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Hertz's ideas on mechanics

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The posthumous work of Hertz *Die Prinzipien der Mechanik in neuem Zusammenhang dargestellt* has given rise to the publication of a certain number of critical studies that have not, at the same time, had the goal of giving a somewhat more complete presentation of it that is intended for those who are interested in the ideas of that great electrician, but which might recoil from reading his book, in which his concern for formal logic seems to have resulted in the accumulation of accessory considerations, arguments that are purely formal and sterile, and definitions of terms that more or less deviate from their usual sense.

The goal that Hertz proposed to pursue was to rid mechanics from the uncontestable difficulties that present themselves in the logical coordination of the principles of that science.

The means for accomplishing that consists of establishing the *unity of force*, while proposing that the hypotheses of nature are only the manifestations of geometric constraints that exist, not only between the masses that are perceptible to our senses, but also between them and the hidden (*verborgene*) masses.

Having admitted that hypothesis, mechanics is then found to be reduced to the study of material systems with constraints and can be constructed by means of a single fundamental principle that consists of the statement of the law of motion of such a system that is devoid of any influence, i.e., any external constraint.

In order to give the statement of that law all of the simplicity of form that is appropriate to its role, Hertz introduced a terminology that reduced a material system with constraints to a geometric point that is confined to remain on a fixed surface.

Hertz's book consists of three divisions:

An introduction that is devoted to a critical examination of the principles of mechanics.

A first part in which the terminology is established.

A second part, which consists of a synthetic presentation of the known propositions of mechanics in the particular form that they get from the hypotheses and terminology that were mentioned.

INTRODUCTION

The *Introduction* was the subject of a study by Poincaré ([1]), in which one will find, along with a presentation of Hertz's ideas, a deep discussion of the principles of mechanics that neatly exhibits the logical difficulties that present themselves in the coordination of those principles.

Referring to that study, we shall confine ourselves to briefly pointing out, without discussion, the objections that Hertz made against the two systems that had been proposed up to then, namely, the *classical system* and the *energetic system*. We will then present Hertz's concept itself.

Classical system. – The objections to that system that Hertz raised were directed, above all, to the notion of force.

First of all, a careful examination of the usual presentation of the principles of mechanics will show that the manner by which that notion is established in it is far from satisfactory.

In addition, there is something fraudulent about the introduction of force into certain questions that gives the impression that it is only useless machinery that pointlessly complicates the intuitive concepts.

Notably, how do you define the notion of force in celestial mechanics, in which observations apply to only the motions, in order for them to first establish the laws, as well as to then verify the deductions that are due to analysis?

A piece of iron resting on a table that is almost horizontal defines something that seems to be extremely simple to us.

Now, in order to account for the fact that it is necessarily at rest according to the laws of mechanics, one must analyze all of the forces to which that piece of iron is subject, such as weight, elastic reaction of the table, friction, molecular, magnetic, electric forces, etc., in order to confirm that those forces will collectively cancel each other.

There are many good reasons for explaining the permanence of a state in the absence of any perceptible cause of perturbation.

A third objection that Hertz made to the motion of force was directed towards its magnitude.

Far from exhausting the notion of force as one finds it in rational mechanics, nature presents us with only forces that are subject to numerous restrictions, such as being decomposable into reciprocal actions between the particles of matter and being independent of the absolute value of time and the absolute location in space.

Some other restrictions still seem to have been assumed, and one is not absolutely fixed in one's choice of them.

Thus, one can demand that the elementary forces consist uniquely of attractions and repulsions along the connecting lines between the active masses if their magnitude depend upon only the distance between those masses or if there is reason to introduce absolute or relative velocities, or even accelerations and higher-order derivatives of the velocity.

Finally, the principles of mechanics do not explain the law of conservation of energy, although its generality would seem to extend from physics to mechanics.

It results from those considerations that mechanics is too vast; it includes more than nature.

([1]) H. POINCARÉ, Revue générale des Sciences (1897), pp. 734.

Energetic system. – That system of mechanics is obtained by substituting the notion of *energy* for that of *force* in the most fundamental notions.

At first glance, one finds oneself in the presence of a difficulty, since energy presents itself in two forms: One of them, viz., kinetic, finds its definition in its analytic expression. The other one, viz., potential, necessitates an experimental definition that seems to be quite difficult to establish suitably.

If one assumes that that obstacle has been overcome then it will become necessary to pose a law that relates the four fundamental notions: space, time, mass, and energy.

The choice of that law among the general theorems of mechanics is arbitrary, to some extent. Hertz chose Hamilton's principle, which expresses the idea that the real motion of a system whose positions at two instants t_0 and t_1 one are given, is the one that minimizes the value of the integral:

$$\int_{t_0}^{t_1} (T - U)\, dt ,$$

in which T represents the kinetic part of the energy, and U is the potential part.

In that conception of mechanics, the notion of force is introduced analytically and will result in a simple definition of the word, like the motion of *vis viva* in classical mechanics.

In a large number of questions, that system presents the advantage of avoiding the use of physical hypotheses that have no other objective than to permit one to apply the principles of mechanics.

The set of facts that it embraces is more restrictive than that of the classical system, since it can be deduced from the latter by means of certain hypotheses.

Indeed, it is too restrictive, and that imperfection is sufficient for one to reject it.

Hamilton's principle is indeed applicable to only those systems whose constraints are expressed by equations in finite terms between the parameters.

Now there exist constraints that are expressed by means of non-integrable differential equations. It suffices to cite the case of rolling and pivoting motions, which is case that was the subject of the work of certain geometers ([1]).

For example, consider the motion of a sphere that rolls without slipping on a horizontal plane.

At each instant, the velocities are determined when one knows that instantaneous axis of rotation around the point of contact, so there will be three quantities, instead of the five that are involved when the sphere can slide on the plane.

When the sphere starts from a given position, it can therefore attain only ∞^3 positions without the intervention of external forces.

However, the constraint equations are not integrable, so the number of different positions that it is kinematically capable of attaining must continue to be represented by ∞^5.

For any two of those ∞^5 positions, Hamilton's principle will obviously determine a trajectory in such a way that when one applies it carelessly, one will find that upon starting from one position, the sphere can attain ∞^5 position without the intervention of external forces, which is a conclusion that is contrary to reality.

([1]) APPELL, HADAMARD, *Mouvements de roulement et de pivotement en dynamique*, Carré et Naud, Paris.

We add that even in the case where the two chosen positions belong to a dynamically-possible trajectory (without the intervention of external forces), Hamilton's principle will generally give a false result.

As a second example of a material system whose parameters are subject to non-integrable differential equations, we cite the case of the bicycle.

We add that Hertz called material systems whose constraints can be represented by a system of equations with finite terms, i.e., ones whose position can be determined when one knows a certain number of independent parameters, *holonomic*.

Hertzian system. – Hertz saw the resolution of the difficulties that presented themselves in coordinating the principles of mechanics in an *explanation* for the concept of force.

Mechanics considers force independently of its cause.

The same is not true in physics, more generally, in which one considers several types of forces that are distinguished by their causes, such as elastic, electric, electrodynamic forces, etc.

Can we not reduce those causes to a single one, i.e., give a common *explanation* for the various forces?

Among the causes of force (in truth, they are quite restricted in number), one finds motion, i.e., inertia.

With Poincaré, we consider a ball-regulator, which we will nonetheless modify with the goal of suppressing the influence of gravity.

Let *ABCD* be an articulated diamond. The upper angle *A* is fixed. The lower angle *C* carries a ring that can slide along a vertical spindle *AX*. A rod *T* is suspended from the ring *C*.

The lower edges of the diamond *CB* and *CD* are prolonged along their own lengths and carry balls at their free extremities whose masses dominate that of the rest of the device considerably. The entire device is animated with a rotational motion around the spindle *AX*, and one sees that the centers of the balls remain in the horizontal plane of the fixed point *A*, in such a way that the effect of the weight is eliminated completely.

The centrifugal force tends to move the balls apart, and consequently, the point *C* will approach *A*.

The system is then capable of exerting a force by the intermediary of the rod *T* due to its own motion.

Suppose that the system is invisible for us. The observer will attribute the traction that is exerted on the rod *T* to a force, namely, an attraction that the point *A* exerts on the that rod.

Hertz's hypothesis can be expressed in the following manner:

All of the forces of nature are due to the motion of masses that are perceptible or latent.

According to Hertz's expression, one will have a *dynamical explanation* for force.

That hypothesis obviously postulates the existence of latent (*verborgene*) material masses, such as the ether of Fresnel and Maxwell.

It remains to establish the particular theory that is appropriate to each type of forces, and one must observe that elasticity itself does not escape the necessity of a dynamical explanation in that way of looking at things.

If we assume Hertz's hypothesis then we will be in a position (at least theoretically) to determine the expressions for the forces that are exerted between the material systems, whether perceptible or latent, by virtue of their constraints, and in order to do that, we appeal to only the principle of inertia, which can be conceived in the absence of any notion of force, as one sees.

It will then result from this that force will be eliminated from mechanics as a fundamental notion, and it will be reduced to the role of an auxiliary notion that is simply defined by an analytical expression.

That is Hertz's conception of things.

Does the hypothesis that is at the basis for it suffice to eliminate the difficulties in the definition of force that have been encountered up to now? We will make a judgment on that later after we have presented the manner by which Hertz introduced that definition.

PART ONE

GEOMETRY AND KINEMATICS OF MATERIAL SYSTEMS

Constraints. – Hertz subjected the constraints to the restriction that they can be represented by systems of linear total differential equations of the form:

$$A_1\,\delta q_1 + A_2\,\delta q_2 + \ldots + A_n\,\delta q_n = 0\,,$$

in which $q_1, q_2, \ldots, q_n$ are parameters that determine the position of the system, and the coefficients $A_1, A_2, \ldots, A_n$ are continuous functions of those parameters.

Equivalently, one can say that the sum of two possible infinitesimal displacements δq and $\delta' q$ that start from the same position is a possible infinitesimal displacement that starts from the same position (viz., it is possible to superpose infinitesimal displacements that are compatible with the constraints).

Hertz attached the linear form of the constraint equations to a property of constraints that he referred to by the name of *continuity in the infinitesimal* and which consists of the idea that *any possible infinitesimal displacements can be obtained by a rectilinear trajectory.*

It first results from the continuity of constraints, which is intended in the ordinary sense of the word, that one can successively perform two infinitesimal displacements δq ($\delta q_1, \delta q_2, \ldots, \delta q_n$) and $\delta' q$ $(\delta' q_1, \delta' q_2, \ldots, \delta' q_n)$ that are supposed to be possible and start from the position q ($q_1, q_2, \ldots, q_n$), because there exists a possible displacement that starts from the position $q + \delta q$ that differs from $\delta' q$ only by infinitesimals of higher order. One can then make the system pass from the position q to the position $q + \delta q + \delta' q$ by way of that trajectory.

Hertz's condition expresses the idea that one can, in addition, pass from the position q to the position $q + \delta q + \delta' q$ by way of a rectilinear trajectory, or rather, since the word "rectilinear" makes no sense in the infinitesimal, that the positions q and $q + \delta q + \delta' q$ belong to the same possible trajectory that has differential elements of first and second order that are continuous at that position.

We prefer the following statement, by which we can characterize the constraints by expressing them as linear equations that are homogeneous in the differentials of the parameters that express a concrete fact.

Any trajectory that is traced through the positions that are obtained by all possible infinitesimal displacements expresses a concrete fact.

Motion of a material point. – A point that is subject to remain on a fixed surface and is devoid of any other influence will traverse a geodesic on the surface with a constant velocity.

That case consists of the one in which the coordinates x, y, z of the point are subject to a linear differential equation:

$$A\,dx + B\,dy + C\,dz = 0\,,$$

which is or is not integrable.

The law of motion is always represented in rectangular coordinates by the formula:

$$\frac{d^2y}{dt^2}\delta x + \frac{d^2y}{dt^2}\delta y + \frac{d^2z}{dt^2}\delta z = 0\,,$$

in which δx, δy, δz represent the variations of the coordinates x, y, z of the point under an arbitrary virtual displacement that is compatible with the constraints.

That formula expresses the idea that the direction of the acceleration is rectangular to all virtual displacements at each positions.

Since the acceleration is, moreover, always contained in the osculating plane of the trajectory, it will be determined by the condition that its osculating plane at each point contains the normal to the surface element that is determined by all of the virtual displacements that relate to this point.

The trajectories thus-determined, which are very analogous to the geodesics of a surface, enjoy the same property of the latter that they have a curvature at each of their points that is less than all other trajectories that are tangent to that point.

For that reason, we call them *trajectories of least curvature*, which then translates into Hertz's expression: *straightest paths*.

Hertz reserved the name of *geodesics* for the trajectories that are determined by the condition that the length between their two points presents a zero variation when one passes to an infinitely-close trajectory that connects the same two points.

Any line of least curvature is obviously a geodesic. However, the converse is true only in the case where the equation of constraint is integrable.

That is because in the contrary case, ∞^1 lines of least curvature and ∞^2 geodesics will pass through an arbitrary point of space since two arbitrary points in space will determine at least one geodesic.

The law of motion of a point that is subject to a condition of the type in question can then be expressed by saying that *the point describes a line of least curvature with a constant velocity.*

Material systems with constraints. – Hertz took a direct approach to this. We believe that by recalling the properties of the motion of the point we can simplify the presentation in what follows.

We let x, y, z denote the coordinates of an arbitrary point of the system. Let m denote its mass, and let ds denote the length of an element of the trajectory.

A *trajectory* of the system is the set of positions that are occupied by the system during a continuous motion.

The *length S* of a trajectory is defined by the formulas:

$$M\ dS^2 = \sum m(dx^2 + dy^2 + dz^2) = \sum m\,ds^2\,,$$

in which one sets:

$$M = \sum m\,.$$

The *velocity V* of the system is defined by the formula:

$$V = \frac{dV}{dt} = \sqrt{\frac{T}{M}}\,,$$

in which T represents the *vis viva*.

Upon indicating the derivative with respect to S by a prime, one will have:

$$\frac{dx}{dt} = x'V\,, \qquad \frac{dy}{dt} = y'V\,, \qquad \frac{dz}{dt} = z'V\,,$$

$$\frac{d^2x}{dt^2} = x''V^2 + x'\frac{dV}{dt} = x''V^2\,, \qquad \frac{d^2y}{dt^2} = y''V^2\,, \qquad \frac{d^2z}{dt^2} = z''V^2$$

in the case where the *vis viva* is constant.

From the fundamental formula that expresses the law of motion:

$$\sum m\left(\frac{d^2x}{dt^2}\delta x + \frac{d^2y}{dt^2}\delta y + \frac{d^2z}{dt^2}\delta z\right) = 0\,,$$

one will then deduce that:

$$\sum m\left(x''\,\delta x + y''\,\delta y + z''\,\delta z\right) = 0\,,$$

in which δx, δy, δz represent an arbitrary virtual displacement of the system.

We have assumed that the equations of constraint have the form:

$$\sum(A\,x' + B\,y' + C\,z') = 0\,.$$

Upon differentiating the left-hand sides with respect to S, one will obtain equations that equate expressions such as $\sum(A\,x'' + B\,y'' + C\,z'')$ to functions of the coordinates x, y, z in such a way that if we consider the trajectories that give rise to the same values x', y', z' for the first derivatives relative to each of the points of the system then we will get equations of constraint for the second derivatives that take the form:

$$\sum (A\delta x'' + B\delta y'' + C\delta z'') = 0 ,$$

i.e., that the variations $\delta x''$, $\delta y''$, $\delta z''$ are subject to the same conditions as the variations δx, δy, δz.

The law of motion can then be expressed by the formula:

$$\sum m (x''\,\delta x'' + y''\,\delta y'' + z''\,\delta z'') = 0$$

or

$$\delta \sum m (x''^2 + y''^2 + z''^2) = 0 .$$

Hertz called the quantity c that was defined by the formula:

$$M\, c^2 = \sum m (x''^2 + y''^2 + z''^2)$$

the *curvature* of a trajectory.

In addition, two trajectories that pass through the same position are called *tangent* there when the derivatives x', y', z' have the same values for both trajectories for each of the points of the system, i.e., when two trajectories are tangent, any material point of the system will describe two tangent trajectories, and the ratio of their velocities will be the same for all points of the system.

The law of motion can be expressed by means of those definitions by saying that a constrained system that is devoid of any other influence *traverses a trajectory of least curvature with a constant velocity* when one says *trajectory of least curvature* to mean a trajectory that presents a curvature at each of the points that comprise it that is less than that of the possible trajectories that are tangent at that position.

Hertz, whose wished to make a didactic presentation of mechanics without employing the usual principles, posed that law as an experimental principle at the head of the second part of his book.

We have preferred to deduce, first of all, the aforementioned law from the principles of d'Alembert and virtual work and show that one will then be naturally led to Hertz's terminology, which is a terminology that we shall now briefly complete.

As is the case for a point, one distinguishes *geodesics* from *trajectories of least curvature*, and those two types of trajectories will coincide when the constraint equations are integrable, i.e., when it is possible to determine the position of the system by means of a certain number of independent parameters.

The *rectilinear* trajectories are the ones whose curvatures are zero, i.e., the ones for which one has:

$$x'' = 0 , \qquad y'' = 0 , \qquad z'' = 0 ,$$

or rather the ones for which the various points of the system describe straight lines, so the spaces that are traversed by all of their points in the same time interval will be proportional.

Two positions determine a rectilinear trajectory, in which one naturally assumes that the material points that comprise the system are not subject to any constraint.

The *distance* between two positions is the length of the rectilinear trajectory that connects the two positions.

That distance R is given by the formula:

$$M\,R^2 = \sum m[(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2] = \sum m r^2 \,.$$

The *angle* ω of two rectilinear trajectories is defined by the following formula:

$$\sqrt{\sum m r^2}\sqrt{\sum m r'^2}\cos\omega = \sum m\, r\, r' \cos\theta$$

or

$$M\,R\,R'\cos\omega = \sum m\, r\, r' \cos\theta \,.$$

One verifies that the curvature of a trajectory is equal to the ratio of the angle between two infinitely-close elements to the length of the corresponding arc.

One defines the *parallelism* of two rectilinear trajectories by the condition that the angle between them is zero.

In other words, the lines that are described by the same material point along the two trajectories will be parallel.

On thus arrives at the notion of *direction.*

Two directions are *rectangular* when the angle between them is equal to $\pi / 2$.

Upon denoting the direction cosines of the displacements of the point mass m along two rectilinear trajectories by α, β, γ and α', β', γ', the condition will be expressed by the formula:

$$\sum m(\alpha\,\alpha' + \beta\,\beta' + \gamma\,\gamma') = 0 \,.$$

∞^{3n-2} rectangular rectilinear trajectories with a given direction pass through a given position.

The notion of *vectorial quantity* includes the notions of directions and length, i.e., it applies to a set of vectors that are each attached to one of the points of the system.

If u, v, w are the components along the coordinate axes of the vector that is attached to a point of m that belongs to the system then the magnitude R of the vectorial quantity will be given by the formula:

$$M\,R^2 = \sum m(u^2 + v^2 + w^2) \,.$$

The set of vectors that represent the velocities of the various points of the system is a vectorial quantity whose magnitude V is given by the preceding formula, namely:

$$M\,V^2 = \sum m\left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2\right],$$

and coincides with the value of the velocity that was given before.

That vectorial quantity itself takes the name of *velocity*.

One will similarly define a vectorial quantity that is composed of the set of vectors that represent the accelerations of the various points and call it the *acceleration*.

The magnitude f of the acceleration will be given by the formula:

$$M\ f^2 = \sum m\left[\left(\frac{d^2x}{dt^2}\right)^2 + \left(\frac{d^2y}{dt^2}\right)^2 + \left(\frac{d^2z}{dt^2}\right)^2\right].$$

One calls a vectorial quantity that has the given direction and a magnitude that is the rectangular projection of the vectorial quantity onto that direction, i.e., the magnitude of the latter multiplied by the cosines of the angle between the two directions, the *component of a vectorial quantity* along a given direction.

Upon decomposing the acceleration of each of the points of the system along the tangent to the trajectory at that point, one will get, by that fact itself, the decomposition of the acceleration of the system into a *tangential component* f_t and a *normal component* f_n .

One easily sees that one has:

$$f_t = \sum \frac{m\,\dfrac{dx}{dt}\,\dfrac{d^2x}{dt^2}}{M\,V} = \frac{dV}{dt} = \frac{d^2S}{dt^2},$$

$$f_n^2 = f^2 - f_t^2 = c^2 V^2, \qquad f_n = cV^2$$

when one takes into account the relations:

$$\frac{d^2x}{dt^2} = x'' V^2 + x' \frac{dV}{dt}, \qquad \frac{d^2y}{dt^2} = y'' V^2 + y' \frac{dV}{dt}, \qquad \frac{d^2z}{dt^2} = z'' V^2 + z' \frac{dV}{dt},$$

$$\sum m(x'^2 + y'^2 + z'^2) = M, \quad \sum m(x'\,x'' + y'\,y'' + z'\,z'') = 0\,.$$

The direction of the vectorial quantity that is characterized by the set of vectors x'', y'', z'' can be assimilated in the direction of the *principal normal* to a curve.

The normal component to the acceleration points along that principal normal.

It is easy to see that in the case where there exist no other forces than the ones that are due to constraints, the law of motion can be expressed thus:

In any position of the system, *the direction of the acceleration is rectangular to all the virtual displacements that relate to that position.*

One can further say that:

The natural motion in the case where there exist no other forces than the ones that are due to constraints is the one that minimizes the magnitude of the acceleration.

It is implicit that the givens for the motion are the position of the system, and the direction and magnitude of its velocity.

Any of those propositions above will suffice to determine the second-order differential elements of the coordinates as functions of those coordinates and their first-order differential elements and will permit one to exhibit the problem of motion in the form of an equation as a result.

Now suppose that the position of the system is determined by means of a certain number of parameters or coordinates $q_1, q_2, \ldots, q_r$.

Start from a position of the system and vary one coordinate q while leaving the others constant. The direction of the trajectory thus-obtained will be called the *direction of the coordinate q* for the position in question.

The magnitude and direction of the velocity of the system at a point are determined completely by its components along the directions of the coordinates, just as the velocity of a point that is subject to move on a fixed surface is determined by its components along the tangents to the curves of the chosen curvilinear coordinates on the surface.

The same thing will not be true for acceleration when the number of coordinates q is less than $3n$, where n is the number of material points.

If one lets T denote the expression for the *vis viva* as a function of the coordinate q and their derivatives q' with respect to time then upon applying the given definitions, one will find that the component f_q of the acceleration f along the coordinate q is given by the formula:

$$M f_q = \frac{d}{dt}\frac{\partial T}{\partial q'} - \frac{\partial T}{\partial q} ,$$

in which the term $\frac{d}{dt}\frac{\partial T}{\partial q'}$ is due to the tangential component of the acceleration and the term $\frac{\partial T}{\partial q}$ is due to its normal component.

The condition that the acceleration must be rectangular with all of the virtual displacements of the system will be written:

$$\sum f_q \,\delta q = 0 ,$$

in which the δq represent an arbitrary virtual displacement.

If the δq are arbitrary then the condition will be written:

$$f_q = 0 \qquad \text{or} \qquad \frac{d}{dt}\frac{\partial T}{\partial q'} - \frac{\partial T}{\partial q} = 0 \qquad (r \text{ equations}).$$

If the δq are subject to some constraint equations of the form:

$$\sum a\,\delta q = 0\,, \qquad \sum b\,\delta q = 0\,, \qquad \ldots$$

then one must have:

$$f_q = \lambda\, a + \lambda' b + \ldots$$

or

$$\frac{d}{dt}\frac{\partial T}{\partial q'} - \frac{\partial T}{\partial q} = \lambda\, a + \lambda' b + \ldots \qquad (r \text{ equations}).$$

Hertz's terminology applies no less fortunately to the case in which the system is subject to forces.

Those forces determine a vectorial quantity that represents, by definition, the *force* that is applied to the material system.

Upon denoting the vectorial quantity to which one gives the name of acceleration by *J*, the general equation of dynamics will express the idea that the *vectorial quantity M J – F is rectangular to all virtual displacements*, where the minus sign indicates an operation on the vectorial quantities whose significance is obvious.

That vectorial quantity can be called the *constraint* (*der Zwang* to Gauss), and one sees that the motion is determined by the condition that *the magnitude of the constraint is a minimum* when the motions that one compares it to exhibit the same velocity in terms of magnitude and direction at the position in question.

One sees that the general propositions of the mechanics of systems with constraints take a particularly simple form when one uses the Hertz terminology.

It is appropriate to observe that an analogous terminology, that is perhaps a little less fortunate, was presented before by Julius Kœnig ([1]).

PART TWO

MECHANICS OF MATERIAL SYSTEMS

In this part of his book, Hertz proposed to construct rational mechanics by means of his hypothesis and his terminology by following a perfectly logical argument.

Indeed, it came down to nothing but suppressing force as a fundamental concept.

The notions of space, time, and mass are supposed to have been acquired already. The constraints between the masses are supposed to be independent of time and subject to the restrictions that were indicated before. Hertz called a material system whose constraints are *internal* a *free* system, i.e., one is interested in only the *relative positions* of the points of the system, or more concisely, one permits any displacement without deformation of the system.

Hertz posed, *a priori*, the following law of motion of a free system as the fundamental law of mechanics:

([1]) JULIUS KŒNIG, "Interpretation der fundamental Gleichungen der Dynamik," Math. Ann. **31** (1888).

Any free system describes a trajectory of least curvature with a constant velocity.

By hypothesis, any system that is not free will belong to a free system, and its motion can consequently be determined from the fundamental law (at least theoretically).

Such is the general economy of the logical edifice that Hertz constructed.

His terminology permitted him to express the properties of motion of free systems in the form of a single law.

His hypothesis on the nature of force, which reduced to only a manifestation of the motion of perceptible or latent masses, permitted him to construct mechanics in terms of only the notions of space, time, and mass.

Hertz easily deduced the Lagrange equations for free systems from the fundamental law, as well as all of the general propositions of mechanics *including the principles of the center of gravity and areas.*

Any material system whose motion is not governed by the fundamental law must be considered to be linked with masses that do not belong to the system.

Let $q_1, q_2, \ldots, q_r$ be the parameters of the partial system in question, and let $p_1, p_2, \ldots, p_s$ be the parameters that determine the positions of the other masses of the total system.

An equation of constraint will have the form:

$$\sum A\,dq + \sum B\,dp = 0\,.$$

A particularly simple case is the one in which the masses that are external to the system q remain fixed, i.e., $dp = 0$.

In that case, the constraint equations express simply the idea that this system is subject to constraints that are independent of time. Although it is not free, it will have its motion ruled by the fundamental law, which is as it should be, since we have started from the usual principles of mechanics and established a law for all systems with constraints, whether free or not, upon which no other forces are exerted besides the ones that are due to constraints, where the word "force" is employed in its usual sense.

It is perhaps not unworthy of interest to point out (notably in the search for the significance of the principles of the center of gravity and areas) that in all cases in nature in which the constraints on a system do not permit all of the displacements without deformation, one can always add some *perceptible* masses to them, i.e., solid, liquid, or gaseous bodies, such that the constraints will become *internal*.

Let us return to the general case.

The term that is due to the constraint considered in the Lagrange equation that relates to the coordinates q will have the form λA.

If one regards the motion of the masses that do not belong to the partial system of q as known then the coefficient λ will be determined in the usual manner by appealing to the constraint equation, which takes the form:

$$\sum A\,dq + C\,dt = 0\,,$$

in which one sets:

$$\sum B \frac{dp}{dt} = C ,$$

and one will easily see that the result agrees with the ones that d'Alembert's principle will give when it is extended to the case in which the constraints are independent of time.

That explains the validity of the latter principle.

The notion of force. – In order to introduce the notion of force, Hertz considered a type of constraint that was capable of being expressed by the equality of one or more parameters in another systems. Hertz called that sort of constraint a *coupling* (*Koppelung*).

If q and p are parameters that are equal in the two systems, respectively, then the constraint equation:

$$q = p \quad \text{or} \quad \delta q - \delta p = 0$$

will give some terms in the Lagrange equations that relate to q and p, respectively, and are equal and opposite in sign in such a way that if Q is the term that is due to the constraint that relates to q, and P is the analogous term in the equation that relates to p then one will have:

$$Q = - P .$$

Here, we have arrived at a crucial point in Hertz's book, namely, the introduction of the notion of force.

At the same time, we fall into darkness. We also believe that the reader could do no better than to cast their eyes upon a literal translation of the text itself, while suppressing only the proofs, as well as certain considerations that are irrelevant to the question that we are addressing.

DEFINITION. – We intend the term "force" to mean the influence that one of two coupled systems exerts upon the other due to the fundamental law.

CONSEQUENCE. – Each force necessarily corresponds to a reaction (*Gegenkraft*).

That is because……………………………………………………………………………………………………...

ANALYTICAL REPRESENTATION OF FORCE. – In agreement with the definition, we can, and would therefore like to, propose that the set of quantities P that are determined for all of the coordinates p must constitute the analytical expression for the force that the system of p exerts on the system of q. The quantities P, or rather Q, are called the components of the form along the corresponding coordinates p or q, or simply the forces along those coordinates.

We establish the agreement with the present terminology in mechanics with that determination, and the necessity of realizing that accord sufficiently justifies the choice that we made from among several possible determinations.

CONSEQUENCE. – The force that a system exerts upon another can be considered to be a vectorial quantity that relates to the second system whose components along the common coordinates are generally non-zero, whose components along the coordinates that are not common are annulled, and finally whose components along the direction that cannot be expressed by the variations of the coordinates employed will remain indeterminate.

SECOND CONSEQUENCE. –The force that a system exerts upon another can just as well be considered to be a vectorial quantity that relates to the first system.

……...

Action and reaction are equal and opposite. One must intend that to means that their components along each of the coordinates that are employed are equal and opposite, and that one considers the force to be a vectorial quantity in one or the other system.

That is because……………………………………………………………………………………………...

REMARK 1. – The preceding proposition corresponds to Newton's *Lex tertia* and is referred to as the principle of reaction. Since its content does not coincide completely with the content of that third law, their exact relationship is as follows:

Newton's law contains our proposition completely, in the view of its author, as the examples that support the law show. However, Newton's law contains more. At least, it is generally applicable to actions at a distance, i.e., to forces that are exerted between two bodies that have no common coordinates.

Nonetheless, our mechanics does not know of such forces. That is why one can, for example, deduce the consequence from our proposition that a planet attracts the Sun with a force that is equal to the one with which the Sun is attracted to the planet, but it is necessary to have some information that is more intrinsic to the nature of the coupling that exists between the two bodies.

REMARK 2. – One might demand to know whether the surplus that principle of reaction presents beyond out proposition might deservedly figure among the fundamental laws of mechanics, or if, in the contrary, the essential part of that principle that is valid in full generality does not consist of our proposition.

As far as the form of it is concerned, the scope of the third law is not expressed clearly in its application to actions at a distance, because if the force and the reaction are applied to different bodies then what one intends the term "opposite direction" to mean is not well-defined. For example, that is what happens when one addresses the reciprocal action between two current elements. As far as the content is concerned, the application of the principle of reaction to the actions at a distance in the usual mechanics obviously represents a fact of experience whose exactness in all cases begins to become doubtful. Thus, one is almost convinced that the reciprocal action between two magnetized particles in motion will satisfy the principle in any case.

A host of questions that one would like to ask Hertz appear.

First of all, what is the physical significance (if one exists) of that sort of constraint that Hertz called a *coupling* of the two systems and that he defined only by an analytical property, namely, the fact that they have common coordinates?

Incidentally, in regard to the law of the equality of action and reaction, Hertz told us that the actions at a distance are the ones that are exerted between two bodies that have no common coordinates, and that his mechanics knows of no such forces.

What sort of incompatibility did Hertz imagine existed between the analytical fact of having common coordinates and the physical fact of acting at a distance?

It is not that one does not perceive the guiding idea in that darkness.

Hertz obviously contemplated the theory of cyclic systems, which he presented by following its brilliant creator, Helmholtz, and more especially, Maxwell's kinetic theory of electrodynamics.

Maxwell's mechanical image, which was also very imprecise, was the *model* for any mechanical explanation for Hertz.

Now, in that theory, one considers a hypothetical material system whose *vis viva* depends upon the coordinates of the perceptible bodies, which amounts to saying that a constraint consists of the fact of having common coordinates (*controllable* coordinates).

We see the origin of the idea of a *coupling* in that, to which Hertz attached much importance, since he could not imagine the notion of force in its absence.

When one reads that statement, it might seem that it is thanks to that idea that one can recover the law of the equality of action and reaction, or at the very least, whatever it is that must replace it.

But that is not so, because the fact that Hertz preserved that law, namely, the principle of the center of gravity and the principle of areas, results immediately from his fundamental hypothesis that all forces are forces of constraint.

Indeed, the forces of constraint do no work under any virtual displacement that is compatible with those constraints, and in particular when one is dealing with a free system, under any displacement without deformation, which constitutes precisely the condition for those forces to be in equilibrium in the sense that the latter work present in the mechanics of undeformable bodies.

The postulate is basically this one:

We do not know how to conceive of constraints that do not permit all displacements without deformation of the set of material masses in question in such a way that the existence of a constraint that prevents a material system from displacing freely without deformation will always demand the presence of material masses that are foreign to the system.

Moreover, the forces of constraint can always be calculated without having recourse to the law of the equality of action and reaction. The latter law will then become completely useless when one assumes, with Hertz, that there exist only forces of constraint.

I add that when one conveniently extends the postulate above, one can *prove* the principles of the center of gravity and areas in the case of arbitrary forces, and in such a general fashion that one can calculate the action that the second one exerts on the first, which constitutes, in total, the role of the law of the equality of action and reaction.

One will undoubtedly conclude, as we did, that the manner by which Hertz introduced the notion of force is clearly not sufficient.

Moreover, we believe that the arbitrariness that persists in the determination of force when one defines it, as Hertz did, by way of certain terms in the Lagrange equations would not be appropriate to a notion that would come to mind naturally apart from any mechanical theory.

Is there not something distasteful in refusing to allow the notion of force the right to exist, although in practice, we have no hesitation about determining the direction of a force and measuring its magnitude?

Certainly, among the objections that are naturally raised when one tries to *define* (but not effectively measure) the magnitude of a force, is it not true that they could also be applied to the more established notions?

For example, consider the notions that are associated with length and displacement without deformation, which one might say serve as the basis for all science.

If one tries to define the equality (or the comparison) of two lengths then one will be reduced to the notion of displacement without deformation. However, the scholars have not ignored the fact that there exist no bodies that displace without deformation. On the contrary, for the ignorant, there exist a great number of them, and it is their crude notion of undeformable body that is at the basis for geometry, which is *pure* science *par excellence*, just the same, and a slightly more subtle analysis might discover some basic physical concepts that are more complicated, which is, in truth, a simple notion, but not because it obtained by the force of abstraction, which it is not, but because it is the entirely unconscious result of our first glance when we turn our attention to nature.

The notion of force, like that of length, is essentially subjective, and it is illusory for one to look for a purely-objective definition.

Cyclic motions. – The study of cyclic motion is due to the ideas that led physicists to identify certain potential energies (heat, electrodynamical potential, etc.) with kinetic energy.

Hertz concluded his book with a presentation of the theory of cyclic systems. Due to the influence that his theory seems to have had on the genesis of Hertz's ideas on mechanics, we believe that we must recall its principles.

Consider a material system whose position depends upon coordinates of two types q and φ that enter into the expression for the *vis viva* of the system only by their derivatives, and the former give rise to a *vis viva* that is negligible with respect to the *vis viva* that corresponds to the latter.

The coordinates φ are called *cyclic*.

When one writes the *vis viva* as a sum:

$$T = T_q + T_{q\varphi} + T_\varphi ,$$

one preserves only the last term and supposes, in addition, that the coefficients of the squares and the products of the derivatives $d\varphi / dt$ depend upon only the coordinates q.

It is clear that one cannot assume with all due rigor that the derivatives of the coordinates q do not enter into the general expression for T. However, one can consider only the motions for which the terms that contain those derivatives are negligible with respect to the terms that contain only the derivatives of the cyclic coordinates.

The Lagrange equation for a coordinate q will have the form:

$$-\frac{\partial T}{\partial q} = P ,$$

and for a coordinate φ:

$$\frac{d}{dt}\frac{\partial T}{\partial \varphi'} = F .$$

Suppose that the cyclic system is *coupled* to a system by the parameters q, i.e., suppose that those parameters q are constantly equal in number to the parameters of the second system.

The term that is due to the motion of the cyclic system in the Lagrange equation relative the parameter of a new system that is considered to be equal to q will be – P or $\partial T / \partial q$, and the work done under a displacement of the latter system will be:

$$d\mathcal{T} = \sum \frac{\partial T}{\partial q}\, dq\ .$$

The expression for T in the two interesting cases will be a function of only the parameters q, namely, in the case where the derivatives $d\varphi / dt = \varphi'$ are constants, and in the case where the quantities $\partial T / \partial \varphi'$ are constants.

In the former case, one has:

$$d\mathcal{T} = dT\ ,$$

and in the latter case:

$$d\mathcal{T} = -\, dT\ .$$

In the former case, the motion is called *isocyclic*. In the second case, it is called *adiabatic*.

In the latter case, one has:

$$\frac{d}{dt}\frac{\partial T}{\partial \varphi'} = 0\ ,$$

i.e., the force exerted on the cyclic system along the coordinate φ is zero.

The system is then called *conservative*.

The kinetic energy T of the cyclic system represents the potential of the forces that are exerted by that system on the system whose parameter is q.

The properties of cyclic systems lead to some interesting formulas that were suggested by the mechanical theories of physical phenomena.

Suppose that the system is monocyclic, i.e., it depends upon only one cyclic coordinate φ.

Let dQ be the work done by the cyclic force, i.e., let:

$$dQ = \frac{d}{dt}\frac{\partial T}{\partial \varphi'}\, d\varphi = \frac{d}{dt}\frac{\partial T}{\partial \varphi'} \times \varphi'\, dt = \varphi'\, d\,\frac{\partial T}{\partial \varphi'}\ .$$

On the other hand, one has:

$$T = \frac{1}{2}\frac{\partial T}{\partial \varphi'}\,\varphi'\ .$$

Hence:

$$\frac{dQ}{T} = 2\,\frac{d\,\dfrac{\partial T}{\partial \varphi'}}{\dfrac{\partial T}{\partial \varphi'}} = d\log\left(\frac{\partial T}{\partial \varphi'}\right)^2 = d\log\left(\frac{\partial T}{\partial \varphi'}\right)^2 .$$

Upon setting:

$$\frac{\partial T}{\partial \varphi'} = \Phi\ ,$$

one will have:

$$\int \frac{dQ}{T} = \log \frac{\Phi^2}{\Phi_0^2} .$$

Upon assuming that the *vis viva T* is proportional to temperature, one will have an expression for entropy.

In these few pages, we proposed to show what appears to us to be important in Hertz's book and leave to the reader the task of reflecting upon the numerous considerations that first inspired its subject and then Hertz's ideas.

To conclude out study, we shall pass along the following appreciation of Hertz's theory that Poincaré expressed ([1]).

> "Although it is certainly interesting, it does not satisfy me entirely, because it depends too much upon hypothesis."
>
> …………………………………………………………………………………………
>
> "Nonetheless, from that fact alone, it is new, so the method of exposition is useful. It forces us to reflect and free ourselves of the old ways of associating ideas. Moreover, we cannot see the monument in its entirety. There is some value in having a new perspective and taking a new viewpoint."

G. COMBEBIAC (Limoges).

([1]) H. POINCARÉ, *loc. cit.*