"Su di una particolare classe di coazione elastica che si incontra nello studio delle resistenza delle artiglierie," Rend. della R. Accad. dei Lincei. Cl. sci fis., mat. e nat. (5) **27** (1918), 112-117.

On a particular class of elastic coaction that one encounters in the study of the resistance of artillery.

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Presented to the Society by GIAN ANTONIO MAGGI ([1])

Translated by D. H. Delphenich

A year ago, I published some considerations in these Rendiconti ([2]) that were intended to exhibit the convenience of giving an organic structure to the theory of those states of elastic coaction in the absence of external forces that do not belong to the category of distortions that has been neatly defined and circumscribed by now.

I then promised that I would initiate a systematic study of the argument; the occupation of the country that the state of war brought about prevented me, and still prevents me, from following up on my proposal, but in compensation, it has offered me more than one occasion to thoroughly convince myself of the importance of the question, even in practice. Meanwhile, I ask that the Academy permit me to return to it, if only in passing, in order to treat a problem in this note that is very interesting from the realm of the theory of the resistance of artillery and to generalize one of the results to which one arrives in that problem in a following note.

Consider a homogeneous, isotropic elastic solid that is bounded by two coaxial cylindrical surfaces of radius r_0 and r_1 . One proposes to characterize the states of elastic coaction that are symmetric with respect to the planes that pass through the axes, as well as with respect to the planes that are normal to it.

To that end, examine the neighborhood of a generic point of the solid and orient the reference axes along the three principal directions: One of them, the z-axis, is parallel to the axis of the cylinder, another, the r-axis, is perpendicular to it, and the third, the t-axis, is normal to the preceding ones.

The elementary elastic potential energy can then be expressed in the well-known form:

$$\varphi = G\left[\varepsilon_z^2 + \varepsilon_r^2 + \varepsilon_t^2 + \frac{1}{m-2}(\varepsilon_z + \varepsilon_r + \varepsilon_t)^2\right],$$

([1]) Received at the Academy on 9 September 1918.

([2]) G. COLONETTI, "Su certi stati di coazione elastica che non dipendono da azioni esterne," Rend. della R. Accad. dei Lincei. Cl. sci fis, mat. e nat. (5) **26** (1917), 43-47.

in which G and m are constants that measure the transverse modulus of elasticity and the coefficient of lateral contraction, respectively, of the material, while ε_x , ε_r , ε_t represent the three principal dilatations, and under the hypotheses that were made, they are considered to be (very small) functions of only the distance r of the generic point of the axis that is considered.

We eliminate any axial deformations from what follows by supposing that:

$$\varepsilon_x , = 0,$$

but not without warning that this restrictive hypothesis, which is justifiable only as long as one treats indefinite cylinders, can be easily removed when one passes to the practical case of cylinders of finite length by means of a know procedure of superposition that was already usefully applied in similar cases by Prof. Volterra ([1]).

The elastic potential energy that relates to a truncated solid that is bounded by two cross-sections, which will be regarded as being situated at a distance of one length unit apart, for simplicity, will then prove to be expressed by:

$$\Phi = 2\pi G \int_{r_0}^{r_1} \left[\varepsilon_r^2 + \varepsilon_l^2 + \frac{1}{m-2} (\varepsilon_r + \varepsilon_l)^2 \right] r\, dr .$$

Having said that, introduce the hypothesis that in the state under examination, the given solid is found to be in stable equilibrium in the absence of external forces. Its elastic potential energy must then be a minimum with respect to all of the virtual variations of the configuration (i.e., small and compatible ones that we can imagine it to be subjected to).

The symmetry conditions remain in place, along with the other restrictions that were imposed above, so we confine ourselves to considering the variations of the configuration that one will obtain by transforming any cylindrical surface that is coaxial with the given solid and has a generic radius of r (which is naturally found between r_0 and r_1) into another surface that also cylindrical and coaxial with it, and has a radius of $r + \rho$ (where ρ is supposed to be a small, uniform, continuous function of the variable r).

It is easy to verify that the principal dilatations ε_r , ε_l of the generic elements must become:

$$\varepsilon_r + \frac{d\rho}{dr}, \qquad \varepsilon_l + \frac{\rho}{r},$$

respectively, after such a variation of the configuration.

It then follows that the first variation of the elastic potential energy is expressed by:

$$\delta\Phi = 4\pi G \int_{r_0}^{r_1} \left[\varepsilon_r \frac{d\rho}{dr} + \varepsilon_l \frac{\rho}{r} + \frac{1}{m-2} (\varepsilon_r + \varepsilon_l) \left(\frac{d\rho}{dr} + \frac{\rho}{r} \right) \right] r\, dr .$$

([1]) V. Volterra, "Sur l'équilibrium des corps élastique multiplement connexes," Annales de l'École Normale **24** (1907), 3.

For equilibrium, one must then succeed in verifying the condition:

$$\int_{r_0}^{r_1}\left[\varepsilon_r\frac{d\rho}{dr}+\varepsilon_l\frac{\rho}{r}+\frac{1}{m-2}(\varepsilon_r+\varepsilon_l)\left(\frac{d\rho}{dr}+\frac{\rho}{r}\right)\right]r\,dr = 0$$

or

$$\int_{r_0}^{r_1} r\left[\varepsilon_r+\frac{1}{m-2}(\varepsilon_r+\varepsilon_l)\right]\frac{d\rho}{dr}dr+\int_{r_0}^{r_1}\rho\left[\varepsilon_l+\frac{1}{m-2}(\varepsilon_r+\varepsilon_l)\right]dr = 0.$$

Under the hypothesis that the function:

$$\varepsilon_r+\frac{1}{m-2}\ (\varepsilon_r+\varepsilon_l)$$

is continuous and endowed with a derivative that is bounded in the entire interval $(r_0\ ,\ r_1)$ ([1]), one can set:

$$r\left[\varepsilon_r+\frac{1}{m-2}(\varepsilon_r+\varepsilon_l)\right]\frac{d\rho}{dr}$$

$$=\frac{d}{dr}\left\{\rho r\left[\varepsilon_r+\frac{1}{m-2}(\varepsilon_r+\varepsilon_l)\right]\right\}-\rho\frac{d}{dr}\left\{r\left[\varepsilon_r+\frac{1}{m-2}(\varepsilon_r+\varepsilon_l)\right]\right\}$$

$$=\frac{d}{dr}\left\{\rho r\left[\varepsilon_r+\frac{1}{m-2}(\varepsilon_r+\varepsilon_l)\right]\right\}-\rho r\frac{d}{dr}\left[\varepsilon_r+\frac{1}{m-2}(\varepsilon_r+\varepsilon_l)\right]-\rho\left[\varepsilon_r+\frac{1}{m-2}(\varepsilon_r+\varepsilon_l)\right],$$

so the equation of condition will become:

$$\int_{r_0}^{r_1}\frac{d}{dr}\left\{\rho r\left[\varepsilon_r+\frac{1}{m-2}(\varepsilon_r+\varepsilon_l)\right]\right\}dr-\int_{r_0}^{r_1}\rho r\frac{d}{dr}\left[\varepsilon_r+\frac{1}{m-2}(\varepsilon_r+\varepsilon_l)\right]dr-\int_{r_0}^{r_1}\rho\,(\varepsilon_r-\varepsilon_l)\,dr$$

$$= 0.$$

Finally, if one performs the indicated integration in the first term and reduces the result then one will have:

$$\left[\rho r\left(\varepsilon_r+\frac{1}{m-2}(\varepsilon_r+\varepsilon_l)\right)\right]_{r_0}^{r_1}-\int_{r_0}^{r_1}\rho\left\{r\left[\varepsilon_r+\frac{1}{m-2}(\varepsilon_r+\varepsilon_l)\right]+(\varepsilon_r-\varepsilon_l)\right\}dr = 0.$$

([1]) That hypothesis does not constitute a limitation in regard to the physical magnitudes that the function under discussion and its representative derivative must necessarily satisfy. [Cf., C. Somigliana, "Sulla teoria delle distorsioni elastiche," Rend. della R. Accad. dei Lincei (5) **23** (1914).]

That condition must be verified identically, no matter what values that one attributes to the function *r*, because it is notoriously necessary that one should have:

$$\text{[I]} \qquad \frac{d}{dr}\left[\varepsilon_r + \frac{1}{m-2}(\varepsilon_r + \varepsilon_l)\right] = \frac{\varepsilon_r - \varepsilon_l}{r}$$

for any *r*, as well as:

$$\text{(II)} \qquad \varepsilon_r + \frac{1}{m-2}\,(\varepsilon_r + \varepsilon_l) = 0 \qquad \text{for } \begin{cases} r = r_0, \\ r = r_1. \end{cases}$$

It is easy to exhibit the mechanical significance of these conditions. It is enough to introduce the principal stresses, which will be denoted by σ_z , σ_r , σ_l , respectively, for obvious reasons of analogy, and to recall that they are coupled to the dilatations by the relations:

$$\sigma_z = \frac{2G}{m-2}(\varepsilon_r + \varepsilon_l),$$

$$\sigma_r = 2G\left[\varepsilon_l + \frac{1}{m-2}(\varepsilon_r + \varepsilon_l)\right],$$

$$\sigma_l = 2G\left[\varepsilon_l + \frac{1}{m-2}(\varepsilon_r + \varepsilon_l)\right].$$

The indefinite equation (1) will then immediately become:

$$\frac{d\sigma_r}{dr} = \frac{\sigma_l - \sigma_r}{r},$$

or, what amounts to the same thing:

$$\text{[III]} \qquad \frac{d(r\sigma_r)}{dr} = \sigma_l .$$

However, the boundary equations (II) reduce, as could easily be predicted, to the double condition:

$$\text{[IV]} \qquad \sigma_r = 0 \qquad \text{for } \begin{cases} r = r_0, \\ r = r_1. \end{cases}$$

The problem thus posed can obviously be solved in an infinitude of ways by choosing σ_r arbitrarily [with the one reservation that it must be continuous and endowed with a derivative that is bounded in all of the interval (r_0 , r_1) and satisfies (IV) at the extremes] and one can then deduce σ_l from (III).

It is well-understood that the states of deformation that can then be defined will not *generally* satisfy the so-called compatibility (or Saint-Venant) equation, which we have specifically abstracted from.

One can also, if desired, specify that it is, indeed, beyond doubt that supposing that the equations are satisfied everywhere could only lead us to Volterra's particular distortions.

Other particular cases (e.g., Somigliana distortions) will be obtained when one supposes that the compatibility equations are satisfied in all of the space that is occupied by the solid, except for only certain well-defined surfaces that, in homage to the symmetry hypothesis that was posed, must be cylindrical and coaxial to that solid. It is easy to see that those particular cases include all of the ones that are realized habitually in the construction of artillery by forcibly placing one tube inside of the other in such a way that the initial external diameter of one of them is slightly larger than the initial internal diameter of the one that has to enclose it.

However, the more typical and general cases are obtained only when one drops the compatibility equations completely; one will then find, among other things, all of the states that are called *strongly continuous*, which are produced spontaneously when one supposes that a hollow cylinder is acted on by internal stresses that are intense enough to determine permanent deformations. Indeed, it can happen that if one operates in that way then the internal layers, which are more deformed, will then keep the external layers in a state of tension (even after the pressure has ceased to act), while at the same time, one will find compressions for the reactions to them.

Such states of coaction were recently predicted by the engineers Jacob and Malaval of the French Naval Artillery as being the ones that will permit modern artillery to support pressures that are far more elevated than the ones that were tolerated in the past ([1]) with no permanent final deformations when one makes a more complete utilization of the resistance properties of the materials.

For obvious reasons, it is not possible for me to enter into a discussion of the technical problems to which I alluded. I shall limit myself to pointing out a noteworthy property that is common to all of those states of coaction, which is a property that had already been discovered for some time in certain particular cases, but which can now be easily established in a much more general way.

To that end, consider the expression for the elementary cubic dilatation:

$$\Theta = \varepsilon_z + \varepsilon_r + \varepsilon_t \, .$$

Taking into account the hypotheses that we made, as well as the results that we deduced from them, we can write:

$$\Theta = \frac{m-2}{2mG}(\sigma_r + \sigma_t) = \frac{m-2}{2mG}\left[\sigma_r + \frac{d(r\sigma_r)}{dr}\right].$$

Consequently, the total cubic dilatation for the usual truncated cylinder of unit length will be measured by:

$$2\pi\int_{r_0}^{r_1}\Theta r\,dr = \frac{m-2}{mG}\pi\int_{r_0}^{r_1}\left[\sigma_r + \frac{d(r\sigma_r)}{dr}\right]r\,dr$$

([1]) Cf., E. Bravetta, *La resistenza delle artigliere*, Ed. Carlo Pasta, Torino, 1913.

$$= \frac{m-2}{mG}\pi\int_{r_0}^{r_1}\frac{d}{dr}(r^2\sigma_r)\,dr = \frac{m-2}{mG}\pi\left[r^2\sigma_r\right]_{r_0}^{r_1}.$$

However, according to (IV), σ_r must vanish for $r = r_0$, as well as $r = r_1$; the total cubic dilatation will then be zero identically.

In other words, the total volume of the tube in the state of elastic coaction that is assumed to exist in it will be equal to the sum of the volumes that would be assigned to the individual elements that it is composed of identically if they remained independent of each other, so that one could assume that they were all in their natural undeformed state.

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