COURSE IN GEOMETRY OF THE SCIENCE FACULTY

LESSONS

ON THE GENERAL THEORY

OF SURFACES

AND THE

GEOMETRIC APPLICATIONS TO THE INFINITESIMAL CALCULUS

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PART ONE

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PREFACE

The first part of the book that I publish today is a summary of the lectures that I gave at the Sorbonne during the Winters of 1882 and 1885. I have begun my exposition of the *Theory of surfaces* with the single objective of finding new applications for the theory of partial differential equations, which is vast and little known. I was planning to devote barely a year to teaching it, but the interest that the subject presented, and also the demands of my listeners, obliged me to impose the limits that I originally established.

The first volume is composed of three distinct parts: The first book treats some *applications of the theory of relative motion to geometry*. I will have to return to the propositions that are presented in it in a later part in which the beautiful formulas of Codazzi will be studied in all the necessary detail. The second book contains the study of the *different systems of curvilinear coordinates*. In it, I successively consider the systems of conjugate lines, whose study has been neglected too often, the asymptotic lines, the lines of curvature, and the orthogonal and isothermal systems.

The volume concludes with the *theory of minimal surfaces*, in which I benefit from the quite remarkable work that has been published by some eminent geometers in recent years; it defines almost half of this volume. Except for the last three chapters, which were re-edited at the time of printing, these lectures were taught with two different repetitions in 1882 and 1885. One or two important questions have been omitted. They would find a better place in what follows after I have given the general propositions to which one can attach them.

Consistent with its usual practice, Gauthier-Villars devoted all of its effort to the printing of this book after receiving it. They receive my most enthusiastic acknowledgements here. I must also extend them to my listeners, who desired to see these lectures published, and more especially, to one of our young geometers – namely, G. Koenigs, Maître de Conférences à l'École Normale – who kindly assisted me in the revision of the proofs.

14 June 1887.

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PART ONE

BOOK I

APPLICATIONS OF THE THEORY OF RELATIVE MOTIONS TO GEOMETRY

CHAPTER I

ONE-PARAMETER DISPLACEMENTS. APPLICATION TO THE THEORY OF SKEW CURVES

Displacement of an invariable system. – Application to the theory of skew curves. Characteristic property of the helix. – Formulas of J.-A. Serret. – Spherical indicatrix. – Search for the curve whose principal normals are also principal normals of another curve. – Developments of skew curves.

1. Consider a solid body or invariable system that moves around a fixed point. One knows that at an arbitrary instant the velocities of the various points of the system are the same as if they turned around a line that passes through the fixed point, which is a line that has been given the name of *instantaneous axis of rotation.* In mechanics, one proves that rotations, like forces, can be represented geometrically by lines and can be composed or decomposed according to the same law; i.e., if one composes or decomposes the rotations like the forces then the velocity that is implied by the resulting rotation at an arbitrary point is the resultant of the velocities that will be communicated to that point by each of the component rotations in isolation. One also knows that if one considers a point that moves with respect to the invariable system then the absolute velocity of that point is the resultant of its relative velocity and its *driven* velocity (*vitesse d'entraînement*). One applies that name to the velocity that a point will have that coincides with the moving point at the instant considered, but remains invariably linked to the solid system.

It results from these propositions that one can construct the velocities of all points of the invariable system at an arbitrary instant once one has the magnitude and direction of the rotation at that instant. It seems natural to determine that rotation at each instant by its components relative to three rectangular axes that are fixed in space and have the fixed point of the solid system for their origin. In reality, the most important elements (which are the only ones that most frequently permit a deeper study of motion) are the components of the rotation relative to the moving axes that is carried by the motion of the invariable system. We quickly recall the method that is employed in mechanics.

Let *OX*, *OY*, *OZ* be three fixed axes that pass through the fixed point *O* of the system, and let *Ox*, *Oy*, *Oz* be three rectangular axes that are coupled invariably with the moving system. We suppose that the two systems of axes have the same disposition – i.e., that they can be made to coincide. Furthermore, we suppose that the sense of the axes has been chosen in such manner that the rotation around *OZ* that displaces *OX* to *OY* will be represented by a line that is directed along the positive half of *OZ*. We determine the moving axes by the cosines of the angle that they form with the fixed axes. In order to do that, we write the table:

	x	y	z
X	a	b	c
Y	a'	b'	c'
Z	a''	b''	c''

which shows the cosines of the angles that each of the fixed axes defined with the moving axes.

One has the relations:

$$
(1)\qquad \left\{\begin{array}{ll}
a^2+b^2+c^2=1, & aa'+bb'+cc'=0, \\
a^2+a'^2+a''^2=1, & ab+a'b'+a''c''=0, \\
a=b'c''-c'b'', & \begin{vmatrix} a & b & c \\ a' & b' & c' \\ a'' & b'' & c'' \end{vmatrix}=1,
\end{array}\right.
$$

to which one must add the ones that one obtains by performing circular permutations of the symbols or their indices. Further recall that the nine cosines can be expressed in terms of the three Euler angles by means of the formulas ([1]):

([1]) In these formulas, ψ denotes the angle between *OX* and the common intersection *ON* of the *xy*-plane with the *XY*-plane, and φ denotes the angle between *Ox* and the same line *ON*. Finally, θ is the angle

$$
(2) \qquad \begin{cases}
a = \cos\theta \sin\varphi \sin\psi + \cos\varphi \cos\psi, \\
b = \cos\theta \sin\psi \cos\varphi - \cos\psi \sin\varphi, \\
c = \sin\theta \sin\psi, \\
a' = \cos\theta \cos\psi \sin\varphi - \sin\psi \cos\varphi, \\
b' = \cos\theta \cos\psi \cos\varphi + \sin\psi \sin\varphi, \\
c' = \sin\theta \cos\psi, \\
a'' = -\sin\theta \cos\psi, \\
b'' = -\sin\theta \cos\varphi, \\
c'' = \cos\theta.
\end{cases}
$$

Now, denote the components of the rotation at the instant t with respect to the moving axes by p, q, r. Consider a point whose coordinates are x, y, z relative to the moving axes and look for the components of its *absolute* velocity with respect to the same axes. Upon writing that the absolute velocity is the resultant of the relative velocity and the ones that are due to the three rotations p, q, r, one will obtain the following expressions for those components:

$$
(3) \qquad \begin{cases}
V_x = \dfrac{dx}{dt} - qz - ry, \\
V_y = \dfrac{dy}{dt} + rx - pz, \\
V_z = \dfrac{dz}{dt} + py - qx,
\end{cases}
$$

which we will often make use of.

We shall now show how one can deduce the expressions for p, q, r as functions of the nine cosines and their derivatives with respect to time. In order to do that, consider the point that is taken on the OX axis at a distance of 1. That point has the relative coordinates (i.e., relative to the moving axes) a, b, c. Upon expressing the idea that its velocity is zero and applying formulas (3), we will obtain the fundamental equations:

$$
(4) \qquad \begin{cases}
\dfrac{da}{dt} = br - cq, \\
\dfrac{db}{dt} = cp - ar, \\
\dfrac{dc}{dt} = aq - bp,
\end{cases}
$$

---

between Oz and OZ. The angle φ measures the magnitude of the rotation that one must impart to ON in the xy-plane, and in the direct sense, in order to make ON coincide with Ox. One can then suppose that it varies from 0 to 180°. Similarly, ψ measures the rotation that one must impart to ON in the XY-plane – always in the direct sense – in order to make that line coincide with OX; that angle will vary from 0 to 360°.

to which, one can add the following ones:

$$(4') \qquad \left\{\begin{aligned} \frac{da'}{dt} &= b'r - c'q, \\ \frac{db'}{dt} &= c'p - a'r, \\ \frac{dc'}{dt} &= a'q - b'p, \end{aligned}\right.$$

$$(4'') \qquad \left\{\begin{aligned} \frac{da''}{dt} &= b''r - c''q, \\ \frac{db''}{dt} &= c''p - a''r, \\ \frac{dc''}{dt} &= a''q - b''p, \end{aligned}\right.$$

which one proves in the same manner.

One deduces the following formulas from that:

$$(5) \qquad \left\{\begin{aligned} p\,dt &= \sum c\,db = -\sum b\,dc, \\ q\,dt &= \sum a\,dc = -\sum c\,da, \\ r\,dt &= \sum b\,da = -\sum a\,db, \end{aligned}\right.$$

which give the desired values of the rotation. If one replaces the cosines with their expressions as functions of the Euler angles then one will have the system:

$$(6) \qquad \left\{\begin{aligned} p &= \sin\varphi\sin\theta\frac{d\psi}{dt} - \cos\varphi\frac{d\theta}{dt}, \\ q &= \cos\varphi\sin\theta\frac{d\psi}{dt} + \sin\varphi\frac{d\theta}{dt}, \\ r &= \frac{d\varphi}{dt} - \cos\theta\frac{d\psi}{dt}, \end{aligned}\right.$$

which is easy to prove geometrically. Upon solving that system for the derivatives of the angles, one will find that:

$$(7) \qquad \left\{\begin{aligned} \frac{d\theta}{dt} &= q\sin\varphi - p\cos\varphi, \\ \sin\theta\frac{d\psi}{dt} &= q\cos\varphi + p\sin\varphi, \\ \frac{d\varphi}{dt} &= r + \cot\theta(p\sin\varphi + q\cos\varphi). \end{aligned}\right.$$

2. Having recalled all of that, we shall study the following problem, which is fundamental in our theory: *Determine the motion completely when one is given p, q, r as functions of time t.*

It is clear that the question will be solved if one has expressions for the nine cosines as functions of time. Now, it results immediately from formulas (4) that if one separates the cosines into three groups that are composed of a, b, c; a', b', c'; a'', b'', c'', respectively, then the three cosines in each group will be the simultaneous solutions of the system:

$$\left\{\begin{aligned} \frac{d\alpha}{dt} &= \beta r - \gamma q, \\ \frac{d\beta}{dt} &= \gamma p - \alpha r, \\ \frac{d\gamma}{dt} &= \alpha q - \beta p. \end{aligned}\right. \tag{8}$$

All of the difficulty then reduces to the integration of that system. The detailed study of that integration will be the subject of the following chapter. For the moment, we shall be content to point out the following properties of system (8):

First, as a result of its linear form, it will always admit one and only one solution for which the initial values of α, β, γ are given.

In the second place, if α, β, γ; α', β', γ' denote two systems of arbitrary solutions then the expressions:

$$\alpha^2 + \beta^2 + \gamma^2, \qquad \alpha\alpha' + \beta\beta' + \gamma\gamma', \qquad \alpha'^2 + \beta'^2 + \gamma'^2$$

will be constants. One easily recognizes that by differentiating them and taking equations (8) into account.

Those properties permit us to establish that there will always be an infinitude of motions under which those three quantities will be the components of the rotation relative to the moving axes, no matter what the expressions for p, q, r are as functions of time.

Indeed, consider a tri-rectangular trihedron (T_0) with the same sense as the trihedron $OXYZ$ that is formed from the fixed axes, and let a_0, b_0, c_0, … be the direction cosines of OX, OY, OZ with respect to the axes of (T_0). Determine the three systems of solutions of equations (8), a, b, c; a', b', c'; a'', b'', c'', which correspond to the following initial values a_0, b_0, c_0 ; a_0', b_0', c_0'; a_0'', b_0'', c_0''.

From the properties of system (8), functions such as:

$$\alpha^2 + \beta^2 + \gamma^2, \qquad \alpha\alpha' + \beta\beta' + \gamma\gamma', \qquad \ldots$$

that have the initial values 1 or 0 and must remain constant will not cease to keep their initial values. Consequently, at each instant, the nine quantities a, a', a'', … will be the direction cosines of the three rectangular lines that are defined by a moving trihedron (T) whose initial position will be (T_0). Since that initial position can be chosen at will, one sees that there exists an infinitude of motions for which p, q, r are given functions of time.

All of those motions, which depend upon three arbitrary constants, basically depend upon just one, but when they are referred to different fixed axes.

Indeed, consider the position that is occupied by the moving trihedron at the initial instant in any of them, and choose the fixed system of axes to which one refers the motion of the moving system. The initial values of the nine cosines are then 1 or 0, so the solution that corresponds to those numerical values will contain no arbitrary constant, and will then be well-defined.

It results from the preceding that when one has obtained one arbitrary solution of the problem – i.e., a system of values for the nine cosines – if one would like to get the most general solution then it would suffice to change the fixed axes, which will introduce three constants, and then suppose that the new formulas are referred to the old axes.

3. We shall now study the case in which the moving system no longer has a fixed point. One must then add the components of the velocity of the origin O of the moving axes, which are always taken relative to the moving axes Ox, Oy, Oz, to the components p, q, r ; denote them by ξ, η, ζ. When combined with the three rotations, they will intervene in all questions that relate to the study of motion. Suppose that one knows the expressions for those six quantities as functions of time, and try to find how one can determine the motion of the moving trihedron. Let (T) denote that moving trihedron, and let (T') be the trihedron whose origin is an arbitrary fixed point and whose axes are parallel to those of (T). At an arbitrary instant, the two trihedra are animated with the same rotation, and consequently, the nine cosines will be determined by means of p, q, r as in the preceding case. Moreover, if X_0, Y_0, Z_0 denote the coordinates of the moving origin O with respect to the fixed axes then one will obviously have:

$$(9) \qquad \begin{cases} \dfrac{dX_0}{dt} = a\xi + b\eta + c\zeta, \\ \dfrac{dY_0}{dt} = a'\xi + b'\eta + c'\zeta, \\ \dfrac{dZ_0}{dt} = a''\xi + b''\eta + c''\zeta \end{cases}$$

upon projecting the velocity of that origin onto the fixed axes.

When one has determined the cosines, these formulas will give one X_0, Y_0, Z_0 by simple quadratures, which will introduce three new constants.

Here again, all of the possible motions that correspond to different values of the six arbitrary constants reduce to one and the same motion that is observed with respect to different axes, because the integration introduces no arbitrary constant and gives only one motion if one supposes that the fixed axes coincide with the initial position of the moving axes.

In regard to the case that we just considered, I recall that if x, y, z are the coordinates of a point relative to the moving axes then the absolute velocity of that point will have the three quantities:

(10)
$$\begin{cases} V_x = \xi + qz - ry + \dfrac{dx}{dt}, \\[2mm] V_y = \eta + rx - pz + \dfrac{dy}{dt}, \\[2mm] V_z = \zeta + py - qx + \dfrac{dz}{dt} \end{cases}$$

for its components relative to those same axes.

For example, consider the points that are invariably linked with the moving system and look for the ones for which the velocity is a minimum. One must determine the values of x, y, z that give a minimum to the sum:

$$(\xi + qz - ry)^2 + (\eta + rx - pz)^2 + (\zeta + py - qx)^2.$$

Upon equating the derivatives with respect to x, y, and z to zero, one will obtain three equations that reduce to the following two:

$$\frac{\xi + qz - ry}{p} = \frac{\eta + rx - pz}{q} = \frac{\zeta + py - qx}{r}.$$

These two equations represent a line – viz., the *central axis* of the motion at the instant considered. One easily finds that the common value of the preceding ratios is:

$$\frac{\xi p + \eta q + \zeta r}{p^2 + q^2 + r^2},$$

which gives:

$$\frac{\xi p + \eta q + \zeta r}{\sqrt{p^2 + q^2 + r^2}}$$

for the minimum value of the velocity.

The necessary and sufficient condition for the motion of the system to reduce to a simple rotation is then the following one:

$$\xi p + \eta q + \zeta r = 0,$$

and the axis of rotation in that case will be represented by the three equations:

(11)
$$\begin{cases} \xi + qz - ry = 0, \\ \eta + rx - pz = 0, \\ \zeta + py - qx = 0, \end{cases}$$

which indeed characterize the points whose velocity is zero, as formulas (10) show.

4. In order to now point out an application of the preceding proposition, consider an arbitrary skew curve and study the motion of the trihedron (*T*) that is defined by the tangent, which we take to be the *x*-axis, the principal normal, which we take to be the *y*-axis, when we suppose that it is, for example, directed towards the center of curvature, and the binormal, which will be the *z*-axis, whose sense is defined by the conventions that were made already.

Take the arc length *s* to be the independent variable, or – what amounts to the same thing – suppose that:

$$\frac{ds}{dt} = 1.$$

Here, one has:

$$\xi = 1, \quad \eta = 0, \quad \zeta = 0,$$

and if *x*, *y*, *z* denote the coordinates with respect to fixed axes of the point of the curve that is the summit of the trihedron then:

$$a = \frac{dx}{ds}, \qquad a' = \frac{dy}{ds}, \qquad a'' = \frac{dz}{ds}.$$

The general formulas (4) give us:

$$(12) \qquad da = (br - cq)\,ds, \qquad db = (cp - ar)\,ds, \qquad dc = (aq - bp)\,ds.$$

We express the idea that the binormal – whose direction cosines are c, c', c'' – is perpendicular to the osculating plane; i.e., to the two lines whose direction cosines are a, a', a, and $a + da$, $a' + da'$, $a'' + da''$. One of the equations will be satisfied by itself, and the other one will give us the condition:

$$\sum c\,da = q\,ds = 0.$$

The component *q* must be zero, and formulas (12) will reduce to the following ones:

$$(13) \qquad \frac{da}{ds} = br, \qquad \frac{db}{ds} = cp - ar, \qquad \frac{dc}{ds} = -\,bp.$$

It is easy to obtain the geometric significance of the rotations *p* and *r*.

In fact, draw parallels to the edges of the trihedron (*T*) through a fixed point. We will obtain a trihedron (T_1) whose rotation is the same at an arbitrary instant as that of the trihedron (*T*). A point that is situated at the distance 1 from the *x*-axis of the trihedron (T_1) will have a velocity whose components are, from formulas (3):

$$0, \quad r, \quad 0,$$

and consequently that point will describe the path $r\,ds$, or – what amounts to the same thing – the tangent to the curve will turn through the angle $r\,ds$ when its point of contact describes the arc ds. Thus, the component r will be equal to the first curvature of the curve.

Upon taking a point that is situated at the distance 1 on the z-axis of the trihedron (T_1), one will likewise see that the components of its velocity will be:

$$0, \quad -p, \quad 0,$$

and consequently, the osculating plane will turn through the angle $-p\,ds$ when the point of the curve describes the arc ds. In other words, $-p$ will be the torsion of the curve. One can then set:

$$r = \frac{1}{\rho}, \qquad p = -\frac{1}{\tau}, \tag{14}$$

in which ρ and τ denote the radii of curvature and torsion, and formulas (13) become:

$$\frac{da}{ds} = \frac{b}{\rho}, \qquad \frac{dc}{ds} = \frac{b}{\tau}, \qquad \frac{db}{ds} = -\frac{c}{\tau} - \frac{a}{\rho}. \tag{15}$$

One recognizes the formulas of J.-A. Serret, which play a very important role in the theory of skew curves.

5. The method that we just established exhibits some propositions that are easy to prove in a different way, and which one makes continual use of in geometric proofs.

If one is given the skew curve (C), and one draws a parallel to the tangent of that curve that has a length equal to 1 through the origin then the extremity of that parallel will describe a spherical curve that we – with P. Serret – call the *spherical indicatrix* of the skew curve. It results from the preceding that the tangent to the spherical indicatrix will be parallel to the principal normal of the curve (C), because the point that describes the indicatrix is the one that is situated at a distance 1 on the x-axis of the trihedron (T_1), and we have seen that the velocity of that point is equal to $1 / \rho$ and parallel to the principal normal.

Similarly, if we draw a line of length 1 through the origin that is parallel to the binormal then the extremity of that line will be the point at a distance 1 on the z-axis of the trihedron (T_1). That point will have a velocity that is equal to $1 / \tau$ and which will be again parallel to the principal normal. The spherical curve that it describes will be parallel to the indicatrix. One obtains it by measuring out a length that is equal to one quadrant on the great circles that are normal to the indicatrix in a convenient sense; in other words, it will be the *polar curve* to the spherical indicatrix.

6. We further point out the following theorem, which is very important ([2]):

([2]) On the subject of this theorem, *see*:

Any curve for which the ratio ρ / τ is constant is a helix that is traced on an arbitrary cylinder.

Indeed, if we consider the motion of the trihedron (T_1) that is parallel to (T) then we will know that since the component q is zero, the instantaneous axis of rotation will always be in the xz-plane. When the ratio ρ / τ or $- p / r$ remains constant, that instantaneous axis will be fixed with respect to the moving axes. Now, one knows that when the instantaneous axis occupies an invariable position with respect to the moving system, it will remain fixed in space. The trihedron (T_1) will then turn around a fixed axis. Its x-axis, which is parallel to the tangent to the curve, will make a constant angle with that fixed line and generate a cone of revolution. One recognizes the characteristic property of the helix that is traced on an arbitrary cylinder.

When ρ and τ are constants, that helix will be traced on a cylinder of revolution. Indeed, in that case, the motion of the trihedron (T) will present a translation and an invariable rotation at each instant. All points of the moving system, and in particular, the origin of the trihedron, will then describe helices that are traced on right circular cylinders.

7. Three of the six quantities $\xi, \ldots, p, \ldots$ will be zero under the motion that we just studied. We shall show that, conversely, if one has:

$$\eta = \zeta = q = 0$$

then the origin of that trihedron will describe a curve that is tangent to the x-axis and admits the y-axis for its principal normal. The first point results immediately from the equations:

$$\eta = \zeta = 0.$$

On the other hand, since the component q is zero, we will have:

$$\sum c\, da = 0.$$

The z-axis of the moving trihedron is then normal to the two consecutive positions of the x-axis. In other words, the xy-plane is the osculating plane of the curve that is described by the origin of the coordinates.

Upon reducing all velocities by the same ratio, in such a manner that ξ becomes equal to 1, one must replace p, r with $p / \xi, r / \xi$. The curvature and the torsion of the curve will then be given by the formulas:

$$\frac{1}{\rho} = \frac{r}{\xi}, \qquad \frac{1}{\tau} = \frac{-p}{\xi}. \tag{16}$$

PUISEAUX, "Problème de Géométrie," Journal de Liouville (1) **7**. BERTRAND, "Sur la courbe dont les deux courbures sont constants," Journal de Liouville (1) **8**. LIOUVILLE, *Application de l'Analyse à la Géométrie*, by Monge, 5th ed., Note I.

8. The kinematic method that we just presented applies in an elegant manner to the complete solution of the following problem, which was solved completely by BERTRAND ([3]): *Find out whether there exists a curve whose principal normals are also principal normals to another curve.*

Let M be a point of the given curve, and let (T) be the trihedron that relates to that point. If one measures out a length $MM' = a$ on the principal normal then the velocity of the point M' will have the components:

$$1 - ra, \quad \frac{da}{ds}, \quad pa$$

along Mx, My, Mz, respectively; that results from formulas (10). If one desires that the curve that is described by the point M' should be normal to MM' then it will be necessary that one must have:

$$\frac{da}{ds} = 0;$$

i.e., a must be constant. That result should have been obvious *a priori*, and we could have supposed it immediately.

The velocity v of M is then perpendicular to My, and if one lets ω denote the angle that it makes with Mx then one will have:

$$(17) \qquad \begin{cases} v \cos \omega = 1 - ra, \\ v \sin \omega = pa. \end{cases}$$

The line $M'M$ will then be normal to the curve that is described by the point M', but it will not be the principal normal, in general. Construct the trihedron (T') that is defined by the tangent $M'x'$ to the curve that is described by the point M', the line $M'y'$, and the perpendicular that is common to those two lines, and remark that the y-axis of that trihedron will coincide with the axis of the same name in (T). One will have a trihedron that has the same orientation as (T') by making the trihedron (T) turn through the angle ω around its y-axis. One will then obtain the instantaneous rotation of the trihedron (T') by composing the two rotations p, r of the trihedron (T) with a rotation $d\omega / dt$ around My.

Now, as we have seen, the necessary and sufficient condition for the line My or $M'y$ to be the principal normal of the curve that is described by the point M' is that the rotation of (T') around $M'y$ must be zero. It will then be necessary that one must have:

$$\frac{d\omega}{dt} = 0,$$

([3]) J. BERTRAND, "Mémoire sur la théorie des courbes à double courbure," Journal de Liouville (1) **15**, pp. 332. *See* also the paper of Bonnet that was included in the 32nd letter of the Journal de l'École Polytechnique, in which the author proved (pp. 134) that if two curves have the same principal normals then their osculating planes at the corresponding points will make a constant angle.

and, in turn, that the angle ω must be constant. Thus, *the osculating planes of the curves that are described by the point M, M′ must cut at a constant angle* ω.

If one now refers to formulas (17) then one will deduce:

$$\frac{\sin\omega}{a} = r\sin\omega + p\cos\omega,$$

by eliminating v or, upon replacing r and p with their geometric expressions:

$$\frac{\sin\omega}{a} = \frac{\sin\omega}{\rho} - \frac{\cos\omega}{\tau}. \tag{18}$$

There is then a linear relation between the two curvatures.

9. Conversely, if there exists a linear relation between the two curvatures:

$$C = \frac{A}{\rho} + \frac{B}{\tau}$$

then the curve will generally enjoy the indicated property. One identifies the preceding relation with equation (18), and one will have:

$$a = \frac{A}{C}, \qquad \cot\omega = -\frac{B}{A}.$$

Meanwhile, we point out two exceptional cases:

If one has $C = 0$, without A being zero, then the relationship between the curvatures will take the form:

$$\frac{\rho}{\tau} = \text{const.},$$

and a will become infinite. Hence, the second curve, which is the locus of M', will be pushed out to infinity. The proposed curve will then be a helix.

If one has $A = 0$ – i.e., if the curve has constant torsion – then a will be zero, and the two curves that are the loci of M and M' will coincide.

One can have more than two curves that have the same principal normals when:

1. The value of a is indeterminate; i.e., if one has $A = C = 0$. In this case, the curve will be planar.

2. There is more than one linear relation between the curvatures; i.e., if the two curvatures are constant. In this case, equation (8) will satisfied for any value of a and will give one ω. There will then be an infinitude of curves that have the same principal normals. The original curve (and consequently, all of the other ones) will be a helix that is traced on a right circular cylinder. The surface that is defined by the principal normals will be the skew helicoid with a director plane.

10. We return to the general case and look for the two curvatures of the curve that is the locus of M'. The trihedron (T') that relates to that curve is invariably linked with the trihedron (T). In order to get the components p', r' relative to the trihedron (T'), it will then suffice to project the rotations p, r onto the axes of (T'). That will give:

$$\begin{aligned} p' &= \ \ p \cos \omega + r \sin \omega, \\ r' &= - p \sin \omega + r \cos \omega. \end{aligned}$$

Now, from formulas (16), one will have:

$$p' = -\frac{v}{\tau'}, \qquad r' = \frac{v}{\rho'},$$

if one denotes the two curvatures of the curve that is the locus of (M) by $1 / \rho'$, $1 / \tau'$.

When one substitutes the expressions for p, q, p', q', the preceding relations will then give us:

$$(19) \qquad \left\{ \begin{aligned} \frac{v}{\tau'} &= \frac{\cos\omega}{\tau} - \frac{\sin\omega}{\rho}, \\ \frac{v}{\rho'} &= \frac{\sin\omega}{\tau} + \frac{\cos\omega}{\rho}, \end{aligned} \right.$$

which are formulas that one add to the following system, which is obtained by replacing r and p with their expressions in formulas (17):

$$(20) \qquad \left\{ \begin{aligned} v\cos\omega &= 1 - \frac{a}{\rho}, \\ v\sin\omega &= -\frac{a}{\tau}. \end{aligned} \right.$$

Formulas (19) and (20) contain all of the relations between the two curves. One deduces from them, for example, that:

$$\frac{\cos\omega}{\tau'} + \frac{\sin\omega}{\rho'} = -\frac{\sin\omega}{a},$$

which is a linear relation between the two curvatures of the new curve whose existence was obvious *a priori*. Moreover, system (19) can be replaced with the following one:

$$(21) \qquad \left\{ \begin{aligned} -\frac{\cos\omega}{v} &= 1 + \frac{a}{\rho'}, \\ \frac{\sin\omega}{v} &= \frac{a}{\tau'}, \end{aligned} \right.$$

which is much simpler (4).

One of the more interesting particular cases was pointed out already by Monge (5): It is the one in which the osculating planes of the two curves are perpendicular; one will then have:

$$\rho = a, \quad \rho' = -a.$$

Each of the two curves will be the locus of the centers of curvature of the other one, and also the locus of the centers of the osculating spheres of the other one.

12. The results that were obtained by Bertrand immediately give the solution to a problem that has been treated quite a lot: *Determine all skew surfaces whose radii of curvature are equal and opposite in sign at each point.*

Indeed, the desired skew surfaces must have an equilateral hyperbola for an indicatrix at each point, and consequently, their curvilinear asymptotic lines must cut the rectilinear generators at a right angle. Since the osculating plane of an asymptotic line is the tangent plane to the surface, one sees that the rectilinear generators must be the principal normals of all the asymptotes. From the result that was proved previously, those asymptotes can only be helices, and the ruled surface must be a helicoid with a director plane. One sees, moreover, (no. **9**) that the surface indeed enjoys the stated property. Thus: *The skew helicoid with a director plane is the only ruled surface whose radii of curvature are equal and opposite in sign at each point.*

12. We conclude our discussion of the subject by giving the determination of the developables of a skew curve.

Consider the trihedron (T) that relates to a point M. The developable must be generated by a point N in the yz-plane, and that point must chosen in such a manner that the tangent to the curve that it described must pass through M at each instant.

Call its coordinates y and z. The components of its velocity are:

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(4) *See* a note "Sur les courbes qui ont les mémes normales principales," that was submitted by Mannheim to the Comptes rendus **85**, pp. 212, in which some relations are proved that one can deduce from the formulas that were established here.

(5) MONGE, "Supplément oú l'on fait voir que les équations aux différences ordinaires, pour lesquelles les conditions d'intégrabilité ne sont pas satisfaites, sont susceptibles d'une véritable intégration et que c'est de cette intégration que dépend celle des équations aux dérivées partielles élevées," Mémoire de l'Académie Royale des Sciences for the year 1784, pp. 536, *et seq.*

As its title implies, this beautiful work completes the celebrated "Mémoire sur le Calcul intégral des équations aux différences partielles," which was published in the same volume (pp. 118), and in which one finds the first research by Monge on the partial differential equation of minimal surfaces. In the *Suppplément*, Monge showed that if a curve has a constant radius of curvature then the locus of centers of curvature will enjoy the same property and will have its centers of curvature on the original curve. Moreover, the osculating planes to the two curves at the corresponding points will be rectangular. However, the process that Monge described for the determination of the equation in finite terms of the curves whose curvature is constant is obviously incorrect. In fact, the finite equations that the illustrious geometer gave contained two arbitrary functions that Monge regarded as independent, although he had proved some pages before that they were coupled to each other by a differential equation.

$$(22) \qquad 1 - ry, \qquad \frac{dy}{ds} - pz, \qquad \frac{dz}{ds} + py.$$

Express the idea that the velocity is directed towards the point M. We will have the two equations:

$$y = \frac{1}{r} = r, \qquad \frac{dy - pz\,ds}{dz + py\,ds} = \frac{y}{z},$$

or

$$\frac{z\,dy - y\,dz}{y^2 + z^2} = p\;ds = -\frac{ds}{\tau}.$$

Upon integrating, we will find that:

$$\text{arc tan}\;\frac{y}{z} = \int \frac{ds}{\tau}.$$

We will then have:

$$(23) \qquad \begin{cases} y = \rho, \\ z = y \tan \int \dfrac{ds}{\tau}. \end{cases}$$

Those equations contain the entire theory of developables. One sees that the angle V that is defined by the principal normal and the line that joins the point of the development to the corresponding point of the curve will have the value:

$$V = \int \frac{ds}{\tau}.$$

Hence, the normals to the curve that envelop two different developments will define a constant angle between them. Conversely, if two normals to the curve make a constant angle, and if one of them envelops a development of the curve then the same thing will be true for the other one. We will make frequent use of these propositions.

The first of formulas (23) further shows us that the developments are traced completely on the *polar surface*, which is the envelope of the normal planes to the proposed curve. Indeed, it results from formulas (22) in relation to the velocity of the point (y, z) that all of the points in the normal plane that are situated on the line $y = \rho$ have their velocities directed into that plane. Hence, that line will be the generator of contact of the plane with its envelope, which is the polar surface. Moreover, the osculating plane of the development that contains the tangent to the proposed curve is, by that fact itself, normal to the polar surface.

CHAPTER II

ON THE INTEGRATION OF THE LINEAR SYSTEM THAT PRESENTED ITSELF IN THE PRECEDING THEORY.

Linear systems that possess a second-degree integral. – Their integration reduces to that of a Riccati equation. – General remarks on that equation.

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13. It remains for us to study the integration of the system:

$$(1) \qquad \begin{cases} \dfrac{d\alpha}{dt} = \beta r - \gamma q, \\ \dfrac{d\beta}{dt} = \gamma p - \alpha r, \\ \dfrac{d\gamma}{dt} = \alpha q - \beta p \end{cases}$$

that the three groups of cosines satisfy in a detailed manner. We have already pointed out one fundamental property of that system: It admits the second-degree integral:

$$(2) \qquad \alpha^2 + \beta^2 + \gamma^2 = \text{const.},$$

and the existence of that integral implies a series of propositions as corollaries that facilitate the integration of that system in some cases.

Before commencing with the study of equation (1), I would first like to show that any linear system of the form:

$$(3) \qquad \begin{cases} \dfrac{d\alpha}{dt} = A\alpha + B\beta + C\gamma, \\ \dfrac{d\beta}{dt} = A'\alpha + B'\beta + C'\gamma, \\ \dfrac{d\gamma}{dt} = A''\alpha + B''\beta + C''\gamma, \end{cases}$$

in which $A, B, C, \dots$ are functions of t, can be reduced to the form (1) whenever it admits a second-degree integral:

$$(4) \qquad \varphi(\alpha, \beta, \gamma) = \text{const.},$$

in which φ denotes a homogeneous function of degree two with constant or variable coefficients.

Indeed, one can convert equation (4) (except in special cases where the function φ is a sum of two squares, which one can easily treat) to the form:

(5) $$\alpha^2 + \beta^2 + \gamma^2 = \text{const.}$$

by a linear substitution that obviously will not change the form of equations (3). If one expresses the idea that the left-hand side of that equation is an integral of system (3) then one will obtain the equations:

$$A = B' = C'' = B + A' = C + A'' = C' + B'' = 0,$$

which indeed shows that the system (3) reduces to the form (1).

The system (1) then appears to have the type – or *reduced form* – of an entire class of systems that exhibit the property of admitting a second-degree integral that one frequently encounters in applications. That particular character of the equations that we shall study deserves to be pointed out, and it will suffice to justify the scope of the developments that we shall pursue.

14. I would first like to show that whenever one knows a particular solution (α_0, β_0, γ_0) of system (1), one can append the first-degree integral:

$$\alpha\,\alpha_0 + \beta\,\beta_0 + \gamma\,\gamma_0 = \text{const.}$$

to the second-degree integral that was given already.

Indeed, if one has an arbitrary solution (α, β, γ) of system (1) then one can deduce a more general solution:

$$\alpha + k\alpha_0, \qquad \beta + k\beta_0, \qquad \gamma + k\gamma_0,$$

in which k denotes an arbitrary constant, from the properties of any linear system. One must then have:

$$(\alpha + k\alpha_0)^2 + (\beta + k\beta_0)^2 + (\beta + k\beta_0)^2 = \text{const.}$$

for all values of k, or upon development:

$$\alpha^2 + \beta^2 + \gamma^2 + 2k\,(\alpha\,\alpha_0 + \beta\,\beta_0 + \gamma\,\gamma_0) + k^2\,(\alpha_0^2 + \beta_0^2 + \gamma_0^2) = \text{const.}$$

Since the first and last terms of the left-hand side are constant, the same thing will be true for:

$$\alpha\,\alpha_0 + \beta\,\beta_0 + \gamma\,\gamma_0,$$

as one can prove.

It obviously results from this that if one knows only two particular solutions (α_0, β_0, γ_0), (α_1, β_1, γ_1) of the system (1) then one can immediately write down the general solutions, which will be defined by the equations:

$$\begin{aligned} \alpha^2 + \beta^2 + \gamma^2 &= \text{const.},\\ \alpha\,\alpha_0 + \beta\,\beta_0 + \gamma\,\gamma_0 &= \text{const.},\\ \alpha\,\alpha_1 + \beta\,\beta_1 + \gamma\,\gamma_1 &= \text{const.} \end{aligned}$$

These equations can be solved for α, β, γ, and give the values:

$$(6)\qquad \left\{ \begin{aligned} \alpha &= c_0\alpha_0 + c_1\alpha_1 + c_2(\beta_0\gamma_1 - \beta_1\gamma_0),\\ \beta &= c_0\beta_0 + c_1\beta_1 + c_2(\gamma_0\alpha_1 - \alpha_0\gamma_1),\\ \gamma &= c_0\,\gamma_0 + c_1\gamma_1 + c_2(\alpha_0\beta_1 - \alpha_1\beta_0) \end{aligned} \right.$$

in which c_0, c_1, c_2 denote arbitrary constants. However, one can obtain a more complete proposition and show that if one knows just one solution of system (1) then one quadrature will suffice to give its general integral.

15. In order to establish this essential result, we remark that the most general values of α, β, γ must satisfy the relation:

$$\alpha^2 + \beta^2 + \gamma^2 = \text{const.}$$

We first begin by discarding the case in which the constant is zero. One can always suppose that one has:

$$(7)\qquad \alpha^2 + \beta^2 + \gamma^2 = 1$$

by dividing those values by a suitable constant.

We likewise remark that in the particular problem that we have to treat, since α, β, γ are three direction cosines, they must necessarily satisfy that relation. It is natural to express α, β, γ as functions of two independent variables in such a manner that the preceding relation is always satisfied, and to seek the differential equations that those two variables must satisfy.

Now, if one regards α, β, γ as the coordinates of a point in space then equation (7) will represent a sphere of radius 1 that has its center at the coordinate origin. Consider that sphere to be a ruled surface that admits a double system of imaginary generators and take the variables to be two quantities that remain constant on the generators of each system, respectively. In order to do that, we set:

$$(8)\qquad \left\{ \begin{aligned} \frac{\alpha + i\beta}{1-\gamma} &= \frac{1+\gamma}{\alpha - i\beta} = x,\\ \frac{\alpha - i\beta}{1-\gamma} &= \frac{1+\gamma}{\alpha + i\beta} = -\frac{1}{y}, \end{aligned} \right.$$

which will give:

$$(9)\qquad \alpha = \frac{1-xy}{x-y}, \qquad \beta = i\,\frac{1+xy}{x-y}, \qquad \gamma = \frac{x+y}{x-y}.$$

We remark that, from formulas (8), x and y will be imaginary when α, β, γ are real, and in addition, the conjugate imaginary to x will be $-1/y$.

If we substitute the values (9) of α, β, γ in the differential equation then those equations will reduce to two, as one would expect, and after some simple calculations, one will obtain the system:

$$(10)\qquad \begin{cases} \dfrac{dx}{dt} = -i\,rx + \dfrac{q-ip}{2} + \dfrac{q+ip}{2}x^2, \\ \dfrac{dy}{dt} = -i\,ry + \dfrac{q-ip}{2} + \dfrac{q+ip}{2}y^2. \end{cases}$$

x and y must then be two different solutions to the same equation in σ:

$$(11)\qquad \frac{d\sigma}{dt} = -\,ir\sigma + \frac{q-ip}{2} + \frac{q+ip}{2}\sigma^2,$$

and the integration of the proposed system will be reduced to that of just that one equation. Two distinct particular solutions of that equation will always give real or imaginary values of α, β, γ that verify the system (1) by the use of formulas (9). We likewise remark that when p, q, r are real functions, it will suffice to know a particular solution σ of equation (11) in order to deduce a solution (α, β, γ) of the proposed system. Indeed, let σ' denote the conjugate imaginary of σ. I would like to show that $-1/\sigma'$ is again a particular solution of equation (11).

In order to do that, change i into $-i$ in that equation and get:

$$\frac{d\sigma'}{dt} = -\,ir\sigma' + \frac{q+ip}{2} + \frac{q-ip}{2}\sigma'^2,$$

and consequently:

$$\frac{d}{dt}\left(-\frac{1}{\sigma'}\right) = -\,ir\left(-\frac{1}{\sigma'}\right) + \frac{q-ip}{2} + \frac{q+ip}{2}\left(-\frac{1}{\sigma'}\right)^2.$$

It will suffice to compare equation (11) in order to recognize that $-1/\sigma'$ is indeed a particular solution to that equation.

16. The equation in σ belongs to the group of equations of the form:

$$(13)\qquad \frac{d\sigma}{dt} = a + 2b\sigma + c\sigma^2,$$

in which a, b, c are arbitrary functions of t. They are the next simplest ones after linear equations. Since one frequently encounters them in the applications, one gives them the name of *Riccati equations*, because they include the equation:

$$\frac{d\sigma}{dt} = a\sigma^2 + b\,t^m$$

as a particular case, which was the only one that was the subject of research of the Italian geometer. We shall rapidly recall their principal properties.

First, they do not change form when one performs a linear substitution on σ – i.e., when one replaces σ with the variable λ that is defined by the equation:

$$\lambda = \frac{P\sigma + Q}{R\sigma + S},$$

in which P, Q, R, S are arbitrary functions of t.

In the second place, one can integrate them as soon as one knows one particular solution. Indeed, let $\sigma = \sigma_0$ be one such solution. Set:

$$\sigma = \sigma_0 + \frac{1}{\lambda},$$

and obtain the linear equation for λ:

$$\frac{d\lambda}{dt} = -c - 2\,(c\sigma_0 + b)\,\lambda, \tag{14}$$

whose integration will require only two quadratures that are performed in succession.

One of the fundamental properties of the Riccati equation results from this. Since the general value of λ is linear in the arbitrary constant C and of the form:

$$PC + Q,$$

one sees that the general integral of the Riccati equation will be of the form:

$$\sigma = \frac{RC + S}{PC + Q},$$

in which P, Q, R, S are functions of the independent variable t. One deduces from this that *the anharmonic ratio of four solutions to the equation is constant and equal to that of the four values of the arbitrary constant that correspond to those solutions.*

17. If one then knows three particular solutions σ_0, σ_1, σ_2 then the general integral will be given by the formula:

$$\frac{\sigma - \sigma_0}{\sigma_1 - \sigma_0} : \frac{\sigma - \sigma_2}{\sigma_1 - \sigma_2} = C,$$

which does not contain any quadrature.

If one knows only two solutions σ_0, σ_1 then one quadrature will suffice. Here is the fastest procedure for obtaining that solution: Set:

$$\lambda = \frac{\sigma - \sigma_0}{\sigma_1 - \sigma_0},$$

so one has:

$$\frac{1}{\lambda}\frac{d\lambda}{dt} = +\frac{1}{\sigma - \sigma_0}\left(\frac{d\sigma}{dt} - \frac{d\sigma_0}{dt}\right) - \frac{1}{\sigma - \sigma_1}\left(\frac{d\sigma}{dt} - \frac{d\sigma_1}{dt}\right),$$

or, upon replacing $\frac{d\sigma}{dt}, \frac{d\sigma_0}{dt}, \frac{d\sigma_1}{dt}$ with their values that are inferred from equation (13):

$$\frac{1}{\lambda}\frac{d\lambda}{dt} = c\,(\sigma_0 - \sigma_1)\,.$$

λ will then be obtained by a simple quadrature, and one will have:

(15)
$$\lambda = \frac{\sigma - \sigma_0}{\sigma - \sigma_1} = C\, e^{\int c(\sigma_0 - \sigma_0)\,dt},$$

in which C denotes an arbitrary constant.

The Riccati equation then possesses one of the fundamental properties of linear equations, and knowing each particular solution will permit one to make one more step towards the general solution. Indeed, it is easy to convert its integration into that of a second-order linear equation.

Here is the procedure for proving that last proposition that seems the most elegant to us:

18. Set:

$$\sigma = \frac{\mu}{\nu},$$

so the equation becomes:

$$\nu\frac{d\mu}{dt} - \mu\frac{d\nu}{dt} = a\,\nu^2 + 2b\mu\nu + c\,\mu^2,$$

and that single equation can obviously be replaced with the following ones:

(16)
$$\begin{cases} \dfrac{d\mu}{dt} = a\nu + (b+h)\mu, \\[2mm] \dfrac{d\nu}{dt} = -c\mu + (b-h)\nu, \end{cases}$$

in which h denotes a function that one chooses arbitrarily ([6]). Now, the elimination of μ or ν will obviously lead to a second-order linear equation.

If one takes $h = b$, for example, then one will have:

$$\mu = -\frac{1}{c}\frac{d\nu}{dt},$$

and ν will satisfy the equation:

$$\frac{d^2\nu}{dt^2} - \left(2b + \frac{c'}{c}\right)\frac{d\nu}{dt} + ac\nu = 0. \tag{17}$$

If ν_1 and ν_2 denote two particular solutions of that equation then one will have:

$$\sigma = -\frac{1}{c}\frac{\nu_1' + C\nu_2'}{\nu_1 + C\nu_2}, \tag{18}$$

in which C denotes an arbitrary constant.

19. We have seen that the anharmonic ratio of four arbitrary particular solutions to the Riccati equation is constant. It is easy to establish that this property is characteristic – i.e., it belongs to just that one equation.

Indeed, if σ_0, σ_1, σ_2 are three particular solutions then the general integral of the equation considered will be given by the formula:

$$\frac{\sigma - \sigma_0}{\sigma - \sigma_1} : \frac{\sigma_2 - \sigma_0}{\sigma_2 - \sigma_1} = C,$$

and the elimination of C by one differentiation will lead to a Ricatti equation.

20. We apply these general propositions that relate to the Riccati equation to our equation (11) in σ. Whenever one knows a solution to the system (1) for which the constant sum $\alpha^2 + \beta^2 + \gamma^2$ is non-zero, one can reduce that sum to unity, and formulas (8) will then show us two particular solutions of the equation in σ; let σ_0, $-1/\sigma_0'^2$ denote those two solutions. In order to determine the general integral of the equation in σ, it will suffice to perform just one quadrature. The application of formula (15) will lead us to the equation:

([6]) It is good to remark that although the complete integration of the system (16) implies that of the Riccati equation, without it being necessary to perform a quadrature, the converse is not true. When one has integrated the Riccati equation once, one will have only the ratio μ / ν; the determination of μ or ν by equations (16) will demand another quadrature.

$$\frac{\sigma-\sigma_0}{1+\sigma\sigma_0'} = C\,e^{-\int\left(ir+\sigma_0'\frac{q-ip}{2}-\sigma_0\frac{q+ip}{2}\right)dt}$$

by some simple transformations, or furthermore:

$$\frac{\sigma-\sigma_0}{1+\sigma\sigma_0'} = C\sqrt{\frac{\sigma_0}{\sigma_0'}}\,e^{-\int\left(\frac{1+\sigma_0\sigma_0'}{?}\right)\left(\frac{q-ip}{\sigma_0}-\frac{q+ip}{\sigma_0'}\right)dt},$$

in which C denotes the arbitrary constant.

The quadrature that appears in these formulas will involve a real function whenever the rotations p, q, r, and the particular solution that one starts with are real, because σ_0, σ_0' will be conjugate imaginary then, and the function under the $\int$ sign in the preceding formulas will have the form $i\,\Theta$, where Θ is real.

It is then proved that whenever one knows one particular solution to the system (1) for which the constant $\alpha^2 + \beta^2 + \gamma^2$ is non-zero, the general solution of that system will be obtained by a simple quadrature.

21. Now, suppose that the particular solutions α, β, γ that are being considered satisfy the relation:

$$\alpha^2 + \beta^2 + \gamma^2 = 0.$$

We begin by remarking that at least one of the quantities α, β, γ must be imaginary. Upon exhibiting the real and imaginary parts, one will then have:

$$\alpha = \alpha' + i\,\alpha'', \qquad \beta = \beta' + i\,\beta'', \qquad \gamma = \gamma' + i\,\gamma''.$$

Having said that, if p, q, r are real functions of t then α', β', γ' and α'', β'', γ'' will obviously constitute two different systems of real solutions of the system (1) for which the sum $\alpha^2 + \beta^2 + \gamma^2$ is non-zero. The application of formulas (6) will then show the complete solution to the system (1) without integration.

Now, suppose that p, q, r are imaginary functions. One can then set:

(19) $$\frac{\alpha+i\beta}{\gamma} = -x = \frac{\gamma}{\alpha-i\beta},$$

and upon introducing a proportionality factor ρ, one will have:

(20) $$\alpha = \rho\,(1-x^2), \qquad \beta = i\rho\,(1+x^2), \qquad \gamma = 2\rho\,x.$$

The substitution of those values in the system (1) leads us to the two equations:

(21)
$$\frac{dx}{dt} = -\,irx + \frac{q-ip}{2} + \frac{q+ip}{2}x^2,$$

(22)
$$\frac{1}{\rho}\frac{d\rho}{dt} = ir - (q + p)\,x.$$

Hence, x must be a solution of equation (11). Moreover, if one sets:

$$\sigma = x - \frac{1}{\lambda}$$

then that equation will take the form:

$$\frac{d\lambda}{dt} = \frac{q+ip}{2} + [ir - (q + ip)\,x]\,\lambda,$$

or, upon taking formula (22) into account:

(23)
$$\frac{d}{dt}\left(\frac{\lambda}{\rho}\right) = \frac{q+ip}{2\rho}.$$

One will then get λ, and consequently σ, by just one quadrature. Hence:

In any case, knowing just one system of solutions to equations (1) *will permit us to obtain the complete integration of those equations by just one quadrature.*

The particular systems of solutions for which the sum $\alpha^2 + \beta^2 + \gamma^2$ is zero play an essential role in the important work of Hermite on the rotation of a solid body ([7]).

22. Euler, who was the first to study the motion of a solid body, proved the preceding result by an entirely different method. We have seen that he expressed the nine cosines by means of just three angles, and we know that the rotations p, q, r are expressed as functions of those angles and their derivatives with respect to time by formulas (6) [pp. 4]. If one then supposes that the rotations are known then those three formulas will constitute a system of differential equations that will replace the system (1) and will suffice to determine the angles θ, φ, ψ. In truth, the lack of symmetry in these equations hardly allows one to employ them in a general manner. Meanwhile, one can deduce the fundamental property of system (1) from them very simply.

Indeed, let a'', b'', c'' be the particular values of α, β, γ that verify system (1), which are assumed to be known. If we take the axis OZ to be the line whose direction cosines are a'', b'', c'' then will get θ and φ from the last three of formulas (2) [pp. 3]. Then, the last of formulas (6) or the second of formulas (7) [pp. 4] will permit us to determine ψ by

([7]) HERMITE, *Sur quelques applications des fonctions elliptiques*, Paris, Gauthier-Villars, 1885.

one quadrature. Knowing the three Euler angles will then give us three particular solutions of the system (1), and consequently, the general solution, as well.

It is easy to see that the quadratures to be performed in the two methods will reduce to each other, and will differ by only some exactly-integrable quantities.

CHAPTER III

GEOMETRIC INTERPRETATION OF THE METHOD THAT WAS DEVELOPED IN THE PRECEDING CHAPTER

Study of the symmetric coordinates in the case of the sphere. – Geometric interpretation of a linear substitution that is performed simultaneously on the two coordinates. – Formulas of Euler and Olinde Rodrigues that relate to coordinate transformation. – Representation of the imaginary variable by a point on the sphere according to Riemann's method.

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23. From the preceding developments, one sees that the integration of any system of linear equations with three unknowns that admits a homogeneous, second-degree integral will be converted into that of a Riccati equation – i.e., into that of the most general linear system in two unknowns. It seems interesting to us to justify and explain that result by some considerations of pure geometry. In order to do that, we shall make a rapid study of the system of curvilinear coordinates x, y that determine the points of the sphere of radius 1, and which are defined by formulas (9).

By an elementary calculation, those formulas will lead us to the following result:

$$d\alpha^2 + d\beta^2 + d\gamma^2 = \frac{4\,dx\,dy}{(x-y)^2}, \tag{1}$$

which gives the differential of the arc length that is described by the point whose curvilinear coordinates are x, y. One sees that this arc length will be zero when one displaces along one or the other of the rectilinear generators of the sphere. That is a well-known result; however, formula (1) will lead us to some other consequences.

24. Its right-hand side enjoys the property of being reproduced when one subjects x and y to the same linear substitution. Indeed, set:

$$x = \frac{ax_1 + b}{cx_1 + d}, \qquad y = \frac{ay_1 + b}{cy_1 + d}, \tag{2}$$

in which a, b, c, d are constants; we will find that:

$$\frac{4\,dx\,dy}{(x-y)^2} = \frac{4\,dx_1\,dy_1}{(x_1-y_1)^2}.$$

It results from this that if one considers two figures that are described on the sphere, one of which (F) is at the point (x, y), while the other one (F_1) is at the point (x_1, y_1), then the distance between two arbitrary, infinitely-close points on one of the figures will be equal to the distance between the corresponding points of the other one. Consequently, the infinitely-small triangles that correspond in the two figures and have three equal sides will be equal or symmetric, and *the two figures will then be equal or symmetric, resp.:* I say that they are equal.

Indeed, vary a, b, c, d in formulas (2) from their present values to the following ones: 1, 0, 0, 1 in a continuous manner. The figure (F_1) will displace in a continuous manner, and since it is always equal or symmetric to (F), it will always remain superposable with its original position. Now, for extreme values of a, b, c, d, the substitution (2) will reduce to the following one:

$$x = x_1, \qquad y = y_1 .$$

The figure (F_1) will coincide with (F), and consequently, the two figures will be equal.

The right-hand side of formula (1) will also be reproduced if one employs the substitution:

$$x = \frac{ay_1 + b}{cy_1 + d}, \quad y = \frac{ax_1 + b}{cx_1 + d}. \tag{3}$$

However, it is clear that this substitution results from the composition of the substitution (2), which replaces any figure (F) with an equal figure, with the following one:

$$y = x_1, \qquad x = y_1 .$$

It suffices to refer to formulas (9) [pp. 18] in order to recognize that the latter substitution replaces a point of the sphere with the diametrically-opposite point – i.e., the figure (F) with a symmetric figure. The same thing will then be true for the more general substitution that is defined by formulas (3).

25. The preceding results have been deduced from equation (1), which gives the distance between two infinitely-close points. However, one can also obtain them by the use of the formula that expresses the distance between two arbitrary points of the sphere in the x, y coordinate system.

Indeed, let M, M' be two points with coordinates x, y ; x', y'. Upon denoting the arc of the great circle that joins them by MM', one will have:

$$\cos MM' = \frac{2xy - 2x'y' - (x - y)(x' - y')}{(x - y)(x' - y')}; \tag{4}$$

hence, one will deduce that:

$$\cos^2 \frac{MM'}{2} = \frac{(x - x')(y - y')}{(x - y)(x' - y')}, \qquad \sin^2 \frac{MM'}{2} = \frac{(x - y')(y - x')}{(x - y)(x' - y')}, \tag{5}$$

Those formulas, which I gave already in 1872 ([8]), along with some other ones, can be further written in the form:

$$\cos^2 \frac{MM'}{2} = R(x, y', x', y), \qquad \sin^2 \frac{MM'}{2} = R(x, x', y', y),$$

in which R (a, b, c, d) denotes the anharmonic ratio of the quantities a, b, c, d. It is clear that these expressions will remain invariable when one applies one or the other of the substitutions (2) or (3) to the coordinates of the two points. One sees that these substitutions do not change the spherical distance between two arbitrary points. They can only replace a figure (F) with an equal or symmetric figure then, which confirms the proposition that was obtained already.

It results from the preceding that if one considers four arbitrary points M, M', M'', M''' on the surface of the sphere then the anharmonic ratio of the values of the coordinate x that relate to those four points will remain constant when one displaces the invariable figure that is defined by those four points in an arbitrary manner. In other words, that anharmonic ratio depends upon only the form of the quadrilateral. One knows various expressions for it that I shall not stop to establish. It will suffice for us to know that it remains constant when the quadrilateral is displaced without deforming.

26. After that, we return to the system (1) of the preceding chapter, and consider α, β, γ to be the coordinates of a point on the sphere in it. Each particular solution of the system (1) will correspond to a certain curve on the sphere that is described by that point. It results from the propositions that were established at the outset (no. **14**) that if two points of the sphere represent two different particular solutions of the system then they will always remain at an invariable distance from each other. Hence, if four points describe curves that correspond to four different particular solutions in their motion then they will define an invariable figure, and the anharmonic ratio of the four particular values of x that corresponds to those four points will be constant; i.e, x must satisfy a Riccati equation (no. **19**) when considered as a function of t.

It remains for us to explain why the second coordinates y satisfies the same equation as the first one. In order to do that, it will suffice to remark that if a point M of the sphere gives a solution of the system (1) then the same thing will be true for the diametrically-opposite point, which corresponds the same values of α, β, γ, but with opposite signs. Now, one passes from one of those points to the other one by switching x and y; those coordinates must then satisfy the same differential equation.

27. The analytical results of the preceding chapter are then explained completely. We shall not pursue the complete study of the x, y coordinate system now, and we shall be content to point out how one determines the displacement that corresponds to a linear substitution that is performed on the two coordinates simultaneously.

([8]) G. DARBOUX, "Mémoire sur une classe remarquable de courbes et de surfaces algébriques," pp. 212.

Recall the formulas:

(6) $$\alpha = \frac{1-xy}{x-y}, \qquad \beta = \frac{1+xy}{x-y}, \qquad \gamma = \frac{x+y}{x-y},$$

which gives the rectangular coordinates α, β, γ as functions of x, y. If one performs the substitution on x and y that is defined by the formulas:

(7) $$x = \frac{mx_1+n}{px_1+q}, \quad y = \frac{my_1+n}{py_1+q},$$

and if one lets α_1, β_1, γ_1 denote the rectangular coordinates that corresponds to x_1, y_1, one will find, from a calculation that offers no difficulty, that:

(8) $$\begin{cases} \alpha_1 = a\alpha + a'\beta + a''\gamma, \\ \beta_1 = b\alpha + b'\beta + b''\gamma, \\ \gamma_1 = c\alpha + c'\beta + c''\gamma, \end{cases}$$

in which a, b, c, … have the following values:

(9) $$\begin{cases} a = \dfrac{q^2+m^2-n^2-p^2}{2B}, & a = i\dfrac{m^2+n^2-p^2-q^2}{2B}, & c = \dfrac{pq-mn}{B}, \\ a' = i\dfrac{q^2+n^2-m^2-p^2}{2B}, & b' = \dfrac{m^2+n^2+p^2+q^2}{2B}, & c' = i\dfrac{pq+mn}{B}, \\ a'' = \dfrac{nq-mp}{B}, & b'' = -i\dfrac{mp+nq}{B}, & c'' = \dfrac{mq+nq}{B}, \end{cases}$$

in which B is the determinant of the substitution:

$$B = mq - np.$$

It is easy to see that these nine quantities are the coefficients of an orthogonal substitution with determinant 1, which proves once more the theorem that was established above (no. **24**).

If one replaces m, n, p, q with the following expressions:

$$\begin{aligned} m &= -\rho + i\nu, & n &= -\mu + i\lambda, \\ q &= -\rho - i\nu, & p &= \mu + i\lambda \end{aligned}$$

then upon setting:

(10) $$B = \lambda^2 + \mu^2 + \nu^2 + \rho^2,$$

to abbreviate, one will find that:

$$
(11) \quad \begin{cases}
Ba \;= \rho^2 + \lambda^2 - \mu^2 - \nu^2, & Bb \;= 2(\mu\lambda + \rho\nu), & Bc \;= 2(\lambda\nu - \rho\mu), \\
Ba' = 2(\mu\lambda - \nu\rho), & Bb' = \rho^2 + \mu^2 - \lambda^2 - \nu^2, & Bc' = 2(\mu\nu + \lambda\rho), \\
Ba'' = 2(\lambda\nu + \mu\rho), & Bb'' = 2(\mu\nu - \lambda\rho) & Bc'' = \rho^2 + \nu^2 - \lambda^2 - \mu^2.
\end{cases}
$$

These are the well-known expressions for the nine cosines in homogeneous form, which are due to Euler and Olinde Rodrigues.

28. Since formulas (7) define a real or imaginary displacement – i.e., a finite rotation – we propose to determine the axis and magnitude of that rotation. One will obtain those two elements in the following manner:

The points where the rotational axis meets the sphere will remain immobile during the motion. They must then satisfy the relations:

$$x = x_1, \qquad y = y_1 ;$$

consequently, x, y will be the roots of the equation:

$$(12) \qquad px^2 + (q - m)\, x - n = 0,$$

which define the double elements of the linear substitution. Let x', y' be the two roots of that equation, which we assume to be different. One sees that the motion will leave the four points

$$
(13) \qquad \begin{cases}
x = x', \;\; x = y', \;\; x = x', \;\; x = y'; \\
y = y', \;\; y = x', \;\; y = x', \;\; y = y'
\end{cases}
$$

invariable.

The first two are at a finite distance and diametrically-opposite to each other: They are the points where the rotational axis cuts the sphere. The other two satisfy the relation $x = y$, and consequently, from formulas (6), they will be on the circle at infinity. That will define a displacement from the projective viewpoint. It is a homographic transformation of the sphere that leaves four points invariable, two of which are at infinity, while the other two are diametrically opposite. Those four points define the summits of a skew quadrilateral that is situated on the sphere entirely.

As for the magnitude of the rotation, one will determine it in the following manner: Write down the equation (7) in the canonical form:

$$(14) \qquad \frac{x - x'}{x - y'} = k\,\frac{x_1 - x'}{x_1 - y'}, \qquad \frac{y - x'}{y - y'} = k\,\frac{y_1 - x'}{y_1 - y'}.$$

That form is obviously preserved if one performs a displacement of the set – i.e., if one subjects all of the variables x, y, x', … *to the same linear substitution.* Suppose that the displacement is chosen in such a manner that the point $x = x'$, $y = y'$ is placed on the positive half of the z-axis. x' will then become equal to ∞, y' will become 0, and formulas (14) will become:

(15) $$x_1 = kx, \qquad y_1 = ky,$$

or, upon returning to the rectangular coordinates and calling the rectilinear coordinates of the two corresponding positions of the same point α, β, γ; $\alpha_1, \beta_1, \gamma_1$:

$$\frac{\alpha_1 + i\beta_1}{1-\gamma_1} = k\,\frac{\alpha + i\beta}{1-\gamma}, \qquad \frac{\alpha_1 - i\beta_1}{1+\gamma_1} = \frac{1}{k}\,\frac{\alpha - i\beta}{1+\gamma}.$$

These formulas obviously agree with a rotation around Oz through an angle θ that is defined by the equation:

(16) $$e^{i\theta} = k.$$

One will recognize that fact immediately upon considering the points in the xy-plane for which one has:

$$\gamma = \gamma_1 = 0.$$

Hence, the magnitude of the rotation will be determined unambiguously by formula (16). As far as the value of k is concerned, as one knows, it will be given by the equation:

$$k = \frac{m - px'}{m - py'},$$

or, if one would like to obtain it without passing to the values of x', y', by the equation:

(18) $$\frac{(1+k)^2}{k} = \frac{(m+q)^2}{mq - np}.$$

29. The displacement that is defined by formulas (7) is not real, in general. However, the various methods in the foregoing permit one to show the conditions under which the displacement will be real. Indeed, we have seen that if two real points have the coordinates x, y and x_1, y_1, respectively, then the imaginary variables x, x_1 will have $-1/y$, $-1/y_1$ for their conjugates, resp. Upon then changing i into $-i$ in the first of equations (7) and denoting the quantities that are conjugate to m, n, p, q by m_0, n_0, p_0, q_0, one will have:

$$-\frac{1}{y} = \frac{-m_0 + n_0 y_1}{-p_0 + q_0 y_1},$$

and before that relation can be true whenever x, y are coordinates of a real point, it must necessarily be identical to the second of formulas (7). That will give the conditions:

$$\frac{p_0}{n} = \frac{-q_0}{m} = \frac{-m_0}{q} = \frac{n_0}{p},$$

which permits one to write down formulas (7) in the form:

(19) $$x = \frac{mx_1 + n}{-n_0 x_1 + m_0}, \qquad y = \frac{my_1 + n}{-n_0 y_1 + m_0},$$

in which m_0, n_0 denote the conjugate imaginaries of m and n.

30. It is easy to recognize that when one represents an imaginary variable by a point of the sphere following Riemann's method, the quantity that we denote by x will be affixed to the point (α, β, γ).

Indeed, Riemann's method consists of first representing the variable $z = x' + i\,y'$ with the point (x', y') in the xy-plane, as Gauss and Cauchy did. One then makes a stereographic projection of that plane onto the sphere of radius 1 that has its center at the origin by taking the pole of the point on that sphere to be situated along the positive half of the z-axis. If we denote the coordinates of that stereographic projection by α, β, γ then an elementary calculation will give us:

$$x = \frac{\alpha + i\beta}{1-\gamma} = x' + iy' = z,$$

which justifies our remark.

31. In the theory of functions and various studies in geometry, it can be advantageous to modify the x, y coordinate system slightly and replace y with the variable:

$$x_0 = -\frac{1}{y}.$$

One will then have the following expressions for α, β, γ:

(20) $$\alpha = \frac{x + x_0}{1 + xx_0}, \quad \beta = i\,\frac{x - x_0}{1 + xx_0}, \quad \gamma = \frac{xx_0 - 1}{xx_0 + 1},$$

and equation (1) will take the form:

(21) $$d\alpha^2 + d\beta^2 + d\gamma^2 = \frac{4\,dx\,dx_0}{(1 + xx_0)^2}.$$

With that new system, the coordinates x, x_0 of any real point will be conjugate imaginaries. However, a displacement will no longer be represented by *the same* linear substitution that is performed on both variables ([9]).

([9]) For everything that concerns the relationship between displacements and linear substitutions, one can consult the important papers of F. Klein that were included in volumes IX and XII of Mathematische Annalen, in which those relationships were expanded upon and applied to the solution of some problems with a more advanced interest to them.

CHAPTER IV

APPLICATIONS OF THE PRECEDING THEORY.

Extension of Poinsot's theory. – Determination of the motions in which there are two relations between the rotations that are given in advance. – Determination of the skew curves whose curvature and torsion satisfy a given relation. – Study of the case in which that relation is linear. – Curves with constant torsion.

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32. Before continuing with the exposition of the general theory, we shall make some applications of the preceding propositions. Recall the system:

(1) $$\frac{d\alpha}{dt} = \beta r - \gamma q, \qquad \frac{d\beta}{dt} = \gamma p - \alpha r, \qquad \frac{d\gamma}{dt} = \alpha q - \beta p$$

that must be satisfied by the cosines of the angles that a fixed axis makes with the moving axes. One knows that when a solid body moves around a fixed point without being subject to any force, the preceding system will be verified if one replaces α, β, γ with the derivatives $\frac{\partial f}{\partial p}, \frac{\partial f}{\partial q}, \frac{\partial f}{\partial r}$ of a function $f(p, q, r)$ that is homogeneous of degree two, and which represents one-half the total *vis viva* of the body.

We look for all of the motions that enjoy an analogous property – i.e., ones for which the system (1) admits the solution:

(2) $$\alpha = \frac{\partial f}{\partial p}, \qquad \beta = \frac{\partial f}{\partial q}, \qquad \gamma = \frac{\partial f}{\partial r}.$$

However, $f(p, q, r)$ will no longer be subject to being homogeneous and of degree two now. Since α, β, γ, and the derivatives of f transform by the same substitution when one performs a change of moving axes, it will be obvious that the preceding property will be independent of the choice of axes.

Upon writing down the idea that the system (1) is verified by the values (2) of α, β, γ, we will have the equations:

(3) $$\begin{cases} \dfrac{d}{dt}\left(\dfrac{\partial f}{\partial p}\right) = r\dfrac{\partial f}{\partial q} - q\dfrac{\partial f}{\partial r}, \\[2ex] \dfrac{d}{dt}\left(\dfrac{\partial f}{\partial q}\right) = p\dfrac{\partial f}{\partial r} - r\dfrac{\partial f}{\partial p}, \\[2ex] \dfrac{d}{dt}\left(\dfrac{\partial f}{\partial r}\right) = q\dfrac{\partial f}{\partial p} - p\dfrac{\partial f}{\partial q}; \end{cases}$$

hence, we will deduce:

$$p\,d\left(\frac{\partial f}{\partial p}\right) + q\,d\left(\frac{\partial f}{\partial q}\right) + r\,d\left(\frac{\partial f}{\partial r}\right) = 0.$$

One then gets:

(4) $$p\frac{\partial f}{\partial p} + q\frac{\partial f}{\partial q} + r\frac{\partial f}{\partial r} + f = \text{const.}$$

upon integrating, which is an equation to which one must append the following one:

(5) $$\left(\frac{\partial f}{\partial p}\right)^2 + \left(\frac{\partial f}{\partial q}\right)^2 + \left(\frac{\partial f}{\partial r}\right)^2 = 1,$$

which expresses the idea that α, β, γ are direction cosines.

Upon substituting the values of p, q as functions of r that are inferred from these equations (4) and (5) in one of equations (3), one will get time by a quadrature. After having thus obtained the expressions for p, q, r as functions of time, one will achieve the integration of the system (1) by means of just one quadrature, and one already knows the particular solution that is provided by equations (2).

As one sees, the solution is entirely analogous to the one that one gives in the study of motion of a solid body that is left to itself. However, the analogy will become even more complete if one supposes that the function f is homogeneous. Equation (4) will then reduce to the following one:

$$f(p, q, r) = \text{const.},$$

and one can represent the motion by rolling the surface that is invariably coupled to the moving axes, and whose equation is:

$$f(x, y, z) = 1$$

with respect to those axes, on a fixed plane. If one suppose that f is entire and of degree two then one will recover Poinsot's solution.

33. As a second application, we propose to determine the motions in which there are two relations between the three rotations that are given in advance:

(6) $$f(p, q, r) = 0, \qquad \varphi(p, q, r) = 0.$$

We shall first give a geometric method for indicating the degree of difficulty of that problem. In the Poinsot representation, the motion is obtained when one rolls the cone (γ), which is the locus of the instantaneous axis of rotation in the moving body, on the fixed cone (C). Now, the preceding two equations, which determine the locus that is described in the body by the extremity of the instantaneous axis, show us the cone (γ) by that fact in its own right. As for the cone (C), we take it arbitrarily, but in such a manner that the section of that cone by the sphere of radius 1 is the spherical development of an arbitrary curve that traced on that sphere, which permits us to obtain the arc length of that section with no quadratures; one must then roll the cone (γ) on the cone (C). The equations that we have to write in order to express that motion will obviously contain the quadrature that gives the arc length of the curve of intersection of the cone (γ) with the sphere of radius 1. For an arbitrary position of the cone (γ), the instantaneous axis will be the generator of contact of that cone with the cone (C), and the ratios of p, q, r will be known. Equations (6) then show us the magnitudes of those rotations. Upon expressing the idea that the cone (γ) rolls with the velocity thus-obtained at each instant, one will have to perform a new quadrature that will determine the time.

Hence, the calculation can be directed in such a manner that one must perform only two quadratures. One will arrive at equivalent results by the following analytical method.

34. Suppose that three of the nine cosines – a, b, c, for example – are expressed as functions of the two variables x and y by formulas (6) [pp. ?]. If a, b, c are real then x and y will be imaginaries of the form:

$$x = h + ki, \qquad y = -\frac{1}{h-ki}.$$

One will get two relations upon expressing the idea that these two quantities verify the Riccati equation (11) [pp. ?]; they are identical to the ones that one obtains by substituting only the value of x and equating the real parts and the imaginary parts:

$$\text{(7)} \qquad \left\{ \begin{aligned} \frac{dh}{dt} &= \quad rk + \frac{q}{2}(1+h^2-k^2) - phk, \\ \frac{dk}{dt} &= -rk + qhk - \frac{p}{2}(1+k^2-h^2). \end{aligned} \right.$$

If we eliminate p, q, r from equations (6) and (7) then we will be led to an equation of the form:

$$F\left(h, k, \frac{dh}{dt}, \frac{dk}{dt}\right) = 0.$$

Moreover, if one takes k to be – for example – an arbitrary function of h then that equation will tell us the time t by one quadrature. We know three of the nine cosines, namely, a, b, c. One last quadrature will tell us the other six. As one sees, the results thus-obtained will coincide with the ones that the geometrical method provides us with.

35. We have seen that if one considers an arbitrary skew curve, and if one studies the motion of the trihedron that is defined by the tangent, the principal normal, and the binormal at a point then upon supposing that the origin of that trihedron describes a unit arc length in a unit time, one will have that:

$$p = -\frac{1}{\tau}, \qquad q = 0, \qquad r = \frac{1}{\rho},$$

in which ρ and τ denote the radii of curvature and torsion. We propose to determine all of the curves for which there is a relation between the curvature and torsion that is given in advance:

$$f\left(\frac{1}{\tau}, \frac{1}{\rho}\right) = 0.$$

That will amount to determining the motion of a trihedron in which one has two relations:

(8) $$q = 0, \qquad f(-p, r) = 0$$

between the two rotations.

Upon applying the general method that was given above, one will get expressions for the nine cosines that determine the position of the moving trihedron as a function of the arc length of the curve, which is equal to time duration, here. One then determines the rectangular coordinates x, y, z of the point on the curve that is the summit of the trihedron by the formulas:

$$\frac{dx}{ds} = a, \qquad \frac{dy}{ds} = a', \qquad \frac{dz}{ds} = a'',$$

which will give:

$$x = \int a\,ds, \qquad y = \int a'\,ds, \qquad z = \int a''\,ds.$$

36. The method that we just pointed out, which is general, is susceptible to some simplifications in certain special cases.

For example, suppose that one demands that the curves have constant torsion. Serret's formulas:

$$\frac{dc}{ds} = \frac{b}{\tau}, \qquad \frac{dc'}{ds} = \frac{b'}{\tau}, \qquad \frac{dc''}{ds} = \frac{b''}{\tau}$$

will give us b, b', b'' as functions of the derivatives of c. If one substitutes those values in the relations between the nine cosines:

$$a = b'c'' - c'b'', \qquad a' = b''c - b''c, \qquad a'' = bc' - cb'$$

then one will find that:

$$\begin{aligned} a &= \tau\left(c''\frac{dc'}{ds} - c'\frac{dc''}{ds}\right), \\ a' &= \tau\left(c\frac{dc''}{ds} - c''\frac{dc}{ds}\right), \\ a'' &= \tau\left(c'\frac{dc}{ds} - c\frac{dc'}{ds}\right). \end{aligned}$$

One will, in turn, have the formulas:

$$x = \int a \quad ds = \tau \int (c'' \, dc' - c' \, dc''),$$
$$y = \int a' \; ds = \tau \int (c \;\, dc'' - c'' \, dc),$$
$$z = \int a'' \, ds = \tau \int (c' \;\, dc - c \;\, dc')$$

for the rectangular coordinates of a point on the curve, in which c, c', c'' are three functions of only one variable that are subject to the single condition:

$$c^2 + c'^{\,2} + c''^{\,2} = 1.$$

If, for example, one sets:

$$\frac{c}{h} = \frac{c'}{k} = \frac{c''}{l} = \frac{1}{\sqrt{h^2 + k^2 + l^2}}$$

then one will have:

$$(9) \qquad \begin{cases} x = \tau \displaystyle\int \frac{l\,dk - k\,dl}{h^2 + k^2 + l^2}, \\[2ex] y = \tau \displaystyle\int \frac{h\,dl - l\,dh}{h^2 + k^2 + l^2}, \\[2ex] z = \tau \displaystyle\int \frac{k\,dh - h\,dk}{h^2 + k^2 + l^2}. \end{cases}$$

These formulas coincide, up to notations, with the ones that J.-A. Serret gave in the 5th edition of *Application de l'Analyse à la Géométrie,* by Monge, pp. 566.

37. An analogous method applies to the determinations of the curves whose radius of first curvature is constant. Indeed, recall the formulas:

$$\frac{dx}{ds} = a, \qquad \frac{da}{ds} = \frac{b}{\rho}.$$

One deduces that:

$$\frac{ds}{\rho} = \sqrt{da^2 + da'^2 + da''^2}$$

from this, and consequently:

$$dx = a\,\rho\,\sqrt{da^2 + da'^2 + da''^2}.$$

One will then have:

$$(10) \qquad \begin{cases} x = \rho \displaystyle\int a \;\; d\sigma, \\[2ex] y = \rho \displaystyle\int a' \; d\sigma, \\[2ex] z = \rho \displaystyle\int a'' \, d\sigma \end{cases}$$

for the three coordinates of a point of the desired curve, in which $d\sigma$ denotes the differential of the arc length of the spherical curve that is described by the point (a, a', a''). The center of curvature will have the following values for its coordinates:

$$
(11) \qquad \begin{cases}
x_1 = x + b\ \rho = \rho \dfrac{da}{d\sigma} + \rho \displaystyle\int a\ \ d\sigma, \\
y_1 = y + b'\rho = \rho \dfrac{da'}{d\sigma} + \rho \displaystyle\int a'\ d\sigma, \\
z_1 = z + b''\rho = \rho \dfrac{da''}{d\sigma} + \rho \displaystyle\int a''\ d\sigma,
\end{cases}
$$

and it is easy to verify that the locus of that point is also a curve whose first curvature is constant and equal to that of the first one, which conforms to the results of no. **10**.

38. Finally, if one seek the curves that enjoy the property that was pointed out by Bertrand (no. **8**), and whose two curvatures are linked by the equation:

$$
(12) \qquad \frac{m}{\rho} + \frac{n}{\tau} = 1,
$$

then one will set:

$$
(13) \qquad \begin{cases}
ma\ + nc\ = \alpha\ \sqrt{m^2 + n^2}, \\
ma' + nc' = \alpha' \sqrt{m^2 + n^2}, \\
ma'' + nc'' = \alpha'' \sqrt{m^2 + n^2},
\end{cases}
$$

in which α, α', α'' are three functions that are obviously subject to the relation:

$$\alpha^2 + \alpha'^2 + \alpha''^2 = 1.$$

The Serret formulas:

$$\frac{da}{ds} = \frac{b}{\rho}, \qquad \frac{dc}{ds} = \frac{b}{\tau}$$

will give us:

$$
(14) \qquad \sqrt{m^2 + n^2}\,\frac{d\alpha}{ds} = b, \qquad \sqrt{m^2 + n^2}\,\frac{d\alpha'}{ds} = b', \qquad \sqrt{m^2 + n^2}\,\frac{d\alpha''}{ds} = b'',
$$

if we take the relation (12) into account.

One will then have:

$$(15)\qquad \begin{cases} na - mc = n\,(b'c'' - c'b'') - m\,(a'b'' - b'a''), \\ \qquad\qquad = (m^2 + n^2)\left(\alpha''\dfrac{d\alpha'}{ds} - \alpha'\dfrac{d\alpha''}{ds}\right), \end{cases}$$

and similarly:

$$(15')\qquad \begin{cases} na' - mc' = (m^2 + n^2)\left(\alpha\dfrac{d\alpha''}{ds} - \alpha''\dfrac{d\alpha}{ds}\right), \\ na'' - mc'' = (m^2 + n^2)\left(\alpha'\dfrac{d\alpha}{ds} - \alpha\dfrac{d\alpha'}{ds}\right). \end{cases}$$

The relations (13) and (15) permit us to determine a, a', a''; c, c', c'', and give us:

$$(16)\qquad \begin{cases} a = \dfrac{m}{\sqrt{m^2+n^2}}\,\alpha + n\left(\alpha''\dfrac{d\alpha'}{ds} - \alpha'\dfrac{d\alpha''}{ds}\right), \\ a' = \dfrac{m}{\sqrt{m^2+n^2}}\,\alpha' + n\left(\alpha\dfrac{d\alpha''}{ds} - \alpha''\dfrac{d\alpha}{ds}\right), \\ a'' = \dfrac{m}{\sqrt{m^2+n^2}}\,\alpha'' + n\left(\alpha'\dfrac{d\alpha}{ds} - \alpha\dfrac{d\alpha'}{ds}\right). \end{cases}$$

Moreover, we will deduce that:

$$ds^2 = (m^2 + n^2)\,(d\alpha^2 + d\alpha'^2 + d\alpha''^2)$$

from formulas (14).

One will then get the rectangular coordinates of a point of the curve by the use of the equations:

$$x = \int a\,ds, \qquad y = \int a'\,ds, \qquad z = \int a''\,ds,$$

which will lead to the definitive result:

$$(17)\qquad \begin{cases} d\sigma = \sqrt{d\alpha^2 + d\alpha'^2 + d\alpha''^2}, \\ x = m\int \alpha\;\;d\sigma + n\int (\alpha''d\alpha' - \alpha'd\alpha''), \\ y = m\int \alpha'\;\;d\sigma + n\int (\alpha\,d\alpha'' - \alpha''d\alpha), \\ z = m\int \alpha''\,d\sigma + n\int (\alpha'\;d\alpha - \alpha\,d\alpha'). \end{cases}$$

One will recover formulas (9) or (10), according to whether one makes m or n equal to zero. Moreover, α, α', α'' are, as we have seen, three functions of just one variable that are subject to the single relation:

$$(18)\qquad \alpha^2 + \alpha'^2 + \alpha''^2 = 1.$$

The preceding formulas also established very easily by the use of geometry.

39. Of the three systems (9), (10), (17), the simplest of them is the system (9), which determines the curves whose torsion is constant. We shall apply it to the search for the curve of constant torsion whose spherical indicatrix is a spherical conic. One will easily recognize that if one draws a parallel to the binormal through the center of the sphere of radius 1 then that parallel will cut the sphere at a point whose coordinates will be c, c', c'', and which will describe a spherical ellipse that is supplementary to the spherical indicatrix.

Upon choosing the axes suitably, one can then obtain very simple expressions for h, k, l:

$$h = \sqrt{\frac{a(a-\rho)}{(a-b)(a-c)}}, \quad k = \sqrt{\frac{b(b-\rho)}{(b-a)(b-c)}}, \quad l = \sqrt{\frac{c(c-\rho)}{(c-a)(c-b)}},$$

which agrees with the curve that is situated on the cone:

$$\frac{x^2}{a} + \frac{y^2}{b} + \frac{z^2}{c} = 0,$$

and upon substituting these values for h, k, l in formulas (9), one will get the system:

$$\text{(19)} \qquad \left\{ \begin{aligned} x &= \frac{\tau}{2}\sqrt{\frac{bc}{(a-c)(c-a)}}\int \frac{d\rho}{\sqrt{(b-\rho)(c-\rho)}}, \\ y &= \frac{\tau}{2}\sqrt{\frac{ac}{(b-c)(a-b)}}\int \frac{d\rho}{\sqrt{(a-\rho)(c-\rho)}}, \\ z &= \frac{\tau}{2}\sqrt{\frac{ab}{(a-c)(c-b)}}\int \frac{d\rho}{\sqrt{(a-\rho)(b-\rho)}}, \end{aligned} \right.$$

which will define the desired curve.

We do not know of any real, algebraic curve whose torsion is constant. It would be interesting to examine whether all of the curves of constant torsion are necessarily transcendental, or if they are algebraic, to determine the simplest ones.

CHAPTER V

DISPLACEMENTS WITH TWO INDEPENDENT VARIABLES.

Differential relations between two systems of rotations. Determination of the motion when those rotations are known. Application to the case in which they are functions of only one variable.

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40. In the preceding chapters, we saw how one could attach the theory of skew curves to the study of the motion of a trihedron. Other research in geometry, and in particular the work that referred to the theory of surfaces, demanded that one should consider moving systems whose various positions depended upon two distinct parameters. We shall undertake the study of such systems, and in order to first study the properties of rotations, we shall begin by supposing that the moving system has a fixed point, which will be the origin of both the fixed and moving axes, as before.

The nine cosines that determine the positions of the moving axes will then be functions of two independent variables u and v. By starting from each of its positions, the moving system can take on an infinitude of motions that correspond to the various relations that one can establish between u and v. We introduce two different systems of rotations here. One of them, which we denote by p, q, r, refers to the displacements under which only u varies. They give rise to the system:

$$(1) \qquad \frac{\partial \alpha}{\partial u} = \beta r - \gamma q, \qquad \frac{\partial \beta}{\partial u} = \gamma p - \alpha r, \qquad \frac{\partial \gamma}{\partial u} = \alpha q - \beta p,$$

which must admit the three cosines from each group as particular solutions. The other ones, which we denote by p_1, q_1, r_1, relate to the case in which only v varies. They likewise give rise to the system:

$$(1) \qquad \frac{\partial \alpha}{\partial v} = \beta r_1 - \gamma q_1, \qquad \frac{\partial \beta}{\partial v} = \gamma p_1 - \alpha r_1, \qquad \frac{\partial \gamma}{\partial v} = \alpha q_1 - \beta p_1,$$

which is entirely similar to the first one. It results immediately from this that if one considers a displacement of the system in which u and v are given functions of t then one will have:

$$\frac{d\alpha}{dt} = \beta R - \gamma Q, \qquad \frac{d\beta}{dt} = \gamma P - \alpha R, \qquad \frac{d\gamma}{dt} = \alpha Q - \beta P,$$

in which P, Q, R have the values:

$$(3) \qquad P = p\frac{du}{dt} + p_1\frac{dv}{dt}, \qquad Q = q\frac{du}{dt} + q_1\frac{dv}{dt}, \qquad R = r\frac{du}{dt} + r_1\frac{dv}{dt},$$

and consequently those three quantities P, Q, R will be rotations relative to the motions considered. The projections onto the moving axes of the path or the infinitely-small arc that is described by a point under that motion whose coordinates relative to those axes are x, y, z will have the values:

$$(4)\qquad \begin{cases} dx + (q\,du + q_1 dv)\,z - (r\,du + r_1\,dv)\,y, \\ dy + (r\,du + r_1 dv)\,x - (p\,du + p_1\,dv)\,z, \\ dz + (p\,du + p_1 dv)\,y - (q\,du + q_1\,dv)\,x. \end{cases}$$

We shall first establish certain partial differential equations that the six rotations must satisfy.

We equate the two values of $\dfrac{\partial^2 \alpha}{\partial u\,\partial v}$ that one can obtain by differentiating the first two equations of the systems (1) and (2). After replacing the derivatives of β, γ with their values that we infer from these two systems, we will have:

$$\beta\left(\frac{\partial r}{\partial v} - \frac{\partial r_1}{\partial u} - pq_1 + qp_1\right) = \gamma\left(\frac{\partial q}{\partial v} - \frac{\partial q_1}{\partial u} - rp_1 + pr_1\right).$$

Since that relation must be true when one replaces β, γ with either b, c, or b', c', or b'', c'', it is necessary that the coefficients of β and γ must be separately zero. We will then have two equations. Upon likewise equating the two values of $\dfrac{\partial^2 \beta}{\partial u\,\partial v}$, $\dfrac{\partial^2 \gamma}{\partial u\,\partial v}$ that are deduced from the systems (1) and (2), one will obtain just one new equation, and one will be led to the system:

$$(5)\qquad \begin{cases} \dfrac{\partial p}{\partial v} - \dfrac{\partial p_1}{\partial u} = qr_1 - rq_1, \\[2mm] \dfrac{\partial q}{\partial v} - \dfrac{\partial q_1}{\partial u} = rp_1 - pr_1, \\[2mm] \dfrac{\partial r}{\partial v} - \dfrac{\partial r_1}{\partial u} = pq_1 - qp_1, \end{cases}$$

which plays a fundamental role in the theory ([10]).

41. Conversely, whenever one knows six quantities p, q, r, p_1, q_1, r_1 that satisfy equations (5), there will exist a motion for which those six quantities are rotations. In order to establish that result, it will obviously suffice to show that one can obtain the

([10]) These equations were obtained by Combescure [Annales de l'École Normale (1) **4**, pp. 108], who was the first to employ kinematic considerations in the proof of formulas that related to the theory of surfaces and orthogonal systems. We personally presented them before the publication of Combescure's paper in a course that we gave in 1866-67 at the Collége de France as a substitute for J. Bertrand.

values of the nine coefficients that satisfy both the systems (1) and (2). The proof of that essential proposition can be deduced from general theorems that relate to partial differential equations, but one can also obtain it directly in the following manner:

I first say that upon supposing that equations (5) are satisfied, one can deduce a new solution of equations (1) from any system of values (α, β, γ) that satisfies equations (1), but not equations (2).

Indeed, set:

$$A = \frac{\partial\alpha}{\partial v} - \beta r_1 + \gamma q_1\,,$$

$$B = \frac{\partial\beta}{\partial v} - \gamma p_1 + \alpha r_1\,,$$

$$C = \frac{\partial\gamma}{\partial v} - \alpha q_1 + \beta p_1\,.$$

We shall show that the quantities A, B, C, which are not all zero, by hypothesis, verify equations (1).

Indeed, one has:

$$\begin{aligned}\frac{\partial A}{\partial u} &= \frac{\partial^2\alpha}{\partial u\,\partial v} - r_1\frac{\partial\beta}{\partial u} - q_1\frac{\partial\gamma}{\partial u} - \beta\frac{\partial r_1}{\partial u} - \gamma\frac{\partial q_1}{\partial u}\\ &= \frac{\partial}{\partial v}(\beta r - \gamma q) - r_1\frac{\partial\beta}{\partial u} - q_1\frac{\partial\gamma}{\partial u} - \beta\frac{\partial r_1}{\partial u} - \gamma\frac{\partial q_1}{\partial u}\,,\end{aligned}$$

or, upon replacing $\frac{\partial\beta}{\partial u}$, $\frac{\partial\gamma}{\partial u}$ with their values and taking equations (5) into account:

$$\frac{\partial A}{\partial u} = Br - Cq. \tag{6}$$

One will likewise have:

$$\left\{\begin{aligned}\frac{\partial B}{\partial u} &= Cp - Ar,\\ \frac{\partial C}{\partial u} &= Aq - Bp,\end{aligned}\right. \tag{6′}$$

upon performing some permutations, and consequently, A, B, C will indeed give a new solution to system (1). Since system (6) has degree one with respect to the derivatives of the functions A, B, C, it will obviously admit just one solution for which the initial values of those functions that correspond to a given value u_0 of u will be quantities A_0, B_0, C_0 that are given in advance ([11]). It is also obvious that if those initial values are zero then

([11]) The proposition that we just assumed here, and in the developments that will follow – namely, that when a first-order system of differential equations is solved with respect to the derivatives of the unknown functions, it will admit just one solution for which the initial values of the unknown functions are given – is due, as one knows, to Cauchy, who proved it in full generality. In truth, he assumed some exceptions that the unique solution that they will correspond to is the one that is determined by the equations:

$$A = B = C = 0.$$

We can then state the following proposition:

If one knows one solution of the system (1) *that will satisfy equations* (2) *when it is substituted in them and one gives a particular value* u_0 *to* u *then it will likewise satisfy them for any value of* u.

42. Having made that point, suppose that one desires to find the most general solutions that are common to both equations (1) and (2). We shall see that there exists an infinitude of values for α, β, γ that satisfy those equations, and that each system of common solutions is determined completely when one gives the values $\alpha_0, \beta_0, \gamma_0$ of α, β, γ that correspond to the initial values u_0, v_0 of u and v.

Indeed, suppose that one replaces u with u_0 in α, β, γ. The desired solutions will reduce to functions α', β', γ' of v. Now, those functions of v are plainly determined by the condition that they must satisfy equations (2) when one replaces u with u_0 and assume the initial values $\alpha_0, \beta_0, \gamma_0$ for $v = v_0$. On the other hand, α, β, γ are functions of u that must satisfy equations (1) and reduce to α', β', γ' for $u = u_0$. They themselves are also determined completely by that double condition, and it will suffice for us to show that those functions α, β, γ likewise satisfy system (2).

Now, that fact is almost obvious, because if one replaces u with u_0 then α, β, γ will reduce to α', β', γ' and satisfy equations (2) for that particular value of u. Consequently, they will satisfy them for all values of u, from the proposition that we just proved.

43. There is then an infinitude of different systems of common solutions, and the most general solution depends on three arbitrary constants, as one sees. On the other hand, if α, β, γ; $\alpha_1, \beta_1, \gamma_1$ denote two different systems of solutions then the functions:

$$\alpha^2 + \beta^2 + \gamma^2, \qquad \alpha\alpha_1 + \beta\beta_1 + \gamma\gamma_1, \qquad \alpha_1^2 + \beta_1^2 + \gamma_1^2$$

will remain constant for all values of u and v. The proof of this is the same as in the case of one variable. As a result, if we take three different systems of solutions a, b, c; a', b', c' ; a'', b'', c'' whose initial values are the nine cosines that determine the position of a tri-rectangular trihedron (T_0) with respect to the fixed axes then we will have relations such as the following ones:

$$a^2 + b^2 + c^2 = 1, \qquad aa' + bb' + cc' = 0$$

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corresponded to the cases in which the derivatives of the unknown functions were presented, wholly or in part, in an indeterminate or infinite form. However, we obviously do not have to be preoccupied with those exceptional cases in the questions that we shall address.

for all values of u and v, and for all values of u and v, our three systems of solutions will define the position of a moving trihedron for which the rotations will be the given quantities p, q, r ; p_1, q_1, r_1 , precisely.

Here again, as in the case of just one variable, all of the solutions that one can obtain will be deduced from one of them by a simple change of coordinates. One will always have the same displacement, but it will be referred to different axes.

41. As an application, we propose to look for the motions under which the six rotations depend upon just one variable v. Equations (5) then become:

$$\text{(7)} \qquad \begin{cases} \dfrac{\partial p}{\partial v} = q r_1 - r q_1, \\ \dfrac{\partial q}{\partial v} = r p_1 - p r_1, \\ \dfrac{\partial r}{\partial v} = p q_1 - q p_1. \end{cases}$$

One deduces from this that:

$$p \frac{\partial p}{\partial v} + q \frac{\partial q}{\partial v} + r \frac{\partial r}{\partial v} = 0,$$

and consequently, since h is a constant:

$$p^2 + q^2 + r^2 = h^2.$$

One already sees that the systems (1) and (2) admit the solution:

$$\alpha = \frac{p}{h}, \qquad \beta = \frac{q}{h}, \qquad \gamma = \frac{r}{h}.$$

We can take that solution to represent a'', b'', c'', and upon appealing to Euler's formulas, we will have:

$$-\sin\theta \sin\varphi = \frac{p}{h}, \qquad -\sin\theta\cos\varphi = \frac{q}{h}, \qquad \cos\theta = \frac{r}{h},$$

which already shows that θ and φ are functions of the single variable v.

If we refer to formulas (7) [pp. ?], which give the rotations, then here we will have:

$$\sin\theta \frac{\partial \psi}{\partial u} = p \sin\varphi + q \cos\varphi,$$

$$\sin\theta \frac{\partial \psi}{\partial v} = p_1 \sin\varphi + q_1 \cos\varphi,$$

and consequently:

$$\frac{\partial \psi}{\partial u} = -\, h, \qquad \frac{\partial \psi}{\partial v} = -\, \frac{pp_1 + qq_1}{h^2 - r^2}\, h,$$

which will give:

$$\psi = -\, hu + V,$$

in which V denotes a function of v. Conversely, formulas (6) [pp. ?] show us that if θ, φ do not contain u, and if ψ contains it only linearly then the six rotations will indeed be functions of the single variable v.

45. In the preceding example, the six rotations were functions of one of the independent variables. We propose to seek, in a more general manner, all of the cases in which they are functions of any of them. p, q, r; p_1, q_1, r_1 can then be regarded as functions of one certain variable θ, which will depend on the variables u and v in an arbitrary manner.

If we denote the derivatives of p, q, … with respect to θ by p', q', …, resp., then equations (5) will give us:

$$(8) \qquad \begin{cases} p' \dfrac{\partial \theta}{\partial v} - p_1' \dfrac{\partial \theta}{\partial u} = qr_1 - rq_1, \\ q' \dfrac{\partial \theta}{\partial v} - q_1' \dfrac{\partial \theta}{\partial u} = rp_1 - pr_1, \\ r' \dfrac{\partial \theta}{\partial v} - r_1' \dfrac{\partial \theta}{\partial u} = pq_1 - qp_1. \end{cases}$$

If one can infer the values of $\dfrac{\partial \theta}{\partial u}$, $\dfrac{\partial \theta}{\partial v}$ from any two of these equations then those values will have the form:

$$\frac{\partial \theta}{\partial u} = f(\theta), \qquad \frac{\partial \theta}{\partial v} = \varphi(\theta),$$

and those two equations will lead to an expression for θ of the form:

$$\theta = F(au - bv),$$

in which a and b are two constants. Upon replacing the variables u and v with the following ones:

$$u_1 = au - bv, \qquad v_1 = av - bu,$$

one will come back to the preceding case.

It only remains for us to examine the case in which the equations (8) can be solved for $\dfrac{\partial \theta}{\partial u}$, $\dfrac{\partial \theta}{\partial v}$, and in which one has, consequently:

(9) $$\frac{p'}{p_1'} = \frac{q'}{q_1'} = \frac{r'}{r_1'}, \qquad \frac{p'}{qr_1 - rq_1} = \frac{q'}{rp_1 - pr_1} = \frac{r'}{pq_1 - qp_1}.$$

One first deduces from the relations that:

$$p'p + q'q + r'r = 0.$$
$$p_1' p_1 + q_1' q_1 + r_1' r_1 = 0,$$

and upon integrating:

$$p^2 + q^2 + r^2 = \text{const.},$$
$$p_1^2 + q_1^2 + r_1^2 = \text{const.}$$

Upon multiplying the variables u and v by suitably-chosen constants, one can write:

$$p^2 + q^2 + r^2 = 1,$$
$$p_1^2 + q_1^2 + r_1^2 = 1,$$

and consequently, the extremities (p, q, r), (p_1, q_1, r_1) of the two rotational axes will describe two curves (C), (C') that are situated on the sphere of radius 1 with respect to the moving axes.

It results from the first of equations (9) that those two curves will have parallel tangents for each value of θ. They are then two mutually-parallel curves, and if one supposes (as is obviously permissible) that one has taken θ to be the arc length of the curve (C) when one starts from a fixed origin then one can set:

(10) $$\begin{cases} p_1 = p\cos h + \sin h\,(qr' - rq'), \\ q_1 = q\cos h + \sin h\,(rp' - pr'), \\ r_1 = r\cos h + \sin h\,(pq' - qp'), \end{cases}$$

in which h is a constant angle.

46. One can deduce a geometric representation of motion from these results. Consider the curve that is described in space by the extremity of one of the instantaneous axes; for example, by the point (p, q, r). If one lets $a, b, c, a', \ldots$ denote the nine cosines that determine the position of the moving system, and lets X, Y, Z denote the coordinates of that point relative to the fixed axes then one will have:

$$X = a\,p + b\,q + c\,r,$$
$$Y = a'p + b'q + c'r,$$
$$Z = a''p + b''q + c''r.$$

Totally differentiate the first of these equations and replace da, db, dc with their values that are deduced from (1), (2). Upon then replacing p_1, q_1, r_1 with their values that one infers from equations (10), we will obtain:

$$dX = (ap' + bq' + cr')\, d\,(\theta + v \sin h).$$

That formula shows us that X, Y, Z depend upon the same variable $\theta + v \sin h$. Consequently, the pole (X, Y, Z) will describe a curve (Γ) in space that is traced on the sphere of radius 1, and which will always be in contact with the curve (C). We will then be led to the following result:

Consider two curves (C) and (Γ) on the sphere of radius 1. If we displace the curve (C), while requiring it to remain tangent to the curve (Γ), then a curve (C') that is parallel to (C) and is carried along by its motion will always remain tangent to a fixed curve (Γ') that is parallel to (Γ). The displacement of the two curves (C) and (C') is precisely the one that we propose to define. When only u varies, the curve (C) will roll with a constant velocity on (Γ), and similarly when v only varies, (C') will roll on (Γ') with a velocity that is also constant.

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CHAPTER VI

SIMULTANEOUS INTEGRATION OF THE LINEAR SYSTEMS THAT WERE ENCOUNTERED IN THE PRECEDING THEORY.

Reduction of the problem to the simultaneous integration of two Riccati equations. – Various propositions that relate those two equations. – Another method of solution that is based upon the determination of *a*, *a′*, *a″*.

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47. Now that we recognize the existence of common solutions to the systems (1) and (2), we shall indicate how one can determine them. Since one must consider only solutions for which one has:

$$\alpha^2 + \beta^2 + \gamma^2 = 1 \tag{1}$$

in the question that we address, one can express α, β, γ as functions of the x, y by means of formulas (9) [pp. ?], and from the results that were obtained in no. **15**, the variables x, y must both satisfy the two equations:

$$\left\{\begin{aligned} \frac{\partial\sigma}{\partial u} &= -ir\sigma + \frac{q-ip}{2} - \frac{q+ip}{2}\sigma^2, \\ \frac{\partial\sigma}{\partial v} &= -ir_1\sigma + \frac{q_1-ip_1}{2} + \frac{q_1+ip_1}{2}\sigma^2, \end{aligned}\right. \tag{2}$$

which are obviously compatible, like the systems that one deduces from them.

48. We are then led to the following problem in analysis: Study the simultaneous integration of the two equations of the form:

$$\left\{\begin{aligned} \frac{\partial\sigma}{\partial u} &= a + 2b\sigma + c\sigma^2, \\ \frac{\partial\sigma}{\partial v} &= a_1 + 2b_1\sigma + c_1\sigma^2, \end{aligned}\right. \tag{3}$$

in which a, b, c ; a_1, b_1, c_1 are given functions of u and v.

Upon equating the two values of $\dfrac{\partial^2\sigma}{\partial u\,\partial v}$ that one can deduce from these equations, one will be led to the relation:

$$2\,(c\sigma + b)(a_1 + 2b_1\,\sigma + c_1\sigma^2) - 2\,(c_1\sigma + b_1)(a + 2b\sigma + c\sigma^2)$$

$$+\ \frac{\partial a}{\partial v}-\frac{\partial a_1}{\partial u}+2\sigma\left(\frac{\partial b}{\partial v}-\frac{\partial b_1}{\partial u}\right)+\sigma^2\left(\frac{\partial c}{\partial v}-\frac{\partial c_1}{\partial u}\right)=0,$$

which has degree two with respect to σ.

If that relation is not true identically then the two equations (3) can admit at most two common solutions. Hence, if one demands that the system (3) should have a solution that contains an arbitrary constant then it will be necessary that the coefficients of the various powers of σ in the preceding equation should be zero, which will give:

$$(4)\qquad \begin{cases} \dfrac{\partial a}{\partial v}-\dfrac{\partial a_1}{\partial u}+2ba_1-2ab_1=0, \\ \dfrac{\partial b}{\partial v}-\dfrac{\partial b_1}{\partial u}+\ ca_1-\ ac_1=0, \\ \dfrac{\partial c}{\partial v}-\dfrac{\partial c_1}{\partial u}+2cb_1-2bc_1=0. \end{cases}$$

When these relations are applied to equations (2), they will reproduce formulas (5) of the preceding chapter. Consequently, we can suppose that they are verified in the rest of our discussion.

49. Conversely, we shall show that when the coefficients a, b, c ; a_1, b_1, c_1 satisfy conditions (4), the proposed equations (3) will admit a common solution that contains an arbitrary constant. In order to do that, we let σ denote an arbitrary solution of the first of those equations and set:

$$(5)\qquad \theta=\frac{\partial\sigma}{\partial v}-a_1-2b_1\sigma-c_1\,\sigma^2.$$

Moreover, one has:

$$\frac{\partial\sigma}{\partial u}-a_1-2b_1\sigma-c_1\,\sigma^2=0.$$

Differentiate the preceding equation with respect to v and equation (5) with respect to u, and subtract the two relations thus-obtained. Upon taking the identities (4) into account, as well as the preceding two equations, we will have:

$$(6)\qquad \frac{\partial\theta}{\partial u}=2\,(c\sigma+b)\ \theta,$$

which will give:

$$(7)\qquad \theta=\theta_0\ e^{2\int_{u_0}^{u_1}(c\sigma+b)\,du}$$

by integration, in which θ_0 denotes the value of θ for $u = u_0$. Hence, if θ is zero for $u = u_0$ then the same thing will be true for all values of u.

Having established that point, set $u = u_0$ in the second of equations (3) and determine the function σ' of v that satisfies that equation and reduces to σ_0 for $v = v_0$. One will then have:

$$\frac{\partial \sigma'}{\partial v} = a_1 + 2b_1\sigma' + c_1\,\sigma'^2$$

for $u = u_0$.

Consider the first of equations (3), in turn, and determine the function σ that satisfies that equation and reduces to σ' for $u = u_0$. From what we just proved, that function σ will satisfy two equations, because the value of the function that we have denoted by θ relative to the solution thus-determined will be zero for $u = u_0$, and consequently, it will remain zero for all values of u. The function σ that satisfies the two equations will contain the arbitrary constant σ_0, moreover.

All of the operations that were just indicated are possible and require no quadrature when one knows how to separately integrate each of equations (3). Thus, one will know how to determine the common solution to the two equations without integration when one has obtained the integral of each of them.

50. We shall now prove some propositions that make the simultaneous integration of equations (3) simpler.

First, suppose that one knows a common solution to those two equations $\sigma = x$. Upon setting:

$$\sigma = x - \frac{1}{\omega},$$

one will be led to two linear equations of the form:

$$\frac{\partial \omega}{\partial u} = P\,\omega + Q,$$
$$\frac{\partial \omega}{\partial v} = P_1\,\omega + Q_1,$$

and the general value of ω will be obtained by the formula:

$$\omega = e^{\int (P\,du + P_1 dv)} \int e^{-\int (P\,du + P_1 dv)} (Q\,du - Q_1\,dv), \tag{8}$$

which contains only exact differentials, as one easily assures oneself.

It is pointless to dwell upon the case in which one has two or three common solutions; the method that we followed in no. **17** will apply without modification. However, it is convenient to examine the hypothesis that one has a particular solution of one of the equations *that does not satisfy the other one.* One can append the following result to the ones that were just established: *One can determine the general integral from those of the equations without integration when one knows a particular solution.*

For example, let x be a value of σ that satisfies the first equation and not the second one. Set:

$$\sigma = x - \frac{1}{\omega}.$$

If we substitute that value of σ in the first of equations (3) then it will become:

$$\frac{\partial \omega}{\partial u} = -2\,(b + cx)\;\omega - c.$$

Introduce the function θ that is defined by equation (5) when one replaces σ with x, and which, from formula (6), will verify the equation:

$$\frac{\partial \theta}{\partial u} = 2\,(b + cx)\;\theta.$$

The equation that ω must satisfy can be exhibited in the following form:

(9) $$\frac{\partial (\omega\theta)}{\partial u} - c\theta = 0.$$

Now, an easy calculation will lead to the identity:

$$c\theta = \frac{\partial}{\partial v}(cx + b) - \frac{\partial}{\partial u}(c_1 x + b_1) = \frac{\partial}{\partial u}\left(\frac{1}{2}\frac{\partial \log \theta}{\partial v} - c_1 x - b_1\right).$$

Upon substituting the value of $c\theta$ in equation (9), we will find that:

$$\frac{\partial}{\partial u}\left(\omega\theta + \frac{1}{2}\frac{\partial \log \theta}{\partial v} - c_1 x - b_1\right) = 0,$$

and upon integrating:

(10) $$\omega\theta = \frac{\theta}{\sigma - x} = c_1 x + b_1 - \frac{1}{2}\frac{\partial \log \theta}{\partial v} + C,$$

in which C denotes the arbitrary constant, which can be a function of v. The formula that shows one the general integral of the first equation contains no quadrature sign, as we have asserted.

The proposition that was just established, when combined with the ones that preceded it, will permit us to conclude that if one knows a particular solution to each of the two equations (3) that does not satisfy the other one then it will be possible to obtain the general solutions to those two equations with no actual quadrature, *and consequently their common solutions, as well.*

51. In conclusion, we will show how one can define various differential equations whose integration will imply that of the system (3) without quadrature by appealing to the latter result.

Add equations (3), after multiplying them by du and dv, respectively. We will have:

$$d\sigma - (a\,du + a_1\,dv) - 2\,(b\,du + b_1\,dv)\,\sigma - (c\,du + c_1\,dv)\,\sigma^2 = 0.$$

Now, imagine the two differential equations that we define by replacing σ with the two functions σ_1, σ_2 in succession, *which are chosen at will:*

$$(11)\qquad \begin{cases} d\sigma_1 - a\,du - a_1 dv - 2(b\,du + b_1 dv)\,\sigma_1 - (c\,du + c_1 dv)\,\sigma_1^2 = 0, \\ d\sigma_2 - a\,du - a_1 dv - 2(b\,du + b_1 dv)\,\sigma_2 - (c\,du + c_1 dv)\,\sigma_2^2 = 0. \end{cases}$$

Suppose that one knows how to integrate each of these two differential equations. I say that one can integrate the system (3) with no quadrature.

Indeed, let:

$$\varphi(u, v) = \alpha$$

be the integral to the first equation (11) and let:

$$\psi(u, v) = \beta$$

be that of the second equation. Make a change of variables and substitute α, β for u and v. From the definition of α and β itself, one will have:

$$(12)\qquad \begin{cases} \dfrac{\partial \sigma_1}{\partial \beta} = a\dfrac{\partial u}{\partial \beta} + a_1\dfrac{\partial v}{\partial \beta} + 2\left(b\dfrac{\partial u}{\partial \beta} + b_1\dfrac{\partial v}{\partial \beta}\right)\sigma_1 + \left(c\dfrac{\partial u}{\partial \beta} + c_1\dfrac{\partial v}{\partial \beta}\right)\sigma_1^2, \\ \dfrac{\partial \sigma_2}{\partial \alpha} = a\dfrac{\partial u}{\partial \alpha} + a_1\dfrac{\partial v}{\partial \alpha} + 2\left(b\dfrac{\partial u}{\partial \alpha} + b_1\dfrac{\partial v}{\partial \alpha}\right)\sigma_2 + \left(c\dfrac{\partial u}{\partial \alpha} + c_1\dfrac{\partial v}{\partial \alpha}\right)\sigma_2^2. \end{cases}$$

As for the system (3), it will take the form:

$$(13)\qquad \begin{cases} \dfrac{\partial \sigma}{\partial \beta} = a\dfrac{\partial u}{\partial \beta} + a_1\dfrac{\partial v}{\partial \beta} + 2\left(b\dfrac{\partial u}{\partial \beta} + b_1\dfrac{\partial v}{\partial \beta}\right)\sigma + \left(c\dfrac{\partial u}{\partial \beta} + c_1\dfrac{\partial v}{\partial \beta}\right)\sigma^2, \\ \dfrac{\partial \sigma}{\partial \alpha} = a\dfrac{\partial u}{\partial \alpha} + a_1\dfrac{\partial v}{\partial \alpha} + 2\left(b\dfrac{\partial u}{\partial \alpha} + b_1\dfrac{\partial v}{\partial \alpha}\right)\sigma + \left(c\dfrac{\partial u}{\partial \alpha} + c_1\dfrac{\partial v}{\partial \alpha}\right)\sigma^2. \end{cases}$$

Equations (12) express the idea that σ_1, σ_2 are particular solutions to the first and second of equations (13), respectively. Consequently, from what we proved, one can integrate the system (13) completely, which is equivalent to the system (3).

In order to apply the last proposition, we return to the proposed system of equations (2), and take:

$$\sigma_1 = 1, \qquad \sigma_2 = -1.$$

The two equations (11) will become:

$$q\,du + q_1\,dv + i\,(r\,du + r_1\,dv) = 0,$$
$$q\,du + q_1\,dv - i\,(r\,du + r_1\,dv) = 0,$$

here.

The integration of these two equations will then imply that of the system (2). We shall explain that result by introducing a new means of first integrating the system of equations (1) and (2) in the preceding chapter that is completely different from the one that we just studied.

52. One always denotes the nine cosines that satisfy these equations (1) and (2) by a, b, … One will have:

$$(14) \qquad \begin{cases} da = (br - cq)\,du + (br_1 - cq_1)\,dv, \\ db = (cp - ar)\,du + (cp_1 - ar_1)\,dv, \\ dc = (aq - bp)\,du + (aq_1 - bp_1)\,dv, \end{cases}$$

and the analogous relations that one will obtain by putting primes on a, b, c. If one combines the first of these equations with the ones that give da', da'' then one will obtain:

$$(15) \qquad da^2 + da'^2 + da''^2 = (q\,du - q_1\,dv)^2 + (r\,du - r_1\,dv)^2 .$$

Now, since a, a', a'' are coupled by the equation:

$$a^2 + a'^2 + a''^2 = 1,$$

they will be the coordinates of a point in a sphere, and it is likewise obvious that the sphere will be the one that is described by a point that is situated at a distance 1 from the moving axis Ox. Here again, if one consider that sphere to be a ruled surface, and if one sets:

$$(16) \qquad \frac{a + ia'}{1 - a''} = x, \qquad \frac{a - ia'}{1 - a''} = -\frac{1}{y}$$

then equation (15) will take the form:

$$(17) \qquad \frac{4\,dx\,dy}{(x - y)^2} = (q\,du + q_1\,dv)^2 + (r\,du + r_1\,dv)^2.$$

If one can deduce x, y from that equation as functions of u and v then one will have a, a', a'', and formulas such as the following ones:

$$\frac{\partial a}{\partial u} = br - cq, \qquad \frac{\partial a}{\partial v} = br_1 - cq_1$$

will show one the six cosines that remain to be determined by elementary calculations. Hence, everything comes down to finding the values of x and y as functions of u and v that satisfy equation (17).

We decompose the left-hand side of that equation into two factors that we equate to zero. We will then have two differential equations:

$$(18) \qquad \begin{cases} q\,du + q_1 dv + i(r\,du + r_1 dv) = 0, \\ q\,du + q_1 dv - i(r\,du + r_1 dv) = 0. \end{cases}$$

Let:

$$(19) \qquad \varphi(u, v) = \alpha, \quad \psi(u, v) = \beta$$

be the integrals of those two equations. If we make a change of variables, and if we replace u and v with the functions α, β then the right-hand side of formula (17) will take the form:

$$4\,\lambda\,d\alpha\,d\beta,$$

in which λ is a known function of α and β, and the equation to be solved will become:

$$(20) \qquad \frac{4\,dx\,dy}{(x-y)^2} = \lambda\,d\alpha\,d\beta.$$

It can clearly admit only one of the following two solutions:

$$x = A, \qquad y = B,$$
$$x = B, \qquad y = A,$$

in which A denotes a function of α, and B denotes a function of β.

λ must then have the value:

$$(21) \qquad \lambda = \frac{A'B'}{(A-B)^2},$$

and it will be necessary to deduce the values of A and B from that equation.

If one eliminates the functions A and B by a well-known process then one will see that λ satisfies the partial differential equation:

$$(22) \qquad \frac{\partial^2 \log \lambda}{\partial\alpha\,\partial\beta} = -\,2\lambda,$$

which will be useful to us. However, if one would like to obtain the expression for A or B then one will be presented with a difficulty upon taking into account that, from a remark that was made before, the expression for λ will not change when one replaces A and B with:

$$\frac{m+nA}{p+qA}, \frac{m+nB}{p+qB},$$

respectively, in which m, n, p, q are constants. It then seems that the most general expression for A that can satisfy equation (21) must contain three constants, and consequently can be determined only by integrating a third-order differential equation.

Indeed, let us attempt to determine A. Upon taking the logarithmic derivative with respect to α of the two sides of equation (21), we will have:

$$\frac{A''}{A'} - \frac{2A'}{A+B} = \frac{\partial \log \lambda}{\partial \alpha}. \tag{23}$$

A new differentiation with respect to a will permit us to eliminate B, and will lead to the equation:

$$\frac{3A''^2}{A'^2} - \frac{2A'''}{A'} = 3\left(\frac{\partial \log \lambda}{\partial \alpha}\right)^2 - \frac{2}{\lambda}\frac{\partial^2 \lambda}{\partial \alpha^2}. \tag{24}$$

We shall see that the integration of that equation can be performed with no difficulty.

Since the right-hand side is equal to the left, it cannot depend upon β, and consequently, it cannot change when one gives an arbitrary constant value β_0 to β.

Let λ_0 be the corresponding value of λ. Equation (24) will obviously admit the solution:

$$A' = \lambda_0, \qquad A = \int \lambda_0 \, d\alpha,$$

or, upon taking equation (22) into account:

$$A = -\frac{1}{2}\frac{\partial \log \lambda}{\partial \beta_0},$$

and it will suffice to substitute that value of A in equation (23) in order to deduce the corresponding value of B.

53. Hence, all of the difficulty in our new method consists of integrating the two equations (18). It is clear that if one has to likewise consider b, b', b'' ; c, c', c'', instead of a, a', a'', then one will have to integrate two equations from one of the following groups:

$$\begin{cases} p\,du + p_1 dv + i(r\,du + r_1 dv) = 0, \\ p\,du + p_1 dv - i(r\,du + r_1 dv) = 0, \end{cases} \tag{25}$$

$$(26) \qquad \begin{cases} p\,du + p_1 dv + i(q\,du + q_1 dv) = 0, \\ p\,du + p_1 dv - i(q\,du + q_1 dv) = 0. \end{cases}$$

There is obviously a great advantage to having a free choice of these three different groups.

54. We add a further remark that purely relates to the form of the two equations (3). *One can reduce their simultaneous integration to that of just one Riccati equation.* Indeed, suppose that one must find the common solution to those equations that reduces to σ_0 for $u = u_0$, $v = v_0$. Set:

$$u = u_0 + u'\,t, \qquad v = v_0 + v'\,t,$$

in which u', v' denote constants. σ will become a function of t that must satisfy the equation:

$$\frac{d\sigma}{dt} = \frac{\partial\sigma}{\partial u}u' + \frac{\partial\sigma}{\partial v}v' = a\,u' + a_1 v' + 2\,(bu' + b_1\,v')\,\sigma + (cu' + c_1\,v')\,\sigma^2,$$

in which the coefficients are now functions of t.

It will then be determined by that condition, combined with that of reducing to σ_0 for $t = 0$. Suppose that one has determined that function:

$$\sigma = F\,(u_0, v_0, u, v, t)\,.$$

In order to obtain the desired solution, it will suffice to set $t = 1$ and replace u', v' with $u - u_0$, $v - v_0$, respectively.

Upon taking that purely theoretical remark into account, one can say that:

The simultaneous integration of the systems (1) *and* (2) *of the preceding chapter will reduce to that of just one Riccati equation.*

CHAPTER VII

DISPLACEMENTS IN TWO VARIABLES IN THE CASE WHERE THE MOVING SYSTEM HAS NO FIXED POINT.

Introduction of six translations. – Differential relations that they must satisfy. – Infinitely-small motions that reduce to rotations. – Theorem of Schönemann and Mannheim. – Particular case in which one has an instantaneous center of rotation. – Ribaucour's theorem.

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55. After having treated the case in which the moving system has a fixed point, it remains for us to examine the hypothesis in which the trihedron (T) moves through space in an arbitrary manner. We must then append six new quantities to the six rotations. We let ξ, η, ζ denote the components of the velocity of the origin of the moving axes relative to those axes when only u varies, and let ξ_1, η_1, ζ_1 denote the same components when only v varies. If one lets X_0, Y_0, Z_0 denote the coordinates of the origin of the moving axes with respect to the fixed axes then one will have:

$$\text{(1)} \qquad \frac{\partial X_0}{\partial u} = a\xi + b\eta + c\zeta, \qquad \frac{\partial X_0}{\partial v} = a\xi_1 + b\eta_1 + c\zeta_1,$$

and analogous equations in Y_0, Z_0. Equate the two values of $\dfrac{\partial^2 X_0}{\partial u\,\partial v}$ that one can deduce from those formulas. After replacing the derivatives of the cosines with their values, we will obtain an equation that must be true when one replaces a, b, c with the other systems a', b', c' ; a'', b'', c'' and will decompose into the following three ([12]):

$$\text{(2)} \qquad \begin{cases} \dfrac{\partial \xi}{\partial v} - \dfrac{\partial \xi_1}{\partial u} = q\zeta_1 - q_1\zeta - r\eta_1 + r_1\eta, \\[2mm] \dfrac{\partial \eta}{\partial v} - \dfrac{\partial \eta_1}{\partial u} = r\xi_1 - r_1\xi - p\zeta_1 + p_1\zeta, \\[2mm] \dfrac{\partial \zeta}{\partial v} - \dfrac{\partial \zeta_1}{\partial u} = p\eta_1 - p_1\eta - q\xi_1 + q_1\xi. \end{cases}$$

56. Conversely, when the twelve quantities ξ, p, … satisfy equations (2), at the same time as equations (5) in chapter V, there will exist a displacement for which they will be rotations and translations, because we already know that one can determine the nine

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([12]) One can compare these formulas with the analogous ones that were given by Kirchhoff in the fourth and fifth lecture in *Vorlesungen über Mathematische Physik*, 1876.

cosines, and furthermore, that equation (1), which will be compatible by virtue of equations (2), will provide the coordinates of the origin of the moving axes by quadratures. It is pointless to repeat here all of the motions that are obtained by reducing to basically just one motion that is observed with respect to different axes.

It is obvious that if, instead of considering all of the positions of the moving system that correspond to the different values of u and v, one supposes that u and v are functions of just one parameter α then the rotations and translations that relate to that motion will be:

$$
(3) \qquad \left\{ \begin{array}{lll} p\dfrac{\partial u}{\partial \alpha} + p_1\dfrac{\partial v}{\partial \alpha} & q\dfrac{\partial u}{\partial \alpha} + q_1\dfrac{\partial v}{\partial \alpha} & r\dfrac{\partial u}{\partial \alpha} + r_1\dfrac{\partial v}{\partial \alpha} \\ \xi\dfrac{\partial u}{\partial \alpha} + \xi_1\dfrac{\partial v}{\partial \alpha} & \eta\dfrac{\partial u}{\partial \alpha} + \eta_1\dfrac{\partial v}{\partial \alpha} & \zeta\dfrac{\partial u}{\partial \alpha} + \zeta_1\dfrac{\partial v}{\partial \alpha} \end{array} \right.
$$

and the projections onto the moving axes of the element of the curve that is described by an arbitrary point M whose coordinates are x, y, z with respect to the moving axes will be:

$$
(4) \qquad \left\{ \begin{array}{l} dx + \xi\, du + \xi_1\, dv + (q\, du + q_1\, dv)\, z - (r\, du + r_1 dv)\, y, \\ dy + \eta\, du + \eta_1\, dv + (r\, du + r_1\, dv)\, x - (p\, du + p_1 dv)\, z, \\ dz + \zeta\, du + \zeta_1\, dv + (p\, du + p_1 dv)\, y - (q\, du + q_1 dv)\, x. \end{array} \right.
$$

In other words, if α is the time then one will get the components of the velocity with respect to moving axes upon dividing the preceding three expressions by $d\alpha$. We will often make use of that remark, which dispenses with many of the calculations and permits one to leave aside everything that is concerned with fixed axes.

Upon concluding these preliminary notions on motion here, we shall be content to remark that the method that is followed will apply without modification to the case in which the position of the moving system depends upon three, or even more, parameters.

57. We shall make use of the preceding results in order to prove an important theorem that relates to displacements in two variables ([13]).

If one considers the moving system in a well-defined position then it can leave that position in an infinitude of ways; the rotations and translations that correspond to the most general motion that it can take on are given by Table (3). We seek to know whether one of the infinitely-small motions that one thus obtains can be reduced to a simple rotation.

In order to do that, it is necessary that one must have (no. **3**):

([13]) This theorem was stated by Mannheim in 1866 in the Journal de Liouville (2) **11**, and since then it has been the subject of deep study. In that epoch, they overlooked the fact that it had been stated eleven years before by Schönemann in an article that was presented to the Berlin Academy by Steiner (Monatsberichte, 1855). It was Geiser whose called attention to the paper of Schönemann in 1880 and had the theorem reprinted in Crelle's Journal **90**, pp. 39-48.

$$(5)\qquad \begin{cases} (p\,du + p_1\,dv)(\xi\,du + \xi_1 dv) \\ \quad + (q\,du + q_1\,dv)(\eta\,du + \eta_1 dv) + (r\,du + r_1\,dv)(\zeta\,du + \zeta_1 dv) = 0. \end{cases}$$

That equation provides two values for du / dv, which are generally different. In general, there will then be two different (real or imaginary) motions that reduce to a rotation. The axis of rotation that corresponds to each of the motions will be defined by the equations:

$$(6)\qquad \begin{cases} \xi\,du + \xi_1 dv + (q\,du + q_1\,dv)\,z - (r\,du + r_1\,dv)\;y = 0, \\ \eta\,du + \eta_1 dv + (r\,du + r_1\,dv)\;x - (p\,du + p_1 dv)\,z = 0, \\ \zeta\,du + \zeta_1 dv + (p\,du + p_1 dv)\,y - (q\,du + q_1 dv)\,x = 0, \end{cases}$$

in which one replaces du / dv with the root of equation (5) that corresponds to the motion considered.

The preceding result has an important consequence that is easy to verify by a direct calculation. Since two of the motions reduce to rotations around two lines, which we call D, Δ:

The normal to the surface that is described by an arbitrary point of the invariable system must meet each of those two lines,

because since that normal is perpendicular to all of the displacements of the point considered, it will be perpendicular to the ones that rotate the point, in particular, and consequently it will necessarily meet the axes of those rotations.

58. Equation (5) has degree two with respect to du / dv. It can then have imaginary roots, and the two motions that reduce to rotations will then be certainly imaginary; it can also have equal roots. We shall not discuss all of the cases that can present themselves, but we shall study the displacements for which one knows *a priori* that there exist two displacements that reduce to rotations around two distinct and concurrent lines. Upon starting with that hypothesis, Ribacour obtained an elegant theorem ([14]) that we shall prove.

It is easy to recognize that in the case that we address, equation (5) can be an identity. Indeed, denote the two values that du / dv can take relative to the rotations considered by du / dv, $\delta u / \delta v$; one will necessarily have:

$$(p\,du + p_1\,dv)\,(\xi\,du + \xi_1\,dv) + \ldots = 0,$$
$$(p\,\delta u + p_1\,\delta v)\,(\xi\,du + \xi_1\,dv) + \ldots = 0.$$

The axes of those rotations are defined by formulas (6). The condition for those axes to intersect can be written in the very symmetric manner:

([14]) RIBAUCOUR, "Sur la déformation des surfaces," Comptes rendus **70**, pp. 330.

$$(p\,du + p_1\,dv)\,(\xi\,du + \xi_1\,dv) + (p\,\delta u + p_1\,\delta v)\,(\xi\,du + \xi_1\,dv) + \ldots = 0.$$

The three equations that we just obtained can reduce to the following type:

$$(7)\qquad \begin{cases} A\,du^2 + 2B\,du\,dv + C\,dv^2 = 0, \\ A\,\delta u^2 + 2B\,\delta u\,\delta v + C\,\delta v^2 = 0, \\ A\,du\,\delta u + B(du\,\delta v + C\,dv\,\delta u) + C\,dv\,\delta v = 0, \end{cases}$$

in which A, B, C have the values:

$$\begin{aligned} A &= p\,\xi + \ldots, \\ 2B &= p_1\,\xi + p\,\xi_1 + \ldots, \\ C &= p_1\,\xi_1 + \ldots \end{aligned}$$

If one considers the three equations (7) as determining the unknowns A, B, C then their determinant will be $(du\,\delta v - dv\,\delta u)^3$, and by hypothesis, it will not be zero. One will then have:

$$A = B = C = 0,$$

and consequently equation (5) will be satisfied identically.

One can arrive at the same conclusions by a much simpler argument. If the two lines D, Δ meet then their point of intersection will have a zero velocity under all displacements. If one lets x', y', z' denote the coordinates of that point relative to the moving axes then one must then have:

$$(8)\qquad \begin{cases} \xi + qz' - ry' = 0, & \xi_1 + q_1 z' - r_1 y' = 0, \\ \eta + rx' - pz' = 0, & \eta_1 + r_1 x' - p_1 z' = 0 \\ \zeta + py' - qx' = 0, & \zeta_1 + p_1 y' - q_1 x' = 0, \end{cases}$$

and it is not difficult to deduce the equations:

$$A = 0,\quad B = 0,\quad C = 0$$

from this. However, here is the consequence of it that constitutes Ribaucour's theorem:

Suppose that there are values of x', y'¸ z' that satisfy equation (8) for all values of u and v. The point (x', y', z'), which is considered to be belong to the moving system, describes a surface (s) that we regard as being a part of that system. If one refers the same point to fixed axes then it will describe a surface (S). Give increments du, dv to u and v. The points will be displaced on the surface (S) and will describe an infinitely-small arc whose projections onto the moving axes, which are given by formulas (4), will be, *upon taking equations* (8) *into account:*

$$dx',\ dy',\ dz'.$$

Now, these are the projections onto the same axes of the path that is described by the point considered on the surface (s). Those two paths always have the same direction and magnitude, so one sees that the two surfaces (s) and (S) are not only tangent, but also roll on each other in such a manner that the paths that are traversed by the point of contact of the two surfaces will always have the same length and direction.

In other words, the two surfaces correspond point-by-point in such a manner that the two corresponding curves will have the same length. One then says that they can be *mapped* to each other.

59. We can add the following properties:

We infer the values of ξ, …, ξ_1, … from equations (8), and substitute them into equations (2). We obtain the relations:

$$\text{(9)} \qquad \left\{ \begin{aligned} q\frac{\partial z'}{\partial v} - r\frac{\partial y'}{\partial v} - q_1\frac{\partial z'}{\partial u} + r_1\frac{\partial y'}{\partial u} &= 0, \\ r\frac{\partial x'}{\partial v} - p\frac{\partial z'}{\partial v} - r_1\frac{\partial x'}{\partial u} + p_1\frac{\partial z'}{\partial u} &= 0, \\ p\frac{\partial y'}{\partial v} - q\frac{\partial x'}{\partial v} - p_1\frac{\partial y'}{\partial u} + q_1\frac{\partial z'}{\partial u} &= 0, \end{aligned} \right.$$

from an easy calculation, which will lead to an important consequence. Multiply these relations by $\frac{\partial x'}{\partial v}, \frac{\partial y'}{\partial v}, \frac{\partial z'}{\partial v}$, and add them. We will have:

$$\begin{vmatrix} p_1 & q_1 & r_1 \\ \frac{\partial x'}{\partial u} & \frac{\partial y'}{\partial u} & \frac{\partial z'}{\partial u} \\ \frac{\partial x'}{\partial v} & \frac{\partial y'}{\partial v} & \frac{\partial z'}{\partial v} \end{vmatrix} = 0.$$

If λ and μ_1 are two conveniently-chosen functions then we can set:

$$\text{(10)} \qquad \left\{ \begin{aligned} p_1 &= -\mu_1\frac{\partial x'}{\partial u} + \lambda\frac{\partial x'}{\partial v}, \\ q_1 &= -\mu_1\frac{\partial y'}{\partial u} + \lambda\frac{\partial y'}{\partial v}, \\ r_1 &= -\mu_1\frac{\partial z'}{\partial u} + \lambda\frac{\partial z'}{\partial v}. \end{aligned} \right.$$

We will similarly have:

$$p = \lambda_1\frac{\partial x'}{\partial u} + \mu\frac{\partial x'}{\partial v},$$

and analogous equations in q, r. Moreover, if we substitute the values of the rotations in equations (9) then we will obtain the complementary condition:

$$\lambda_1 = -\lambda, \tag{11}$$

in such a way that we will get the following values for p, q, r:

$$\left\{\begin{aligned} p &= -\lambda\frac{\partial x'}{\partial u} + \mu\frac{\partial x'}{\partial v}, \\ q &= -\lambda\frac{\partial y'}{\partial u} + \mu\frac{\partial y'}{\partial v}, \\ r &= -\lambda\frac{\partial z'}{\partial u} + \mu\frac{\partial z'}{\partial v}, \end{aligned}\right. \tag{12}$$

which will replace formulas (4) and (9) when we combine them with equations (10).

60. The geometric interpretation of equations (10) and (12) is obvious. Since the derivatives of x', y', z' with respect to u and v are proportional to the direction cosines of the two tangents to the surface that is the locus of instantaneous centers, the two rotations whose components are p, q, r and p_1, q_1, r_1 will be in the tangent plane to that surface. The same thing will obviously be true for the rotation that corresponds to an arbitrary simultaneous rotation of u and v; one will get that from formulas (3), which give the components of that rotation. We can then state the following proposition:

If it happens that two infinitely-close motions that take the system to an infinitely-close position reduce to two rotations around axes that are concurrent at that point for each position of the moving system then any other infinitely-close motion of the system will reduce to a rotation around an axis that passes through the same point, which one can call the *instantaneous center of rotation.* The two surfaces that are loci of the instantaneous centers in the moving system and in space can be mapped to each other. They will always be in contact, in such a manner that any motion of the moving system will reduce to the rolling of one of the surfaces on the other one, while the instantaneous axis of rotation will pass through the point of contact of the two surfaces at each instant and will be found in their common tangent plane.

It remains for us to give the geometric interpretation of equation (11). One easily recognizes that it expresses the idea that the relationship between the direction of the curve that is followed by the instantaneous pole and that of the corresponding axis of rotation is reciprocal. That is, if, after considering an infinitely-small displacement for which the axis of instantaneous rotation has a certain direction, one can imagine the displacement under which that direction becomes that of the path that is followed by the instantaneous center then the axis of rotation for that new displacement will have the same direction as the path of the center under the first one. Here, one can construct a theory that is entirely similar to that of conjugate tangents and the Dupin indicatrix. One will likewise find two series of lines that are analogous to the asymptotic lines and are characterized by the property that when the rolling of the two surfaces on each other is performed in such a manner that the instantaneous center describes one of those curves, the rotation will be directed along the tangent to the curve at each instant. Consequently, the two corresponding curves that are then the paths of the center on the two surfaces will have the same curvature and osculating plane at each instant. We leave the task of developing those suggestions to the reader.

61. Conversely, whenever one knows two surfaces (*S*), (*s*) that can be mapped to each other, if one locates the surface (*s*) in such a manner that one of its points coincides with the homologous point of (*S*), and the homologous curves of the two surfaces that pass through that point are tangent, then all of the positions that one will thus obtain for the surface (*s*) will depend upon two parameters, and the displacement in two variables that is defined by these various positions will enjoy all of the properties that we just pointed out.

For example, consider all of the surfaces (*s*) that are symmetric to (*S*) with respect to its tangent planes. They obviously constitute all of the positions of a surface (*s*) that rolls on (*S*). One can apply all of the preceding propositions to that motion. The surfaces that are trajectories of the various points of the moving system will then be homothetic to the (*podaires*) of the various points of space with respect to (*S*) when the homothety ratio is 2.

CHAPTER VIII

FIRST NOTIONS ON CURVILINEAR COORDINATES

Surfaces of revolution. – Alysseid. – Pseudo-spherical surface. – Isothermal systems. – Ruled surfaces. – Developable surfaces. – Determination of all surfaces that can be mapped to the plane by the method of O. Bonnet.

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62. In the first chapter, we saw that one could attach the theory of skew surfaces to the study of the motion of a trihedron. Similarly, the propositions that relate to the displacements in two variables find an important application in the theory of surfaces. However, before developing that application, we shall give some extended notions on the systems of curvilinear coordinates that Gauss employed in a systematic manner for the first time in the fundamental paper "Disquisitiones generales circa superficies curvas" that was published in 1828 in v. VI of Nouveaux Mémoires de la Sociétés de Goettingue.

As one knows, there are two ways of defining a surface. One can determine it by its equation – i.e., by the relation that exists between the coordinates of any of its points. However, one can also suppose that those three coordinates have been expressed in terms of two independent variables that we call u and v. That second way of defining the surface is even more general than the first one, because if one takes u and v to be two of the three rectangular coordinates then the expression for the third one will be precisely the equation for the surface when it is solved for that coordinate.

A system of rectangular coordinates can be represented geometrically. It suffices to trace the two families of curves on the surface that are the loci of points for which one or the other of the variables u, v remains constant. However, it is important to remark that the system of coordinates is not defined completely if one gives only the two families of coordinate curves. Without changing those curves, one can obviously replace u and v with the variables u_1, v_1, which are arbitrary functions of the first ones:

$$u_1 = \varphi(u), \qquad v_1 = \psi(u).$$

That is a remark that one makes frequent use of, and which sometimes permits great simplifications.

63. Gauss's method rests essentially upon the expression for the arc length of an arbitrary curve that is traced on the surface.

Suppose that the rectangular coordinates x, y, z of a point of the surface are expressed as functions of the two variables u and v. The expression of an arc of the curve that is traced on the surface will be given by the formula:

$$(1) \qquad ds^2 = E\, du^2 + 2\, D\, du\, dv + G\, dv^2,$$

in which one has:

$$(2)\qquad \begin{cases} E = \left(\dfrac{\partial x}{\partial u}\right)^2 + \left(\dfrac{\partial y}{\partial u}\right)^2 + \left(\dfrac{\partial z}{\partial u}\right)^2, \\ F = \dfrac{\partial x}{\partial u}\dfrac{\partial x}{\partial y} + \dfrac{\partial y}{\partial u}\dfrac{\partial y}{\partial y} + \dfrac{\partial z}{\partial u}\dfrac{\partial z}{\partial y}, \\ G = \left(\dfrac{\partial x}{\partial v}\right)^2 + \left(\dfrac{\partial y}{\partial v}\right)^2 + \left(\dfrac{\partial z}{\partial v}\right)^2. \end{cases}$$

We call the line element *ds*, to abbreviate. We also put it into the form:

$$(3)\qquad ds^2 = A^2\, du^2 + 2AC \cos \alpha\, du\, dv + C^2\, dv^2,$$

and we will consequently have:

$$(4)\qquad A = \sqrt{E}, \qquad C = \sqrt{G}, \qquad \cos \alpha = \frac{F}{\sqrt{E}\sqrt{G}}.$$

Formulas (2) show that $A\,du$ is the arc length of the curve v = const., $C\,dv$ is the arc length of the curve u = const., and finally, α is the angle at which those two curves intersect at the point considered. One will then have:

$$F = 0$$

whenever one employs rectangular curvilinear coordinates. It likewise results from formulas (2) that the surface element of the surface will have the expression:

$$AC \sin \alpha\, du\, dv = \sqrt{EG - F^2}\, du\, dv.$$

Before going further, we shall give some examples of this mode of representation.

64. First, consider the surfaces of revolution, and suppose that one has taken the z-axis to be the axis of the surface. If one calls the distance from a point on the meridian to the axis r then the equation of the surface will be:

$$z = f(r).$$

Introduce the angle v that the xz-plane makes with the meridian that passes through the point considered. We will have the following expressions for the x, y coordinates:

$$x = r \cos v, \qquad y = r \sin v,$$

and we will deduce the following formula for the line element from them:

(5) $$ds^2 = dr^2 (1 + f'^2) + r^2 dv^2.$$

Here, the curves r = const. are parallels; the curves v = const. are the meridians. If one sets:

$$dr\sqrt{1+f'^2} = du$$

then r will become a function of u and equation (5) will take the form:

(6) $$ds^2 = du^2 + \varphi(u)\, dv^2.$$

The significance of u is obvious: It is the arc length of the meridian when it is measured by starting from a fixed parallel.

One can put that expression for the element into a somewhat different form. Set:

$$\frac{du}{\sqrt{\varphi(u)}} = du_1 = \frac{dr}{r\sqrt{1+f'^2}}.$$

$\varphi(u)$ will become a function $F(u_1)$ and u_1, and equation (6) will give us:

$$ds^2 = F(u_1)\,(du_1^2 + dv^2).$$

65. Whenever the line element of a surface can be converted into the form:

$$ds^2 = \lambda\,(d\alpha^2 + d\beta^2),$$

one says that the coordinate curves form an *isothermal* or *isometric* net. The former term is borrowed from the theory of heat, while the latter, which is due to Bonnet, is explained by the following remarks:

Imagine that one traces all of the coordinate curves on a surface that correspond to values of the parameters α, β that increase according to an arithmetic progression with an infinitely-small increment:

$$\begin{array}{llll} \alpha, & \alpha + d\alpha, & \alpha + 2\,d\alpha, & \ldots, \\ \beta, & \beta + d\beta, & \beta + 2\,d\beta, & \ldots \end{array}$$

One will thus decompose the surface into a series of infinitely-small rectangles whose edges will be equal if one takes $d\alpha = d\beta$. One then says that *the surface is divided into infinitely-small squares.* Without a doubt, that is not rigorously exact, but the ratio of their adjacent edges of the curvilinear rectangles that are defined by the coordinate lines considered will get closer to unity when the increment $d\alpha$ is chosen to be smaller.

In the case of surfaces of revolution, one sees that the meridians and the parallels constitute an isothermal system.

66. In particular, consider the surface of revolution that is generated by the revolution of a hanging chain (*chaînette*) around its base. Here, one will have:

$$r = \frac{a}{2}\left(e^{z/a} + e^{-z/a}\right),$$

and one will find, with no difficulty, that:

$$u = \sqrt{r^2 - a^2}\ .$$

Consequently, formula (6) will give us:

$$ds^2 = du^2 + (u^2 + a^2)\, dv^2 \tag{7}$$

for the line element of the surface.

That very remarkable surface has received the name of *alysseid* or *catenoid*. Since the hanging chain is the only curve whose radii of curvature are equal and opposite in sign to the normal, the alysseid is the only surface of revolution for which the principal radii of curvature at each point are equal and opposite in sign. One gives the name of *minimal surfaces* to all of the ones whose radii of curvature are coupled by that relation. The alysseid is then the only minimal surface of revolution.

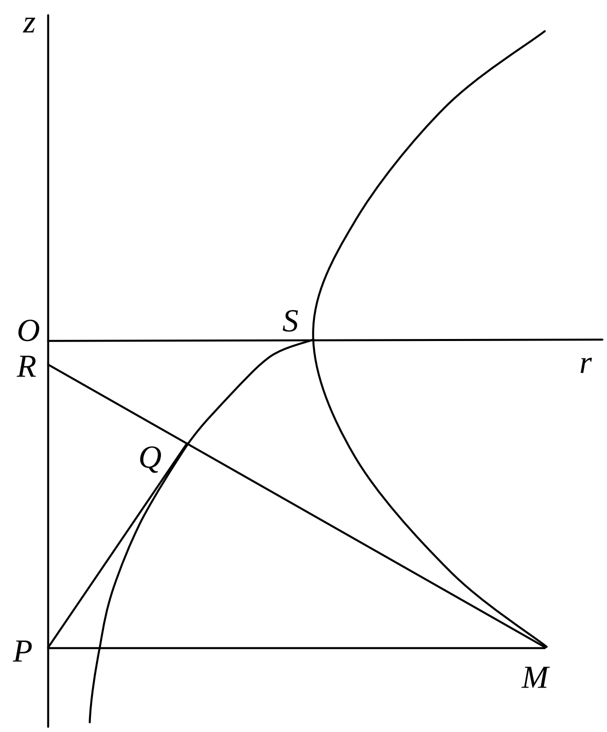

Figure 1.

One knows that if one considers a hanging chain whose base is Oz and one drops a perpendicular PQ from the foot P of the ordinate of the point M to the tangent at M then the arc length of the chain, when measured by starting at the summit S, will be equal to the line segment MQ; consequently, the point Q will describe one of the developments of the chain.

Since PQ is constant and equal to the parameter a of the chain, the locus of the point Q will be *the curve of equal tangents* or *tractrix*. Upon denoting the angle PMQ by φ, the arc that is described by the point Q when φ is increased by $d\varphi$ will have the value:

$$d\sigma = MQ\, d\varphi = a \cot\varphi\, d\varphi .$$

Since the perpendicular that is dropped from Q to Oz will have the expression:

$$r = a \sin\varphi,$$

moreover, one will see that the line element of the surface that is generated by the revolution of the curve of equal tangents around Oz will be given by the formula:

$$ds^2 = d\sigma^2 + r^2\, dv^2 = a^2 (\cot^2\varphi\, d\varphi^2 + \sin^2\varphi\, dv^2).$$

Set:

$$\cot\varphi\, d\varphi = du,$$

which will give:

$$\sin\varphi = e^u,$$

and we will have:

(8) $$ds^2 = a^2 (du^2 + e^{2u}\, dv^2).$$

We remark, moreover, that in the rectangular triangle RPM, one will have:

$$MQ \cdot QR = a^2.$$

The principal centers of curvature of the surface are obviously the points M and R; therefore, the principal radii of curvature will satisfy the relation:

$$RR' = -a^2.$$

However, although that property in no way characterizes the surface, it is easy to prove.

Indeed, we propose to determine all the surfaces of revolution whose radii of curvature are coupled by the preceding relation. By a calculation that we shall not insist upon here, one finds that the variables z and r must satisfy the differential relation:

(9) $$dz = \sqrt{\frac{b^2 - r^2}{r^2 + a^2 - b^2}}\; dr,$$

in which b denotes an arbitrary constant that can take all possible values. The line element of the surface is given by the formula:

$$ds^2 = \frac{a^2 dr^2}{r^2 + a^2 - b^2} + r^2\, dv^2.$$

First, suppose that b is equal to a. One then recovers the surface whose meridian is the curve of equal tangents. We give it the name of the *pseudo-spherical surface.*

If b is smaller than a then the radius r can take all values that are lower than b, and one will obtain a surface that, like the preceding one, also has a parallel of regression BC. However, all of the meridians must *cut* the axis of the surface at the same point A at a finite angle whose tangent has the value:

$$\sqrt{\frac{a^2 - b^2}{b^2}}.$$

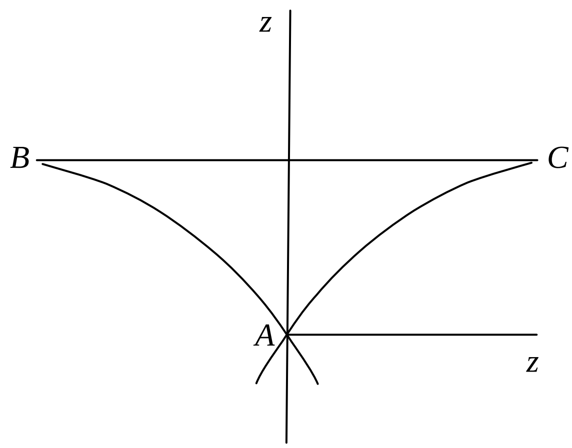

Figure 2.

If one then sets:

$$r = \sqrt{a^2 - b^2}\,\frac{e^u - e^{-u}}{2}, \qquad v = \frac{av'}{\sqrt{a^2 - b^2}}$$

then one will obtain the expression:

$$ds^2 = a^2 \left[du^2 + \left(\frac{e^u - e^{-u}}{2} \right)^2 dv'^2 \right] \tag{10}$$

for the line element.

On the contrary, if b is larger than a then r will have a minimum $\sqrt{b^2 - a^2}$, and the meridians will no longer meet the axis. The surface will then admit two parallels of regression and a throat circle (*cercle de gorge*) DE (Fig. 3). If one sets:

$$r = \sqrt{b^2 - a^2}\,\frac{e^u + e^{-u}}{2}, \qquad v = \frac{av'}{\sqrt{b^2 - a^2}}$$

then the line element will become:

$$ds^2 = a^2 \left[du^2 + \left(\frac{e^u + e^{-u}}{2} \right)^2 dv'^2 \right]. \tag{11}$$

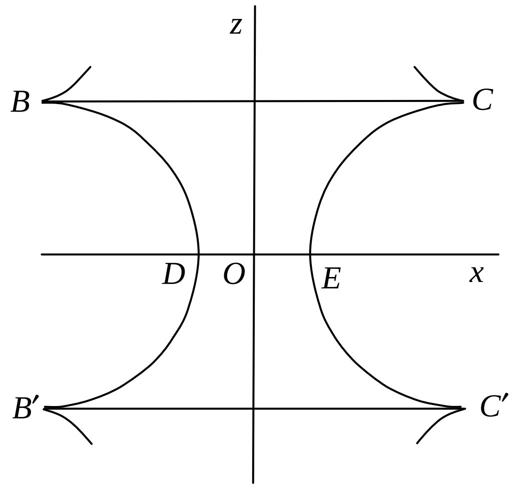

Figure 3.

67. After surfaces of revolution, we examine the very important group that is composed of ruled surfaces. They can be defined by the three equations:

$$\left\{ \begin{aligned} x &= a_1 u + b_1, \\ y &= a_2 u + b_2, \\ z &= a_3 u + b_3, \end{aligned} \right. \tag{12}$$

in which a, b, … must be considered as functions of one parameter v. If one supposes that:

$$a_1^2 + a_2^2 + a_3^2 = 1$$

then u will denote the length measured along each rectilinear generator of the surface when one starts from the curve that is defined by the equations:

$$x = b_1, \qquad y = b_2, \qquad z = b_3 . \tag{13}$$

One deduces the following expression for the line element from formulas (12):

$$ds^2 = du^2 + 2D\, du\, dv + (A\, u^2 + 2B\, u + C)\, dv^2, \tag{14}$$

in which A, B, C, D are functions of v that are defined by the relations:

$$A = a_1'^2 + a_2'^2 + a_3'^2, \qquad C = b_1'^2 + b_2'^2 + b_3'^2,$$

$$B = a_1' b_1' + a_2' b_2' + a_3' b_3' \qquad D = a_1 b_1' + a_2 b_2' + a_3 b_3'.$$

If one supposes that the curve $u = 0$ that is defined by equations (13) has been chosen from the orthogonal trajectories of the generators then one will have:

$$D = 0,$$

and the line element will take the simpler form:

(15) $$ds^2 = du^2 + (A\,u^2 + 2B\,u + C)\,dv^2.$$

In this case, the system of coordinates will be defined by the rectilinear generators $v =$ const. and their orthogonal trajectories.

In order to convert the general line element that is given by formula (14) to the form (15), it will suffice to substitute the following variable for u:

$$u' = u + \int D\,dv.$$

One also recognizes that the orthogonal trajectories of the rectilinear generators will be determined by a simple quadrature for all ruled surfaces.

68. For example, consider the surface that is defined by the principal normals to the helix, or the *skew helicoid with director plane.* It is defined by the three equations:

(16) $$\begin{cases} z = av, \\ x = u\cos v, \\ y = u\sin v; \end{cases}$$

one then deduces that:

(17) $$ds^2 = (a^2 + u^2)\,dv^2 + du^2.$$

The comparison of formulas (7) and (17) shows that if one makes the points on the helicoid and the alysseid for which the values of u and v are the same correspond then the arcs of the two corresponding curves will be rigorously equal. In that case, one says that *the two surfaces can be mapped to each other.* Indeed, it is clear that if one considers a surface to be a flexible and inextensible membrane, and if one admits the possibility of deforming that surface without tearing or duplicating it then the length of each curve that is traced on the surface will remain invariable under the deformation. Without examining the question of knowing whether it is possible to make the first surface coincide with the second one by a continuous sequence of deformations, one says that two surfaces can be mapped to each other when they satisfy the geometric definition that we just gave. The problem of the search for surfaces that can be mapped to a given surface is one of the more interesting ones (but also one of the more difficult ones) that one encounters in the application of analysis to geometry. In the case that we are treating, the helicoid can be mapped to the alysseid, so the rectilinear generators of the first surface will correspond to the meridians of the second one, and the helices, to the parallels.

69. We return to the ruled surfaces. When one has only the expression (15) for the line element, it is easy to distinguish the skew surfaces from the developable surfaces.

Indeed, one has identically:

$$A\,u^2 + 2Bu + C = (a_1' u + b_1')^2 + (a_2' u + b_2')^2 + (a_3' u + b_3')^2 .$$

One then sees that the trinomial on the left-hand side will be a sum of squares, and one will have:

$$B^2 - AC < 0,$$

although the equations:

$$\frac{a_1'}{b_1'} = \frac{a_2'}{b_2'} = \frac{a_3'}{b_3'} \tag{18}$$

will not be satisfied. Now, these latter equations will be true only in the case where the surface is developable.

Indeed, equations (12) will give us:

$$\begin{aligned} dx &= a_1\, du + (a_1' u + b_1')\; dv, \\ dy &= a_2\, du + (a_2' u + b_2')\; dv, \\ dz &= a_3\, du + (a_3' u + b_3')\; dv. \end{aligned}$$

Express the idea that there exists a point on the generator that describes a curve that is tangent to the generator. It is necessary that one must have:

$$\frac{dx}{a_1} = \frac{dy}{a_2} = \frac{dz}{a_3},$$

when one takes u to be a suitable function of v, which will give:

$$\frac{a_1' u + b_1'}{a_1} = \frac{a_2' u + b_2'}{a_2} = \frac{a_3' u + b_3'}{a_3}$$

upon replacing dx, dy, dz with their values.

If one takes the equation $D = 0$ into account then one will find that the common value of the three preceding ratios must be equal to zero.

One must then have:

$$-u = \frac{b_1'}{a_1'} = \frac{b_2'}{a_2'} = \frac{b_3'}{a_3'},$$

and consequently equations (18) will express the necessary and sufficient conditions for the surface to be developable.

Finally, one can reduce the coefficient A to unity by replacing v with a suitable function of v. One will then have:

$$ds^2 = du^2 + [(u - \alpha)^2 + \beta^2]\, dv^2 \tag{19}$$

for the reduced form of the element in the case of skew surfaces and:

$$ds^2 = du^2 + (u - \alpha)^2\, dv^2 \tag{20}$$

in the case of developable surfaces, in which α and β denote functions of v ([15]).

70. One easily deduces that the developable surfaces can be mapped to the plane from the form of the line element.

Indeed, recall formula (20) and decompose the right-hand side into two factors:

$$du + i\,(u - \alpha)\, dv,$$
$$du - i\,(u - \alpha)\, dv.$$

If one multiplies these factors by e^{iv}, e^{-iv}, respectively, then they will become exact differentials, and one can set:

$$\left\{\begin{aligned} e^{iv}[du + i(u-\alpha)\,dv] &= dx + i\,dy, \\ e^{-iv}[du - i(u-\alpha)\,dv] &= dx - i\,dy. \end{aligned}\right. \tag{21}$$

One will have, upon integrating:

$$\left\{\begin{aligned} x + iy &= u\,e^{iv} - i\int \alpha\, e^{iv} dv, \\ x - iy &= u\,e^{-iv} + i\int \alpha\, e^{-iv} dv. \end{aligned}\right. \tag{22}$$

Moreover, upon multiplying corresponding sides of formulas (21), one will obtain:

$$ds^2 = dx^2 + dy^2$$

for the line element of the developable surface, which will prove the stated property.

Formulas (22) can be replaced with the following ones:

([15]) However, that theory passes over the imaginary ruled surfaces, for which one has:

$$a_1^2 + a_2^2 + a_3^2 = 0,$$

and which are generated by lines that meet the imaginary circle at infinity.

(23) $$\begin{cases} x = u\cos v + \int \alpha \sin v\, dv, \\ y = u \sin v - \int \alpha \cos v\, dv. \end{cases}$$

One sees that the rectilinear generators of the surface that are defined by the relation v = const. correspond to the lines in the plane. Consequently, the orthogonal trajectories of the generators have parallel curves for transforms that are orthogonal trajectories to the lines in the plane. The edge of regression $u = \alpha$ of the developable corresponds to the envelope of all lines in the plane. All of these results, which we shall not stop to verify, are in good agreement with the mechanical operations by which one realizes the developments of that class of very important surfaces.

71. We just indicated a way of mapping a developable surface onto a plane. It is natural to demand to know whether there is no other one, and whether, for example, one cannot realize that map of the surface by making the rectilinear generators of the surface correspond to curved lines in the plane. The following argument will give the answer to that question:

Suppose that one has mapped the developable onto the plane in two different ways – i.e., that one has exhibited its line elements in the two forms:

$$ds^2 = dx^2 + dy^2, \qquad ds'^2 = dx'^2 + dy'^2;$$

one will then deduce that:

(24) $$dx^2 + dy^2 = dx'^2 + dy'^2.$$

It is easy to solve that equation in the most general way. Indeed, one can replace it by one or the other of the systems:

(25) $$\begin{cases} dx' = \cos\alpha\, dx - \sin\alpha\, dy, & dx' = \cos\alpha\, dx + \sin\alpha\, dy, \\ dy' = \sin\alpha\, dx + \cos\alpha\, dy, & dy' = \sin\alpha\, dx - \cos\alpha\, dy, \end{cases}$$

in which α is an auxiliary unknown, and which are deduced from each other by changing y into $-y$. For example, consider the first one. Upon writing down the idea that the right-hand sides of the equations are exact differentials, we will obtain the relations:

$$\sin\alpha \frac{\partial \alpha}{\partial y} = \cos\alpha \frac{\partial \alpha}{\partial x}, \qquad \cos\alpha \frac{\partial \alpha}{\partial y} = -\sin\alpha \frac{\partial \alpha}{\partial x},$$

which will give:

$$\frac{\partial \alpha}{\partial x} = \frac{\partial \alpha}{\partial y} = 0;$$

α will then be constant. Upon integrating formulas (25), and denoting the two new constants by x_0, y_0, one will have:

$$x' = x \cos \alpha - y \sin \alpha + x_0 ,$$
$$y' = x \sin \alpha + y \cos \alpha + y_0 .$$

These are the formulas for the coordinate transformation. As a result, x, y and x', y' can be considered to be the coordinates of the same point in the plane when referred to different axes, and the two representations of the developable surface cannot be regarded as truly distinct.

72. Another very interesting question presents itself here. We just saw that the envelope of a moving plane can be mapped to the plane. Is the converse is also true, and is any surface that can be mapped to the plane itself the envelope of a moving plane? That proposition was already assumed by Monge and the other geometers of his epoch to be the justification for the name *developable surface* itself, which was originally given as the envelope of a moving plane ([16]). That is an immediate corollary of the general propositions that we will develop in what follows. However, at present, we can prove it by giving a very simple, direct proof that is due to O. Bonnet ([17]).

Let x, y, z be the coordinates of a point on the desired surface that can be mapped to a plane, and let α, β be those of the corresponding point on the plane. One must have:

$$dx^2 + dy^2 + dz^2 = d\alpha^2 + d\beta^2, \tag{26}$$

which gives the three equations:

$$\left\{\begin{aligned} &\left(\frac{\partial x}{\partial \alpha}\right)^2 + \left(\frac{\partial y}{\partial \alpha}\right)^2 + \left(\frac{\partial z}{\partial \alpha}\right)^2 = 1, \\ &\left(\frac{\partial x}{\partial \beta}\right)^2 + \left(\frac{\partial y}{\partial \beta}\right)^2 + \left(\frac{\partial z}{\partial \beta}\right)^2 = 1, \\ &\frac{\partial x}{\partial \alpha}\frac{\partial x}{\partial \beta} + \frac{\partial y}{\partial \alpha}\frac{\partial y}{\partial \beta} + \frac{\partial z}{\partial \alpha}\frac{\partial z}{\partial \beta} = 0. \end{aligned}\right. \tag{27}$$

Differentiate the first two of these equations; one will get:

([16]) *See*, in particular, the chapter on developable surfaces in Monge's *Application de l'Analyse à la Géométrie.*

([17]) Annali di Matematica (2) **7**, pp. 61.

(28)
$$\begin{cases} \dfrac{\partial x}{\partial \alpha}\dfrac{\partial^2 x}{\partial \alpha^2} + \dfrac{\partial y}{\partial \alpha}\dfrac{\partial^2 y}{\partial \alpha^2} + \dfrac{\partial z}{\partial \alpha}\dfrac{\partial^2 z}{\partial \alpha^2} = 0, \\ \dfrac{\partial x}{\partial \beta}\dfrac{\partial^2 x}{\partial \beta^2} + \dfrac{\partial y}{\partial \beta}\dfrac{\partial^2 y}{\partial \beta^2} + \dfrac{\partial z}{\partial \beta}\dfrac{\partial^2 z}{\partial \beta^2} = 0, \\ \dfrac{\partial x}{\partial \alpha}\dfrac{\partial^2 x}{\partial \alpha \partial \beta} + \dfrac{\partial y}{\partial \alpha}\dfrac{\partial^2 y}{\partial \alpha \partial \beta} + \dfrac{\partial z}{\partial \alpha}\dfrac{\partial^2 z}{\partial \alpha \partial \beta} = 0, \\ \dfrac{\partial x}{\partial \beta}\dfrac{\partial^2 x}{\partial \alpha \partial \beta} + \dfrac{\partial y}{\partial \beta}\dfrac{\partial^2 y}{\partial \alpha \partial \beta} + \dfrac{\partial z}{\partial \beta}\dfrac{\partial^2 z}{\partial \alpha \partial \beta} = 0. \end{cases}$$

Now, differentiate the last of equations (27); upon taking the preceding into account, we will find:

(29)
$$\begin{cases} \dfrac{\partial x}{\partial \alpha}\dfrac{\partial^2 x}{\partial \beta^2} + \dfrac{\partial y}{\partial \alpha}\dfrac{\partial^2 y}{\partial \beta^2} + \dfrac{\partial z}{\partial \alpha}\dfrac{\partial^2 z}{\partial \beta^2} = 0, \\ \dfrac{\partial x}{\partial \beta}\dfrac{\partial^2 x}{\partial \beta^2} + \dfrac{\partial y}{\partial \beta}\dfrac{\partial^2 y}{\partial \beta^2} + \dfrac{\partial z}{\partial \beta}\dfrac{\partial^2 z}{\partial \beta^2} = 0. \end{cases}$$

Upon comparing equations (28) and (29), we will see that one must have:

(A)
$$\frac{\dfrac{\partial^2 x}{\partial \alpha^2}}{\dfrac{\partial^2 x}{\partial \alpha \partial \beta}} = \frac{\dfrac{\partial^2 y}{\partial \alpha^2}}{\dfrac{\partial^2 y}{\partial \alpha \partial \beta}} = \frac{\dfrac{\partial^2 z}{\partial \alpha^2}}{\dfrac{\partial^2 z}{\partial \alpha \partial \beta}},$$

(B)
$$\frac{\dfrac{\partial^2 x}{\partial \alpha \partial \beta}}{\dfrac{\partial^2 x}{\partial \beta^2}} = \frac{\dfrac{\partial^2 y}{\partial \alpha \partial \beta}}{\dfrac{\partial^2 y}{\partial \beta^2}} = \frac{\dfrac{\partial^2 z}{\partial \alpha \partial \beta}}{\dfrac{\partial^2 z}{\partial \beta^2}}.$$

System (A) tells us that $\dfrac{\partial x}{\partial \alpha}, \dfrac{\partial y}{\partial \alpha}, \dfrac{\partial z}{\partial \alpha}$ are functions of the same variable, and system (B) establishes that the same thing is true for $\dfrac{\partial x}{\partial \beta}, \dfrac{\partial y}{\partial \beta}, \dfrac{\partial z}{\partial \beta}$. However, as a result of the last equation (27) or one of equations (29), the two variables upon which these two derivatives depend will be functions of each other, and consequently, the six derivatives of x, y, z will be functions of the same variable, which we will denote by t.

Having said that, if one denotes the derivatives of z (which is considered to be a function of x and y) by p and q then p and q will be determined by the equations:

$$\frac{\partial z}{\partial \alpha} = p\frac{\partial x}{\partial \alpha} + q\frac{\partial y}{\partial \alpha}, \qquad \frac{\partial z}{\partial \beta} = p\frac{\partial x}{\partial \beta} + q\frac{\partial y}{\partial \beta},$$

which shows that p and q are functions of t. Hence, p is a function of q, which characterizes the enveloping surface of a moving plane, as one knows.

The term *developable* that is given to those surfaces is then plainly justified.

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CHAPTER IX

SURFACES THAT ARE DEFINED BY KINEMATIC PROPERTIES.

General helicoids. – Bour's theorem. – Surfaces of revolution that can be mapped to each other. – Surfaces that are generated by the motion of an invariable curve. – Rolling surfaces. – Maurice Levy's spiral surfaces.

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73. Surfaces of revolution enjoy an important kinematic property: They do not cease to slide on themselves when one imagines that a rotational motion around the axis has been imposed upon them. That property that they possess – viz., of being able to be displaced without ceasing to coincide with themselves – also belongs to cylinders and a more extensive class of surfaces, namely, the helicoids, which include cylinders and surfaces of revolution as limiting cases.

Indeed, consider a solid system that is animated with a helicoidal motion. We know that all points of that system describe helices with the same axis and the same pitch. Each of those helices is animated with a motion under which it will not cease to slide to its original position. Thus, if one associates all helices that meet a given curve (C) then they will form a surface that can be generated by the helicoidal motion of the curve (C), and which will obviously possess the property of sliding on itself under the motion. That surface is a *general helicoid*. We shall give the equations that determine it and look for its line element.

The helices that are described by the permanent helicoidal motion are defined by the equations:

$$(1) \qquad \begin{cases} x = \rho \cos v_1, \\ y = \rho \sin v_1, \\ z = z_0 + h v_1, \end{cases}$$

in which h denotes the common pitch of the helices, divided by 2π. If one takes z_0 to be an arbitrary function of ρ then the coordinates x, y, z will become functions of two variables r and v_1 . The preceding formulas define the most general helicoid. Set:

$$z_0 = \varphi(\rho),$$

and calculate the line element of the helicoid. We find:

$$ds^2 = (1 + \varphi'^2)\, d\rho^2 + 2h\, \varphi'\, d\rho\, dv_1 + (\rho^2 + h^2)\, dv_1^2 .$$

Upon transforming the right-hand side, one will obtain:

(2) $$ds^2 = \left(1 + \frac{\rho^2 \varphi'^2}{\rho^2 + h^2}\right) d\rho^2 + (\rho^2 + h^2)\left(dv_1 + \frac{h\varphi' d\rho}{\rho^2 + h^2}\right)^2.$$

If one introduces the two new variables u, v that are defined by the following quadratures:

(3) $$\begin{cases} du = d\rho\sqrt{1 + \dfrac{\rho^2 \varphi'^2}{\rho^2 + h^2}}, \\ dv = dv_1 + \dfrac{h\varphi' d\rho}{\rho^2 + h^2} \end{cases}$$

then $\rho^2 + h^2$ will become a function of u that we denote by U^2, and the line element will be converted into the form:

(4) $$ds^2 = du^2 + U^2\, dv^2.$$

The curves u = const. are the helices that are traced in the surface, and in turn, the curves v = const. are the orthogonal trajectories of the helices. One also recognizes that those trajectories are determined by a simple quadrature.

74. The form (4) of the line element is identical to the one that we have obtained already for the surfaces of revolution (no. **64**). Moreover, one knows that the latter surfaces can be considered to be limiting forms of the general helicoids that correspond to the case in which the common pitch of the helices becomes zero. One can then foresee that the helicoids should be capable of being mapped to surfaces of revolution. That beautiful theorem is due to Bour, who established it in his "Mémoire sur la déformation des surfaces," Journal de l'École Polytechnique 39th letter, pp. 82. In order to prove it, we will see that the form (4) of the line element, which is given *a priori*, is suited to an infinitude of helicoids, among which, one always finds surfaces of revolution.

Formulas (1), in which one considers z_0 to be a function of ρ, define the most general helicoid, and formula (2) exhibits the line element of that surface. In order make it identical to the line element that is given by equation (4), *it will obviously suffice* to set:

$$\left(1 + \frac{\rho^2 \varphi'^2}{\rho^2 + h^2}\right) d\rho^2 = du^2,$$

$$(\rho^2 + h^2)\left(dv_1 + \frac{h\varphi' d\rho}{\rho^2 + h^2}\right)^2 = U^2\, dv^2,$$

or, upon extracting the square roots:

$$
(5) \qquad \begin{cases} du = \pm\sqrt{d\rho^2 + \dfrac{\rho^2 d\varphi^2}{\rho^2 + h^2}}, \\[2ex] dv_1 + \dfrac{h\,d\varphi}{\rho^2 + h^2} = \dfrac{\pm U}{\sqrt{\rho^2 + h^2}}\,dv. \end{cases}
$$

The first of these formulas shows that ρ is a function of u. In order for the second one to be true, it is obviously necessary that the ratio $\dfrac{\pm U}{\sqrt{\rho^2 + h^2}}$ must be equal to a constant, which we denote by $1 / m$. One will then have:

$$
(6) \qquad \begin{cases} \sqrt{\rho^2 + h^2} = \pm mU, \\[2ex] dv_1 + \dfrac{h\,d\varphi}{\rho^2 + h^2} = \dfrac{dv}{m}. \end{cases}
$$

Formulas (5) and (6) lead us to the following values for ρ, $d\varphi$, dv_1 by some simple eliminations:

$$
(7) \qquad \begin{cases} d\varphi = \dfrac{m^2 U du}{m^2 U^2 - h^2}\sqrt{U^2(1 - m^2 U'^2) - \dfrac{h^2}{m^2}}, \\[2ex] dv_1 = \dfrac{dv}{m} - \dfrac{h\,d\varphi}{m^2 U^2} = \dfrac{dv}{m} - \dfrac{h\,du}{U(m^2 U^2 - h^2)}\sqrt{U^2(1 - m^2 U'^2) - \dfrac{h^2}{m^2}}, \\[2ex] \rho = \sqrt{m^2 U^2 - h^2}. \end{cases}
$$

All of the quantities that appear in formulas (1) are thus expressed as functions of u and v; the proposed question is then resolved completely.

75. The helicoids that we just determined depend upon two arbitrary parameters h and m. It is easy to insure that they are not superposable. In particular, consider the case in which one has:

$$U^2 = u^2 + a^2,$$

and suppose that $m = 1$. The preceding formulas become:

$$
\begin{aligned}
\rho &= \sqrt{u^2 + a^2 - h^2}, \\
d\varphi &= \frac{\sqrt{a^2 - h^2}}{u^2 + a^2 - h^2}\sqrt{u^2 + a^2}\; du, \\
dv_1 &= dv - \frac{h\,d\varphi}{u^2 + a^2}.
\end{aligned}
$$

If one gives all of the values between 0 and a to h then one will get a continuous series of helicoids that can all be mapped to each other. They all present forms that are intermediate to the alysseid and the skew helicoid with a director plane, which are the extreme terms in that series, and they correspond to the values 0 and a of the arbitrary h, respectively.

76. Return to the general formula (7). If one sets $h = 0$ then one will obtain surfaces of revolution that can all be mapped to each other, as well as to general helicoids that are defined by those equation. They are determined by the very simple system:

$$(8)\qquad \left\{\begin{aligned} x &= aU\cos\frac{v}{a}, \\ y &= aU\sin\frac{v}{a}, \\ z &= \int\sqrt{1-a^2U'^2}\,du, \end{aligned}\right.$$

in which a replaces m.

When one varies the parameter a, one obtains a continuous sequence of surfaces; we point out, without proof, the following properties, which we will attach to some general theorems later on.

If one considers the points that correspond to the same values of u on all of those surfaces:

1. The product of the principal radii of curvature at those points will be the same for all surfaces.

2. The tangent to the meridian will also have the same length for all points considered when it is prolonged to the point at which it meets the axis.

Later on, we shall point out an application of the latter property, and we shall study two particular examples of it.

77. First set:

$$U = \sin u,$$

which will give surfaces of revolution that map to the sphere.

Here, formulas (8) will become:

$$(9)\qquad \left\{\begin{aligned} x &= a\sin u\cos\frac{v}{a}, \\ y &= a\sin u\sin\frac{v}{a}, \\ z &= \int\sqrt{1-a^2\cos^2 u}\,du. \end{aligned}\right.$$

First, suppose $a^2 < 1$; u can take on all possible values without the expression for z ceasing to be real. The portion OCA of the meridian that corresponds to all values of u that are found between 0 and π will have the form depicted in (Fig. 4).

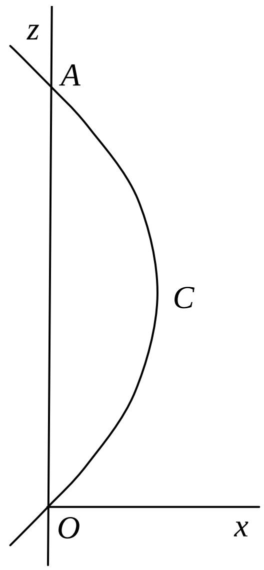

Figure 4.

The angles that the meridian makes with the axis at O and A are finite and become right angles only when a is equal to 1. The meridian will then become a semi-circle, and it will generate the sphere.

The variables u and v, which determine a point on the surface of the sphere, have a very simple significance: They are the *colatitude* and *longitude* of that point. From that remark, we can easily determine the limits and the form of the portion of the sphere that can be mapped precisely onto any surface that is generated by the complete revolution of the branch OCA of the meridian.

Indeed, for an arbitrary value of a, formulas (9) show that the angle v_1 that the meridian that passes through the point (u, v) of the surface makes with a fixed meridian has the value:

$$v_1 = \frac{v}{a}. \tag{10}$$

Consequently, when v_1 varies from 0 to 2π, v will increase to $2\pi a$. Hence, the surface that is generated by the complete revolution of the arc ACO can be mapped to the wedge of the sphere that is included between two meridian planes that make an angle of $2\pi a$. One sees that the wedge will become infinitely thin for very small values of a.

If a^2 is great than unity then the meridian will change form completely, because u can take on only values that satisfy the inequality:

$$\cos^2 u < \frac{1}{a^2}.$$

Let λ_0 be the acute angle that is defined by the formula:

$$\cos \lambda_0 = \frac{1}{a}.$$

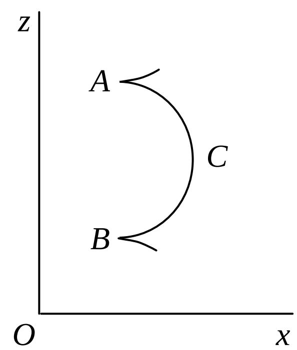

Figure 5.

One must then make u vary between λ_0 and $\pi - \lambda_0$. The meridian will have the form that is represented in (Fig. 5). The surface that is generated by that meridian can be mapped onto the zone of the sphere that is found between the two colatitude circles λ_0 and $\pi - \lambda_0$. However, formula (10) shows us that it will suffice to rotate the meridian ACB through an angle that is equal to $2\pi / a$ in order to obtain all of the portion of the surface that maps point-by-point onto the spherical zone that we just defined.

If a gets larger then that zone will diminish indefinitely and reduce to an infinitely-thin band about the equator of the sphere.

The detailed discussion that we just made had the objective of exhibiting one interesting fact: Consider a piece (of arbitrary form, moreover) of the surface of the sphere. When the sphere is deformed in such a manner as to coincide successively with the various surfaces that are defined by formulas (9), that portion of the spherical surface that we have chosen will displace and deform in a continuous manner while remaining mappable to its initial position. However, that motion cannot be indefinitely continuous without producing a rip, because we have seen that if the parameter a increases after starting from 1 and growing indefinitely then the only portion of the sphere that can be mapped onto the surface that corresponds to those increasing values of a will reduce to a zone that surrounds the equator whose area is as small as one pleases. Consequently, if one considers a finite portion of the sphere then the motion of deformation of that portion will cease to be possible once a has attained an upper limit, which obviously depends upon the form of that portion.

78. At least in the example that we just studied, all of the surfaces that are defined by formulas (9), and for which a^2 is less than or equal to unity, enjoy the property that they represent the line element completely – i.e., they are all real points that correspond to all real values of u and v. That will no longer be true in the following example:

Suppose that one sets:

$$U = e^u$$

in the general formulas.

The line element will have the expression:

$$(11) \qquad ds^2 = du^2 + e^{2u}\, dv^2,$$

and equations (8) will give us:

$$(12) \qquad x = a\, e^u \cos\frac{v}{a}, \qquad y = a\, e^u \sin\frac{v}{a}, \qquad z = \int \sqrt{1 - a^2 e^{2u}}\; du,$$

here.

These formulas define surfaces that are all equal, because if one sets:

$$(13) \qquad a\, e^u = \sin\varphi, \quad v = a\, v_1$$

then they will become:

$$(14) \qquad \begin{cases} x = \sin\varphi \cos v_1, \\ y = \sin\varphi \sin v_1, \\ z = \cos\varphi + \log\tan\dfrac{\varphi}{2}, \end{cases}$$

and no longer contain a. We will then have only one surface, which can be mapped to itself in an infinitude of ways, and the formulas that realize that map are:

$$a\, e^u = e^{u'}, \qquad v = a\, v',$$

in which u', v' denote the coordinates of the point that corresponds to the point (u, v). That result is obvious, moreover, from the form itself of the line element (11). It follows from that and the properties that we pointed out above (no. **76**) that:

1. The meridian can only be the *curve of equal tangents* or *tractrix*.
2. The product of the principal radii of curvature is the same at all points of the surface.

One recognizes the properties of the pseudo-spherical surface that we have just studied directly, and upon comparing the expressions for the line element with the one that was given (no. **65**), one sees that the product of the radii of curvature of the surface is equal to -1.

Here, is an important fact that should be mentioned: In order for the values of x, y, z that are given by formulas (12) to be real, it is necessary that the angle φ should be real; i.e., that one should have:

$$a^2\, e^{2u} < 1.$$

No matter what the given value of the parameter a is, there will always be sufficiently large values of u that are associated with arbitrary values for v, and *which do not correspond to any point of the surface.* Consequently, if it is true that the pseudo-sphere can be mapped to itself in an infinitude of ways then none of the solutions that one chooses can give a complete geometric representation of the line element.

79. The surfaces that we just studied have a common property: They can all be considered to be generated by an invariable curve of a form that moves according to a given law. We now propose to study the most general surfaces that satisfy that definition.

Consider a curve (*C*) and a moving system of axes that are coupled invariably with the curve. Suppose that the position of the curve and the moving axes depends upon a parameter v that plays the role of time, and let ξ, η, ζ; p, q, r be the translations and rotations of the moving system. Those six quantities are functions of v. Let x, y, z denote the coordinates of an arbitrary point M of the curve with respect to the moving axes. x, y, z will be given functions of a parameter u.

If the moving axes are displaced, and at the same time the point M displaces along the curve in an arbitrary manner then the projections of the infinitely-small arc that is described by the point will be:

$$(15)\qquad \begin{cases} dx + (\xi + qz - ry)\,dv, \\ dy + (\eta + rx - pz)\,dv, \\ dz + (\zeta + py - qx)\,dv. \end{cases}$$

Set:

$$\frac{dx}{du} = x', \qquad \frac{dy}{du} = y', \qquad \frac{dz}{du} = z',$$

to abbreviate. The square of the line element of the surface that is generated by the curve will be expressed by the sum of the squares of those three projections; i.e.:

$$(16)\qquad \begin{cases} ds^2 = (x'^2 + y'^2 + z'^2)\,du^2 \\[1ex] \qquad + 2\left(x'\xi + y'\eta + z'\zeta + \begin{vmatrix} x & y & z \\ x' & y' & z' \\ p & q & r \end{vmatrix} \right) du\,dv \\[1ex] \qquad + [(\xi + qz - ry)^2 + (\eta + rx - pz)^2 + (\zeta + py - qx)^2]\,dv^2. \end{cases}$$

80. It will suffice to introduce some conveniently-chosen special hypotheses into that formula in order to recover all the preceding results.

Suppose, for example, that one would like to obtain the line element of ruled surfaces. One takes the z-axis of the moving trihedron to be the rectilinear generator of the surface, and one describes the origin of the trihedron by an orthogonal trajectory of the generator. That will give the conditions:

$$x = y = 0, \qquad z = u, \quad \zeta = 0,$$

and in turn, formula (16) will reduce to the following one:

(17) $$ds^2 = du^2 + [(\xi + qu)^2 + (\eta - pu)^2]\, dv^2.$$

If one would like to express the idea that the surface is developable then one must consider the projections (15) of the arc that is described by an arbitrary point of the surface. Here, they become:

$$(\xi + qu)\, dv, \qquad (\eta - pu)\, dv, \qquad du.$$

The tangent plane to the point $z = u$ will then have the equation:

$$\frac{x}{y} = \frac{\xi + qu}{\eta - pu}$$

with respect to the moving axes.

In order for the same thing to be true at all points of a generator, it is necessary that one must have:

$$\frac{\xi}{\eta} = -\frac{q}{p};$$

i.e., the coefficient of dv^2 in formula (17) should be a perfect square. That is the result that was established already (no. **69**).

81. Now, consider the new case in which the motion of the moving curve (C) reduces to a translation. One must then set:

$$p = q = r = 0$$

in formula (16), and consequently:

(18) $$ds^2 = (x'^2 + y'^2 + z'^2)\, du^2 + 2\, (x'\xi + y'\eta + z'\zeta)\, du\, dv + (\xi^2 + \eta^2 + \zeta^2)\, dv^2 .$$

One will arrive at an identical result by the following direct method: Let:

$$x = U, \qquad y = U_1, \qquad z = U_2$$

be the equations that determine the curve with respect to the moving axes, in which U, U_1, U_2 denote functions of the same parameter u. Suppose that the fixed axes have been chosen to be parallel to the moving axes, and let V, V_1, V_2 denote the coordinates of the origin of the moving axes with respect to the fixed axes; V, V_1, V_2 will be functions of a parameter v. The coordinates of an arbitrary point of the desired surface with respect to the fixed axes will obviously be the following expressions:

(19) $$\left\{ \begin{aligned} X &= U + V, \\ Y &= U_1 + V_1, \\ Z &= U_2 + V_2. \end{aligned} \right.$$

The symmetry of these formulas shows us immediately that the surface can be generated in two different ways by the translation of an invariable curve, and that the coordinate curves of each of the two systems (u) and (v) are deduced from each other by a simple motion of translation.

82. One can further interpret formulas (19) in the following manner: Consider the two curves that are defined by the equations:

$$x = 2U, \qquad y = 2U_1, \qquad z = 2U_2,$$

and

$$x = 2V, \qquad y = 2V_1, \qquad z = 2V_2 .$$

The locus of the midpoints of all the chords that join a point of the first one to a point of the second is the surface in question. That definition, which is due to Lie, exhibits the double mode of generation of the surface quite nicely. It suffices to associate all of the chords that pass through either a point of the first curve or a point of the second one in order to recover the two systems of invariable generators.

Suppose, to fix ideas, that the functions U, V are real, and that one has taken arc lengths on the two curves:

$$x = U, \qquad y = U_1, \qquad z = U_2,$$
$$x = V, \qquad y = V_1, \qquad z = V_2$$

to be the parameters u, v. The line element of the surface will take the form:

$$ds^2 = du^2 + dv^2 + 2\,(UV + U_1'V_1' + U_2'V_2')\,du\,dv.$$

If one then sets:

$$u = \frac{\alpha+\beta}{2}, \qquad \alpha = u + v,$$
$$v = \frac{\alpha-\beta}{2}, \qquad \beta = u - v$$

then the expression for the line element will become:

$$ds^2 = \frac{1+U'V'+U_1'V_1'+U_2'V_2'}{2}\,d\alpha^2 + \frac{1-U'V'-U_1'V_1'-U_2'V_2'}{2}\,d\beta^2 . \tag{20}$$

That formula will exhibit a system of rectangular coordinates on the surface, because the line element will reduce to the form:

$$ds^2 = A\,d\alpha^2 + C\,d\beta^2, \tag{21}$$

and similarly, with the condition:

$$A^2 + C^2 = 1.$$

83. We have to further mention a geometric property of the surfaces that we consider that is completely general. However, in order to prove it, we must begin by recalling a theorem that relates to conjugate tangents.

We say that two families of lines that are traced on a surface are *conjugate* when the tangents to the lines of the two families that pass through an arbitrary point of the surface are conjugate (with Dupin's definition). Here is how one can express that relation:

Let u and v be the two parameters of the two families of curves, and suppose that one knows the expressions for the rectilinear coordinates x, y, z of an arbitrary point of the surface as functions of u and v. If one lets X, Y, Z denote the current coordinates then the equation of the tangent plane to the surface at the point M (x, y, z) will be:

$$Z - z = p\,(X - x) + q\,(Y - y), \tag{22}$$

in which p and q denote the derivatives of z with respect to x and y, according to custom. Suppose that one displaces along the line v = const. From the theory of envelopes, the intersection of the tangent plane with its infinitely-close position will be defined by equation (22), combined with the one that one obtains from it by differentiation with respect to u; i.e.:

$$-\frac{\partial z}{\partial u} = \frac{\partial p}{\partial u}(X - x) + \frac{\partial q}{\partial u}(Y - y) - p\frac{\partial x}{\partial u} - q\frac{\partial y}{\partial u},$$

or, upon suppressing the terms that cancel:

$$\frac{\partial p}{\partial u}(X - x) + \frac{\partial q}{\partial u}(Y - y) = 0. \tag{23}$$

In order to express the idea that the curves (u) and (v) are conjugate, one must write down that equations (22), (23) are verified when one replaces $X - x$, $Y - y$, $Z - z$ with $\frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial y}{\partial v}$ in them. That will give the two equations:

$$\left\{\begin{aligned} \frac{\partial z}{\partial v} &= p\frac{\partial x}{\partial v} + q\frac{\partial y}{\partial v}, \\ \frac{\partial p}{\partial u}\frac{\partial x}{\partial v} &+ \frac{\partial q}{\partial u}\frac{\partial y}{\partial v} = 0. \end{aligned}\right. \tag{24}$$

The first of them is always satisfied, because it expresses the obvious fact that the tangent to the curve u = const. is found in the tangent plane. As for the second one, it is identical to the following one:

$$\frac{\partial}{\partial u}\left(p\frac{\partial x}{\partial v} + q\frac{\partial y}{\partial v}\right) - p\frac{\partial^2 x}{\partial u\,\partial v} - q\frac{\partial^2 y}{\partial u\,\partial v} = 0,$$

or, more simply:

(25) $$\frac{\partial^2 z}{\partial u\,\partial v} - p\frac{\partial^2 x}{\partial u\,\partial v} - q\frac{\partial^2 y}{\partial u\,\partial v} = 0.$$

If one now remarks that p and q are determined by the equations:

(26) $$\frac{\partial z}{\partial u} = p\frac{\partial x}{\partial u} + q\frac{\partial y}{\partial u}, \qquad \frac{\partial z}{\partial v} = p\frac{\partial x}{\partial v} + q\frac{\partial y}{\partial v}$$

then one can eliminate p and q from equations (25) and (26), and one will be led to the equation:

(27) $$\begin{vmatrix} \frac{\partial^2 x}{\partial u\,\partial v} & \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial^2 y}{\partial u\,\partial v} & \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \\ \frac{\partial^2 z}{\partial u\,\partial v} & \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} \end{vmatrix} = 0,$$

which is symmetric with respect to the three coordinates. That relation – which is, moreover, necessary – is also sufficient, because it expresses the idea that the values of p and q satisfy equations (25) and (26), and from formulas (26), p and q will be the derivatives of z when it is considered to be a function of x and y.

84. One can formulate the condition that we found in a different form. Equation (27) is obviously the result of the elimination of A and B from the three equations:

(28) $$\begin{cases} \frac{\partial^2 x}{\partial u\,\partial v} - A\frac{\partial x}{\partial u} - B\frac{\partial x}{\partial v} = 0, \\ \frac{\partial^2 y}{\partial u\,\partial v} - A\frac{\partial y}{\partial u} - B\frac{\partial y}{\partial v} = 0, \\ \frac{\partial^2 z}{\partial u\,\partial v} - A\frac{\partial z}{\partial u} - B\frac{\partial z}{\partial v} = 0. \end{cases}$$

We then obtain the following proposition:

The necessary and sufficient condition for the lines (u) *and* (v) *to be conjugate is that the expressions for the three rectangular coordinates as functions of u and v must satisfy the same linear equation:*

(29) $$\frac{\partial^2 \theta}{\partial u\,\partial v} = A\frac{\partial \theta}{\partial u} - B\frac{\partial \theta}{\partial v},$$

in which A, B denote arbitrary functions of u and v.

That proposition plays a very important role in the theory of surfaces, and we shall have to return to it in order to complete and generalize it.

If we apply it to the surfaces that we are dealing with then we will see immediately that the three coordinates will satisfy the equation:

$$\frac{\partial^2 \theta}{\partial u \, \partial v} = 0, \tag{30}$$

and consequently the two systems of invariable curves that generate the surface here will be conjugate lines. Moreover, geometry also permits one to obtain that result very simply.

In fact, consider the two families of curves (u) and (v). Under the translation of a curve (u), each point M of that curve will describe a curve (v). On the other hand, the tangent to the curve (u) at M will keep an invariable direction during the translation. It will then follow that the developable that is circumscribed by the surface at all points of the curve (v) that is described by the point M will be the cylinder that is generated by the tangent to the curve (u) at M. That simple remark will suffice to prove that the two families of curves (u) and (v) form a conjugate system, and one sees, moreover, that:

The developables that are circumscribed by the surface at all points of one of those curves are cylinders that are generated by the tangents to the curve of the other family that are drawn from the point where they meet the curve considered ([18]).

85. Finally, we treat the case in which the moving curve (C) that generates the surface is a plane, and in which the velocities of all of its points are normal to the plane. We suppose that one has taken the plane of the curve to be the xy-plane of the moving trihedron. One must then introduce the hypotheses:

$$z = 0, \quad \zeta = \eta = r = 0 \tag{31}$$

into formulas (15) and (16).

If one assumes, moreover, that one has chosen u to be the arc length of the curve (C) then one will again have:

$$x'^2 + y'^2 = 1,$$

and the expression for the line element will become:

$$ds^2 = du^2 + (\zeta + py - qx)\, dv^2. \tag{32}$$

As for the projections of the arc that is described by an arbitrary point M of the surface, from formulas (15), they will be:

([18]) S. LIE, "Beiträge sur Theorie der Minimalflächen," Math. Ann. **14**, pp. 332-337.

$$dx, \qquad dy, \qquad (\zeta + py - qx)\, dv.$$

The normal to the surface will be in the plane of the curve, and that plane will roll on a certain developable surface.

One recognizes the surfaces that were studied by Monge in a detailed manner ([19]).

The lines of curvature of one of the systems are the various positions of the moving curve; those of the second system are the trajectories of the various points of that curve.

86. In particular, consider the case in which the plane of the moving curve rolls on a cylinder. If we suppose that the x-axis of the moving trihedron has been taken to be parallel to the generators of the cylinder then the rotation of the system will take place around a parallel to the x-axis, and one will have:

$$q = 0.$$

The line element that is given by formula (32) will take the form:

$$ds^2 = du^2 + \left(\frac{\zeta}{p} + y\right)^2 p^2\, dv^2,$$

or, upon changing the notations:

$$ds^2 = du^2 + (U - V)^2\, dv^2, \tag{33}$$

in which U and V denote functions that depend upon u and v, respectively.

One can give rolling surfaces another definition that is simpler than the preceding one, in some respects.

When the plane of the curve (C) rolls on the cylinder, the trajectories of its various points will obviously be planar curves whose planes are parallel to cross sections of the cylinder, and those trajectories will be normal to the plane of the curve (C) at each instant, moreover. From this, it is obvious that their projections onto the plane of the cross section of the cylinder will constitute a family of parallel plane curves that admit the cross section of the cylinder for their common developable. The following mode of generating these rolling surfaces will result from that:

One gives a family of parallel curves in a plane. If one imparts a finite translation to each of those curves that is normal to the plane and varies according to a given law when one passes from one curve to the other one then the new positions of all of those curves will define the rolling surface.

87. Upon appealing to that definition, one can show that the form (33) of the line element always agrees with an infinitude of rolling surfaces.

([19]) MONGE, *Application de l'Analyse à la Géométrie*, 5th ed., pp. 322: "De la surface courbe dont toutes les normales sont tangentes à une même surface développable quelconque."

Indeed, write down the expression for ds^2 in the form:

$$ds^2 = dU^2 + (U - V)^2\, dv^2 + (1 - U'^2)\, du^2.$$

The first two terms, taken in isolation, constitute the line element of a developable surface, and we have seen (no. **70**) that upon setting:

$$(34)\qquad \begin{cases} x = U \cos v + \int V \sin v\, dv, \\ y = U \sin v - \int V \cos v\, dv, \end{cases}$$

one will have:

$$dx^2 + dy^2 = dU^2 + (U - V)^2\, dv^2.$$

The surface that is defined by formulas (34), combined with the following one:

$$(34)' \qquad z = \int \sqrt{1 - U'^2}\, du,$$

will then have the line element that is expressed by formula (33).

If one remarks that this line element will not change in form when one replaces v with av then one will recognize the possibility of introducing an arbitrary constant into the preceding formulas, and one will find, upon repeating the calculations, that:

$$(35)\qquad \begin{cases} x = aU \cos \dfrac{v}{a} + \displaystyle\int V \sin \dfrac{v}{a}\, dv, \\ y = aU \sin \dfrac{v}{a} - \displaystyle\int V \cos \dfrac{v}{a}\, dv, \\ z = \displaystyle\int \sqrt{1 - a^2 U'^2}\, du. \end{cases}$$

These formulas define a family of rolling surfaces that can be mapped to each other completely ([20]).

88. The kinematic method that we just applied to numerous examples extends to the case in which one considers a curve that varies in form at the same time that it is carried along by the motion of the moving axes. Indeed, it will suffice to regard x, y, z, no longer as functions of only the variable u, but as functions of u and v in formulas (15), which give the projections of the displacement onto the moving axes.

For example, we propose to apply that method to the surfaces that are generated by the motion of a circle. The plane of that circle will envelop a developable surface. We study the motion of the trihedron that is defined by the tangent, the principal normal, and

([20]) BOUR, "Théorie de la déformation des surfaces," Journal de l'École Polytechnique, 39th Letter, pp. 89.

the binormal at a point on the edge of regression of that developable. Upon taking the independent variable to be the arc length of the curve, one will have (no. **4**):

$$\xi = 1, \qquad \eta = 0, \qquad \zeta = 0,$$
$$p = -\frac{1}{\tau}, \qquad q = 0, \qquad r = \frac{1}{\rho}$$

here, in which ρ and τ are the radii of curvature and torsion of the curve. The projections of the displacement of a point whose coordinates are x, y, z relative to the moving axes will have the expressions:

$$\begin{aligned} &dx + (1 - ry)\, dv, \\ &dy + (rx - pz)\, dv, \\ &dz + py\, dv, \end{aligned}$$

in which v denotes the arc length of the edge of regression.

The circle that generates the surface is found in the xy-plane, so one can express the coordinates of one of its points by the formulas:

$$\begin{aligned} x &= a + R \cos \varphi, \\ y &= b + R \sin \varphi, \\ z &= 0, \end{aligned}$$

in which a, b, R are functions of v, and in which φ is the variable that determines a point on each circle. Upon substituting those values for x, y, z, one will have:

$$\begin{gathered} -\ R \sin \varphi\, d\varphi + (a' + 1 - br + R \cos \varphi - rR \sin \varphi)\, dv, \\ R \cos \varphi\, d\varphi + (b' + ra + R'\sin \varphi + rR \cos \varphi)\, dv, \\ (pb + pR \sin \varphi)\, dv \end{gathered}$$

for the projections of the displacement, and the sum of the squares of those projections will give the line element of the surface in the form ([21]):

$$\begin{aligned} ds^2 = R\, d\varphi^2 &+ 2R\, [rR + (b' + ra) \cos \varphi - (a' - br + 1) \sin \varphi]\, du\, dv \\ &+ [(pb + pR \sin \varphi) + (a' + 1 - br + R'\cos \varphi - rR \sin \varphi)^2 \\ &\qquad + (b' + ra + R'\sin \varphi + rR \cos \varphi)^2]\, dv. \end{aligned}$$

89. To conclude this chapter – in which we have studied, above all, surfaces that enjoy kinematic properties – we shall give the definition of a class of surfaces that are consistent with the preceding viewpoint, and which were first studied by Maurice Lévy ([22]).

([21]) Surfaces with a circular generator have been studied recently by Demartres [Annales de l'École Normale (3) **2**, pp. 123].

([22]) MAURICE LÉVY, "Sur le développement des surfaces dont l'élément linéaire est exprimable par une fonction homogène," Comptes rendus **87**, pp. 788.

Consider a system that is displaced, but at the same time varies in magnitude while remaining similar to itself, and propose to seek the law for the velocities at all of its points at an arbitrary instant. Let P_0, P_1 be two infinitely-close positions. Construct the figure P_1', which is homothetic to P_1, by taking the origin of the coordinates to be the center of homothety, while the ratio of homothety is such that P_1' is equal to P_0. One can pass from P_0 to P_1 by:

1. An infinitely-small displacement that takes P_0 to P_1'.

2. A homothetic transformation that has the origin for its center of homothety and transforms P_1' to P_1.

It follows from this that the velocities of all points of the system will be the resultant of the ones that are produced in the displacement and the ones that are due to the homothety transformation. The former will have the well-known expressions:

$$\alpha + qz - ry, \qquad \beta + rx - pz, \qquad \gamma + py - qx.$$

As for the ones that are due to the homothetic transformation, since they have the effect of reducing the coordinates by the same ratio, they will have the expression:

$$hx, \qquad hy, \qquad hz.$$

In summary, the components of the velocities of a point of the system under the motion considered will have the values:

$$(36) \qquad \left\{ \begin{aligned} V_x &= \alpha + hx + qz - ry, \\ V_y &= \beta + hy + rx - pz, \\ V_z &= \gamma + hz + py - qx. \end{aligned} \right.$$

If we let k denote the ratio of similitude of the moving system taken in its present position to the same system taken in a well-defined position then we will obviously have:

$$(37) \qquad h = \frac{1}{k}\frac{dk}{dt}.$$

As long as the parameter h is not zero – i.e., as long as the system varies in magnitude – one can transport the origin of coordinates at a point such that the terms α, β, γ disappear from formulas (36). The interpretation of those formulas will then exhibit the following result: The velocities are the same as if the body turned around a line and, at the same time, experienced a homothetic transformation with respect to a point of that line. If one chooses the axis of rotation to be the new z-axis then formulas (36) will simplify and reduce to the following form:

(38)
$$\begin{cases} V_x = hx - ry, \\ V_y = hy + rx, \\ V_z = hz. \end{cases}$$

90. Let us study the case in which the axis of rotation and the center of homothety remain fixed during all of the motion, while the parameters h and r remain constant. The successive positions of a well-defined point of the moving system will be defined by the differential equations:

$$\frac{dx}{dt} = hx - ry, \qquad \frac{dy}{dt} = hy + rx, \qquad \frac{dz}{dt} = hz.$$

Upon integrating them, one will have:

(39)
$$\begin{cases} z = z_0 e^{ht}, \\ x = r_0 e^{ht} \cos(\omega_0 + rt), \\ y = r_0 e^{ht} \sin(\omega_0 + rt). \end{cases}$$

Each point of the system will describe a curve that is traced on a cone of revolution:

$$\frac{\sqrt{x^2 + y^2}}{z} = \text{const.}$$

that has the origin for its summit and the axis of rotation for its axis. The projection of the trajectory onto the xy-plane will be a logarithmic spiral that has the origin of the coordinates for its pole. If one considers the skew spiral that is described by the point to belong to the moving system and vary in magnitude with it then it will slide on itself during all of the motion in precisely the same way as the helices that are described by the various points of an invariable system under the helicoidal motion.

As a result, the surfaces that admit the curves that are defined by the formulas (39) for their generators are obviously the analogues of the helicoidal surfaces and surfaces of revolution in the theory that we are addressing.

Take r_0, ω_0, z_0 to be arbitrary functions of a parameter θ. Formulas (39), which give expressions for x, y, z as functions of t and θ, define the surface that we have proposed to study. If we seek its line element then we will find a result of the form:

(40)
$$ds^2 = e^{2ht} (A\, dt^2 - 2B\, dt\, d\theta + C\, d\theta^2),$$

in which A, B, C are functions of θ that are defined by the equations:

(41)
$$\begin{cases} A = r_0^2 h^2 + r^2 r_0^2 + h^2 z_0^2, \\ B = h\, z_0 z_0' + h\, r_0 r_0' + r\, r_0^2 \omega_0', \\ C = r_0^2 \omega' + z_0'^2 + r_0'^2. \end{cases}$$

We shall transform that expression for the line element. Set:

$$dt + \frac{B}{A}\, d\theta = \frac{1}{h}\, dv,$$

which will give, upon integrating:

$$t + \int \frac{B}{A}\, d\theta = \frac{v}{h}.$$

The line element will become:

$$ds^2 = e^{2v}\,(A'\, dv^2 + C'\, d\theta^2),$$

in which A' and C' are also functions of θ. Finally, replace θ with the variable u that is defined by the relation:

$$u = \int \sqrt{C'}\ d\theta,$$

in which A is a function U^2 of u, and we will have:

(42)
$$ds^2 = e^{2v}\,(du^2 + U^2\, dv^2)$$

for the definitive form of the line element.

We call the surfaces that we just defined *spiral surfaces.* They agree with the logarithmic spiral in one essential property that results from their definition: Like that curve, they can be enlarged by an arbitrary ratio without ceasing to be superposable with themselves.

Maurice Lévy showed that Bour's theorem extends to those surfaces, so there is an infinitude of them that admit the same line element, and consequently can be mapped to each other. One establishes that proposition by a calculation that we shall omit, because it would be entirely analogous to the one that we developed in the case of the helicoids.