From O. D. Chwolson, *Traité de physique*, v. 1, trans. by E. Davaux, 2nd French ed., Hermann et fils, Paris, 1912, pp. 328-365.

NOTE

ON THE

DYNAMICS OF A POINT AND AN INVARIABLE BODY

BY

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1. Introduction. – For a long time now, the principles of mechanics, whose role is preeminent in physics, have been the subject of discussions that touch upon their origin, nature, and implications. However, under the influence of the profound criticism to which the principles of mathematical analysis and geometry have been subjected, and due to the remarkable development that has been made in experimental research in our time, the interest in those studies has been renewed, and the aspects of the question have been modified from many points of view.

H. POINCARÉ, who has already discussed the principles of analysis, geometry, and mechanics most incisively ([1]), was assigned to report on the difficulties that appear today in mathematical physics at a conference that was held at St. Louis in 1904 ([2]), notably in the application of the principles of mechanics to natural phenomena.

Those difficulties, which one supposes can be discarded at a given moment, will undoubtedly reappear later on, and one might be permitted to think, with LORD KELVIN ([3]), that they amount, on the one hand, to some ideas about the constitution of the physical world that are too simplistic, and, on the other hand, with H. POINCARÉ ([4]), that they result from an application of the theory of the astronomical universe to all phenomena that is too faithful.

([1]) The majority of the articles of H. POINCARÉ have been reproduced in the following book: H. POINCARÉ. – *La Science et l'Hypothèse*, Paris, 1903, a German translation of which appeared in 1904 with notes by LINDEMANN, and in the recent book: H. POINCARÉ. – *La Valeur de la Science*, Paris, 1905.

([2]) H. POINCARÉ. – *L'état actuel et avenir der la Physique mathématique*, talk presented on 24 September 1904 at the Congress of Art and Science in St. Louis. That remarkable talk, which consisted of a critical review of various principles that are at the basis for physics, first appeared in the *Revue des Idées* on 15 November 1904, and then in the issue for December 1904 of the *Bulletin des Sciences mathématique*, second series, t. XXVIII, pp. 302-324, and finally with slight modifications in *La Valeur de la Science*. An English translation was included in *The Monist* **15** (1905), 1-24. The citations that we shall make refer to *La Valeur de la Science*.

([3]) LORD KELVIN. – *Baltimore Lectures on molecular Dynamics and the wave theory of light*, London, 1904.

([4]) St. Louis talk, pp. 171, *et seq.*

It is not pointless to observe that if the founders of celestial mechanics believed very firmly that the study of discrete ensembles of points would reveal the secrets of nature then they did not perhaps have as much faith in NEWTON's laws of motion. In order to see that, it would suffice to read the rather curious chapter, which seems to have been mostly forgotten at present, in LAPLACE's *Traité de Mécanique celeste*, which he entitled: "*Des lois du movement d'un système de corps, dans toutes the relations mathématiquement possibles entre la force et la vitesse* (1)."

In these survey notes, which CHWOLSON has very graciously authorized us to append to the French translation of his Traité de Physique, we would like to show the reader the difficulties that we just spoke of in a precise manner and to inspire them to take part in the open discussion on the fundamentals of the physical sciences (2).

In order to achieve that goal, we shall attempt to sketch out the general features of an exposition, in which some effort will be made to get around the preceding obstacles. To do that, we shall not abandon any of the older acquisitions, and our tendency will be to establish a theory with a more comprehensive form, as H. POINCARÉ recommended (3).

No matter what the interest might be in such an attempt, it would not perhaps be sufficient to introduce a set of theoretical considerations that would present a certain complexity in this treatise. However, we thought that it would be a rapid means of confronting the reader with some controversial questions and to give them, in an orderly manner, the sources that they can draw upon in order to take part in the arduous task that research of theoreticians and experimenters have imposed in our time.

It will also be less frightening, to make contact with the criticism of H. POINCARÉ (4) or that of É. PICARD (5) of all that seems ruinous at the present bases for natural philosophy, and one will better appreciate the extent to which one can respond to the appeal of DE FREYCINET (6) in favor of the older doctrines. Undoubtedly, one can likewise sense how one can reconcile the

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(1) P.-S. LAPLACE. - *Traité de Mécanique celeste*, T. I, 1re Partie, Livre I, Chap. VI; this chapter occupies pages 74-49 of tome I in LAPLACE's *Œuvres completes*.

(2) Fortunately, the first section of the Société scientifique de Brixelles has taken the initiative in 1904 of making the discussion of the principles of mechanics the order of the day in its future sessions (*Annales de la Société scientifique de Bruxelles*, 18th year, pp. 98). That decision was preceded and followed by a certain number of works that one will find in the preceding collection (T. XVI, XVIII, XIX, XX, XXIV, XXV, XXVI, 1891-1902).

The Société française de Philosphie, which was founded in 1901, has also been engaged in some interesting studies of the same subject on several occasions. In particular, we shall point out the theses of LE ROY, "De la valeur objective des Lois physique" (Bulletin for May 1901) and PAINLEVÉ, "Les axioms de la Mécanique et le principe de causalité" (Bulletin for February 1905).

(3) St. Louis talk, pp. 210.

(4) Books and articles that were cited before.

(5) É. PICARD. – *Quelques réflexions sur la Mécanique, suivies d'une première leçons de Dynamique*, Paris, 1902. That volume, which is taken from the first lesson on dynamics, is the reproduction of the three paragraphs in Chap. II of a report that was written in the context of a universal exposition on 1900. The first two paragraphs appeared in 1901 in the *Bulletin des Sciences mathématiques*, series 2, v. XXV, pp. 17-27. See also the article that É. PICARD wrote in the *Revue générale des Sciences*, t. XV, pp. 1063-1066, in order to present the book by E. MACH, *La Mécanique, Etude historique et critique de son développement*, translation by E. BERTRAND, Paris, Hermann, 1904.

(6) C. DE FREYCINET. – *Essai sur la philosophie des sciences, Analyse, Mécanique*, Paris, 1900. – *Sur les principes de la Mécanique rationelle*, Paris, 1902. Some of the analyses in the latter book were made by MAURICE LÉVY, Journal de Savants (1902), 245-252 and Revue générale des Sciences **13** (1902), pp. 394.

agnostic attitude of the searchers who, like DUHEM ([1]), demanded that quality should return to science and the confidence that other scholars like HELMHOLTZ ([2]), HERTZ ([3]), BOLTZMANN ([4]) have had in the success of constructing of theoretical physics in the deductive manner ([5]).

We shall add a few words on the survey that we would like to present.

We shall adopt the general features and mode of deduction of the *energetic system*. However, from the beginning, we shall give it a broader form than the one that one usually considers. In addition, we shall introduce some motions, among which we will confine ourselves to pointing out the following ones, for the present:

First of all, we shall extend to the entire edifice of mechanics the idea that, whereas we saw a definitive theory at a not-so-distant time, we can only speak of a first approximation today. Therefore, we shall point to the notions of *natural state* and *state infinitely-close to the natural state* everywhere. We shall thus make the notions of *rapid motion* and *slow motion* in dynamics more precise, which were sometimes glimpsed by HELMHOLTZ, and we shall make them play the same role that the notions of *finite deformation* and *infinitely-small deformation* play in statics. As a result, we can show how one can conceive of the *separation of energies* in an energetic system.

In the second place, we shall place the *notion of group* at the basis for mechanics. That notion has already played a significant role in the study of the principles of geometry. It also presents itself in a manner that is worthy of attention in crystallographic theories. For us, it seems that it must take on an importance in the discussion of the fundamentals of the physical sciences that is no less considerable because one can regard them an adequate translation of the idea of measurement due to the invariance that it implies.

Finally, one knows that under the various systems on which physics rests, one introduces continuous media, either *a priori*, as G. GREEN did ([6]) in the theory of light, or as a limiting case of molecular media, for ease of applications ([7]). It would not be pointless to observe that in the former case, the properties that one attributes to the continuous system are always deduced, more or less consciously, from a passage to the limit ([8]). Now, under that passage, it does not seem that one endeavors to define the constitution of the continuous medium from the original model as

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([1]) P. DUHEM. – "L'évolution de la Mécanique," Revue générale des Sciences **14** (1903). It has also appeared separately in print under the same title.

([2]) "Principien der Statik monocyklischer Systeme," J. reine angew Math. **97** (1884), pp. 111-140, 317-336. – "Ueber die physikalische Bedeutung des Princips der kleinsten Wirkung," *ibid.*, **100** (1887), 137-166, 213-222.

([3]) HERTZ. – *Die Prinzipien der Mechanik*, etc., 1894. That book gave rise to the analyses of PICARD (*loc. cit.*) and POINCARÉ (note cited at the end). See also COMBEBIAC, "Les idées de Hertz sur la Mécanique," l'Enseignement mathématique **4** (1902), 247-271, and the D'ADHÉMAR, "Les principes de la Mécanique et les idées de Hertz," Revue des questions scientifique (3) **1** (1902), 173-204.

([4]) L. BOLTZMANN. – *Vorlesungen über die Prinzipe der Mechanik*, I. Teil, 1897; II Teil, 1904.

([5]) STUART MILL. – *A system of Logic, ratiocinative and inductive*, see Chap. XII, "Of the explanation of Laws of Nature," vol. I, 5th edition, 1862, pp. 510-536.

([6]) G. GREEN. – *Mathematical Papers*, London, 1871. See also LORD KELVIN, *loc. cit.*, Lesson 1.

([7]) In particular, consult the exposition that BOUSSINESQ made in his *Leçons synthétiques de Mécanique Générale,* and notably see nos. 4 and 55 (pps. 5 and 68) for what was just said.

([8]) Here, one only needs to confirm, once and for all, the influence that the theories of the one had on those of the others and the importance that the various viewpoints might imply for a parallel development. One will find an example of that parallelism in LORD KELVIN's *Baltimore Lecture*. Each of the twenty lectures of that famous scholar is generally devoted in part to continuous media and in part to molecular dynamics.

completely as possible. For example, if one considers the simplest molecular system, in which one finds small invariable solids, then it will be clear that the passage to the limit can be expressed by means of *material points* only by virtue of hypotheses that must be stated explicitly. In that case, one is more generally led to imagine, in place of the material point, something that we can call a *material trihedron*, which is obtained by completing the notion of material point by the addition of three rectangular directions that issue from that point. A continuous medium is then generated by a moving material trihedron, and the difference between that trihedron and the point occupied by its summit is the same as the difference that the geometers understand to exist between a sphere of radius zero and the center of that sphere. Moreover, it is obvious that the more complex models that are proposed by the molecular theories ([1]) can give rise to corresponding models for continuous media.

In this first Note ([2]), we shall begin our examination of the fundamentals of dynamics ([3]) of the point and the invariable body, while leaving aside the motion of the systems, which will be envisioned in a later note. We shall only speak of the fundamentals of statics, and we shall insist upon the conservation of energy only at the point when the authors study liquids and solids. The dynamics of the point or trihedron, in which one supposes that all of the energy that belongs to the medium in which the motion is produced, will find its place in the volume of this treatise that is dedicated to Electricity and Magnetism.

We shall first recall some notions that relate to the kinematics of an invariable body and to the continuous group of Euclidian displacements.

2. The theory of the moving trihedron and the motion of an invariable body. – Assume that we possess an invariable system: The notion of the motion of a point or the motion of an invariable body is acquired by comparing it with the given invariant system. The most convenient system of comparison consists of a tri-rectangular trihedron *Oxyz* ([4]).

The moving body can itself be referred to a tri-rectangular trihedron $O'x'y'z'$ that is invariably coupled with it. We suppose that this moving trihedron has the same disposition as the trihedron *Oxyz*, and we define it by giving the coordinates a, b, c to O' at each instant t, and the direction cosines α, α', α''; β, β', β''; γ, γ', γ'' of the three directions $O'x'$, $O'y'$, $O'z'$ with respect to the trihedron *Oxyz*. Those direction cosines are coupled by the relations:

([1]) One finds the most interesting ideas on those models in the works of LORD KELVIN (*Mathematical and physical Papers*, vol. I, II, and III, Cambridge, 1882, 1884, 1890.)

([2]) The main results that are contained in this Note were presented to the Academy of Sciences, C. R. Acad. Sci. Paris **140** (3 April 1905), pp. 932.

([3]) From the viewpoint that we shall assume in the Note that is devoted to the fundamentals of *statics*, it will seem better to employ the term *kinetics* instead of the term *dynamics*, as LORD KELVIN and TAIT did in the *Natural Philosophy*. It is regretful that this new terminology has not come into regular use more completely in France.

([4]) We suppose that the axes *Ox*, *Oy*, *Oz* are oriented in such a fashion a rotation of 90° in the positive sense around *Oz* (the sense of left to right for an observer that has their feet at *O* and their head at *z*) takes *Ox* to *Oy*. That convention is made in such a manner as to give an interpretation to the formulas in which rotations appear with the usual representation of a rotation by a vector with three elements (viz., axis, angular velocity, sense). (APPELL, *Traité de Mécanique rationelle*, 1st edition, t. I, no. 3, 6, 41; 2nd edition, t. I, nos. 2, 7, 43).

$$(1)\qquad \begin{cases} \sum \alpha^2 = \sum \beta^2 = \sum \gamma^2 = 1, \\ \sum \alpha\,\beta = \sum \gamma\,\alpha = \sum \alpha\,\beta = 0. \end{cases}$$

and the following one:

$$(2)\qquad \sum \pm \alpha\,\beta'\,\gamma'' = 1\,,$$

which expresses the idea that the two trihedra have the same disposition, and that will bear upon only the interpretation of the rotations.

Consider a point that moves in an absolutely-arbitrary fashion: its coordinates x', y', z' at the instant t, with respect to the corresponding position of the trihedron $O'x'\,y'\,z'$, are coupled to the coordinates x, y, z with respect to the trihedron $Oxyz$ by the relations:

$$(3)\qquad \begin{cases} x = a + \alpha\;x' + \beta\;\,y' + \gamma\;\,z', \\ y = b + \alpha'\,x' + \beta'\,y' + \gamma'\,z', \\ z = c + \alpha''\,x' + \beta''\,y' + \gamma''\,z'. \end{cases}$$

The coordinates x', y', z' are not necessarily constant.

The velocity of the point in question at the instant t with respect to the comparison trihedron $Oxyz$ will have components v_x, v_y, v_z along the axes of that trihedron that one can calculate from the formulas:

$$(4)\qquad v_x = \frac{dx}{dt}, \qquad v_y = \frac{dy}{dt}, \qquad v_z = \frac{dz}{dt}.$$

Suppose that one has replaced x, y, z in those formulas by their values in (3), and one then looks for the projections of the velocity considered onto the axes $O'x'$, $O'y'$, $O'z'$ at those positions at the instant. One immediately finds that if one sets:

$$(5)\qquad \begin{cases} \xi = \alpha\,\dfrac{da}{dt} + \alpha'\,\dfrac{db}{dt} + \alpha''\,\dfrac{dc}{dt}, \\[2ex] \eta = \beta\,\dfrac{da}{dt} + \beta'\,\dfrac{db}{dt} + \beta''\,\dfrac{dc}{dt}, \\[2ex] \zeta = \gamma\,\dfrac{da}{dt} + \gamma'\,\dfrac{db}{dt} + \gamma''\,\dfrac{dc}{dt}, \end{cases} \qquad (6)\qquad \begin{cases} p = \sum \gamma\,\dfrac{d\beta}{dt} = -\sum \beta\,\dfrac{d\gamma}{dt}, \\[2ex] q = \sum \alpha\,\dfrac{d\gamma}{dt} = -\sum \gamma\,\dfrac{d\alpha}{dt}, \\[2ex] r = \sum \beta\,\dfrac{d\alpha}{dt} = -\sum \alpha\,\dfrac{d\beta}{dt} \end{cases}$$

then one will have:

$$(7)\qquad \begin{cases} v'_x = \xi + q\,z' - r\,y' + \dfrac{dx'}{dt}, \\ v'_y = \eta + r\,x' - p\,z' + \dfrac{dy'}{dt}, \\ v'_z = \zeta + p\,y' - q\,x' + \dfrac{dz'}{dt}, \end{cases}$$

which are fundamental formulas that dominate all of geometrical kinematics.

In particular, if one applies those formulas to the points of the invariable body that is coupled with the trihedron $O'x'y'z'$ then the coordinates x', y', z' will be constants, and formulas (7) will become the following:

$$(8)\qquad v'_x = \xi + q\,z' - r\,y', \quad v'_y = \eta + r\,x' - p\,z', \quad v'_z = \zeta + p\,y' - q\,x',$$

which express the idea that the velocity of an arbitrary point of the body at the instant t is the geometric sum of a velocity (ξ, η, ζ), which is the same for all points of the body and equal to the velocity of the point O' and a velocity that is due to a rotation ([1]) whose representative vector that issues from the point O' has p, q, r for its projections onto the axes $O'x'$, $O'y'$, $O'z'$.

One deduces from that result that the projections v_x, v_y, v_z of the velocity of a point of the invariable body with respect to the trihedron $Oxyz$ onto the axes of that trihedron are determined by the formulas:

$$(8')\qquad v_x = l + \omega_2\,z - \omega_3\,y, \quad v_y = m + \omega_3\,x - \omega_1\,z, \quad v_z = n + \omega_1\,y - \omega_2\,x,$$

in which ω_1, ω_2, ω_3 denote the projections of representative vector of the instantaneous rotation onto Ox, Oy, Oz, and one sets:

$$l = \frac{da}{dt} - \omega_2\,z_0 + \omega_3\,z_0, \qquad m = \frac{db}{dt} - \omega_3\,x_0 + \omega_1\,z_0, \qquad n = \frac{dc}{dt} - \omega_2\,z_0 + \omega_2\,y_0.$$

Furthermore, one can deduce formulas (4) from those formulas directly in the present case.

We shall not develop the theory of the motion of an invariable body any further ([2]). In conclusion, we shall simply indicate, without proof, how the state of the velocity at the instant t is the same as if the body were animated with a *helicoidal motion* at that instant around a line ([3]) that is called the *instantaneous axis of rotation and sliding*. The equations of that line:

([1]) Due to that result, in what follows, we shall give the name of the *instantaneous rotation* of the trihedron $O'x'y'z'$ to the vector that issues from O' and whose projections onto $O'x'$, $O'y'$, $O'z'$ are p, q, r.

([2]) For the development of that theory, see G. KŒNIGS, *Leçons de Cinématique, Cinématique théorie*, Paris, 1897; P. APPELL, *Traité de Mécanique rationelle*, t. I.

([3]) A trihedron $O'x'y'z'$ and a solid body that is invariable with it are said to be animated with a *continuous helicoidal motion* around a line D that is invariably coupled with the comparison trihedron and to which the axis $O'z'$

$$\frac{\xi + q\,z' - r\,y'}{p} = \frac{\eta + r\,x' - p\,z'}{q} = \frac{\zeta + p\,y' - q\,x'}{r}$$

at the instant t with respect to the corresponding position of the trihedron is $O'x'\,y'\,z'$ are obtained by seeking the locus of the points of the body whose velocity is parallel to the direction of the instantaneous rotation or rather the points of the body whose velocity at the instant t is a minimum.

We further add that knowing the quantities ξ, η, ζ, p, q, r as functions of time will permit us to completely define the motion of the trihedron $O'x'\,y'\,z'$ with respect to a trihedron with which it coincides at a given epoch.

3. The groups of motions and displacements. The continuous group of Euclidian displacements. – The word *motion* takes on a new significance in the theory of groups ([1]) and in crystallography ([2]). That amounts to saying that, on the one hand, one ignores time, and on the other hand, under the influence of the notion of symmetry that is introduced in crystallography, one extends the simple geometric sense to which one is first led.

Consider formulas (3), in which a, b, c, α, β, …, γ'' are supposed to be given and to verify the relations (1) and (2). We can interpret those formulas, no longer by supposing that x, y, z and x', y', z' refer to the same point, but to a transformation that makes the point whose coordinates with respect to $Oxyz$ are x', y', z' correspond to a new point whose coordinates with respect to the same trihedron $Oxyz$ are x, y, z. That transformation is called a *proper motion* or *motion of the first type*, or even *displacement*.

If one now supposes that α, β, …, γ'' always satisfy the relations (1), but verify the relation $\sum \pm \alpha\,\beta'\,\gamma'' = -1$, instead of (2), then formulas (3) will again define a transformation that makes the point (x, y, z) correspond to (x', y', z'). That second transformation is called an *improper motion*, or *motion of the second type*. An improper motion is obviously a *displacement* followed by a *reflection* through a point or a plane, which is what one calls the transformation that replaces each point with its symmetric image relative to the point or plane, resp. The transformations that replace the point (x', y', z') with the point $(-x', -y', -z')$ or the point $(x', y', -z')$, respectively, are indeed reflections through the point O or the plane Oxy.

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will remain attached when a point on the axis $O'x'$ describes an ordinary helix with D for its axis. Under that helicoidal motion, which is realized by a screw that moves along its thread, everything happens at each instant as if the moving body were animated with a rotation around D and a translation $h\omega$ (h is a constant) along D. See G. KŒNIGS, *Leçons de Cinématique*, pp. 103-105.

([1]) J.-A. DE SÉGUIER. – *Eléments de la théorie des groupes abstraits*, Paris, 1904; consult the note entitled: *Sur les groupes de mouvements*, which occupy pages 150-159 of that book.

([2]) A. SCHOENFLIES, *Krystallsysteme und Krystallstructure*, Leipzig, 1891: H. HILTON, *Mathematical Crystallography and the theory of groups of movements*, Oxford, 1903. In § **7** of Chap. 11, pp. 28, of the latter author, the word *movement* is employed in the most general sense that was just indicated, and what we, with DE SÉGUIER, call a *motion of the first type* and *motion of the second type* are what he called *operation of the first sort* and *operation of the second sort*, resp.

Consider two motions m and m_1 that are defined by formulas (3) and (1) in the first case and by the formulas:

$$(3') \qquad \left\{ \begin{aligned} x_1 &= a_1 + \alpha_1 x_1' + \beta_1 y_1' + \gamma_1 z_1' , \\ y_1 &= b_1 + \alpha_1' x_1' + \beta_1' y_1' + \gamma_1' z_1' , \\ z_1 &= c_1 + \alpha_1'' x_1' + \beta_1'' y_1' + \gamma_1'' z_1' \end{aligned} \right.$$

in the second, in which α_1, α_1', …, γ_1'' will verify the relations (1) if one substitutes them for α, α', …, γ'', respectively. The first motion makes each point M (x', y', z') correspond to a point M (x, y, z). The second motion makes the point M correspond to a point M_1 whose coordinates (x_1, y_1, z_1) are calculated by formulas (3′) when one replaces x_1', y_1', z_1' in the right-hand side with x, y, z, respectively. If one replaces x_1', y_1', z_1' in formulas (3′) with the expressions (3) for x, y, z, respectively, then one will get formulas that define a transformation that one can call the *composed transformation* or *resultant*, and also the *product* of μ and μ_1, which will be the component transformations. Now, one sees immediately that the new transformation is again a motion: One expresses that by saying that when one composes the motions by performing them in succession, they will form a *group* that one calls the *group of motions*.

We remark that the *inverse* transformation of a given motion, i.e., the transformation that will reduce the figure that is transformed by the latter motion to the original figure when it is applied, will be defined by solving equations (3), and it will once more be a motion of the same type as the motion that one considers. As a result, the group of motions will contain the transformation that is called the *identity transformation*, or *unity* transformation, which is obtained by the product of two transformations that are inverse to each other.

One likewise sees that when one performs displacements in succession, they will also form a group that will be called the *group of displacements:* One says that the latter is contained in the former or that it is a *subgroup* of the former. The inverse transformation to a displacement is again a displacement, as one has observed for the motions: Therefore, the group of displacements also contains the identity transformation.

In addition to the preceding two groups, one also applies the name of a *group of motions* to any set of motions such that the *product* of two arbitrary motions of the set, as defined before, is again a motion of the same set. A group of motion is called *improper* or *proper* according to whether it does or does not contain an improper motion, respectively. In the second case, one can say that one has a *group of displacements* ([1]). The proper motions of an improper group obviously form a proper group that is *contained* in the former.

One can propose to put an arbitrary motion into a *reduced form* that is the resultant of the composition of particular motions. To that effect, one introduces the notions of rotation, translation, helicoidal displacement, reflection, improper rotation, and improper translation. The translation and rotation do not need to be defined. They are the transformations that correspond to the operations of that name in kinematics. The helicoidal displacement is the product of a translation and a rotation. It can be realized by supposing that the translation is rectilinear and

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([1]) The groups of displacements were envisioned and determined in a general fashion by JORDAN (C. R. Acad. Sci. Paris **65** (1867), 229-232; Annali di Matematica (2) **2** (1868), pp. 167-215, 322-348.

parallel to the axis of rotation. It then corresponds, to a certain degree, to the helicoidal motion of kinematics. The improper rotation is the product of a reflection in a plane and a rotation whose axis is normal to that plane. The improper translation is the product of a reflection in a plane and a translation parallel to that plane. One easily establishes that any motion has a *reduced form* that is either a helicoidal displacement or an improper rotation or an improper translation. In particular, any displacement will have a reduced form that is a helicoidal displacement.

We complete the preceding notions with a note in regard to crystallography, for which they will serve as preparatory: In conclusion, we introduce the motion of a *continuous group* ([1]) *of Euclidian displacements*, which is more of interest in the present note, in particular, and which we just reduced to the kinematical notion of motion.

Consider the group of Euclidian displacements. We can suppose that we express the nine cosines $\alpha, \ldots, \gamma''$ of formulas (3) in terms of three independent parameters, for example, the three parameters of OLINDE RODRIGUES, and those formulas (3) will then have the form:

$$\begin{aligned} x &= f(x', y', z', a_1, a_2, a_3, a_4, a_5, a_6)\,, \\ y &= \varphi(x', y', z', a_1, a_2, a_3, a_4, a_5, a_6)\,, \\ z &= \psi(x', y', z', a_1, a_2, a_3, a_4, a_5, a_6)\,, \end{aligned}$$

in which $a_1, a_2, a_3, a_4, a_5, a_6$ are six parameters, and f, φ, ψ are continuous functions of the letters that appear in them. Instead of directing our attention to simply the various transformations and their combinations, as before, we suppose the notion of continuity, in addition, by varying the parameters in a continuous fashion. We arrive at the motion of the *continuous group of Euclidian displacements*. The number of parameters $a_1, a_2, \ldots, a_6$ cannot be reduced to a lower one, so one says that they are *essential*, and that the group considered is a *continuous group with six parameters* that is composed of the set of ∞^6 Euclidian displacements.

Let us return to the qualifier of *continuous* in order to make it more precise and to attach it to the notion of *infinitesimal transformation* of the group of Euclidian displacements, which will be paramount in what follows. Suppose that the six independent parameters that figure in the transformation formulas are continuous functions of a variable t, or what amounts to the same thing, that the symbols $a, b, c, \alpha, \ldots, \gamma''$ in formulas (3) represent continuous functions of a variable t that verify the relations (1) and (2). Observe that a displacement μ_2 that corresponds to the value t_2 of t can be deduced from the displacement μ_1 that corresponds to the value t_1 of t by superposing μ_1 with a displacement that is nothing but the product of the inverse of μ_1 with μ_2. Under those conditions, we can consider the displacement that corresponds to the value t' of t to be the result of the superposition of the displacement that corresponds to the value t with a sequence of

([1]) The notion of continuous group in its general form was developed by SOPHUS LIE, which was devoted to numerous works. One will find French presentations of them that serve to introduce the research of SOPHUS LIE in the *Traité d'Analyse* of PICARD, t. III, Chap. XVII, § **1**, pp. 492-508, and in the following book: G. VIVANTI, *Leçons élémentaires sur la théorie des groupes de transformations*, Paris, 1904. As far as the continuous groups of displacements are concerned, one might consult: SOPHUS LIE, *Vorlesungen über Differentialgleichungen mit bekannten infinitesimalen Transformationen*, pp. 15-17. *Vorlesungen über continuierliche Gruppen*, etc., pp. 100-112, pp. 666-716.

displacements that realize the displacements that successively realize the displacements that correspond to values of the parameter t that are found between t' and t by the addition of their effects. Now, suppose that the number of intermediate values of t increase indefinitely, while the difference between two consecutive values tends to zero. In addition, associate t with time, and consider an invariable system of points that are subject to the transformations successively. We then recover the kinematical notion of motion.

In addition, that will lead us to envision the transformation that makes the point (x, y, z) correspond to the point $(x + \delta x, y + \delta y, z + \delta z)$, in which δx, δy, δz denote the principal parts of the increases that the coordinates of the point (x, y, z) submit to when the value of t that corresponds to those coordinates passes to the infinitely-close value $t + \delta t$, at which one imagines the corresponding point to be. That transformation, which one calls, with SOPHUS LIE, an *infinitesimal transformation* of the group of Euclidian displacements, is defined by virtue of the relations (8′), by the formulas:

$$\delta x = (l + \omega_2\, z - \omega_3\, y)\, dt\,, \qquad \delta y = (m + \omega_3\, x - \omega_1\, z)\, dt\,, \qquad \delta z = (n + \omega_1\, y - \omega_2\, x)\, dt\,. \tag{9}$$

The application of a sequence of such infinitesimal transformation whose number increases indefinitely will permit us to reconstitute an arbitrary transformation of the group in the limit. Moreover, without going into the analytical details, that remark will suffice here to explain why in the search for expressions that remain unaltered by the transformation of the group of Euclidian displacements, we confine ourselves to considering the most-general infinitesimal transformation of that group.

In conclusion, we remark that along with the continuous of Euclidian displacements, we shall further consider its subgroups. We shall have occasion to meet up with them in some later notes.

4. The action for a point motion, considered to be an invariant under the group of Euclidian displacements. – Consider a function W of *two infinitely-close positions* of a point in motion with respect to a well-defined comparison trihedron $Oxyz$, i.e., a function of the coordinates x, y, z of the point at the instant t and their first derivatives dx / dt, dy / dt, dz / dt. We propose to determine what the form of W must be in order for it to remain *invariant* under any infinitesimal transformation of the group of Euclidian displacements, i.e., in order for its variation δW to be zero when x, y, z experience the infinitely-small variations δx, δy, δz that are defined by formulas (9), in which $l, m, n, \omega_1, \omega_2, \omega_3$ are six arbitrary constants, and δt is an infinitely-small quantity ([1]):

$$\delta W = \frac{\partial W}{\partial x}\delta x + \frac{\partial W}{\partial y}\delta y + \frac{\partial W}{\partial z}\delta z + \frac{\partial W}{\partial \frac{dx}{dt}}\delta\frac{dx}{dt} + \frac{\partial W}{\partial \frac{dy}{dt}}\delta\frac{dy}{dt} + \frac{\partial W}{\partial \frac{dz}{dt}}\delta\frac{dz}{dt}\,,$$

and from formulas (9):

([1]) Compare SOPHUS LIE. – *Vorlesungen über contiuierliche Gruppen mit geometrischen und anderen Anwendungen*, pp. 675.

$$\delta \frac{dx}{dt} = \frac{d\,\delta x}{dt} = \left(\omega_2 \frac{dz}{dt} - \omega_3 \frac{dy}{dt} \right) \delta t\,,$$
$$\delta \frac{dy}{dt} = \frac{d\,\delta y}{dt} = \left(\omega_3 \frac{dx}{dt} - \omega_1 \frac{dz}{dt} \right) \delta t\,,$$
$$\delta \frac{dz}{dt} = \frac{d\,\delta z}{dt} = \left(\omega_1 \frac{dy}{dt} - \omega_2 \frac{dx}{dt} \right) \delta t\,,$$

hence:

$$\delta W = \left[l \frac{\partial W}{\partial x} + m \frac{\partial W}{\partial y} + n \frac{\partial W}{\partial z} + \omega_1 \left(y \frac{\partial W}{\partial z} - z \frac{\partial W}{\partial y} + \frac{dy}{dt} \frac{\partial W}{\partial \frac{dz}{dt}} - \frac{dz}{dt} \frac{\partial W}{\partial \frac{dy}{dt}} \right) + \cdots \right] \delta t\,.$$

The variation δW must be zero, no matter what the constants $l, m, n, \omega_1, \omega_2, \omega_3$ might be, so we will get:

$$\frac{\partial W}{\partial x} = 0\,, \qquad \frac{\partial W}{\partial y} = 0\,, \qquad \frac{\partial W}{\partial z} = 0\,,$$

$$\frac{dy}{dt} \frac{\partial W}{\partial \frac{dz}{dt}} - \frac{dz}{dt} \frac{\partial W}{\partial \frac{dy}{dt}} = 0\,, \qquad \frac{dz}{dt} \frac{\partial W}{\partial \frac{dx}{dt}} - \frac{dx}{dt} \frac{\partial W}{\partial \frac{dz}{dt}} = 0\,, \qquad \frac{dx}{dt} \frac{\partial W}{\partial \frac{dy}{dt}} - \frac{dy}{dt} \frac{\partial W}{\partial \frac{dx}{dt}} = 0\,.$$

If we then *suppose that the point* (x, y, z) *can describe all possible trajectories* then we will arrive at some identities that are verified by the function W of the six arguments $x, y, z, dx / dt, dy / dt, dz / dt$ and imply the conclusion that W must not depend upon x, y, z and it must be a function of only the expression:

$$v^2 = \left(\frac{dx}{dt} \right)^2 + \left(\frac{dy}{dt} \right)^2 + \left(\frac{dz}{dt} \right)^2 . \tag{10}$$

In place of the function v^2, we can introduce a function v here that we suppose to be continuous, like x, y, z, and their derivatives, and which is defined by the relation (10). If the right-hand side of (10) is not annulled in the interval that is defined by the two values of t then such a function v will be defined in that interval by specifying the value that it takes for a particular value of t. If the right-hand side of (10) is annulled in the interval that one considers then that will no longer be true. In any case, we will get an example one such function v by considering the velocity of the point on its trajectory, which is defined in the usual way, after making a choice of a *sense of traversal* ([1]).

([1]) A first immediate advantage of introducing the function v is the possibility of interpreting values with different signs that might be introduced for v in what follows.

Having thus chosen a function v that is deduced from the motion of the point ([1]), we can imagine, under the name of a *material point*, a point to which we associate a function $W(v)$ that we call the *Euclidian action* on the point *at the instant t*. The product $W(v)\,dt$ of W with dt, which we call the *elementary Euclidian action relative to the time interval dt*, is also an *invariant* with regard to the group of Euclidian motions. As a particular case, it includes the invariant that, when imagined from the geometric viewpoint by the name of *linear element*, is produced by considering the distance between two infinitely-close points of a curve.

Just as the *fundamental notion of Euclidian geometry is the linear element* $ds = \sqrt{dx^2+dy^2+dz^2}$, the elementary Euclidian action can be considered to be the fundamental notion of dynamics. We say "Euclidian action" because it is clear that we can establish the notion of a non-Euclidian action in an entirely analogous way, as RIEMANN did for the linear element of geometry ([2]). We shall pass over that order of ideas entirely, which has no immediate interest.

5. The Hamiltonian method and the variable action. Definitions of the quantity of motion, external force, work, impulse, and kinetic energy. – One of the most fruitful methods of theoretical physics is the *Hamiltonian method of variable action* ([3]). We shall not employ it here in the usual way, in parallel to the *principle of least action*. We shall appeal to it in order to define the quantity of motion, the external force, the work, and the kinetic energy.

With the hypotheses on the continuity of functions that we have made here, at least implicitly, we can say that the length of a curve between two points M_0 and M_1 is, by definition, the integral:

$$\int_{M_0}^{M_1} \sqrt{dx^2+dy^2+dz^2}\ .$$

Similarly, if we consider the trajectory of a point in motion then we can say that due to the close analogy that was indicated above between the linear element and elementary action, the action A from the instant t_0 and the instant t_1, is by definition, the integral:

$$A = \int_{t_0}^{t_1} W\,dt\ .$$

([1]) The function v must be associated with not only the geometric consideration of the trajectory, but also the *motion* of the point.

([2]) RIEMANN. – *Ueber die Hypothesen, welche der Geometrie zu Grunde liegen*, 1854. The translation of J. HOUËL, which was published in 1870 in the Annali di Matematica, (3), t. III, pp. 309-326, was inserted in the French translation of *Œuvres mathématique de RIEMANN*, Paris, 1898.

([3]) W. R. HAMILTON. – "On a general method in dynamics," Trans. London Phil. Soc., Part II of (1834), 247-308, *ibid.* Part I of (1835), 95-144. See also: W. THOMSON and TAIT. – *Treatise on Natural Philosophy*, vol. I, Part I, no. 326, pp. 337. G. KIRCHHOFF. – *Vorlesungen über mathematische Physik, Bd. I. Mechanik*, 4th ed., Leipzig, 1897. J.-J. THOMSON. – "On some Applications of Dynamical Principles to Physical Phenomena," Trans. London Phil. Soc., vol. **176**, Part II (1885). HELMHOLTZ and HERTZ (articles and work cited before).

If the coordinates x, y, z of each of the points of the trajectory submit to arbitrary variations δx, δy, δz, which are functions of t, then the action that a variation δA will experience will have the value:

$$\delta A = \int_{t_0}^{t_1} \frac{1}{v}\frac{dW}{dv}\left(\frac{dx}{dt}\cdot\frac{d\,\delta x}{dt}+\frac{dy}{dt}\cdot\frac{d\,\delta y}{dt}+\frac{dz}{dt}\cdot\frac{d\,\delta z}{dt}\right)dt$$

by virtue of the relation (10). Integrate by parts and set:

$$(11)\qquad F = \frac{dW}{dv}\frac{1}{v}\frac{dx}{dt}, \qquad G = \frac{dW}{dv}\frac{1}{v}\frac{dy}{dt}, \qquad H = \frac{dW}{dv}\frac{1}{v}\frac{dz}{dt},$$

$$(12)\qquad X = \frac{dF}{dt}, \qquad Y = \frac{dG}{dt}, \qquad Z = \frac{dH}{dt}.$$

We will have:

$$(13)\qquad \delta A = \left[F\,\delta x + G\,\delta y + H\,\delta z\right]_{t_0}^{t_1} - \int_{t_0}^{t_1}(X\,\delta x + Y\,\delta y + Z\,\delta z)\,dt\,.$$

We give the name of the *quantity of motion* at the instant t to the vector (F, G, H) that issues from the point (x, y, z) and the name of *external force* at that point to the vector (X, Y, Z) that issues from the same point: F, G, H are the projections of a vector that one obtains by multiplying the vector that represents the velocity by the expression $\frac{1}{v}\frac{dW}{dv}$. Finally, we say that the expression $X\,\delta x + Y\,\delta y + Z\,\delta z$ is the work done by the external force relative to the infinitely-small displacement (δx, δy, δz) that is applied to the point (x, y, z) and that $F\,\delta x + G\,\delta y + H\,\delta z$ is *work* done by the quantity of motion.

Parallel to the elementary action, we can now consider the vector ($X\,dt$, $Y\,dt$, $Z\,dt$), which we call the *elementary impulse* performed by the force (X, Y, Z) during the infinitely-small time dt, and parallel to the action, the vector $\left(\int_{t_0}^{t_1} X\,dt, \int_{t_0}^{t_1} Y\,dt, \int_{t_0}^{t_1} Z\,dt\right)$ will be called the *impulse* performed by that force during the time interval (t_0, t_1). Since δA must be identically zero, by virtue of the *invariance of the action in the group of Euclidian displacements*, when δx, δy, δz are given by formulas (9), and since that must be true for any value of the constants l, m, n, ω_1, ω_2, ω_3, we conclude that:

$$\left[F\right]_{t_0}^{t_1} = \int_{t_0}^{t_1} X\,dt, \qquad \left[G\right]_{t_0}^{t_1} = \int_{t_0}^{t_1} Y\,dt, \qquad \left[H\right]_{t_0}^{t_1} = \int_{t_0}^{t_1} Z\,dt,$$

$$\left[yH - zG\right]_{t_0}^{t_1} = \int_{t_0}^{t_1}(yZ - zY)\,dt, \qquad \left[zF - xH\right]_{t_0}^{t_1} = \int_{t_0}^{t_1}(zX - xZ)\,dt,$$

$$\left[xG - yF\right]_{t_0}^{t_1} = \int_{t_0}^{t_1}(xY - yX)\,dt\,.$$

In other words, *the impulse is a vector that measures the geometric variation of the quantity of motion, and the limit of the sum of the moments of the elementary impulses with respect to an axis is equal to the geometric variation of the moment of the quantity of motion with respect to that axis.*

Observe, moreover, that if one supposes that x, y, z are known as functions of time then after all of the calculations have been completed, formulas (11) and (12) will give F, G, H, X, Y, Z as functions of t. However, by virtue of the formulas that determine x, y, z as functions of t, one can obviously imagine that one has expressed F, G, H, X, Y, Z in terms of t, x, y, z, and their derivatives dx / dt, …, up to whatever order that one chooses. Hence, if one imagines a question in which X, Y, Z, for example, are given then one can imagine that the given values of X, Y, Z include not only t, but also x, y, z, and the derivatives of the latter.

Form the expression $X\,dx + Y\,dy + Z\,dz$. It will immediately become:

$$X\,dx + Y\,dy + Z\,dz = d\left(v\frac{dW}{dv} - W\right). \tag{13'}$$

Set:

$$E = v\frac{dW}{dv} - W .$$

We say that E is the *kinetic energy* of the point in motion, and we see that, as in classical dynamics, *the variation of the energy during an arbitrary finite time interval is equal to the work done by the external force during that interval.*

6. Normal form for the equations of motion when X, Y, Z **are given functions of** t, x, y, z. **The Hamiltonian, H. Poincaré's Maupertusian action, and the various forms for the principle of least action**. – In conformity with the suggestions in the preceding section, suppose that X, Y, Z are given functions of t, x, y, z: Equations (11) and (12) can be considered to be a system of differential equations that define the six unknowns x, y, z, F, G, H. Put that system into the *normal form*, i.e., ([1]), solve it for the derivatives dx / dt, …, dH / dt of the unknowns. To that effect, observe that v is defined as a function of F, G, H by the relation:

$$\left(\frac{dW}{dv}\right)^2 = F^2 + G^2 + H^2 ,$$

and that, on the other hand, if we substitute the value of v that we obtain in $E = v\dfrac{dW}{dv} - W$ then we will have a function of F, G, H that depends upon the last three quantities only by the intermediary of the function V that is defined by the formula ([2]):

([1]) C. JORDAN. – *Cours d'analyse de l'École Polytechnique*, t. III, pp. 4.
([2]) Here, one can repeat what was said about v in § **4** in the context of V.

$$V^2 = F^2 + G^2 + H^2 ,$$

and whose total differential is ([1]):

$$\delta E = \frac{v}{\frac{dW}{dv}} (F\,dF + G\,dG + H\,dH) .$$

The normal form of the system considered is then:

$$\frac{dx}{dt} = \frac{\partial E}{\partial F}, \qquad \frac{dy}{dt} = \frac{\partial E}{\partial G}, \qquad \frac{dz}{dt} = \frac{\partial E}{\partial H},$$
$$\frac{dF}{dt} = X, \qquad \frac{dG}{dt} = Y, \qquad \frac{dH}{dt} = Z .$$

We have then supposed that, by virtue of the formulas that define *x*, *y*, *z* as functions of time, we can express *X*, *Y*, *Z* as functions of *t*, *x*, *y*, *z*. That is possible in an infinitude of ways, and one can always choose the new forms of *X*, *Y*, *Z* in such a fashion that they are the partial derivatives $\frac{\partial U}{\partial x}$, $\frac{\partial U}{\partial y}$, $\frac{\partial U}{\partial z}$, respectively, of the same function *U*, which is or is not independent of time *t*. Suppose that this is true, and let $\mathcal{H}$ denote the function of *x*, *y*, *z*, *F*, *G*, *H* (and of *t*, if *U* depends upon *t*) that is defined by the formula:

$$\mathcal{H} = E - U .$$

The preceding system will then take the form:

$$\frac{dx}{dt} = \frac{\partial \mathcal{H}}{\partial F}, \qquad \frac{dy}{dt} = \frac{\partial \mathcal{H}}{\partial G}, \qquad \frac{dz}{dt} = \frac{\partial \mathcal{H}}{\partial H},$$
$$\frac{dF}{dt} = -\frac{\partial \mathcal{H}}{\partial x}, \qquad \frac{dG}{dt} = -\frac{\partial \mathcal{H}}{\partial y}, \qquad \frac{dH}{dt} = -\frac{\partial \mathcal{H}}{\partial z} .$$

We then have the equations of motion that define a generalization of those of HAMILTON. If we suppose, in particular, that the new forms for *X*, *Y*, *Z* are chosen in such a way that *t* does not appear in them and they are the partial derivatives of a function *U* of *x*, *y*, *z* (as is always possible) then the latter equations will bear the name of *canonical equations* in a more specialized sense ([2]).

Upon recalling the remarkable treatise of POINCARÉ ([3]), it is easy to recover the various forms of the *principle of least action*.

In order to get POINCARÉ's *Hamiltonian action*, one must define:

([1]) We preserve the symbol *E* for the expression as a function of *F*, *G*, *H*.

([2]) C. JORDAN. – *Cours d'Analyse de l'Ecole Polytechnique*, t. III, pp. 51. H. POINCARÉ, *Les Nouvelles méthodes de la Mécanique céleste*, t. I, pp. 8; same author, *Leçons de Mécanique céleste*, t. I, pp. 1.

([3]) H. POINCARÉ. – *Les Nouvelles méthodes de la Mécanique céleste*, t. III, pp. 249 *et seq.*

$$-(E-U)+ F\frac{\partial E}{\partial F}+G\frac{\partial E}{\partial G}+H\frac{\partial E}{\partial H},$$

and if one takes into account the fact that:

$$F\frac{\partial E}{\partial F}+G\frac{\partial E}{\partial G}+H\frac{\partial E}{\partial H} = v\frac{\partial W}{\partial v}$$

then one will come down to the integral:

$$\int_{t_0}^{t_1}(W+U)\,dt,$$

which served as our point of departure, by definition.

In order to get POINCARÉ's *Maupertuisian action*, one must express:

$$F\frac{\partial E}{\partial F}+G\frac{\partial E}{\partial G}+H\frac{\partial E}{\partial H}$$

as a function of $\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}$, which gives $v\frac{\partial W}{\partial v}$. One then proceeds in a symmetric fashion and replaces the variation t with the arc-length of the trajectory, which one can suppose verifies the relation $\frac{ds}{dt} = v$. One finally gets:

$$\int \varphi\,ds$$

for the Maupertuisian action, in which φ denotes a function x, y, z that is obtained by replacing v in dW / ds with its value that one infers from the relation:

$$v\frac{\partial W}{\partial v} - W = U + h,$$

which one presently deduces from (13′), and in which h is a constant.

We recover the result that LAPLACE pointed out ([1]) *precisely*, as one can verify *a posteriori* by a known argument ([2]).

7. Some particular motions. The Maupertuisian mass. Critical values of the velocity. Notion of natural state. Absolute space and absolute time. – A function v and a function $W(v)$ are defined by a motion of a well-defined material point. We can imagine some other motions of

([1]) LAPLACE. – *Traité de Mécanique céleste*, t. I, and his *Œuvres complètes*, pp. 78.

([2]) APPELL. – *Traité de Mécanique rationelle*, t. I, 1st edition, pp. 529, *et seq.*, 2nd edition, pp. 530, *et seq.*

the same material point that can be performed at the same instants. In general, the function v will be modified then. If we consider different material points then will be led to modify the function $W(v)$, in addition.

In the study of those modifications, one of the first questions to ask is the following one: What will NEWTON's first law become for the general dynamics of the material point that we consider, i.e., what are the cases in which the projections F, G, H of the quantity of motion are constants? If we first suppose that the three projections F, G, H are identically zero then we will get either the material points for which $\frac{1}{v}\frac{\partial W}{\partial v}$ is defined for $v = 0$ and which are in the rest state, or the points in motion for which the function v will be a non-zero constant that annuls dW / dv. If we suppose that the three projections F, G, H are constants that are not all zero then we will get either the material points for which W is a linear function of v and which are animated with a rectilinear motion or rather points for which the motion is uniform and rectilinear.

Those first results lead us to make several essential remarks.

In the first place, in the context of the rest state, we were just led to consider only the material points for which $\frac{1}{v}\frac{\partial W}{\partial v}$ was defined for $v = 0$. In a general fashion, that led us to imagine the values of v for which the function $\frac{1}{v}\frac{\partial W}{\partial v}$, which call the *Maupertuisian mass* at the instant t and which features in equations (11) and (12), which define F, G, H, X, Y, Z, is not *defined and continuous*. In the preceding paragraphs, we had supposed implicitly that such values would not come under consideration for the motion we imagined. However, the notion of the critical values of v permits us to regard the present discussion as a construction of the type that POINCARÉ indicated at the end of his talk in St. Louis ([1]), and in which, for example, the speed of light might figure.

In the second place, we have just exhibited some special cases of motion. By analogy with what we do in the study of deformable bodies, we will then be led to compare the other arbitrary motions that correspond to the same function W to those special cases, and more generally, to introduce the motion of *natural state of motion*, by which, we mean a state of motion to which we compare the other motions that refer to the same function W. In order to perform that comparison, here is how we can proceed:

Consider a moving point whose coordinates are functions of not only time t, but also a second parameter h, and suppose that for $h = 0$, those functions reduce to the coordinates of the point in the natural state of motion that one envisions. It is clear that for a different well-defined value of h, those functions can reduce to the coordinates of the point that is in a well-defined, but arbitrary, state of motion. Now, formulas (11) and (12) will then give well-defined functions of t and h for F, G, H, X, Y, Z. That being the case, one can either study the latter six functions while supposing that x, y, z are known as functions of t and h, or one can study the functions of x, y, z, and t and h while supposing the some of the functions F, G, H, X, Y, Z are given, along with certain conditions at the limits. In the latter case, one can even suppose that the givens are provided in the general form that was indicated at the end of § **5**.

([1]) *Loc. cit.*, pp. 210.

We finally remark that among the particular cases that were exhibited above, one finds the *rest state* and the *state of uniform rectilinear motion*. That leads us to insist upon the fact that we have introduced *a trihedron Oxyz that plays a particular role*.

No matter what idea that we have in mind as our representation of the physical world that might imply mechanics, we must demand to know what the trihedron *Oxyz* might correspond to as its image in the applications. Here, one might answer that in two ways ([1]): Either one attributes the value of only an approximation to the principles of mechanics and the description of the external world that they permits one to obtain, and one can then be content to establish the correspondence ([2]) between reality and the fixed trihedron *Oxyz* that seems most *convenient* and *practical*, or one can regard the laws of mechanics as possessing a *rigorous exactitude*, to use the expression of DE FREYCINET ([3]), and in that case, with NEWTON, one will assume the existence of an *absolute time* and an *absolute space* ([4]).

On the subject of absolute space, LAPLACE ([5]) expressed his views in the following manner:

> "To us, a body appears to move when it changes its situation with respect to a system of bodies that we judge to be at rest. However, like all bodies, the same ones that seem to play the role of absolute rest can be in motion. One imagines an unbounded space that is immobile and penetrable as follows: In our mind, we refer the position of the body to the parts of the real or ideal space, and we imagine that it is in motion when it successively corresponds to various locations in space."

If we ignore the medium in which the body moves for the construction of dynamics, as we shall do in the present Note, then the immobile space that LAPLACE spoke of can be only an ideal, and its existence must be regarded as a postulate in which mechanics, as it is envisioned from the second viewpoint that was indicated above, does not exist.

On the contrary, one can deal with the reality of that existence if one considers the dynamics in the *extrinsic* manner, to which we alluded at the end of the Introduction. For the moment, we leave aside that second aspect ([6]) of absolute space. We observe only that being given the trihedron *Oxyz* is linked with being given the medium and that one will again confront the alternative of whether to regard mechanics as an approximation or not when one passes over to the applications ([7]).

([1]) Compare the analysis in the book by MACH with that of DUHEM in Bull. Sci. math. **27** (1903), 261-283.

([2]) That question was considered in recent years, in particular. In addition to some of the articles that were pointed out in § **1**, one might cite MACH, *La Mécanique*…, French translation, pp. 234, and the very interesting presentation that one finds in J. PERRIN, *Traité de Chimie physique*, *Les Principes*, Paris, 1903.

([3]) C. DE FREYCINET. – *Les principes de la Mécanique rationelle*, pp. 78.

([4]) Consult A. VOSS. – "Die Prinzipien der rationellen Mechanik," *Enzyklopädie der mathematischen Wissenschaften mit Einschluss ihrer Anwendungen*, Band IV 1, Leipzig, 1901, pp. 3-121.

([5]) P. S. LAPLACE. – *Traité de Mécanique céleste*, Part I, Book I, Chap. 1, pp. 5 of the last edition.

([6]) Compare MACH, pp. 224. LARMOR, *Aether and Matter*, Cambridge, 1900. P. LANGEVIN, "La physique des électrons" [Report to the Congress at St. Louis that was read on 22 September 1904; Revue générale des Sciences **16** (1905).]

([7]) POINCARÉ. – *St. Louis talk*, pp. 186.

8. Ordinary mechanics, considered to be the study of the state of motion that is infinitely-close to the state of rest. The Hamiltonian mass, the Maupertuisian mass, and the kinetic mass. – We will recover the relations of ordinary mechanics by supposing that:

$$W = \tfrac{1}{2} m v^2$$

in the formulas of § **5**, in which m is a constant. However, as we pointed out in the Introduction, we shall adopt a different viewpoint and show that we can consider the formulas of classical mechanics to be a first approximation to the formulas of § **5**. To that effect, it will suffice to study the state of motion that is infinitely-close to the natural state that consists of the state of rest that was discussed in the preceding section.

Suppose that $\frac{1}{v}\frac{dW}{dv}$ is defined for $v = 0$. The function $W(v)$ is then developable ([1]) in the neighborhood of $v = 0$ in positive integer powers of v. The coefficient of v in the development of W is zero, and we assume that the value of W for $v = 0$ is zero ([2]), which is a value that does not enter into the formulas (11) and (12), moreover. Under those conditions, we will have:

$$W = \tfrac{1}{2} m v^2 + W_3 + \ldots , \tag{15}$$

in which m denotes a constant, and W_i ($i \geq 3$) represents the term that contains v to the power i.

If the fixed coordinates of the point in the rest state are x_0, y_0, z_0 then consider a moving point whose coordinates are functions of t and h that are developed in powers of h by way of the formulas:

$$\begin{aligned} x &= x_0 + x_1 + \ldots + x_i + \ldots , \\ y &= y_0 + y_1 + \ldots + y_i + \ldots , \\ z &= z_0 + z_1 + \ldots + z_i + \ldots , \end{aligned}$$

in which x_i, y_i, z_i denote the terms that include h^i as a factor.

Formulas (11) and (12) permit us to calculate the developments of F, G, H, X, Y, Z in powers of h. The terms that are independent of h are zero, and the terms F_1, G_1, H_1, X_1, Y_1, Z_1, which contain h to the first power, are given by the formulas:

$$F_1 = m\frac{dx^{(1)}}{dt}, \qquad G_1 = m\frac{dy^{(1)}}{dt}, \qquad H_1 = m\frac{dz^{(1)}}{dt},$$
$$X_1 = m\frac{d^2x^{(1)}}{dt^2}, \qquad Y_1 = m\frac{d^2y^{(1)}}{dt^2}, \qquad Z_1 = m\frac{d^2z^{(1)}}{dt^2},$$

in which one sets:

([1]) To abbreviate the presentation, we shall introduce series developments in what follows, and we shall assume that those series lend themselves to the usual procedures of calculus.

([2]) In the natural state of rest, the kinetic energy will then be zero, like the quantity of motion and the external force.

$$x^{(1)} = x_0 + x_1 , \qquad y^{(1)} = y_0 + y_1 , \qquad z^{(1)} = z_0 + z_1 .$$

If we consider the motion of the point $x^{(1)}$, $y^{(1)}$, $z^{(1)}$, under the name of the *state of motion that is infinitely-close to the natural state of rest*, and the vectors (F_1, G_1, H_1) and (X_1, Y_1, Z_1), under the names of *quantity of motion* and *force, relative to that state*, then we will arrive at the result that *the ordinary dynamics of the material point* (or NEWTONIAN dynamics, if one prefers) *appears to be the dynamics of the point in the state of motion that is infinitely-close to the natural rest state.*

When dynamics is presented in the preceding form, one will no longer have the notion of an *invariable mass* for a state of motion that is not infinitely close to the natural state of rest. Previously, we gave the name of *Maupertuisian mass* at the instant t to the expression $\frac{1}{v}\frac{dW}{dv}$, by which one must multiply the velocity vector in order to obtain the *quantity of motion*. With the hypothesis that was made on W, formula (15) will give:

$$\frac{1}{v}\frac{dW}{dv} = m + \dots$$

in the neighborhood of the value $v = 0$, in which the unwritten terms represent terms in v, v^2, etc.

We can remark that the *action W at the instant t*, in its own right, leads us to imagine the expression $2W / v^2$, to which we can given the name of the *Hamiltonian mass*, and whose development in powers of v, when performed as in formula (15), will also begin with the constant m.

The same remark applies once more to the expression $2E / v^2$, which we can call the *Leibnizian mass* or *kinetic mass*. From the relation:

$$E = v\frac{dW}{dv} - W ,$$

we see that *the Maupertuisian mass is equal to the mean of the Hamiltonian and Leibnizian masses*.

9. The dynamics of the invariable body. Maupertuisian center and axes of inertia. – We shall now consider an invariable body in motion that we define by associating it with a moving trihedron $O'x'y'z'$, to which it is invariably linked. The motion of the trihedron $O'x'y'z'$ with respect to the fixed trihedron $Oxyz$ is defined entirely by the six arguments ξ, η, ζ, p, q, r of § **2**. We associate each point of the invariable body whose coordinates are x', y', z' with respect to the trihedron $O'x'y'z'$ with a function w of the position of that point in the body and the expression for v that is defined by consideration that are analogous to the ones in § **4**, by means of the following relation that determines v^2 and results from formulas (8):

(16) $$v^2 = (\xi + q\,z' - r\,y')^2 + (\eta + r\,x' - p\,z')^2 + (\zeta + p\,y' - q\,x')^2 .$$

We call the integral:

$$\mathrm{S}\, w\, d\varpi ,$$

which extends over a portion of the invariable body the action over that portion at the instant t. The action over the volume element $d\varpi$ that surrounds the point (x', y', z') will be $[w(x', y', z') + \varepsilon]\,d\varpi$, where ε tends uniformly to zero with $d\varpi$.

In particular, consider the action W at the instant t over the total body: It is calculated from the formula:

$$W = \mathrm{S}\, w\, d\varpi ,$$

in which the integral extends over the entire body, and by virtue of formula (16), it is a function of ξ, η, ζ, p, q, r that is defined as soon as one knows the function w and the form of the body.

We call the integral:

(17) $$\mathcal{M} = \mathrm{S}\, \frac{1}{v}\frac{\partial w}{\partial v}\, d\varpi ,$$

which extends over the volume of the body, the *Maupertuisian mass* of the invariable body at the instant t. When the same integral is extended over an arbitrary portion of the body, it will give the Maupertuisian mass of that portion. The Maupertuisian mass of the volume element $d\varpi$ that surrounds (x', y', z') will be $\left[\frac{1}{v}\frac{\partial w}{\partial v} + \varepsilon_1\right] d\varpi$, in which ε_1 will tend to zero uniformly with $d\varpi$.

It results immediately from that definition of Maupertuisian mass that one has the definitions of the Maupertuisian center of inertia at the instant t, the moments of inertia, the ellipsoid of inertia, and the central ellipsoid of inertia, which all refer to the instant t and to geometric elements that are or are not fixed in the body.

The coordinates at the instant t of the *center of inertia* with respect to the trihedron $O'\,x'\,y'\,z'$ will be:

(18) $$x_1' = \frac{\mathrm{S}\,\frac{1}{v}\frac{\partial w}{\partial v}\, x'\, d\varpi}{\mathcal{M}} , \quad y_1' = \frac{\mathrm{S}\,\frac{1}{v}\frac{\partial w}{\partial v}\, y'\, d\varpi}{\mathcal{M}} , \quad z_1' = \frac{\mathrm{S}\,\frac{1}{v}\frac{\partial w}{\partial v}\, z'\, d\varpi}{\mathcal{M}} .$$

The *moment of inertia* at the instant t with respect to a line in the body that issues, for example, from O' and whose direction cosines relative to the axes $O'x'$, $O'y'$, $O'z'$ are α, β, γ, is:

$$\mathcal{A}\,\alpha^2 + \mathcal{B}\,\beta^2 + \mathcal{C}\,\gamma^2 - 2\mathcal{D}\,\beta\,\gamma - 2\mathcal{E}\,\gamma\,\alpha - 2\mathcal{F}\,\alpha\,\beta ,$$

in which one sets:

$$(19)\quad \left\{\begin{aligned} &\mathcal{A} = \mathrm{S}\frac{1}{v}\frac{\partial w}{\partial v}(y'^2 + z'^2)\,d\varpi, \quad &&\mathcal{B} = \mathrm{S}\frac{1}{v}\frac{\partial w}{\partial v}(z'^2 + x'^2)\,d\varpi, \quad &&\mathcal{C} = \mathrm{S}\frac{1}{v}\frac{\partial w}{\partial v}(x'^2 + y'^2)\,d\varpi, \\ &\mathcal{D} = \mathrm{S}\frac{1}{v}\frac{\partial w}{\partial v}\,y'\,z'\,d\varpi, &&\mathcal{E} = \mathrm{S}\frac{1}{v}\frac{\partial w}{\partial v}\,z'\,x'\,d\varpi, &&\mathcal{F} = \mathrm{S}\frac{1}{v}\frac{\partial w}{\partial v}\,x'\,y'\,d\varpi, \end{aligned}\right.$$

and the *ellipsoid of inertia* at the instant *t* relative to the point O' will have the equation:

$$\mathcal{A}\,x'^2 + \mathcal{B}\,y'^2 + \mathcal{C}\,z'^2 - 2\mathcal{D}\,y'\,z' - 2\mathcal{E}\,z'\,x' - 2\mathcal{F}\,x'\,y' = 1$$

relative to the trihedron $O'\,x'\,y'\,z'$.

We call the integral:

$$A = \int_{t_0}^{t_1} W\,dt$$

the *action over the invariable body* in question from the instant t_0 to the instant t_1.

We apply the method of variable action here. Suppose that the expressions $a, b, c, \alpha, \beta, \ldots, \gamma''$, that determine the position of the trihedron $O'\,x'\,y'\,z'$ at the instant *t* submit to the variations $\delta a, \delta b, \delta c, \delta\alpha, \delta\beta, \ldots, \delta\gamma''$. Introduce the auxiliary variables $\delta I'$, $\delta J'$, $\delta K'$, which are defined by the formulas:

$$\delta I' = \sum \gamma\,\delta\beta = -\sum \beta\,\delta\gamma, \quad \delta J' = \sum \alpha\,\delta\gamma = -\sum \gamma\,\delta\alpha, \quad \delta K' = \sum \beta\,\delta\alpha = -\sum \alpha\,\delta\beta,$$

which are the analogues of the expressions $p\,dt$, $q\,dt$, $r\,dt$ of § **2**. In addition, introduce the variations $\delta a'$, $\delta b'$, $\delta c'$ of the projections a', b', c' of the segment OO' onto $O'x'$, $O'y'$, $O'z'$, and the auxiliary variables $\delta_1 a'$, $\delta_1 b'$, $\delta_1 c'$, which are defined by the formulas:

$$\delta_1 a' = \delta a' + c'\,\delta J' - b'\,\delta K', \quad \delta_1 b' = \delta b' + a'\,\delta K' - c'\,\delta I', \quad \delta_1 c' = \delta c' + b'\,\delta I' - a'\,\delta J',$$

and which represent the projections of the displacement $(\delta a, \delta b, \delta c)$ that the point O' submits to onto $O'x'$, $O'y'$, $O'z'$, resp.

An easy calculation will give the following formulas for determining the variations that $\xi, \eta, \zeta, p, q, r$ are subjected to:

$$(20)\quad \left\{\begin{aligned} \delta\xi &= \eta\,\delta K' - \zeta\,\delta J' + \frac{d\,\delta_1 a'}{dt} + q\,\delta_1 c' - r\,\delta_1 b', \\ \delta\eta &= \zeta\,\delta I' - \xi\,\delta K' + \frac{d\,\delta_1 b'}{dt} + r\,\delta_1 a' - p\,\delta_1 c', \\ \delta\zeta &= \xi\,\delta J' - \eta\,\delta I' + \frac{d\,\delta_1 c'}{dt} + p\,\delta_1 b' - q\,\delta_1 a', \end{aligned}\right. \quad (21)\quad \left\{\begin{aligned} \delta p &= \frac{d\,\delta I'}{dt} + q\,\delta K' - r\,\delta J', \\ \delta q &= \frac{d\,\delta J'}{dt} + r\,\delta I' - p\,\delta K', \\ \delta r &= \frac{d\,\delta K'}{dt} + p\,\delta J' - q\,\delta I'. \end{aligned}\right.$$

Having said that, the function $W(\xi, \eta, \zeta, p, q, r)$ will submit to a variation:

$$\delta W = \frac{\partial W}{\partial \xi}\delta\xi + \frac{\partial W}{\partial \eta}\delta\eta + \frac{\partial W}{\partial \zeta}\delta\zeta + \frac{\partial W}{\partial p}\delta p + \frac{\partial W}{\partial q}\delta q + \frac{\partial W}{\partial r}\delta r ,$$

and the action A will submit to a variation δA that is defined by the formula:

$$\delta A = \int_{t_0}^{t_0} \delta W \, dt .$$

By virtue of the formulas (20) and (21), and after an integration by parts, that variation will take form:

$$\delta A = \left[F'\delta_1 a' + G'\delta_1 b' + H'\delta_1 c' + I'\delta I' + J'\delta J' + K'\delta K'\right]_{t_0}^{t_1}$$
$$- \int_{t_0}^{t_0} (X'\delta_1 a' + Y'\delta_1 b' + Z'\delta_1 c' + L'\delta I' + M'\delta J' + N'\delta K')\, dt$$

when one sets:

$$(22) \qquad F' = \frac{\partial W}{\partial \xi}, \quad G' = \frac{\partial W}{\partial \eta}, \quad H' = \frac{\partial W}{\partial \zeta}, \quad I' = \frac{\partial W}{\partial p}, \quad J' = \frac{\partial W}{\partial q}, \quad K' = \frac{\partial W}{\partial r},$$

$$(23) \qquad \left\{ \begin{array}{ll} X' = \dfrac{dF'}{dt} + q H' - r G', & L' = \dfrac{dI'}{dt} + q K' - r J' + \eta H' - \zeta G', \\ Y' = \dfrac{dG'}{dt} + r F' - p H', & M' = \dfrac{dJ'}{dt} + r I' - p K' + \zeta F' - \xi H', \\ Z' = \dfrac{dH'}{dt} + p G' - q F', & N' = \dfrac{dK'}{dt} + p J' - q I' + \xi G' - \eta F'. \end{array} \right.$$

Those results lead us to pose some definitions that are analogous to the ones that we established for the dynamics of the point.

First of all, recall the following definitions from the theory of vectors ([1]):

If one is given arbitrary vectors then one says *general resultant* and *resultant moment relative to a point P* to mean the resultant of the vectors that have P for their origin, are equal and parallel to the given vectors, and the resultant of the moments of the given vectors relative to the point P, resp. If one varies the position of the point P then the magnitude and direction of the general resultant will remain invariable. The resultant with respect to a point P' that is different from the point P is the geometric sum of the resultant moment with respect to the point P and the moment with respect to P' of the general relative to the point P. Two systems of vectors are called *equivalent* when their general resultants and resultant moments with respect to the same point in

([1]) P. APPELL. – *Traité de Mécanique rationelle*, t. I; G. KŒNIGS, *Leçons de Cinématique*, *Cinématique théorique*.

space are identical. Their resultant moments will then be identical with respect to any point in space. A system of vectors is called *equivalent to zero* when its general resultant and its resultant moment with respect to a point are zero. Those quantities will then be zero for any point in space.

Having recalled those definitions, first consider the two segments that issue from the point O' whose projections onto $O'x'$, $O'y'$, $O'z'$ are F', G', H' and I', J', K', respectively. We consider them to be the general resultant and the resultant moment, resp., of an infinitude of systems of equivalent vectors. We give the name to *system of quantities of motion* to the latter systems. The origin of that term is provided by the following remark: Set:

$$f' = \frac{1}{v}\frac{\partial w}{\partial v}(\xi' + q\,z' - r\,y')\ , \quad g' = \frac{1}{v}\frac{\partial w}{\partial v}(\eta' + r\,x' - p\,z')\ , \quad h' = \frac{1}{v}\frac{\partial w}{\partial v}(\zeta' + p\,y' - q\,x')\ .$$

Formulas (22) can be written:

$$F' = \mathrm{S}\, f'\,d\varpi\,, \qquad G' = \mathrm{S}\, g'\,d\varpi\,, \qquad H' = \mathrm{S}\, h'\,d\varpi\,,$$

$$I' = \mathrm{S}(y'\,h' - z'\,g')\,d\varpi\,, \qquad J' = \mathrm{S}(z'\,f' - x'\,h')\,d\varpi\,, \qquad K' = \mathrm{S}(x'\,g' - y'\,f')\,d\varpi\,.$$

If we suppose that we have replaced each element $d\varpi$ of the body with a material point of the same Maupertuisian mass that is situated at a point (x', y', z') that is interior to that element then in the limit, the general resultant and resultant moment relative to the point O' of the system of vectors that are composed of the quantities of motion of the various material points will be (F', G', H') and (I', J', K'), respectively.

That remark leads us to the conclusion that if we replace the trihedron $O'x'y'z'$ with a trihedron to which it is invariable linked then the systems of quantities of motion that are defined by the second trihedron will be the same as they are for the trihedron $O'x'y'z'$. The general resultant (F', G', H') and the resultant moment (I', J', K') relative to the trihedron $O'x'y'z'$ determine systems that are equivalent to the ones that are generated by the analogous elements that relate to another trihedron that is invariably linked with the first one.

Now consider the two segments that issue from the point O' whose projections onto $O'x'$, $O'y'$, $O'z'$ are F', G', H' and I', J', K', respectively. We consider them to be the general resultant and resultant moment, respectively, of an infinitude of systems of equivalent vectors. We give the name of the *system of external forces* that is applied to the body to any of the latter systems.

Here again, if one replaces the trihedron $O'x'y'z'$ with a trihedron to which it is invariably linked then the systems of forces that are defined by means of the second trihedron will be the same as for the trihedron $O'x'y'z'$: The general resultant (X', Y', Z') and the resultant moment (L', M', N') relative to the trihedron $O'x'y'z'$ define systems that are equivalent to the ones that are generated by the analogous elements relative to another trihedron that is invariably linked with the first one.

An immediate consequence of the following interpretation of formulas (23), which are considered *independently* of the relations (22) that determine F', G', H', I', J', K' as functions of ξ, η, ζ, p, q, r, is that:

If one constructs the general resultant $O\mu$, OM and the resultant moments $O\alpha$, OS relative to the point O, respectively, of the system of quantities of motion and the system of external forces then equations (23) *will express the idea that the velocities of the geometric points μ, α are equal and parallel to the segments OM and OS, respectively.*

As one can see, that proposition, which can be proved immediately, can be stated in an identical way to the one in ordinary mechanics ([1]) that one deduces upon supposing that w has the form $\frac{1}{2}\rho v^2$, where ρ is a well-defined function of x', y', z', or even better, upon passing over to the notion of the state that is infinitely-close to the rest state, which we will speak of later on.

We just stated a proposition that transforms into a fundamental proposition of classical mechanics either upon supposing that w has the form $\frac{1}{2}\rho(x', y', z')v^2$ or upon considering the notion of the state that is infinitely-close to the rest state.

We must now examine the other generalizations of the more interesting propositions of ordinary mechanics.

If we preserve the usual definition of the work done by a system of forces then the sum relative to the time dt of the elementary works done by the external forces that are applied to the invariable body is:

$$(\xi X' + \eta Y' + \zeta Z' + p L' + q M' + r N')\, dt\,.$$

Replace X', Y', Z', L', M', N' with their values that are defined by means of formulas (22) and (23). It will immediately become:

$$(\xi X' + \eta Y' + \zeta Z' + p L' + q M' + r N')\, dt = dE\,, \tag{24}$$

in which E denotes the function of ξ, η, ζ, p, q, r that is defined by the formula:

$$E = \xi \frac{\partial W}{\partial \xi} + \eta \frac{\partial W}{\partial \eta} + \zeta \frac{\partial W}{\partial \zeta} + p \frac{\partial W}{\partial p} + q \frac{\partial W}{\partial q} + r \frac{\partial W}{\partial r} - W\,. \tag{25}$$

([1]) The geometric representation of the theorem of moments of quantities of motion in classical mechanics seems to be due to SAINT-GUILHEM, who gave it in 1854 in Tome XIX of the *Journal de Liouville*. Likewise consult the following work by the same author: "Nouvelle solution synthétique du problème de la rotation des corps," Mémoires de l'Académie des Sciences de Toulouse (4) **5** (1855), 338-350. One can include the latter article, which was reproduced in the Nouvelles Annales de Mathématiques (1) **15** (1856), 63-76, with the following one: R. B. HAYWARD, "On a Direct Method of estimating Velocities, Accelerations, and all similar Quantities with respect to Axes moveable in any manner in space, with Applications," Trans. Camb. Phil. Soc., vol. X. The work of R. B. HAYWARD, upon which KLEIN and SOMMERFELD insisted in a very particular fashion on pages 114 and 115 of their book *Ueber die Theorie des Kreisels*, includes the suggestion that it was read on 25 February 1856.

If we give the name of the *kinetic energy* for the body in question to E then equation (24) will be stated thus:

The infinitely-small variation of the kinetic energy is equal to the sum of the elementary works done by the external forces.

One concludes from this that:

The variation of the kinetic energy over a finite time interval is equal to the sum of the works done by the external forces during that interval.

In particular:

If the body is not acted on by any external force then its kinetic energy will remain constant.

We shall presently examine the propositions in which the *center of Maupertuisian energy* can be involved.

First, observe that if one introduces the coordinates x_1', y_1', z_1' that are defined by formulas (18) of the Maupertuisian center of inertia at the instant t, the Maupertuisian mass $\mathcal{M}$ that is defined by formula (17), and the expressions $\mathcal{A}$, $\mathcal{B}$, …, $\mathcal{F}$ that are defined by formulas (19) then one can give the relations (22) the following forms:

$$(26) \qquad F' = \mathcal{M}(\xi + q\, z_1' - r\, y_1'), \qquad G' = \mathcal{M}(\eta + r\, x_1' - p\, z_1'), \qquad H' = \mathcal{M}(\zeta + p\, y_1' - q\, x_1'),$$

$$(27) \qquad \left\{ \begin{aligned} I' &= +\mathcal{A}\, p - \mathcal{F}\, q - \mathcal{E}\, r + \mathcal{M}(\eta\, z_1' - \zeta\, y_1'), \\ J' &= -\mathcal{F}\, p + \mathcal{B}\, q - \mathcal{D}\, r + \mathcal{M}(\zeta\, x_1' - \xi\, z_1'), \\ K' &= -\mathcal{E}\, p - \mathcal{D}\, q + \mathcal{C}\, r + \mathcal{M}(\xi\, y_1' - \eta\, x_1'). \end{aligned} \right.$$

Formulas (26) are stated in the following form:

The general resultant of the quantities of motion is a vector that is equipollent to the quantity of motion of a material point that is placed at the point of the invariable body that coincides with the center of inertia at the instant t, and which will have a Maupertuisian mass at that instant that is equal to the value of $\mathcal{M}$.

If one now imagines, aside from the proposition that was just stated, the properties by which one introduces the center of inertia into classical mechanics then it will be easy to see that they are attached to two distinct orders of ideas:

1. The center of inertia enjoys some special properties in regard to the motion of the *invariable body that is not subject to any external force.*

2. It enjoys some special properties in regard to the study of *relative motion*. We shall adopt the first viewpoint.

10. Some particular motions. Helicoidal motions and spontaneous rotation, spontaneous axes. Motions of permanent rotation, permanent axes. The critical values of $\xi, \eta, \zeta, p, q, r$. **Ordinary dynamics, considered to be the study of the state of motion infinitely-close to the state of rest.** – The motions of an invariable body for which the six expressions X', Y', Z', L', M', N' are identically zero can be considered to be the ones that give the system of differential equations:

$$
(28)\quad
\begin{cases}
\dfrac{d\frac{\partial W}{\partial \xi}}{dt} + q\dfrac{\partial W}{\partial \zeta} - r\dfrac{\partial W}{\partial \eta} = 0, & \dfrac{d\frac{\partial W}{\partial p}}{dt} + q\dfrac{\partial W}{\partial r} - r\dfrac{\partial W}{\partial q} + \eta\dfrac{\partial W}{\partial \zeta} - \zeta\dfrac{\partial W}{\partial \eta} = 0, \\[2ex]
\dfrac{d\frac{\partial W}{\partial \eta}}{dt} + r\dfrac{\partial W}{\partial \xi} - p\dfrac{\partial W}{\partial \zeta} = 0, & \dfrac{d\frac{\partial W}{\partial q}}{dt} + r\dfrac{\partial W}{\partial p} - p\dfrac{\partial W}{\partial r} + \eta\dfrac{\partial W}{\partial \xi} - \xi\dfrac{\partial W}{\partial \zeta} = 0, \\[2ex]
\dfrac{d\frac{\partial W}{\partial \zeta}}{dt} + p\dfrac{\partial W}{\partial \eta} - q\dfrac{\partial W}{\partial \xi} = 0, & \dfrac{d\frac{\partial W}{\partial r}}{dt} + p\dfrac{\partial W}{\partial q} - q\dfrac{\partial W}{\partial p} + \xi\dfrac{\partial W}{\partial \eta} - \eta\dfrac{\partial W}{\partial \xi} = 0
\end{cases}
$$

that define the six unknowns $\xi, \eta, \zeta, p, q, r$, which are characteristics of a motion.

The study of the case in which an invariable body is not subject to any external force then coincides with the study of the preceding system.

If we suppose that the Hessian of the function W is not identically zero then the system (28) can be put into the normal form immediately, which is comprised of six equations here that give the first derivatives of the unknowns $\xi, \eta, \zeta, p, q, r$ as well-defined functions $f_1, f_2, \ldots, f_6$ of *only those unknowns*. We immediately conclude the following proposition:

If the functions $f_1, f_2, \ldots, f_6$ *are regular in the neighborhood of certain values* $\xi_0, \eta_0, \zeta_0, p_0, q_0, r_0$ *of their arguments and are annulled for those particular values then the functions* $\xi, \eta, \zeta, p, q, r$ *that verify the system* (28) *and take the values* $\xi_0, \eta_0, \zeta_0, p_0, q_0, r_0$ *at a certain instant will be constant.*

Now, here one encounters a fact that is worthy of note, namely, that the system of six equations $f_1 = 0, f_2 = 0, \ldots, f_6 = 0$ that defined the six unknowns $\xi, \eta, \zeta, p, q, r$ presents a relatively-considerable indeterminacy. We shall specify that indeterminacy and interpret the solutions.

We first defined the particular solution ([1]) by setting $f_1 = f_2 = \ldots = f_6 = 0$, i.e., the rest state. In that case, from formulas (26) and (27), the six expressions F', G', H', I', J', K' will be likewise zero ([2]).

We further ignore the case in which the functions p, q, r are identically zero and ξ, η, ζ are constants that are not all zero. Equations (28) are verified as a result of the forms (26) that one can give to the first three of formulas (22), and by virtue of which the values of $\frac{\partial W}{\partial \xi}$, $\frac{\partial W}{\partial \eta}$, $\frac{\partial W}{\partial \zeta}$ are presently proportional to ξ, η, ζ. We have an *arbitrary motion of uniform translation* and the following particular proposition, which corresponds to the first law of motion in NEWTON's mechanics:

If an invariable body that is not acted on by any external force begins to be animated by a uniform translatory motion then it will continue to be animated by that uniform translatory motion.

Finally, consider the case in which the six functions ξ, η, ζ, p, q, r are constants ξ_0, η_0, ζ_0, p_0, q_0, r_0, and the values p_0, q_0, r_0 of p, q, r are not all zero. The motion of the body is then a *helicoidal motion*, or as a special case, *a rotational motion.* If we express the idea that equations (28) are verified upon setting $\xi = \xi_0$, $\eta = \eta_0$, $\zeta = \zeta_0$, …, $r = r_0$ then that will give the six conditions:

$$
(29) \qquad \left\{
\begin{aligned}
& q\frac{\partial W}{\partial \zeta} - r\frac{\partial W}{\partial \eta} = 0, \quad q\frac{\partial W}{\partial r} - r\frac{\partial W}{\partial q} + \eta\frac{\partial W}{\partial \zeta} - \zeta\frac{\partial W}{\partial \eta} = 0, \\
& r\frac{\partial W}{\partial \xi} - p\frac{\partial W}{\partial \zeta} = 0, \quad r\frac{\partial W}{\partial p} - p\frac{\partial W}{\partial r} + \eta\frac{\partial W}{\partial \xi} - \xi\frac{\partial W}{\partial \zeta} = 0, \\
& p\frac{\partial W}{\partial \eta} - q\frac{\partial W}{\partial \xi} = 0, \quad p\frac{\partial W}{\partial q} - q\frac{\partial W}{\partial p} + \xi\frac{\partial W}{\partial \eta} - \eta\frac{\partial W}{\partial \xi} = 0,
\end{aligned}
\right.
$$

which must be verified by the initial values, and which can replace the equations $f_1 = 0$, $f_2 = 0$, …, $f_6 = 0$. Now, the six conditions (29) obviously reduce to four relations and thus determine a double infinitude of helicoidal motions that call *spontaneous helicoidal motions* ([3]) of the body considered, and in regard to which we have the following proposition:

If an invariable body that is not acted on by any external force begins to be animated with a spontaneous helicoidal motion that is appropriate to it then it will continue to be animated with that helicoidal motion.

([1]) In all of what follows, we will suppose that the various functions of ξ, η, ζ, p, q, r that figure in the equations (26), (27), (28), along with f_1, f_2, …, f_6, are regular for the particular values that we consider.

([2]) We can repeat a discussion that is analogous to the one that we developed for the system of forces in regard to the system of quantities of motion.

([3]) Those motions include a simple infinitude of *spontaneous rotational motions*.

The spontaneous helicoidal motions of an invariable body are, by the very nature of the question that give rise to them, independent of the choice of trihedron $O'x'\,y'\,z'$ in the body. We give the name of *spontaneous axes* (of helicoidal motion or rotation, in particular) to the lines of the body that are the axes of those helicoidal motions. When the function *w* is arbitrary, those lines will form a congruence of lines in the body, and each line will be associated with a well-defined helicoidal motion. When the function *w* has the form that is suitable to ordinary mechanics, as we shall see, the spontaneous axes will no longer depend upon any parameters and will coincide with the lines that presently bear the name of *spontaneous axes of rotation*. Each line is then associated with a double infinitude of helicoidal motions.

In order to quickly attach the spontaneous helicoidal motions to the notion of Maupertuisian center of inertia, we remark that upon choosing the trihedron $O'x'\,y'\,z'$ in the body ([1]) conveniently, we may suppose that:

$$\xi_0 = 0\,, \qquad \eta_0 = 0\,, \qquad p_0 = 0\,, \qquad q_0 = 0\,.$$

To that effect, it suffices to take the axis $O'z'$ to be the axis of helicoidal motion or the axis of rotation. Since r_0 is not zero, the conditions (29) will then reduce to the following four:

$$\frac{\partial W}{\partial \xi} = 0\,, \qquad \frac{\partial W}{\partial \eta} = 0\,, \qquad \frac{\partial W}{\partial p} = 0\,, \qquad \frac{\partial W}{\partial q} = 0\,,$$

which must be verified by the initial values. Now, from the forms (26) and (27) of formulas (22), the latter conditions are equivalent to:

$$x_1' = 0\,, \qquad y_1' = 0\,, \qquad \mathcal{D} = 0\,, \qquad \mathcal{E} = 0\,.$$

Those relations express that property of any spontaneous helicoidal motion that *its axis is one of the axes of the central ellipsoid of inertia that corresponds to it* ([2]).

Furthermore, consider the case that corresponds to the EULER problem in classical mechanics that relates to the *invariable body that has a fixed point*, i.e., the case in which if one supposes that the point O' is fixed then the resultant moment relative to the point O' will be zero, in such a way that one will have:

$$\xi = 0\,, \qquad \eta = 0\,, \qquad \zeta = 0\,, \qquad L' = 0\,, \qquad M' = 0\,, \qquad N' = 0\,.$$

The first three equations (23) define X', Y', Z', and the last three are differential equations that define *p*, *q*, *r*, namely:

([1]) We have already remarked, in a different form, that if one replaces the trihedron $O'x'\,y'\,z'$ with another tri-rectangular trihedron to which it is invariably coupled then the functions ξ, η, ζ, *p*, *q*, *r* relative to that new trihedron will be constant when that is true relative to the first one, and conversely.

([2]) According to the formula in § **9**, each helicoidal motion will correspond to a well-defined center of inertia, central ellipsoid of inertia, etc.

$$\frac{d\dfrac{\partial W}{\partial p}}{dt}+q\frac{\partial W}{\partial r}-r\frac{\partial W}{\partial q}=0\,,$$

$$\frac{d\dfrac{\partial W}{\partial q}}{dt}+r\frac{\partial W}{\partial p}-p\frac{\partial W}{\partial r}=0\,,$$

$$\frac{d\dfrac{\partial W}{\partial r}}{dt}+p\frac{\partial W}{\partial q}-q\frac{\partial W}{\partial p}=0\,.$$

W denotes $W(0, 0, 0, p, q, r)$ here.

We can repeat what we said about the system (28) for those equations. Suppose that we can solve them for the derivatives of $p, q, r,$ and imagine the motions for which p, q, r are constants. They are the ones for which the values p, q, r at a given instant verify the conditions:

$$q\frac{\partial W}{\partial r}-r\frac{\partial W}{\partial q}=0\,,\qquad r\frac{\partial W}{\partial p}-p\frac{\partial W}{\partial r}=0\,,\qquad p\frac{\partial W}{\partial q}-q\frac{\partial W}{\partial p}=0\,.$$

If we ignore the rest case then those conditions, which obviously reduce to two relations, will lead us to associate the point O' of the body with an infinitude of rotational motions that we call *permanent motions* of the rotation of the body considered relative to the point O' and in regard to which we have the following proposition:

If an invariable body is subject to the geometric condition that it must have a fixed point O', and it is found to be subject to an external force that passes through that point, and if the body begins to be animated with one of the permanent rotational motions that are appropriate to it and are relative to the point O' then it will continue to be animated with that rotational motion.

The permanent rotational motions relative to the point O' are, by the very nature of the question that gave rise to them, independent of the directions of the axes of the chosen trihedron $O'x'y'z'$ in the body. We give the name of *permanent axes of rotation* to the lines of the body that issue from O', which are the axes of those rotational motions. When the function w is arbitrary and the point O' varies, those lines will depend upon four parameters, and each line will be associated with a well-defined rotational motion. When the function w has the form that is appropriate to ordinary mechanics, as we shall see, the permanent axes relative to a well-defined point O' will no longer depend upon any parameter and will coincide with the lines that presently bear the same name ([1]): Each of the lines will then be associated with an infinitude of rotational motions, and the set of all of them will define a complex of lines ([2]).

We remark that upon choosing the trihedron $O'x'y'z'$ conveniently in the body, we can suppose that the constant values of p and q are zero. To that effect, it suffices to take the axis $O'z'$ to be the axis of rotation. If r is non-zero then one must have:

$$\frac{\partial W}{\partial p} = 0\,, \qquad \frac{\partial W}{\partial q} = 0$$

for the initial values, i.e., upon utilizing the form (27) for the last three formulas in (22):

$$\mathcal{D} = 0\,, \qquad \mathcal{E} = 0\,.$$

Those relations express that property of any permanent rotational motion relative to the point O' that *its axis is one of the axes of the ellipsoid of inertia that corresponds to it*.

We can further consider the case in which the axis $O'z'$ slides along Oz, and more particularly, the case in which O' is fixed and $O'z'$ points along Oz. In the latter case, which is the analogue of the motion around a fixed axis Oz of ordinary mechanics ([3]), we will have:

$$\xi = 0\,,\ \eta = 0\,,\ \zeta = 0\,,\ p = 0\,,\ q = 0\,.$$

The functions F', G', H', I', J', K' reduce to functions of r. In particular, K' is equal to dW/dt, while we continue to let W denote the function W (0, 0, 0, 0, 0, r). Formulas (26) and (27) give:

$$\begin{aligned} F' &= -\mathcal{M}\,r\,y_1'\,, & G' &= \mathcal{M}\,r\,x_1'\,, & H' &= 0\,, \\ I' &= -\mathcal{E}\,r\,, & J' &= -\mathcal{D}\,r\,, & K' &= \mathcal{C}\,r\,, \end{aligned}$$

and the equations that define X', Y', Z', L', M', N' will be:

([1]) The discovery of permanent axes of rotation seems to be due to SEGNER (*Specimen theoriae turbinum*, Halle, 1755). However, it was, above all, in the hands of EULER that they acquired all of their importance. In Tome I of the second edition of his *Mécanique analytique* (2nd Partie, Sect. III, art. 29), LAGRANGE, after defining the principal axes by considering the products of inertia, remarked that it was by imagining the largest and smallest moments of inertia that EULER had found the principal axes. It was only in 1834 that the ellipsoid of inertia was introduced by POINSOT in his "Théorie nouvelle de la rotation des corps," which appeared only in 1851 in tome XVI of the Journal de Liouville and in the *Connaissance des temps*.

([2]) That complex of lines was first considered in the following articles: J. BINET, "Mémoire sur la théorie des axes conjugués et des moments d'inertie," J. Ec. Polyt. **16** (1813) [1811], 41-67. AMPÉRE, "Mémoire sur quelques Nouvelles propriétés des axes permanents de rotation des corps et des plans directeurs de ces axes, Mém. de l'Acad. Fr. **5** (1826) [1821], 86-152. GUIBERT, "Note sur les axes principaux des corps," J. Ec. Polyt. **25** (1837), 118-122. G. GASCHEAU, "Remarques sur la théorie géométrique des axes permanents de rotation," J. de Liouville (1) **6** (1841), 241-266. Certain results in the preceding works have given rise to the following well-known presentations: MOIGNO, *Leçons de Mécanique analytique*, Lecture 16, Paris, 1868. OTTO HESSE, *Vorlesungen über analytische Geometrie des Raumes*, etc., 2nd ed., Leipzig, 1869, Lecture 25.

([3]) P. APPELL. – *Traité de mécanique rationelle*, T. II, 1st ed., pp. 127, *et seq.*; 2nd ed., pp. 76, *et seq.*

$$X' = \frac{dF'}{dt} - r\,G', \qquad Y' = \frac{dG'}{dt} + r\,F', \qquad Z' = 0,$$

$$L' = \frac{dI'}{dt} - r\,J', \qquad M' = \frac{dJ'}{dt} + r\,I', \qquad N' = \frac{dK'}{dt}.$$

The formula $K' = \mathcal{C}\,r$ is the analogue of the well-known formula in classical mechanics ([1]), and the equation $N' = dK'/dt$ is also equivalent to the theorem that replaces the *vis viva* theorem ([2]), because upon multiplying by $r\,dt$, it can be written:

$$r\,N'\,dt = d\left(r\,\frac{dW}{dr} - W\right).$$

To conclude, we add that we can make some remarks that are analogous to the ones in § **7** by either imagining the *systems of critical values of* ξ, η, ζ, p, q, r or by considering the notion of the *natural state*.

Upon supposing that the function w in the formulas of § **9** has the form $\frac{1}{2}\rho v^2$, in which ρ is a function of only the variables x', y', z', one recovers the proposition of classical dynamics. However, as in § **8**, it is convenient to adopt another point of view and consider ordinary dynamics to be the study of a *state infinitely-close to the rest state*. We shall not stop to develop that suggestion by analogy to what was done in § **8**, which relates to the present case.

11. The material trihedron. The hidden trihedron. – With an eye towards constructing the material trihedron that was in question in the Introduction, consider a point O', to which we attach a tri-rectangular trihedron $O'x'y'z'$. Suppose that this trihedron is referred to a fixed trihedron $Oxyz$ and preserve the notations of § **2**.

We can first establish the notion of *Euclidian action* while imagining a function of two infinitely-close positions of the trihedron $O'x'y'z'$, i.e., a function of a, b, c, α, β, …, γ'', and their first derivatives da/dt, …, $d\gamma''/dt$. We determine that function in such a way that it will remain *invariant under the group of Euclidian displacements*, i.e., in such a fashion that its variation will be zero when, on the one hand, the coordinates a, b, c of the origin O' of the trihedron $O'x'y'z'$ submit to infinitely-small variations that are defined by the formulas:

$$\delta a = (l + \omega_2\,c - \omega_3\,b)\,\delta t, \qquad \delta b = (m + \omega_3\,a - \omega_1\,c)\,\delta t, \qquad \delta c = (n + \omega_1\,b - \omega_2\,a)\,\delta t,$$

([1]) APPELL. – T. II, 1st ed., pp. 79-80; 2nd ed., pp. 28.
([2]) APPELL. – T. II, 1st ed., pp. 128; 2nd ed., pp. 77.

in which $l, m, n, \omega_1, \omega_2, \omega_3$ are six arbitrary constants, and δt is an infinitely-small quantity, and that on the other hand, the trihedron $O'x'y'z'$ will submit to an infinitely-small rotation whose components along the axes *Ox*, *Oy*, *Oz* have the principal parts:

$$\omega_1\,\delta t, \quad \omega_2\,\delta t, \quad \omega_3\,\delta t\,.$$

Observe that in the present case, the variations $\delta\xi, \delta\eta, \delta\zeta, \delta p, \delta q, \delta r$ of the six expressions $\xi, \eta, \zeta, p, q, r$ are zero, as would result from the theory of the moving trihedron, and since we can verify that immediately, moreover, by means of formulas (20) and (21) by replacing $\delta_1 a'$, $\delta_1 b'$, $\delta_1 c'$, $\delta I'$, $\delta J'$, $\delta K'$ with their present values. It will then result that we will get a solution to the problem by taking *W* to be a function of the six expressions $\xi, \eta, \zeta, p, q, r$. An easy calculation that is entirely similar to the one in § **4**, except for a few details, leads to the *final remarkable result that one will then get the general solution to the problem when one assumes that the moving trihedron can take all possible motions.*

Having done that, we can imagine a trihedron $O'x'y'z'$, to which we associate a function *W* of the six arguments $\xi, \eta, \zeta, p, q, r$ that are determined by the motion of the trihedron $O'x'y'z'$ and which will inversely permit one to determine them in the known manner.

The function *W* will be what we call the *Euclidian action* of the moving trihedron at the instant *t*, and we shall call the product *W dt* of *W* with *dt* the *Euclidian action relative to the time interval dt*.

Before pursuing that idea, we first remark that if we would like to constitute an *invariable continuous body* in a manner that is analogous to what was done in § **9** by means of material points by speaking of the material trihedron then we should associate each point of the invariable body whose coordinates are x', y', z' with respect to the trihedron $O'x'y'z'$ with a function *w* of the position of that point in the body and six quantities $\xi, \eta, \zeta, p, q, r$, and we give the name of *Euclidian action over the body at the instant t* to the integral $\mathrm{S}\; w\,dw$ that extends over the volume of the body. Now, having made all calculations, we will then find a function *W* of $\xi, \eta, \zeta, p, q, r$.

We will not increase the generality of our discussion by passing to the material trihedron of the invariant body, and everything that we shall about the former will consequently apply to the latter.

Starting from the function *W*, we shall call the integral over the invariable body in question from the instant t_0 to the instant t_1 :

$$A = \int_{t_0}^{t_1} W\,dt$$

the *action* of that body.

Apply the method of variable action. *By the calculations of* § **9**, we will recover the formula:

$$\delta A = \left[F'\,\delta_1 a' + G'\,\delta_1 b' + H'\,\delta_1 c' + I'\,\delta I' + J'\,\delta J' + K'\,\delta K'\right]_{t_0}^{t_1}$$
$$- \int_{t_0}^{t_1} (X'\,\delta_1 a' + Y'\,\delta_1 b' + Z'\,\delta_1 c' + L'\,\delta I' + M'\,\delta J' + N'\,\delta K')\,dt\,,$$

along with formulas (22) and (23). We easily verify that the systems of quantities of motion and external forces, which are defined as in § **9**, are again independent of the chosen trihedron $O'x'y'z'$ here.

We will once more have formulas (24) and (25), which introduce the *kinetic energy E* of the material trihedron or the invariable body.

Formulas (17), (18), (19), (26) and (27), which are based upon the particular form of the function *W* of § **9**, will vanish. We will then have some formulas of a higher degree of generality than then ones in § **9**.

We likewise remark that if we suppose that *W* depends upon ξ, η, ζ, *p*, *q*, *r* only by the intermediary of the single quantity $\xi^2+\eta^2+\zeta^2$, which, from § **2**, represents the square v^2 of the velocity of the point O' with respect to the trihedron *Oxyz*, then we can take *W* to have the form *W* (*v*) after associating the function *v* in § **4** to the motion.

Upon taking the relation $v^2 = \xi^2+\eta^2+\zeta^2$ into account, formulas (22) and (23) will then give us:

$$F' = \frac{1}{v}\frac{dW}{dv}\,\xi\,, \quad G' = \frac{1}{v}\frac{dW}{dv}\,\eta\,, \quad H' = \frac{1}{v}\frac{dW}{dv}\,\zeta\,, \quad I' = 0\,, \quad J' = 0\,, \quad K' = 0\,,$$

$$\begin{aligned} X' &= \frac{dF'}{dt} + q\,H' - r\,G'\,, \qquad & L' &= 0\,, \\ Y' &= \frac{dG'}{dt} + r\,F' - p\,H'\,, \qquad & M' &= 0\,, \\ Z' &= \frac{dH'}{dt} + p\,G' - q\,F'\,, \qquad & N' &= 0\,. \end{aligned}$$

If we define the *quantity of motion* (F', G', H') and the *external force* (X', Y', Z') that issue from the point O' by their projections (*F*, *G*, *H*) and (*X*, *Y*, *Z*) onto the axes *Ox*, *Oy*, *Oz* then we will get back to equations (11) and (12) of § **5**.

Upon considering the general case of a material trihedron, that will lead us to the notion of *hidden trihedron*, whether directly or upon replacing it with an invariable body of very small dimensions. To that effect, it will suffice for us to assume that we direct our attention to *only* the motion of the point O' of the solid body or to *ignore* the existence of the trihedron that is attached to the point O' of the material trihedron. Now, upon supposing, to fix ideas, that the components of the vectors (X', Y', Z'), (L', M', N') that issue from O' with respect to the fixed trihedron *Oxyz* are determined as functions of time, and upon introducing the coordinates *a*, *b*, *c* of the point O' and three parameters λ, μ, ν by means of which one expresses the cosines $\alpha, \beta, \ldots, \gamma''$, equations (22) and (23) will lead to a system of differential equations that include the six expressions *a*, *b*, *c*, λ, μ, ν, and their first derivatives. If we have only *a*, *b*, *c* in mind then we can imagine that we have eliminated λ, μ, ν, and one will be led to some questions of the same type that were treated by KŒNIGSBERGER [1], as a result of some remarks that were made by HELMHOLTZ [2] and HERTZ [3].

12. Relative motion of the invariable body. – In addition to the moving trihedron $O'x'y'z'$ that is invariably coupled with an invariable body, consider an auxiliary trihedron $O_1 x_1 y_1 z_1$ that has the same disposition as *Oxyz*, like $O'x'y'z'$, and we define its position at the instant *t* by giving the coordinates *l*, *m*, *n* of the point O_1 with respect to the axes *Ox*, *Oy*, *Oz* at that instant and the direction cosines λ, λ', λ''; μ, μ', μ''; ν, ν', ν'' of the axes $O_1 x_1$, $O_1 y_1$, $O_1 z_1$, respectively.

Instead of referring the trihedron $O'x'y'z'$ to the trihedron *Oxyz* at each instant, we can refer to the corresponding position of the trihedron $O_1 x_1 y_1 z_1$. We call the coordinates of the point O' with respect to the latter trihedron a_1, b_1, c_1, and let α_1, α_1', α_1''; β_1, β_1', β_1''; γ_1, γ_1', γ_1'' denote the direction cosines of $O'x'$, $O'y'$, $O'z'$, respectively.

According to the usual terminology, we give the name of *absolute motion* and *relative motion* to the motions of the invariable body with respect to the trihedron *Oxyz* and the trihedron $O_1x_1y_1z_1$, respectively. Those motions will be defined by means of the trihedron $O'x'y'z'$. We give the name of *dragging motion* to the motion of the trihedron $O_1 x_1 y_1 z_1$ with respect to the trihedron *Oxyz*.

We preserve the notations of § **2** and **9** for absolute motions, and once more introduce the six expressions ξ, η, ζ, *p*, *q*, *r* that are defined by formulas (5) and (6) of § **2**, and that we shall presently denote by ξ_a, η_a, ζ_a, p_a, q_a, r_a. We let ξ_r, η_r, ζ_r, p_r, q_r, r_r and ξ_e, η_e, ζ_e, p_e, q_e, r_e denote the analogous quantities relative to the trihedra $O'x'y'z'$ and $O_1 x_1 y_1 z_1$, respectively, in the relative motion and the dragging motion, resp.

It is easy to determine ξ_a, η_a, ζ_a, p_a, q_a, r_a in terms of the analogous quantities ξ_e, η_e, ζ_e, p_e, q_e, r_e; ξ_r, η_r, ζ_r, p_r, q_r, r_r, and in terms of the quantities a_1, b_1, c_1, α_1, β_1, …, γ_1'' that define the motion of the trihedron $O'x'y'z'$ with respect to the trihedron $O_1 x_1 y_1 z_1$.

ξ_a, η_a, ζ_a are the projections of the absolute velocity of the point $O'x'$, $O'y'$, $O'z'$, resp. Now, from the formulas (7) in § **2**, the projections of the latter onto $O_1 x_1$, $O_1 y_1$, $O_1 z_1$, respectively, are:

[1] LEO KŒNIGSBERGER. – "Ueber die Principien der Mechanik," Berl. Ber. (1896), 899-944, 1173-1182, J. reine angew. Math. **118** (1897), 275-300; *ibid.* **119** (1898), 35-49. "Ueber verborgene Bewegung und unvollständige Probleme," Berl. Ber. (1897), 158-178. "Ueber die Darstellung der Kraft in der analytischen Mechanik," Berl. Ber. (1897), 885-900. "Ueber die erweiterte Laplace'sche Differentialgleichung für die allgemeine Potentialfunction," Berl. Ber. (1898), 5-18. "Ueber die erweiterte Laplace-Poisson'sche Potentialgleichung," Berl. Ber. (1898), 93-101. "Ueber das erweiterte Princip der Erhaltung der Flächen, etc.," Berl. Ber. (1898), 148-158. "Ueber die Erniedrigung der Anzahl der unabhängigen Parameter, etc.," Math. Ann. **51** (1898), 584-607; Berl. Ber. (1898), 491-495. "Ueber die allgemeinen kinetischen Potentiale," J. reine angew. Math. **121** (1899), 141-167. "Sur les principes de la Mécanique," Acta Math. **23** (1899), 63-83. *Die Principien der Mechanik, Mathematische Untersuchungen*, Leipzig, 1901.

[2] Cited articles; likewise consult: LEO KŒNIGSBERGER, Hermann von Helmholtz's *Untersuchungen über die Grundlagen der Mathematik und Mechanik*, Leipzig, 1896.

[3] H. HERTZ. – *Die Principien der Mechanik*, Leipzig, 1894.

$$\frac{da_1}{dt} + \xi_e + q_e\,c_1 - q_e\,c_1, \qquad \frac{db_1}{dt} + \eta_e + r_e\,a_1 - p_e\,c_1, \qquad \frac{dc_1}{dt} + \zeta_e + p_e\,b_1 - q_e\,a_1\,.$$

If one projects onto $O'x'y'z'$ and uses formulas that are analogous to formulas (5), which give ξ_r, η_r, ζ_r, then one will have:

$$\begin{aligned}
\xi_a &= \xi_r + \alpha_1(\xi_e + q_e\,c_1 - q_e\,c_1) + \alpha_1'(\eta_e + r_e\,a_1 - p_e\,c_1) + \alpha_1''(\zeta_e + p_e\,b_1 - q_e\,a_1)\,,\\
\eta_a &= \eta_r + \beta_1(\xi_e + q_e\,c_1 - q_e\,c_1) + \beta_1'(\eta_e + r_e\,a_1 - p_e\,c_1) + \beta_1''(\zeta_e + p_e\,b_1 - q_e\,a_1)\,,\\
\zeta_a &= \zeta_r + \gamma_1(\xi_e + q_e\,c_1 - q_e\,c_1) + \gamma_1'(\eta_e + r_e\,a_1 - p_e\,c_1) + \gamma_1''(\zeta_e + p_e\,b_1 - q_e\,a_1)\,.
\end{aligned}$$

In order to obtain p_a, q_a, r_a, it suffices to replace $\alpha, \beta, \ldots, \gamma''$ with their values as functions of $\lambda, \lambda', \ldots, \nu''$ and $\alpha_1, \beta_1, \ldots, \gamma_1''$ in formulas (6) of § **2**, and by virtue of the hypothesis that the trihedra $O'x'y'z'$ and $O_1\,x_1\,y_1\,z_1$ have the same disposition, the following formulas will result immediately, whose interpretation is well-known:

$$\begin{aligned}
p_a &= p_r + \alpha_1\,p_c + \alpha_1'\,q_e + \alpha_1''\,r_e\,,\\
q_a &= q_r + \beta_1\,p_c + \beta_1'\,q_e + \beta_1''\,r_e\,,\\
r_a &= r_r + \gamma_1\,p_c + \gamma_1'\,q_e + \gamma_1''\,r_e\,.
\end{aligned}$$

Having done that, suppose that we replace $\xi_a, \eta_a, \ldots, r_a$ with their values that were given by the preceding formulas in the function $W(\xi_a, \eta_a, \zeta_a, p_a, q_a, r_a)$ that was defined either as it was in § **9** or more generally in § **11**. We will get a function of $\xi_a, \eta_a, \zeta_a, p_a, q_a, r_a$ and $a_1, b_1, c_1, \alpha_1, \beta_1, \ldots, \gamma_1'', \xi_r, \eta_r, \zeta_r, p_r, q_r, r_r$, and with the hope of avoiding any confusion, we shall employ the symbol W_r for it. Formulas (22) become:

$$(30)\qquad F' = \frac{\partial W_r}{\partial \xi_r}, \quad G' = \frac{\partial W_r}{\partial \eta_r}, \quad H' = \frac{\partial W_r}{\partial \zeta_r}, \quad I' = \frac{\partial W_r}{\partial p_r}, \quad J' = \frac{\partial W_r}{\partial q_r}, \quad K' = \frac{\partial W_r}{\partial r_r}.$$

Formulas (23) then determine X', Y', Z', L', M', N'.

If we suppose that we have replaced $\xi_e, \eta_e, \zeta_e, p_e, q_e, r_e$ with their values as functions of time in the function W_r then we will finally get a function [1] of $t, a_1, b_1, c_1, \alpha_1, \beta_1, \ldots, \gamma_1'', \xi_r, \eta_r, \zeta_r, p_r, q_r, r_r$ that we will continue to denote by W_r, and equations (30) and (23) will determine $F', G', H', \ldots, X', Y', \ldots, N'$.

The formulas that one obtains, which can be used to resolve the inverse question of determining the motion, give rise to the following remark:

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[1] One notes that any function of t, as well as $a_1, \ldots, \alpha_1, \ldots, \gamma_1''$, and their first derivatives will be a function of $t, a_1, b_1, c_1, \alpha_1, \ldots, \gamma_1'', \xi_r, \ldots, r_r$, and inversely.

Suppose that we replace W with its value W_r as a function of t, a_1, b_1, c_1, α_1, β_1, …, γ_1'', ξ_r, …, r_r in the calculation of the action from the instant t_0 to the instant t_1 and then seek to apply the method of variable action to the expression that we obtain:

$$A = \int_{t_0}^{t_1} W\,dt\,,$$

upon supposing that a_1, b_1, c_1, α_1, β_1, …, γ_1'' submit to variations δa_1, …, $\delta\gamma_1''$. Upon introducing the auxiliary variables $\delta I_r'$, $\delta J_r'$, $\delta K_r'$, $\delta_1 a_1'$, $\delta_1 b_1'$, $\delta_1 c_1'$, which are analogous to the ones in § **9**, one will get the formula:

$$\delta A = \left[F'\,\delta_1 a_1' + \cdots + K'\,\delta K_r'\right]_{t_0}^{t_1} - \int_{t_0}^{t_1} (X'\,\delta_1 a_1' + \cdots + N'\,\delta K')\,dt\,,$$

which would result *a priori* from the calculations in § **9**, and which one can verify directly, moreover. That formula is analogous to the one in § **9**, and in which F', …, N' are the expressions that were defined by formulas (30) and (23).

We infer the conclusion from that formula that it will amount to the same thing as either referring the motion to the trihedron $Oxyz$ and attaching the body to the function W of ξ, η, ζ, p, q, r or referring the motion to the trihedron $O_1\,x_1\,y_1\,z_1$ and attaching the body to the function W_r of t, a_1, b_1, c_1, α_1, β_1, …, γ_1'', ξ_r, …, r_r : The method of variable action leads to the same system of quantities of motion and the same system of external forces in both cases.

The function W_r that we just envisioned has a particular form with respect to its arguments. Much later, that will lead us to envision the case in which we apply the method of variable action to the integral:

$$A = \int_{t_0}^{t_1} W\,dt\,,$$

in which W depends upon t, a, b, c, α, β, …, γ'', ξ, η, ζ, p, q, r, and the body being dragged by the trihedron $O'\,x'\,y'\,z'$ is referred to the trihedron $Oxyz$.

13. Some bibliographic information. – The article that VOSS wrote in the *Encyklopädie der mathematischen Wissenschaften*, which is currently being published in Germany ([1]). includes very complete bibliographic information that extends up to 1901. We can do no better than to refer the reader to it. Here, we shall be content to simply add the following suggestions to the ones that VOSS gave, which refer to the present treatise, in particular.

([1]) A. VOSS. – "Die Prinzipien der rationellen Mechanik," *Encyklopädie der mathematischen Wissenschaften mit Einschluss ihrer Anwendungen*, Band IV_1, pp. 3-121, Leipzig, 1901.

R. BLONDLOT. – "Exposé des principes de la Mécanique," Bibliotheque du Congrés international de Philosophie **3** (1901) [1900], 445-455.

DE SAINT-VENANT. – "Mémoire sur les sommes et les differences géométriques, et sur leur usage pour simplifier la Mécanique," C. R. Acad. Sci. Paris **21** (1845), 620-625; *Principes de Mécanique fondés sur la Cinématique*, Paris, 1851; "Notice sur LOUIS-JOSEPH comte DU BUAT," Mémoires de la Société des Sciences de Lille (3) **2** (1866) [1865], 669-679. "De la constitution des atomes," Annales de la Société scientifique de Bruxelles T. II, 2nd Partie (1878), 417-456.

EULER. – *Letters to a German princess on various topics in physics and philosophy*, Saint Petersburg, 1768-1772.

HARTMANN. – "Definition physique de la force," L'Enseignement mathématique **6** (1904), 425-438; Revue de Métaphysique et de Morale **12** (1904), 935-948.

T. KÖRNER. – "Der Begriff des materiallen Punktes in der Mechanik des achtzehnten Jahrhunderts," Biblioteca Mathematica (3) **5** (1904), 935-948.

E. PASQUIER. - *Cours de Mécanique analytique*, t. I, Louvain, Paris, 1901. On page 183 of that book, one will find a list of works in which mass is considered to be a function of time and that can be compared to reference 136, pp. 52 in the VOSS article.

H. POINCARÉ. – "Les idées de HERTZ sur la Mécanique," Revue générale des Sciences **8** (1897), 734-743; "Les rapports de l'analyse pure et de la physique mathématique," Revue générale des Sciences **8** (1897), 857-861; *Verhandlungen des ersten internationalen Mathematiker-Kongresses*, etc., 1898, pp. 81-90. "La mesure du temps," Revue de Mécanique et de Morale **6** (1898), 1-13. "Sur les principes de la Mécanique," Bibliotheque du Congrés international de Philosophie **3** (1901) [1900], 457-499. "Relations entre la physique expérimentale at la physique mathématique," Rapports présentés au Congrés international de Physique de 1900, t. I, pp. 1-29. An English translation of the last article appeared in *The Monist*, t. XII, pp. 516-543. Those various articles were reproduced almost completely in the two works that were cited in § **1**.