0$ then (a) will become:

$$m_{\mu\nu} = 0, \qquad n_{\mu\nu} = 0,$$

and one will deduce from (5) that:

$$\frac{1}{rs} = 0,$$

which expresses the characteristic property of developable surfaces.

There are only two groups of surfaces for which those lines deform like lines of curvature then: Namely, the surfaces with lines of curvature in one system that are situated in parallel planes and the developable surfaces.

Pavia, 14 January 1857.

N. B.: The publication of the Annali for 1856 was delayed somewhat, so the paper was published in the November issue with the date of January 1857.

B. T.