COURSE IN GEOMETRY OF THE SCIENCE FACULTY

LESSONS

ON THE GENERAL THEORY

OF SURFACES

AND

GEOMETRIC APPLICATIONS TO THE INFINITESIMAL CALCULUS

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PART TWO

CONGRUENCES AND LINEAR PARTIAL DIFFERENTIAL EQUATIONS. LINES TRACED ON SURFACES

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TABLE OF CONTENTS

BOOK V

LINES THAT ARE TRACED ON SURFACES

CHAPTER V

CHAPTER VI

CHAPTER VII

CHAPTER VIII

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BOOK V

LINES TRACED ON SURFACES

CHAPTER I

GENERAL FORMULAS.

Definition of a tri-rectangular trihedron (*T*) that is linked to each element of the surface. – Application of the formulas that were given in Book I in regard to two-parameter displacements. – Systems of formulas (*A*) and (*B*) – Conjugate directions. – Asymptotic lines. – Lines of curvature. Equations of the radii of principal curvature. – Kinematic property of the lines of curvature. – Formulas that relate to an arbitrary curve that is traced on a surface. – Meusnier's theorem. – Normal curvature. Geodesic curvature. – Third-order elements. – Formulas of O. Bonnet and Laguerre. – Osculating sphere.

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484. We now propose to recall the study of surfaces by connecting it directly with the developments that were given in Book I. We first present various systems of formulas, among which, we find those of Codazzi.

Consider an arbitrary surface. One can link the study of that surface to the study of the motion of a moving system by operating in the following manner:

Let M denote a point on the surface, and construct a tri-rectangular trihedron (*T*) whose summit is at M and whose z-axis is the normal at M; the x and y axes will be, in turn, situated in the tangent plane to the surface. Those axes will be determined perfectly when one knows the angle between the x-axis and one of the coordinate lines for each position of the point M; for example, the x-axis and the tangent to the curve v = const. Without saying anything more precise in regard to their positions in the tangent plane, for the moment, we shall show how the properties of the surface and the curves that are traced on it can be deduced from the study of the motion of the trihedron (*T*).

We first remark that if one preserves all the notations of Chapter VII [I, pp. 61] then that motion will be characterized by the equations:

$$\zeta = 0, \qquad \zeta_1 = 0,$$

which express the idea that the surface that is described by the summit of the trihedron is tangent to the xy-plane.

The formulas in Book I [I, pp. 43 and 61] will then give the following system:

$$(A)\quad \begin{cases} \dfrac{\partial p}{\partial v}-\dfrac{\partial p_1}{\partial u}=qr_1-rq_1, & \dfrac{\partial \xi}{\partial v}-\dfrac{\partial \xi_1}{\partial u}=\eta r_1-r\eta_1, \\ \dfrac{\partial q}{\partial v}-\dfrac{\partial q_1}{\partial u}=rp_1-pr_1, & \dfrac{\partial \eta}{\partial v}-\dfrac{\partial \eta_1}{\partial u}=r\xi_1-\xi r_1, \\ \dfrac{\partial r}{\partial v}-\dfrac{\partial r_1}{\partial u}=pq_1-qp_1, & p\eta_1-\eta p_1+\xi q_1-q\xi_1=0, \end{cases}$$

and it will obviously result from the propositions that were established in Book I that:

Any system of values for the quantities p, …, ξ, … that satisfies these equations will correspond to a perfectly-determined motion, and consequently just one surface.

If a point has the coordinates x, y, z when it is referred to the trihedron (T) then upon applying formulas (4) [I, pp. 60], one will get:

$$(B)\quad \begin{cases} dx+\xi\,du+\xi_1\,dv+(q\,du+q_1\,dv)\,z-(r\,du+r_1 dv)\,y, \\ dy+\eta\,du+\eta_1\,dv+(r\,du+r_1\,dv)\,x-(p\,du+p_1 dv)\,z, \\ dz \qquad\qquad\qquad +(p\,du+p_1 dv)\,y-(q\,du+q_1 dv)\,x \end{cases}$$

for the projections of its displacement onto the axes of the moving trihedron when u and v take on the increments du and dv.

485. In particular, consider the proposed surface, which is traversed by the origin of the moving trihedron. If ds denotes the differential of the arc length of the curve that is described by that origin, and ω is the angle that the tangent to that curve makes with the x-axis of the moving trihedron then one will have:

$$(1)\qquad ds\cos\omega = \xi\,du+\xi_1\,dv, \qquad ds\sin\omega = \eta\,du+\eta_1\,dv\,.$$

These formulas exhibit the line element of the surface, which will have the expression:

$$(2)\qquad ds^2 = (\xi\,du+\xi_1\,dv)^2+(\eta\,du+\eta_1\,dv)^2.$$

Imagine that one draws lines parallel to the axes of the trihedron (T) through a fixed point O in space. One then defines a trihedron (T_1) whose rotations will be the same as those of the trihedron (T). If one considers the point m at a distance 1 on the z-axis of that trihedron then it will describe a sphere (S) of radius 1. It will obviously be the point that corresponds to M when one performs the spherical representation of the proposed surface on the sphere (S) according to the rule that we have described.

Furthermore, if we apply formulas (4) [I, pp. 43], which relate to the displacement of a trihedron that has a fixed point, then we will find the following values:

$$q\,du+q_1\,dv, \qquad -p\,du-p_1\,dv, \qquad 0$$

for the projections of the displacement of the point m onto the axes of the trihedron (T_1), or – what amounts to the same thing – onto those of the trihedron (T).

As a result, if we let $d\sigma$ denote the arc length of the curve that is described by the point m and let θ denote the angle that the arc makes with the x-axis of the trihedron (T) then we will have:

$$(3) \qquad d\sigma \cos\theta = q\,du + q_1\,dv, \qquad d\sigma \sin\theta = -(p\,du + p_1\,dv).$$

The line element of the sphere on which one performs the representation of the surface will then have the value:

$$(4) \qquad ds^2 = (p\,du + p_1\,dv)^2 + (q\,du + q_1\,dv)^2.$$

Finally, the angle $\omega - \theta$ between a curve that is traced on a surface and its spherical representation will be determined by one or the other of the two equations:

$$(5) \qquad \begin{cases} d\sigma \sin(\omega - \theta) = (p\,du + p_1 dv)\cos\omega + (q\,du + q_1 dv)\sin\omega, \\ d\sigma \cos(\omega - \theta) = (q\,du + q_1 dv)\cos\omega - (p\,du + p_1 dv)\sin\omega. \end{cases}$$

These formulas will be very useful to us. We shall now answer some of the most important questions that present themselves in the applications.

486. We first propose to establish the relation that must exist between two conjugate tangents. If the point M of the surface describes a curve then one will obtain the conjugate of the tangent to that curve by taking the intersection of the tangent plane at M with the tangent plane at an infinitely-close point of the curve; in other words, the conjugate is the characteristic of the tangent plane under the motion of the trihedron. It is the locus of points in that plane whose velocity is directed in that plane. Formulas (B) give the components of that velocity. If one writes down the idea that the component that relates to Mz is zero then one will obtain the equation:

$$(p\,du + p_1\,dv)\,y - (q\,du + q_1\,dv)\,x = 0,$$

which represents the conjugate tangent. Let ω' denote the angle that it makes with the x-axis of the trihedron (T). x and y will be proportional to $\cos\omega'$, $\sin\omega'$, and the preceding equation will become:

$$(6) \qquad (p\,du + p_1\,dv)\sin\omega' - (q\,du + q_1\,dv)\cos\omega' = 0.$$

Let δ denote the differentials that relate to a displacement along the conjugate direction. One will have:

$$\delta s \cos\omega' = \xi\,du + \xi_1\,\delta v, \qquad \delta s \sin\omega' = \eta\,du + \eta_1\,\delta v.$$

Upon substituting those values for sin ω', cos ω' in the equation that we just found, we will find:

$$(7)\qquad \begin{cases} (p\,\eta - q\,\xi)\,du\,\delta u + (p_1\eta_1 - q_1\xi_1)\,dv\,\delta v \\ +(p\eta_1 - q\xi_1)\,du\,\delta v + (p_1\,\eta - q_1\,\xi)\,\delta u\,dv = 0. \end{cases}$$

That relation is, as was to be expected, perfectly symmetric with respect to the differentials d, δ, because the coefficients of $du\,\delta v$ and $\delta u\,dv$ are equal by virtue of the last of formulas (A). It follows from this that relation (6) can also be written in the following form:

$$(6')\qquad (p\,\delta u + p_1\,\delta v)\sin\omega - (q\,\delta u + q_1\,\delta v)\cos\omega = 0.$$

One can then establish a relation between two conjugate tangents as follows: One deduces from the preceding formulas (3):

$$(8)\qquad d\sigma\cos(\omega' - \theta) = (q\,du + q_1\,dv)\cos\omega' - (p\,du + p_1\,dv)\sin\omega'.$$

Now, if the two directions that are defined by the angles ω, ω' are conjugate then one will have, from a property that was proved already [I, pp. 174]:

$$\omega' - \theta = \frac{\pi}{2}.$$

Upon introducing that hypothesis into equation (8), one will again be led to the relation (6).

487. If one supposes that the two conjugate directions coincide then one must replace δ with d and ω with ω' everywhere, and one will get the differential equation for the asymptotic lines in that following two forms:

$$(9)\qquad \begin{cases} (p\eta - q\xi)\,du^2 + (p\eta_1 - q\xi_1 + p_1\eta - q_1\xi)\,du\,dv + (p_1\eta_1 - q_1\xi_1)\,dv^2 = 0, \\ (p\,du + p_1\,dv)\sin\omega - (p\,du + p_1\,dv)\cos\omega = 0. \end{cases}$$

Upon comparing the second of these equations to one of formulas (5), one will immediately recognize a characteristic property of asymptotic lines:

At each point, they define a right angle with the corresponding element of their spherical representation.

As we shall see later on, the second equation in (9) also expresses the idea that the osculating plane to the asymptotic line is tangent to the surface.

488. We now look for the differential equation of the lines of curvature. One obtains all of the essential properties that relate to those lines by assuming various viewpoints that we shall examine in succession.

One can first look for the displacement of the moving trihedron for which the normal to the surface (which is the z-axis of the trihedron) generates a developable surface.

In order for that to be true, it is necessary that there must exist a variable point:

$$x = 0, \quad y = 0, \quad z = \rho$$

on the z-axis of the moving trihedron that describes a curve that is constantly tangent to that axis during the motion in question. Now, from formulas (B), the projections of the displacement of that point when u and v take on increments du, dv are:

$$\begin{gathered} \xi\, du + \xi_1\, dv + (q\, du + q_1\, dv)\, \rho, \\ \eta\, du + \eta_1\, dv - (p\, du + p_1\, dv)\, \rho, \\ d\rho\,. \end{gathered}$$

In order that the curve that is described should be tangent to the z-axis, it is necessary and sufficient that the first two projections should be zero:

$$(10) \qquad \left\{ \begin{aligned} \xi\, du + \xi_1 dv + \rho\,(q\, du + q_1 dv) &= 0, \\ \eta\, du + \eta_1 dv - \rho\,(p\, du + p_1 dv) &= 0. \end{aligned} \right.$$

These two equations exhibit both du / dv and ρ. The last quantity is obviously the radius of principal curvature that corresponds to the line of curvature considered.

If one eliminates ρ then one will obtain the differential equation:

$$(11) \qquad (p\, du + p_1\, dv)(\xi\, du + \xi_1\, dv) + (q\, du + q_1\, dv)(\eta\, du + \eta_1\, dv) = 0,$$

which characterizes the two lines of curvature. One can give them the following form:

$$(11a) \qquad (p\, du + p_1\, dv) \cos \omega + (q\, du + q_1\, dv) \sin \omega = 0,$$

which, when compared with formulas (5), will show that:

The tangents to a line of curvature and its spherical image are parallel.

On the contrary, if one eliminates du / dv then one will obtain the equation of the radii of principal curvature:

$$(12) \qquad \rho^2\, (pq_1 - qp_1) + \rho\,(q\, \eta_1 - q_1\, \eta - \xi\, p_1 + \xi_1\, p) + \xi\eta_1 - \eta\xi_1 = 0.$$

489. One again recovers the lines of curvature by studying one of the fundamental questions that relate to the displacement of the trihedron (T). We have seen that among the infinitely-small motions that are produced by starting with a given position, there are two of them (which can be either real or imaginary) that reduce to rotations. The value of du / dv and the axis of rotation that relates to those motions are defined by equations (6) [I, pp. 61], which reduce to the following ones here:

$$(13) \qquad \begin{cases} \xi\, du + \xi_1 dv + (q\, du + q_1\, dv)\, z - (r\, du + r_1\, dv)\, y = 0, \\ \eta\, du + \eta_1 dv + (r\, du + r_1\, dv)\, x - (p\, du + p_1 dv)\, z = 0, \\ \qquad\qquad (p\, du + p_1 dv)\, y - (q\, du + q_1 dv)\, x = 0. \end{cases}$$

One first deduces the equation:

$$(p\, du + p_1\, dv)(\xi\, du + \xi_1\, dv) + (q\, du + q_1\, dv)(\eta\, du + \eta_1\, dv) = 0,$$

which define the values of du / dv that correspond to the two rotations. Now, in the case that we are dealing with, the preceding equation is that of the lines of curvature.

Moreover, the axis of rotation that relates to each line of curvature will meet the normal to the surface at the corresponding center of the curvature. That will result from a comparison of formulas (10) and (13). Upon combining these results, we can state the following proposition:

Under the displacement of the trihedron (T) *that is linked to the surface, the infinitely-small motions that reduce to rotations are always real. They correspond to the displacements of the origin that are performed along the lines of curvature of the surface. In addition, the axes that correspond to those rotations (which are obviously situated in the normal plane to each line of curvature) must pass through the corresponding center of principal curvature.*

490. It now remains for us to study the properties of an arbitrary curve that is traced on the surface. We have already obtained the formulas that relate to the tangent:

$$(14) \qquad \begin{cases} \cos\omega = \dfrac{\xi\, du + \xi_1 dv}{ds}, \\[2mm] \sin\omega = \dfrac{\eta\, du + \eta_1 dv}{ds}. \end{cases}$$

We shall now indicate the ones that are concerned with the principal normal.

We know [I, pp. 9] that if one draws a parallel to the tangent to the curve through a fixed point in space that has a length equal to unity then upon supposing that the arc length of the curve is equal to time, the velocity of the extremity to that parallel will be equal in magnitude to the curvature $1 / \rho$ of the curve and will have the direction and sense of the principal normal.

Now, if one draws parallels to the edges of the moving trihedron through the fixed point then one will once more define the trihedron (T_1) that was defined already whose rotations will be:

$$\frac{p\,du + p_1 dv}{ds}, \quad \frac{q\,du + q_1 dv}{ds}, \quad \frac{r\,du + r_1 dv}{ds}$$

when one displaces the curve.

The extremity of the parallel to the tangent will have the relative coordinates:

$$\cos\omega, \ \sin\omega, \ 0.$$

Upon applying formulas (4) [I, pp. 43], which give the projections of the velocity onto the moving axes, one will obtain the formulas:

$$(15)\qquad \begin{cases} \dfrac{ds}{\rho}\cos\xi' = -\sin\omega(d\omega + r\,du + r_1\,dv), \\ \dfrac{ds}{\rho}\cos\eta' = +\cos\omega(d\omega + r\,du + r_1\,dv), \\ \dfrac{ds}{\rho}\cos\zeta' = +\sin\omega(p\,du + p_1\,dv) - \cos\omega(q\,du + q_1 dv), \end{cases}$$

in which ξ', η', ζ' denote the angles between the principal normal and the *x*, *y*, and *z*-axes of the trihedron (T_1) or the trihedron (T).

These relations prove that one can take:

$$(16)\qquad \cos\xi' = -\sin\omega\sin\varpi, \qquad \cos\eta' = \cos\omega\sin\varpi, \qquad \cos\zeta' = \cos\varpi,$$

in which ϖ denotes the angle between the normal to the surface and osculating plane of the curve, and formulas (15) can be replaced by the following two:

$$(17)\qquad \frac{ds\cos\varpi}{\rho} = \sin\omega(p\,du + p_1\,dv) - \cos\omega(q\,du + q_1\,dv),$$

$$(18)\qquad \frac{ds\sin\varpi}{\rho} = d\omega + r\,du + r_1\,dv\,.$$

Those formulas merit some remarks:

The first formula shows immediately that $\cos\varpi / \rho$ remains the same for all curves that have the same tangent. We then recover Meusnier's theorem, and we see that our first formula gives what one can call the *normal curvature* – i.e., the curvature of the normal section to the tangent to the curve.

As for the second formula, it defines an element that, as we will see, plays an important role in the theory of the deformation of surfaces. Consider the cylinder that projects the curve onto the tangent plane. From Meusnier's theorem, $\sin \varpi / \rho$ will be the curvature of the normal section of the cylinder that is tangent to the curve – i.e., the curvature of the projection of the curve onto the tangent plane.

Liouville, who addressed the subject after O. Bonnet, gave it the name of *geodesic curvature* ([1]), which is accepted by all geometers.

We call the center of curvature of the tangent normal plane section of the curve the *center of normal curvature* and the center of curvature of the projection of the curve onto the tangent plane the *center of geodesic curvature*.

From Meusnier's theorem, the two centers are found on the axis of the osculating circle of the curve considered.

One knows that one calls any line whose osculating plane is normal to the surface at each point a *geodesic line*. The differential equation of the geodesic lines is then:

$$d\omega + r\,du + r_1\,dv = 0. \tag{19}$$

491. Formula (17) permits us to obtain the differential equation of the lines of curvature in a new manner. Indeed, one knows that these lines are tangent to the normal sections of greatest or least curvature. They are then determined by the equation:

$$\frac{\partial}{\partial\omega}\left(\frac{\cos\varpi}{\rho}\right) = 0,$$

in which one regards $\cos \varpi / \rho$ as a function of u, v, and ω. Now, one has:

$$\frac{\cos\varpi}{\rho} = \sin\,\omega\,\frac{p\,du + p_1 dv}{ds} - \cos\,\omega\,\frac{q\,du + q_1 dv}{ds}.$$

The expressions for $\frac{du}{ds}, \frac{dv}{ds}$ as functions ω are deduced from formulas that were given already (14). One will have:

$$\frac{du}{ds} = \frac{\eta_1 \cos\omega - \xi_1 \sin\omega}{\eta_1\xi - \xi_1\eta}, \qquad \frac{dv}{ds} = \frac{\xi \sin\omega - \eta\cos\omega}{\eta_1\xi - \xi_1\eta}.$$

Upon making use of these equations in order to calculate the derivatives of $\frac{du}{ds}, \frac{dv}{ds}$, when they are considered to be functions of only the variable ω, and subtracting the quantity:

$$(p\,\eta_1 - p_1\,\eta - q\,\xi_1 + q_1\,\xi)\,(\sin^2\omega + \cos^2\omega)$$

([1]) O. BONNET, "Mémoire sur la théorie générale des surfaces," Journal de l'École Polytechnique **32** (1848), pp. 1. Presented to the Academy of Sciences in 1844.

J. Liouville, "Sur la théorie générale des surfaces," Journal de Liouville **16** (1851), pp. 130.

[which is zero, by virtue of the last of equations (A)] from the right-hand side after derivation, we will obtain the identity:

$$\frac{\partial}{\partial\omega}\left(\frac{\cos\varpi}{\rho}\right) = 2\cos\omega\,\frac{p\,du + p_1 dv}{ds} + 2\sin\omega\,\frac{q\,du + q_1 dv}{ds}, \tag{20}$$

which we will have to make use of. Upon equating the right-hand side to zero, we will indeed recover the differential equation of the lines of curvature.

492. We shall now pass on to the third-order elements. We first remark that the angles λ', μ', ν' between the binormal and the axes of the trihedron (T) are known, since we have already determined them from the tangent to the curve and the principal normal. Upon applying formulas (1) [I, pp. 2], one will have:

$$\cos\lambda' = \sin\omega\cos\varpi, \qquad \cos\mu' = -\cos\omega\cos\varpi, \qquad \cos\nu' = \sin\varpi. \tag{21}$$

We also know [I, pp. 9] that if we draw a line of length equal to 1 through a fixed point that is parallel to the binormal then the extremity of that line will have a displacement that is equal to $ds\,/\,\tau$ when one displaces it along the curve, and the direction of that displacement will be that of the principal normal. We then recall the trihedron (T_1) that was considered already that has its origin at the fixed point and is parallel to the trihedron (T). The extremity of the parallel to the binormal that is drawn through the origin will have the coordinates:

$$\sin\omega\cos\varpi, \quad -\cos\omega\cos\varpi, \quad \sin\varpi,$$

and the projections of the velocity of that point onto the moving axes must be:

$$\frac{-\sin\omega\sin\varpi}{\tau}, \quad \frac{\cos\omega\sin\varpi}{\tau}, \quad \frac{\cos\varpi}{\tau}$$

when one takes into account the values that were given already for the direction cosines of the principal normal.

Upon applying any of the formulas (4) [I, pp. 43] that give the projection of the velocity (the last one, for example), one will get:

$$\frac{\cos\varpi}{\rho} = \cos\varpi\,\frac{d\varpi}{ds} - \frac{p\,du + p_1 dv}{ds}\cos\varpi\cos\omega - \frac{q\,du + q_1 dv}{ds}\cos\varpi\sin\omega,$$

and upon dividing by $\cos\varpi$:

$$\frac{1}{\tau} - \frac{d\varpi}{ds} = -\frac{p\,du + p_1 dv}{ds}\cos\omega - \frac{q\,du + q_1 dv}{ds}\sin\omega. \tag{22}$$

One sees that:

The left-hand side remains the same for all curves that have the same tangent.

This important result is due to O. Bonnet. One can further give the preceding equation the following form:

(23) $$\frac{1}{\tau} - \frac{d\varpi}{ds} = -\frac{1}{2}\frac{\partial}{\partial\omega}\left(\frac{\cos\varpi}{\rho}\right),$$

in which the derivative with respect to ω will have the same significance as in equation (20).

493. In order to obtain everything that refers to order three, one must know $d\rho / ds$. In order to do that, we differentiate the formula that gives $\cos \varpi / \rho$.

Since that quantity is always considered to be a function of u, v, and ω, we will have:

$$d\left(\frac{\cos\varpi}{\rho}\right) = \frac{\partial}{\partial u}\left(\frac{\cos\varpi}{\rho}\right)du + \frac{\partial}{\partial v}\left(\frac{\cos\varpi}{\rho}\right)dv + \frac{\partial}{\partial\omega}\left(\frac{\cos\varpi}{\rho}\right)d\omega,$$

or, upon replacing $d\omega$ with its value:

$$\frac{\sin\varpi\, ds}{\rho} - r\, du - r_1\, dv,$$

which is deduced from the formula (18), we will have:

$$d\left(\frac{\cos\varpi}{\rho}\right) - \frac{\partial}{\partial\omega}\left(\frac{\cos\varpi}{\rho}\right)\frac{\sin\varpi\, ds}{\rho} = \left[\frac{\partial}{\partial u}\left(\frac{\cos\varpi}{\rho}\right) - r\frac{\partial}{\partial\omega}\left(\frac{\cos\varpi}{\rho}\right)\right] du + \left[\frac{\partial}{\partial v}\left(\frac{\cos\varpi}{\rho}\right) - r_1\frac{\partial}{\partial\omega}\left(\frac{\cos\varpi}{\rho}\right)\right] dv.$$

One sees that the right-hand side can be written in the form:

(25) $$\left[\frac{\partial}{\partial u}\left(\frac{\cos\varpi}{\rho}\right) - r\frac{\partial}{\partial\omega}\left(\frac{\cos\varpi}{\rho}\right)\right] du + \left[\frac{\partial}{\partial v}\left(\frac{\cos\varpi}{\rho}\right) - r_1\frac{\partial}{\partial\omega}\left(\frac{\cos\varpi}{\rho}\right)\right] dv = K\, ds,$$

in which K depends upon only the direction of the tangent to the curve. As for the left-hand side, if one replaces $\frac{\partial}{\partial\omega}\left(\frac{\cos\varpi}{\rho}\right)$ with its value that is inferred from formula (23) then it will contain only quantities that have a simple geometric significance, and one will obtain the equation:

(25) $$d\left(\frac{\cos\varpi}{\rho}\right)+\frac{2\sin\varpi}{\rho}\left(\frac{ds}{\tau}-d\varpi\right) = K\,ds,$$

which will obviously permit one to calculate $d\rho / ds$.

Upon dividing the two sides of equation (25) by $\cos \varpi / \rho$, one will have:

(26) $$-\frac{1}{\rho}\frac{d\rho}{ds} + \tan v\left(\frac{2}{\tau}-3\frac{d\varpi}{ds}\right) = \frac{K\rho}{\cos\varpi}.$$

Since K and $\rho / \cos \varpi$ do not depend upon any second-order elements, but only on the direction of the tangent to the curve, one will now see that the left-hand side of equation (26), and also that of equation (25) divided by ds, will remain the same for two curves that have the same tangent at the given point, even though the elements that they depend upon have orders two and three. We need to develop a consequence of that result that is due to Laguerre ([2]).

In order to conclude the discussion of third-order topics, we shall determine the center of the osculating sphere. Upon applying the known formulas, and upon denoting the coordinates of the center of that sphere by x_0, y_0, z_0, we will find that:

(27) $$\begin{cases} x_0\cos\omega + y_0\sin\omega = 0, \\ \sin\varpi(-x_0\sin\omega + y_0\cos\omega) + z_0\cos\varpi = \rho, \\ \cos\varpi(\quad x_0\sin\omega - y_0\cos\omega) + z_0\sin\varpi = -\tau\dfrac{d\rho}{ds}, \end{cases}$$

so we will deduce that:

(28) $$\begin{cases} \dfrac{-x_0}{\sin\omega} = \dfrac{y_0}{\cos\omega} = \rho\sin\varpi + \tau\dfrac{\partial\rho}{\partial s}\cos\varpi, \\ z_0 = \rho\cos\varpi - \tau\dfrac{\partial\rho}{\partial s}\sin\varpi. \end{cases}$$

The first two equations (27) represent the axis of the osculating circle. Upon taking the intersection of that axis with either the normal plane or the tangent plane, one will have:

1. The center of normal curvature:

(29) $$x_1 = y_1 = 0, \qquad z_1 = \frac{\rho}{\cos\varpi}.$$

([2]) LAGUERRE, "Sur une propriété relative aux courbes tracées sur une surface quelconque," Bulletin de la Société philomathematique **7** (1870), pp. 49.

2. The center of geodesic curvature:

$$(30) \qquad x_2 = -\frac{\rho \sin \omega}{\sin \varpi}, \qquad y_2 = \frac{\rho \cos \omega}{\sin \varpi}, \qquad z_2 = 0.$$

494. After third order, there no longer remain any geometric elements to calculate. The derivatives of the elements ρ, ϖ, and τ are obtained by the simple differentiation of the preceding formulas. It nonetheless seems good for us to remark that if one so desires then one can write two formulas for the differential elements of arbitrary order that are analogous to relations (23) and (25). Indeed, suppose that one has an equation of the form:

$$\Phi = K,$$

in which Φ contains the differential elements of the curve up to order n, and K is a function of only u, v, and ω. The differentiation of that equation will give us:

$$d\Phi = \frac{\partial K}{\partial u} du + \frac{\partial K}{\partial v} dv + \frac{\partial K}{\partial \omega} d\omega,$$

and one will deduce from this that:

$$\frac{d\Phi}{ds} - \frac{\partial K}{\partial \omega} \frac{\sin \varpi}{\rho} = \frac{\partial K}{\partial u} \frac{du}{ds} + \frac{\partial K}{\partial v} \frac{dv}{ds} - \frac{\partial K}{\partial \omega} \left(r \frac{du}{ds} + r_1 \frac{dv}{ds} \right).$$

That formula will preserve the form of the one from which one has deduced it: The right-hand side will depend upon only u, v, and ω, but the left-hand side will contain differential elements of the curve up to order $n + 1$.

If one applies that method to the two formulas (23) and (25) then one will deduce two new equations by referring to the fourth-order elements, and one can continue in that way up to an arbitrary order.

For example, one can deduce the following formula from (23), which we shall be content to merely state:

$$\frac{d}{ds}\left(\frac{1}{\tau} - \frac{d\varpi}{ds} \right) - \frac{\sin \varpi}{\rho} \left(\frac{2 \cos \varpi}{\rho} - \frac{1}{R} - \frac{1}{R'} \right) = K_1,$$

in which K_1 will remain the same for two curves that have the same tangent, and R, R' denote the radii of principal curvature of the surface.

CHAPTER II

THE CODAZZI FORMULAS

Formulas that relate to oblique coordinates, whose line element is determined by the equation:

$$ds^2 = A^2\, du^2 + C^2\, dv^2 + 2AC \cos \alpha\, du\, dv.$$

Angle between two curves. – Condition for two directions to be conjugate. – Asymptotic lines. – Lines of curvature. – Gauss's theorem. – Total and mean curvature. – Rectangular curvilinear coordinates. Codazzi formulas. – Special study that relates to the coordinate system that is defined by the lines of curvature. – Application of the general method to the coordinate system that is defined by the lines of null length. – Determination of the quantities p, q, r, p_1, q_1, r_1, ξ, η, ξ_1, η_1 when one knows the expressions for the rectangular coordinates x, y, z as functions of the two parameters u, v. – Application to an ellipsoid that one assumes to be referred to its lines of curvature.

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495. Up to now, we have been content to suppose that the z-axis of the trihedron (T) is the normal to the surface. In the questions that involve the line element of the surface, it is important to define the relationship between the trihedron and the surface in a more precise manner in order to recognize the quantities that will remain invariable when one deforms the surface.

In truth, one can arrive at that result with the preceding notations. It will suffice to remark that if the surface is deformed while carrying the trihedron (T) with it then the translations ξ, η, ξ_1, η_1 will remain invariable, and consequently the rotations r, r_1, as well, by virtue of the fourth and fifth of formulas (A). One thus recognizes immediately that the geodesic curvature of an arbitrary curve and the product of the radii of principal curvature at each point of the surface will preserve their values after any deformation of the surface. However, those results, and others, will take on a neater form if one starts with a form for the line element that is given *a priori*. We shall then consider the various coordinate systems successively.

First, suppose that the points of the surface are referred to oblique coordinates for which the line element takes the form:

$$ds^2 = A^2\, du^2 + C^2\, dv^2 + 2\, AC \cos \alpha\, du\, dv. \tag{1}$$

In order to succeed in defining the position of the trihedron (T), we give the angle m that the x-axis makes with the tangent to the curve v = const. – i.e., with the infinitely-small arc $A\, du$. If one similarly lets n denote the angle that the same axis makes with the infinitely-small arc $C\, dv$ then one will obviously have:

$$n - m \equiv \alpha.$$

Since the angle α intervenes in the line element only by its cosines, one can take:

$$n - m \equiv \alpha. \tag{2}$$

It is easy to understand why we do not give a particular value to the angle m. In the case of rectangular coordinates, it would be natural to make the x and y axes of the moving trihedron coincide with the tangents to the coordinate curves. However, if those curves do not cut at a right angle then geometry will not single out any special position for the axes of the moving trihedron. To make one of them coincide with one of the tangents to the coordinates would destroy the symmetry that must exist in the formulas between the two variables u and v. True, that symmetry would be preserved if one takes the axes to be the bisectors of the tangents to the coordinate curves. However, the choice will have the inconvenience of not coinciding with the one that is most natural when the coordinates become rectangular. It then seems preferable to us to preserve that arbitrary m, but give it the value that would be the most advantageous in the study of that question.

When only u varies, the origin of the trihedron will describe an arc $A\,du$ in the tangent plane that makes an angle m with the x-axis. One will then have:

$$\xi = A\cos m, \qquad \eta = A\sin m, \tag{3}$$

and similarly:

$$\xi_1 = C\cos m, \qquad \eta_1 = C\sin m. \tag{4}$$

Introduce those values for the translations into the formulas of the preceding chapter. System (A) will then take on the form:

$$(A')\qquad \left\{\begin{aligned}
&\frac{\partial p}{\partial v}-\frac{\partial p_1}{\partial u}=qr_1-rq_1,\\
&\frac{\partial q}{\partial v}-\frac{\partial q_1}{\partial u}=rp_1-pr_1,\\
&\frac{\partial r}{\partial v}-\frac{\partial r_1}{\partial u}=pq_1-qp_1,\\
&r=-\frac{\partial n}{\partial u}-\frac{1}{C\sin\alpha}\left(\frac{\partial A}{\partial v}-\frac{\partial C}{\partial u}\cos\alpha\right),\\
&r_1=-\frac{\partial m}{\partial u}+\frac{1}{A\sin\alpha}\left(\frac{\partial C}{\partial u}-\frac{\partial A}{\partial v}\cos\alpha\right),\\
&A(p_1\sin m-q_1\cos m)=C(p\sin n-q\cos n).
\end{aligned}\right.$$

The fourth and fifth equation have been solved with respect to r and r_1.

496. The angle ω between the tangent to a curve that is traced on a surface and the differential ds of arc length of that curve will now be defined by the formulas:

$$(5)\qquad \left\{\begin{aligned}
&ds\cos\omega=A\cos m\,du+C\cos n\,dv,\\
&ds\sin\omega=A\sin m\,du+C\sin n\,dv,
\end{aligned}\right.$$

which will give:

$$(6)\qquad A\,\frac{du}{ds}=\frac{\sin(n-\omega)}{\sin\alpha}, \qquad\qquad C\,\frac{dv}{ds}=\frac{\sin(\omega-m)}{\sin\alpha}.$$

From that, if one considers two different curves that pass through the same point of the surface, and if one lets the letter δ denote the differentials that relate to the second curve and lets ω' denote the angle that is analogous to ω then the angle between the two curves will be given by the formulas:

$$(7)\qquad \begin{cases} \cos(\omega-\omega')=\dfrac{A^2du\,\delta u+AC\cos\alpha(du\,\delta v+dv\,\delta u)+C^2dv\,\delta v}{ds\,\delta s}, \\ \sin(\omega-\omega')=\dfrac{AC\sin\alpha(dv\,\delta u-du\,\delta v)}{ds\,\delta s}. \end{cases}$$

As one sees, that angle depends upon only the expression for the line element. Consequently, it will not change when one deforms the surface. That result was pointed out already (no. **119**).

The condition for the two directions to be conjugate will become:

$$(8)\qquad \begin{cases} A(q\cos m-p\sin m)\,du\,\delta u+C(q_1\cos n-p_1\sin n)\,dv\,\delta v \\ \quad +A(q_1\cos m-p_1\sin m)\,\delta u\,dv+C(q\cos n-p\sin n)\,dv\,\delta v=0, \end{cases}$$

here.

Consequently, the differential equation of the asymptotic lines will be:

$$(9)\qquad \begin{cases} A(q\cos m-p\sin m)\,du^2+C(q_1\cos n-p_1\sin n)\,dv^2 \\ \quad +[A(q_1\cos m-p_1\sin m)+C(q\cos n-p\sin n)]\,dv\,du=0. \end{cases}$$

Finally, the two equations that define the lines of curvature will become:

$$(10)\qquad \begin{cases} A\cos m\,du+C\cos n\,dv+(q\,du+q_1dv)\rho=0, \\ A\sin m\,du+C\sin n\,dv-(p\,du+p_1dv)\rho=0. \end{cases}$$

The differential equation of those developed lines can be written:

$$(11)\qquad \begin{cases} A(p\cos m+q\sin m)\,du^2+C(p_1\cos n+q_1\sin n)\,dv^2 \\ \quad +[C(p\cos n+q\sin n)+A(p_1\cos m+q_1\sin m)]\,du\,dv=0. \end{cases}$$

However, we must above all insist upon the new form that the second-degree equation that determines the radii of principal curvature will take. Here, it will become:

$$(12)\qquad \rho^2\,(pq_1-qp_1)-\rho\,[A\,(p_1\cos m+q_1\sin m)-C\,(p\cos n+q\sin n)]+AC\sin\alpha=0.$$

Consequently, if one lets R, R' denote the two principal radii of curvature then one will deduce that:

$$AC \sin a \left(\frac{1}{R}+\frac{1}{R'}\right) = A\,(p_1 \cos m + q_1 \sin m) - C\,(p \cos n + q \sin n), \tag{13}$$

$$\frac{AC\sin\alpha}{RR'} = pq_1 - qp_1 . \tag{14}$$

Replace $pq_1 - qp_1$ with its expression that is deduced from the third of formulas (A'); it will become:

$$\frac{AC\sin\alpha}{RR'} = \frac{\partial r}{\partial v} - \frac{\partial r_1}{\partial u}, \tag{15}$$

or, upon replacing r and r_1 with their values:

$$\frac{AC\sin\alpha}{RR'} = -\frac{\partial^2\alpha}{\partial u\,\partial v} - \frac{\partial}{\partial u}\left[\frac{\dfrac{\partial C}{\partial u} - \dfrac{\partial A}{\partial v}\cos\alpha}{A\sin\alpha}\right] - \frac{\partial}{\partial v}\left[\frac{\dfrac{\partial A}{\partial v} - \dfrac{\partial C}{\partial u}\cos\alpha}{C\sin\alpha}\right]. \tag{16}$$

That formula immediately gives the beautiful theorem of Gauss: The product of the radii of principal curvature depends upon only the expression for the line element and will persist when the surface is deformed without tearing or folding.

497. The expression $\dfrac{1}{RR'}$ has been given the name of *total curvature* of the surface; the name of *mean curvature* was given to the sum $\dfrac{1}{2}\left(\dfrac{1}{R}+\dfrac{1}{R'}\right)$.

Some papers have been written on the search for two quantities that could serve to measure the curvature of a surface at a given point. The geometers who treated that subject did not glimpse the fact that it revives, in a different form, the celebrated question of *vis viva*, and that it raises a question that must be resolved by a definition of the word. At best, one can attempt to reason by analogy by examining the properties that relate to the curvature of the planar lines that are susceptible to being generalized in the theory of surfaces. If all of those generalizations referred to, for example, the quantity that we have called the *total curvature* then the geometers would have some reason to reserve the name of *curvature* for that element. However, that indirect means of resolution avoids the question completely. Among the properties that relate to curvature in the planar lines, the ones that admit a generalization are the ones in which one employs the total curvature. On the contrary, for other ones, one must appeal to the mean curvature. Some of them even admit different generalizations in which one sometimes employs the mean curvature and sometimes the total curvature.

Following Gauss, we shall first show that one can adopt a definition of total curvature that is completely analogous to that of curvature for planar lines.

If one is given an arc of a planar curve, and if one draws parallels to the normals to the extremities of that arc through the center of a circle of radius 1 then those parallels will intercept an arc of the circle that is equal to the angle between the tangents to the two extremities of the curve, and which consequently measures what one calls the *curvature* of the arc of the curve.

Similarly, if one considers a region of the surface that is limited by a closed curve, and one draws parallels to the normals to the surface at all points of a limited curve through the center of a sphere of radius 1 then those parallels will cut the sphere along a likewise-closed curve. There will be a portion of the sphere that is limited by that curve and which will contains all of the points of the sphere that correspond to the various points of the segment of the surface considered. The area of that portion of the sphere will be called *total curvature of the segment of the surface:*

Return to the planar curve. If one divides the total curvature of an arc of the curve by the length of that arc then one will prove that the quotient will tend to a finite, well-defined limit when the area diminishes indefinitely and reduces to a point; by definition, that limit will be the curvature at that point.

If one likewise envisions a segment of the surface around a point M of the surface, and if one supposes that the extent of that segment diminishes independently, and in all senses, around the point M then the total curvature of the segment will diminish indefinitely; however, if one divides it by the area of that region then, as we shall prove, the quotient that one obtains will tend to a finite, well-defined limit that is independent of the form of the segment. That limit is $1 / RR'$; i.e., the element to which we gave the name of *total curvature at the point M.*

If we take the preceding viewpoint then it will seem that the analogy is complete between the curvature of curves and the total curvature of surfaces. However, one can point out some other propositions in which that analogy breaks down, and that one can generalize by replacing the curvature of a planar line with the mean curvature of the surface and no longer the total curvature.

For example, imagine that one measures out infinitely-small lengths along the normals to a curve in such a manner as to obtain a neighboring curve. If h denotes the length that is measured out on each normal then the increase in length when one passes from the first curve to the second one will be represented by the integral:

$$\int h \frac{ds}{\rho},$$

in which ds denotes the differential of the arc length, and ρ denotes the radius of curvature.

If one operates similarly on a portion of the surface then the increase in area when one passes to the infinitely-close surface will be (no. **185**):

$$\iint h \left(\frac{1}{R} + \frac{1}{R'} \right) d\sigma,$$

in which $d\sigma$ denotes the area element, and R, R' denote the radii of principal curvature. As one sees, the element that substitutes for the curvature in the generalized proposition is no longer the total curvature here; it is twice the mean curvature.

It is pointless to insist upon that example and on other ones that one can invoke. One can say that the total curvature is the most important kind in geometry. Since it depends upon only the line element, it will intervene in all questions that relate to the deformation of surfaces. On the contrary, in mathematical physics, it is the mean curvature that seems to play the dominant role.

498. It now remains for us to make the definition of total curvature more precise and to prove the proposition of Gauss that we stated above.

Let M be a point on the surface, and let M' be the corresponding point of the sphere of radius 1 on which one performs the representation. Draw parallels to the axes of the trihedron (T) through M'. One will then have a trihedron (T') whose rotations will obviously be the same as those of the trihedron (T), and which will play the same role with respect to the sphere that the trihedron (T) does with respect to the surface, because its z-axis will be the normal to the sphere. Let:

$$ds^2 = A'^2\, du^2 + 2\, A'\, C' \cos \alpha'\, du\, dv + C'^2\, dv^2$$

be the expression for the line element of the sphere. The surface element will then have the value:

$$A'\, C' \sin \alpha'\, du\, dv$$

in both magnitude and sign. Having said that, apply formula (14) to the sphere.

Since the rotations of the trihedron (T') are the same as those of the trihedron (T), we will have:

$$A'\, C' \sin \alpha' = pq_1 - qp_1,$$

and as a result, upon taking formula (14) into account:

$$A'\, C' \sin \alpha' = \frac{AC \sin \alpha}{RR'}.$$

Hence, the total curvature of a portion of the surface, which from Gauss's definition itself is represented by the integral:

$$\iint A'\, C' \sin \alpha'\, du\, dv,$$

which is extended over that portion of the surface, will also be represented by the integral:

$$\iint \frac{AC \sin \alpha}{RR'}\, du\, dv,$$

which is extended over the same region.

Hence, if one divides the total curvature of a segment of a surface by the area of that segment then the quotient will be:

$$\frac{\iint \frac{AC\sin\alpha}{RR'}du\,dv}{\iint AC\sin\alpha\,du\,dv}.$$

In order to suppress any difficulty that relates to the choice of coordinates, let $d\sigma$ denote the surface element and write the preceding quotient in the form:

$$\frac{\iint \frac{d\sigma}{RR'}}{\iint d\sigma}.$$

It is obvious that it will have the value:

$$\left(\frac{1}{RR'}\right)_0,$$

in which $\left(\frac{1}{RR'}\right)_0$ denotes a mean over all values of $\frac{1}{RR'}$ in the interior of the segment considered. If the extent of that segment diminishes in such a manner that the distances from all of its points to a point M in its interior tend to zero then one will see that the preceding ratio will have the total curvature of the surface at that point for its limit. When the segment reduces, not to a point, but to a given line by the reduction of one of its two dimensions, contrary to hypothesis, that ratio will have no well-defined limit.

499. The general formulas simplify greatly in the case where the chosen curvilinear coordinates are rectangular. One can then make the x-axis of the trihedron (T) coincide with the tangent to the $A\,du$ – i.e., with the tangent to the curve v = const. That will give:

$$n = \frac{\pi}{2}, \qquad m = 0, \qquad \alpha = \frac{\pi}{2}.$$

Formulas (A') will take the much simpler form:

$$(A'') \qquad \begin{cases} Aq_1 + Cp = 0, & \dfrac{\partial p}{\partial v} - \dfrac{\partial p_1}{\partial u} = qr_1 - rq_1, \\ r = -\dfrac{1}{C}\dfrac{\partial A}{\partial v}, & \dfrac{\partial q}{\partial v} - \dfrac{\partial q_1}{\partial u} = rp_1 - pr_1, \\ r_1 = \dfrac{1}{A}\dfrac{\partial C}{\partial v}, & \dfrac{\partial r}{\partial v} - \dfrac{\partial r_1}{\partial u} = pq_1 - qr_1. \end{cases}$$

Up to notations, they coincide with the one that were given for the first time by D. Codazzi ([3]).

([3]) CODAZZI (D.), "Mémoire relative à l'application des surfaces les unes sur les autres, envoyé au Concours ouvert sur cette question, en 1859, par l'Académie des Sciences," t. XXVII of the *Mémoires présentés par divers savants à l'Académie des Sciences*, printed in 1882.

Codazzi also gave some formulas that related to oblique coordinates in an appendix to his paper. Those formulas, which are different from the system (A'), are equivalent to some relations that we shall make known later on (no. **508**). Formulas (A'), which contain an arbitrary number m, and in which all of the quantities are defined by their kinematic properties, were given in 1866 in a course that I had the honor of teaching as a substitute for Joseph Bertrand at the Collège de France. Combescure had already applied some considerations of kinematics to the proof of the Codazzi formulas in an unedited paper that was presented to the Académie des Sciences in 1864. Combescure's paper treated *functional determinants and curvilinear coordinates*. It was published in 1867 in the Annales de l'École Normale (1) t. IV.

O. Bonnet was the first to exhibit all of the interest and utility that the Codazzi formulas have relative to rectangular coordinates. After proving that geometrically in a "Note sur la théorie de la deformation des surfaces gauches" that was included in Comptes rendus **58** (1863), pp. 805, the eminent geometer made a profound study in a 120-page *Addition* to his "Mémoire sur la théorie des surfaces applicables sur une surface donnée," that was included in le Journal de l'École Polytechnique **42** (1867), 31-151. That part of the paper contained a complete proof and numerous applications; we shall have to cite it often. Bonnet can be identified with the same viewpoint as Codazzi, and he defined all of the elements that entered into his formulas by considerations of pure geometry.

Since 1867, a great number of works have been published on the same subject. We cite, first of all, those of Codazzi, which were included in a great paper: "Sulle coordinate curvilinee d'una superficie e dello spazio," Annali di Matematica de Milan t. I, pp. 93-316; t. II, pp. 101-119 and 269-287; t. IV, pp. 16-25 (1867-69). We have also borrowed a very elegant formula from a paper by Laguerre "Sur les formules fondamentales de la théorie des surfaces" that was published in Nouvelles Annales de Mathématiques (2) **11** (1872), pp. 60. The methods of Laguerre were developed by Ch. Brisse in a paper that was entitled: "Exposition analytique de la théorie des surfaces. That paper, whose first part appeared in 1874 in the Annales de l' École Normales (2) **3** (1874), pp. 87, was continued, but not concluded, in Journal de l' École Polytechnique **53** (1883), pp. 213.

However, I must, above all, point out that the closest analogy with the method that is followed in this part of my lectures is offered by the work that Ribaucour developed, in a more or less complete manner, in several of his papers, and which is found to be presented in a detailed manner under the name of *perimorphy*in the paper that was presented to l'Académie de Bruxelles: "Étude des élassoïdes ou surfaces à courbute moyenne nulle," Mémoires couronnes et Mémoires des Savants étrangers publiés par l'Académie Royale de Belgique **44** (1881). Nonetheless, Ribaucour considered only rectangular curvilinear coordinates, and he did not give a kinematic definition of the quantities that enter into his formulas. The two systems of formulas that take the place of our systems (A) and (B) in his theory seem less simple to us and have a less precise significance. At the basis, Ribaucour employed the theory of relative motion, but without saying that explicitly and without utilizing all of the resources that the theory presents.

The Codazzi formulas are hardly the only ones that permit a deep study of the theory of surfaces. Later on, we shall show all of the advantages that one can derive from the beautiful paper by Gauss "Disquisitiones generales circa superficies curvas," which is associated with almost all of the work by

Formulas (B) [pp. 2], which give the projections of the displacement of a point, take a simpler form here and become:

$$(B'') \qquad \begin{cases} dx + A\,du + (q\,du + q_1 dv)\,z - (r\,du + r_1\,dv)\,y, \\ dy + C\,dv + (r\,du + r_1\,dv)\,x - (p\,du + p_1 dv)\,z, \\ dz \qquad\quad + (p\,du + p_1 dv)\,y - (q\,du + q_1 dv)\,x. \end{cases}$$

When one has to apply the formulas of the preceding chapter, one must adopt the following values for the translations:

$$(17) \qquad \begin{cases} \xi = A, & \xi_1 = 0, \\ \eta = 0, & \eta_1 = C. \end{cases}$$

If one considers an arbitrary curve that is traced on the surface then one will have:

$$(18) \qquad \cos\omega = \frac{A\,du}{ds}, \qquad \sin\omega = \frac{C\,dv}{ds},$$

in which ω now denotes the angle between the tangent and the arc $A\,du$.

The lines of curvature will be defined by the two equations:

$$(19) \qquad \begin{cases} A\,du + \rho\,(q\,du + q_1 dv) = 0, \\ C\,dv - \rho\,(q\,du + p_1 dv) = 0. \end{cases}$$

The elimination of ρ will lead to the differential equation of those lines:

$$(20) \qquad Ap\,du^2 + C\,q_1\,dv^2 + (C\,q + Ap_1)\,du\,dv = 0,$$

and the elimination of du / dv will lead to the equation of the radii of principal curvature:

$$(21) \qquad \rho^2 \left(\frac{\partial r}{\partial v} - \frac{\partial r_1}{\partial u}\right) - \rho\,(A\,p_1 - C\,q) + AC = 0.$$

In particular, the total curvature of the surface will be given by the formula:

$$(22) \qquad \frac{AC}{RR'} = \frac{\partial r}{\partial v} - \frac{\partial r_1}{\partial u} = -\frac{\partial}{\partial u}\left(\frac{1}{A}\frac{\partial C}{\partial u}\right) - \frac{\partial}{\partial v}\left(\frac{1}{C}\frac{\partial A}{\partial v}\right).$$

Finally, the relation between two conjugate tangents will take the form:

German geometers. The relations that are established in it permit one to treat all of the essential questions completely.

(23) $$A\,q\,du\,\delta v - C\,p_1\,dv\,\delta v + A\,q_1\,du\,\delta v - C\,p\,dv\,\delta u = 0,$$

and the differential equation of the asymptotic lines will become:

(24) $$A\,q\,du^2 - C\,p_1\,dv^2 + (A\,q_1 - C\,p)\,du\,dv = 0.$$

500. One can make an even more specialized hypothesis and imagine the case (which is very important for the theory and applications) in which the two systems of coordinate lines are lines of curvature on the surface. We shall rapidly point out the formulas that refer to that hypothesis.

In that case, the differential equation (20) of the lines of curvature must be devoid of terms in du^2, dv^2. One must then have:

(25) $$p = q_1 = 0.$$

The Codazzi formulas then reduce to the following ones:

(A''') $$\left\{\begin{array}{ll} r = -\dfrac{1}{C}\dfrac{\partial A}{\partial v}, & \dfrac{\partial p_1}{\partial u} = -q r_1, \\ r_1 = \dfrac{1}{A}\dfrac{\partial C}{\partial v}, & \dfrac{\partial q}{\partial v} = r p_1, \end{array}\right. \qquad \frac{\partial r}{\partial v} - \frac{\partial r_1}{\partial u} = -q\,p_1\,.$$

Six of the twelve rotations or translations of the trihedron will become zero.

One sees that the elimination of p_1, q, r, r_1 from the preceding equations must lead to a differential relation between A and C. One cannot therefore choose the line element for a surface arbitrarily when it is referred to its lines of curvature.

The line element $d\sigma$ of the spherical representation takes the very simple form here:

(26) $$ds^2 = q^2\,du^2 + p_1^2\,dv^2.$$

One then recognizes that the lines of the sphere that serve as the images of the lines of curvature also intersect at right angles.

Let R denote the principal radius of curvature that corresponds to the arc $A\,du$, and let R' denote the other radius that corresponds to the arc $C\,dv$. Formulas (19) give us:

(27) $$R = -\frac{A}{q}, \qquad R' = \frac{C}{p_1}.$$

The line element of the surface can then be written:

(28) $$ds^2 = R^2\,q^2\,du^2 + R'^2\,p_1^2\,dv^2.$$

The differential equation of the asymptotic lines takes one or the other of the two forms:

$$(29)\qquad \begin{cases} A q\,du^2 - C\,p_1\,dv^2 = 0, \\ \dfrac{\cos^2\omega}{R} + \dfrac{\sin^2\omega}{R'} = 0. \end{cases}$$

We further note that upon introducing R, R' in place of A and C, resp., in the first two equations (A'''), one will obtain the formulas:

$$(C)\qquad \begin{cases} \dfrac{\partial R}{\partial v} = \dfrac{1}{q}\dfrac{\partial q}{\partial v}(R' - R), \\ \dfrac{\partial R'}{\partial v} = -\dfrac{1}{p_1}\dfrac{\partial p_1}{\partial u}(R' - R), \end{cases}$$

which constitute the relations between the radii of curvature and the spherical representation. Since one can deduce the last three formulas in (A''') from the following relation between p and p_1:

$$(D)\qquad \frac{\partial}{\partial v}\left(\frac{1}{p_1}\frac{\partial q}{\partial v}\right) + \frac{\partial}{\partial u}\left(\frac{1}{q}\frac{\partial p_1}{\partial u}\right) + q\,p_1 = 0,$$

one sees that it will be impossible to take arbitrary functions of u and v for R and R'.

The system (B'') takes the following form here:

$$(B''')\qquad \begin{cases} dx + A\,du + q\,z\,du - (r\,du + r_1\,dv)\,y, \\ dz + C\,dv + (r\,du + r_1\,dv)\,x - p_1 z\,dv, \\ dz + p_1 y\,dv - qx\,du. \end{cases}$$

501. The systems of curvilinear coordinates that we just employed are all real. In the most important research that relates to surfaces, it often happens that one will be led to appeal to symmetric coordinates for which the line element has the reduced form:

$$(\alpha)\qquad ds^2 = 4\,\lambda^2\,du\,dv.$$

In that case, as well, it is good to point out the formulas that can replace those of Codazzi.

Here is how we determine the position of the trihedron (T) that admits the normal to the surface for its z-axis. At each point, we take the x-axis of the trihedron to be the tangent to the curve:

$$u - v = \text{const.},$$

and the y-axis to be the tangent to the orthogonal curve:

$$u + v = \text{const.}$$

Those hypotheses already give the relations:

$$\xi = \xi_1, \qquad \eta + \eta_1 = 0$$

between the translations.

Now, identify the line element of the surface to be the one that was given by formula (2) of the preceding chapter; we will have the relations:

$$\xi^2 + \eta^2 = 0, \quad \xi^2 - \eta^2 = 2\lambda^2.$$

We take:

$$\xi = \lambda, \qquad \eta = -\lambda i,$$

and consequently the values of the four translations will be given by the following formulas:

$$\tag{30} \begin{cases} \xi = \lambda, & \eta = -\lambda i, \\ \xi_1 = \lambda, & \eta_1 = +\lambda i. \end{cases}$$

It suffices to substitute those values in the formulas of the preceding chapter in order to obtain all of the ones that refer to symmetric coordinates. Here, the system (A) will give us:

$$(A^{\mathrm{IV}}) \quad \begin{cases} p + p_1 = i(q_1 - q), & \dfrac{\partial p}{\partial v} - \dfrac{\partial p_1}{\partial u} = qr_1 - rq_1, \\ r = -i\dfrac{\partial \log \lambda}{\partial u}, & \dfrac{\partial q}{\partial v} - \dfrac{\partial q_1}{\partial u} = rp_1 - pr_1, \\ r_1 = \ \ i\dfrac{\partial \log \lambda}{\partial v}, & \dfrac{\partial r}{\partial v} - \dfrac{\partial r_1}{\partial u} = pq_1 - qp_1. \end{cases}$$

The expressions that are given by formulas (B) become:

$$(B^{\mathrm{IV}}) \quad \begin{cases} dx + \lambda(du + dv) + (q\,du + q_1\,dv)\,z - (r\,du + r_1\,dv)\,y, \\ dy + i\lambda(dv - du) + (r\,du + r_1\,dv)\,x - (p\,du + p_1 dv)\,z, \\ dz \qquad\qquad\qquad + (p\,du + p_1\,dv)\,y - (q\,du + q_1 dv)\,x. \end{cases}$$

The asymptotic lines of the surface have the differential equation:

$$\tag{31} (q + ip)\,du^2 + (q - ip_1)\,dv^2 + (ip_1 - ip + q + q_1)\,du\,dv = 0.$$

The system of equations (10) (no. **488**), which defines the lines of curvature, takes the form:

(32) $$\begin{cases} \lambda(du+dv)+\rho(q\,du+q_1\,dv)=0, \\ i\lambda(du-dv)+\rho(p\,du+p_1\,dv)=0, \end{cases}$$

in which ρ always denotes the radius of principal curvature.

The elimination of du / dv leads to the equation:

(33) $$\rho^2(pq_1 - qp_1) - \lambda\rho(p_1 - p - iq - iq_1) + 2i\lambda^2 = 0,$$

which tells one the radii of principal curvature. From this, one deduces that:

(34) $$\frac{1}{RR'} = -\frac{1}{\lambda^2}\frac{\partial^2 \log\lambda}{\partial u\,\partial v},$$

which is a formula that one will make frequent use of.

The elimination of ρ from equations (32) leads to the differential equation of the lines of curvature:

(35) $$du^2(p - iq) + dv^2(p_1 + iq_1) = 0.$$

The absence of the term in $du\,dv$ shows immediately that the lines of curvature are orthogonal.

502. In summary, we have four different systems of formulas at our disposal that refer to oblique coordinates, rectangular coordinates, coordinates that are determined by the lines of curvature, and symmetric coordinates, respectively. We shall study some of the questions in which those formulas play an essential role. However, before concluding, we shall make a general remark: In any of the systems thus-obtained, the expressions for r and r_1 depend exclusively upon the line element. It follows from this that the geodesic curvature, whose expression contains only rotations r, r_1, will not change when one deforms the surface. In particular, the geodesic lines, for which the geodesic curvature is zero, and whose differential equation is defined by the line element of the surface exclusively, will preserve their definition when one passes from the proposed surface to any other surface that can be mapped to the first one.

When the research that we have to undertake must be applied to all of the cases, we can employ the formulas of Chapter I upon remarking that ξ, ξ_1, η, η_1, r, r_1 depend upon the line element exclusively in all systems of formulas, and will remain the same when the surface is deformed while carrying along the trihedron (T).

503. In order to complete these developments, it remains for us to point out how one can determine the various quantities that enter into those systems of formulas when the surface is known and well-defined; for example, when one has expressions for the rectangular coordinates x, y, z of a point of the surface as functions of the parameters u and v. Consider the first system of formulas, which are the ones from which one can derive all of the other ones, and the ones that contain the translations ξ, ξ_1, η, η_1. If E, F, G denote Gauss's three functions then one will first have:

$$
(36)\qquad
\begin{cases}
E = \left(\dfrac{\partial x}{\partial u}\right)^2 + \left(\dfrac{\partial y}{\partial u}\right)^2 + \left(\dfrac{\partial z}{\partial u}\right)^2 = \xi^2 + \eta^2, \\[2ex]
F = \dfrac{\partial x}{\partial u}\dfrac{\partial x}{\partial v} + \dfrac{\partial y}{\partial u}\dfrac{\partial y}{\partial v} + \dfrac{\partial z}{\partial u}\dfrac{\partial z}{\partial v} = \xi\xi_1 + \eta\eta_1, \\[2ex]
G = \left(\dfrac{\partial x}{\partial v}\right)^2 + \left(\dfrac{\partial y}{\partial v}\right)^2 + \left(\dfrac{\partial z}{\partial v}\right)^2 = \xi_1^2 + \eta_1^2.
\end{cases}
$$

Those three equations will permit one to determine the four translations, as long as they exist. Any hypothesis in the manner by which the trihedron (T) is *attached* to the surface will give a relation that one can combine with the preceding ones. We can then consider the translations to be known.

Having said that, in order to determine the nine cosines that determine the position of the trihedron (T), keep all the notations of Book I [I, pp. 2]. The consideration of the displacements along the coordinate curves will lead us to the six equations:

$$
(37)\qquad
\begin{cases}
\xi a + \eta b = \dfrac{\partial x}{\partial u}, \\[2ex]
\xi a' + \eta b' = \dfrac{\partial y}{\partial u}, \\[2ex]
\xi a'' + \eta b'' = \dfrac{\partial z}{\partial u},
\end{cases}
$$

and

$$
(37')\qquad
\begin{cases}
\xi_1 a + \eta_1 b = \dfrac{\partial x}{\partial v}, \\[2ex]
\xi_1 a' + \eta_1 b' = \dfrac{\partial y}{\partial v}, \\[2ex]
\xi_1 a'' + \eta_1 b'' = \dfrac{\partial z}{\partial v},
\end{cases}
$$

which will exhibit the six cosines a, b, a, … One then finds that:

$$
(38)\qquad
\begin{cases}
\Delta a = \eta_1 \dfrac{\partial x}{\partial u} - \eta \dfrac{\partial x}{\partial v}, \\[2ex]
\Delta a' = \eta_1 \dfrac{\partial y}{\partial u} - \eta \dfrac{\partial y}{\partial v}, \\[2ex]
\Delta a'' = \eta_1 \dfrac{\partial z}{\partial u} - \eta \dfrac{\partial z}{\partial v};
\end{cases}
$$

(38′)
$$\left\{\begin{aligned} \Delta b &= -\xi_1 \frac{\partial x}{\partial u} + \xi \frac{\partial x}{\partial v}, \\ \Delta b' &= -\xi_1 \frac{\partial y}{\partial u} + \xi \frac{\partial y}{\partial v}, \\ \Delta b'' &= -\xi_1 \frac{\partial z}{\partial u} + \xi \frac{\partial z}{\partial v}; \end{aligned}\right.$$

in which Δ denotes the determinant $\xi\eta_1 - \eta\xi_1$, which, from formulas (36), will have the value:

(39)
$$\Delta = \xi\eta_1 - \eta\xi_1 = \pm \sqrt{EG - F^2}\,.$$

As for the direction cosines c, c', c'' of the normal to the surface, one can deduce their known expressions [I, pp. 2] as functions of six other ones. One then finds that:

(40)
$$\left\{\begin{aligned} \Delta c &= \frac{\partial y}{\partial u}\frac{\partial z}{\partial v} - \frac{\partial y}{\partial v}\frac{\partial z}{\partial u}, \\ \Delta c' &= \frac{\partial z}{\partial u}\frac{\partial x}{\partial v} - \frac{\partial z}{\partial v}\frac{\partial x}{\partial u}, \\ \Delta c'' &= \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}. \end{aligned}\right.$$

It remains for us to determine the rotations. One will obtain them, for example, by differentiating formulas (37) and (37′), which will then lead to the following ones:

(41)
$$\left\{\begin{aligned} \mathrm{S}\, c\, d\frac{\partial x}{\partial u} &= \xi\, \mathrm{S}\, c\, da + \eta\, \mathrm{S}\, c\, db \\ &= -\xi(q\,du + q_1 dv) + \eta(p\,du + p_1 dv), \\ \mathrm{S}\, c\, d\frac{\partial x}{\partial v} &= \xi_1\, \mathrm{S}\, c\, da + \eta_1\, \mathrm{S}\, c\, db \\ &= -\xi_1(q\,du + q_1 dv) + \eta_1(p\,du + p_1 dv), \\ \mathrm{S}\frac{\partial x}{\partial v}\, d\frac{\partial x}{\partial u} &= \xi_1 d\xi_1 + \eta_1 d\eta + (\xi\eta_1 - \eta\xi_1)(r\,du + r_1 dv). \end{aligned}\right.$$

Upon replacing c, c', c'' by their values that were given above, and upon introducing the determinants D, D', D'' that are defined by the identity:

$$(42)\qquad D\,du^2 + 2D'\,du\,dv + D''\,dv^2 = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial^2 x}{\partial u^2}du^2 + 2\frac{\partial^2 x}{\partial u\,\partial v}du\,dv + \frac{\partial^2 x}{\partial v^2}dv^2 \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial^2 y}{\partial u^2}du^2 + 2\frac{\partial^2 y}{\partial u\,\partial v}du\,dv + \frac{\partial^2 y}{\partial v^2}dv^2 \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial^2 z}{\partial u^2}du^2 + 2\frac{\partial^2 z}{\partial u\,\partial v}du\,dv + \frac{\partial^2 z}{\partial v^2}dv^2 \end{vmatrix},$$

one will obtain the following values for the rotations by solving the preceding equations:

$$(43)\qquad \begin{cases} \Delta^2(p\,du + p_1 dv) = \xi(D'du + D''dv) - \xi_1(D\,du + D'dv), \\ \Delta^2(q\,du + q_1\,dv) = \eta(D'du + D''dv) - \eta_1(D\,du + D'dv), \\ \Delta\ \ (r\,du + r_1\,dv) = -\xi_1 d\xi - \eta_1 d\eta + \frac{1}{2}\frac{\partial G}{\partial u}dv + \left(\frac{\partial F}{\partial u} - \frac{1}{2}\frac{\partial E}{\partial u}\right)du \\ \qquad\qquad = +\xi_1 d\xi + \eta_1 d\eta - \frac{1}{2}\frac{\partial E}{\partial v}du - \left(\frac{\partial F}{\partial u} - \frac{1}{2}\frac{\partial G}{\partial u}\right)dv. \end{cases}$$

We will see later on that the determinants D, D', D'' play an essential role in Gauss's theory. From the last of the preceding formulas, one recognizes that the rotations r and r_1 depend upon only the line element of the surface, which conforms to some results that were pointed out already (no. **502**). As for the rotations p, q, p_1, q_1, they are expressed as linear functions of D, D', D''. Indeed, one has:

$$(44)\qquad \begin{cases} \Delta^2 p\ = \xi D' - \xi_2 D, \quad \Delta^2 q = \eta D' - \eta_2 D, \\ \Delta^2 p_1 = \xi D'' - \xi_1 D', \ \Delta^2 q_1 = \eta D'' - \eta_2 D', \\ \Delta r = +\xi\frac{\partial \xi_1}{\partial u} + \eta\frac{\partial \eta_1}{\partial u} - \frac{1}{2}\frac{\partial E}{\partial u}, \\ \Delta r_1 = -\xi_1\frac{\partial \xi}{\partial v} - \eta_1\frac{\partial \eta}{\partial v} + \frac{1}{2}\frac{\partial G}{\partial u}. \end{cases}$$

If one substitutes the values of p, q, p_1, q_1 in the third of relations (5) [I, pp. 43] then one will be led to the identity:

$$(45)\qquad \Delta^2\,(qp_1 - qp_1) = DD'' - D'^2 = \Delta^2\left(\frac{\partial r}{\partial v} - \frac{\partial r_1}{\partial u}\right),$$

which establishes a relation between D, D', D'' that depends upon only the line element, and to which we shall have to return after we have discussed Gauss's theory.

504. In order to indicate at least one application, suppose that the surface is an ellipsoid that is referred to its lines of curvature. Here, one will have:

$$(46)\qquad \begin{cases} x = \sqrt{\dfrac{a(a-u)(a-v)}{(a-b)(a-c)}}, \\ y = \sqrt{\dfrac{b(b-u)(b-v)}{(b-a)(b-c)}}, \\ z = \sqrt{\dfrac{c(c-u)(c-v)}{(c-a)(c-b)}}, \end{cases}$$

$$(47)\qquad E = \frac{u\,(u-v)}{f\,(u)}, \qquad G = \frac{v\,(v-u)}{f\,(v)},$$

in which $f(u)$ denotes the function:

$$(48)\qquad f\,(u) = 4\,(a-u)\,(b-u)\,(c-u).$$

The calculation gives:

$$(49)\qquad \begin{cases} D = \dfrac{-4xyz\,(u-v)^2(a-b)(a-c)(b-c)}{f^2(u)\,f\,(v)} = -\dfrac{\sqrt{abc}\,(u-v)^2}{f\,(u)\sqrt{-f\,(u)\,f\,(v)}}, \\ D' = 0, \\ D'' = \dfrac{\sqrt{abc}\,(u-v)^2}{f\,(v)\sqrt{-f\,(u)\,f\,(v)}}. \end{cases}$$

Suppose that one takes:

$$(50)\qquad \xi_1 = 0, \qquad \eta = 0,$$

which amounts to making the x and y axes of the trihedron (T) coincide with the tangents to the coordinates lines. One will have:

$$(51)\qquad \xi = \sqrt{E}, \qquad \eta_1 = \sqrt{G},$$

and formulas (43) will give:

$$EG\,(p\,dv + p_1\,dv) = -\sqrt{E}\,D''\,dv,$$
$$EG\,(q\,dv + q_1\,dv) = -\sqrt{G}\,D\,du.$$

One will deduce from this that:

(52)
$$\begin{cases} p = 0, & q_1 = 0, \\ q = \dfrac{\sqrt{abc}}{u}\sqrt{\dfrac{u-v}{v\,f(u)}}, & p_1 = -\dfrac{\sqrt{abc}}{v}\sqrt{\dfrac{v-u}{u\,f(v)}}, \end{cases}$$

As for the rotations r, r_1, they are deduced from the line element by the third of formulas (43), which will give us:

$$r\,du + r_1\,dv = \frac{1}{2\sqrt{EG}}\frac{\partial G}{\partial u}dv - \frac{1}{2\sqrt{EG}}\frac{\partial E}{\partial u}du\,,$$

and, in turn:

(53)
$$\begin{cases} r = -\dfrac{1}{2\sqrt{EG}}\dfrac{\partial E}{\partial v} = \dfrac{1}{2(u-v)}\sqrt{\dfrac{-u\,f(v)}{v\,f(u)}}, \\ r_1 = -\dfrac{1}{2\sqrt{EG}}\dfrac{\partial G}{\partial u} = \dfrac{1}{2(u-v)}\sqrt{\dfrac{v\,f(u)}{-u\,f(u)}}, \end{cases}$$

Formulas (27) give us, for example, the following values for the radii of principal curvature:

(54)
$$R = -\frac{\sqrt{E}}{q} = -\frac{u^{3/2}\,v^{1/2}}{(abc)^{1/2}}, \qquad R' = \frac{\sqrt{G}}{p_1} = -\frac{u^{1/2}\,v^{3/2}}{(abc)^{1/2}},$$

from which, one will deduce that:

(55)
$$\frac{R}{R'^3} = \frac{abc}{v^4}, \quad \frac{R'}{R^3} = \frac{abc}{u^4}.$$

Therefore:

On each line of curvature, the corresponding principal radius is proportional to the cube of the other radius of principal curvature.

It results from formulas (54) that one has:

(56)
$$RR' = \frac{u^2\,v^2}{abc}.$$

As a result, the lines for which the total curvature remains constant are defined by the equation:

$$uv = \text{const.}$$

Now, if one lets δ denote the distance from the center of the ellipsoid to the tangent plane at a point whose coordinates u, v then a simple calculation will give:

(57)
$$\frac{1}{\delta^2} = \frac{uv}{abc} = \sqrt{\frac{RR'}{abc}}\,.$$

It follows from this that the curves considered are the ones to which Poinsot gave the name of *polhodes*, and which are the loci of the points for which the distance from the center of the ellipsoid to the tangent plane preserves a constant value.

Some considerations of geometry that are inferred from the theory of conjugate diameters, and which the reader can easily supply, will exhibit a remarkable property of those curves:

If one draws a normal plane to the ellipsoid through each tangent to the polhode then the section of the ellipsoid through that plane will have one if its summits at the point of contact of the tangent.

In other words:

The polhodes are curves such that each normal section that is tangent to the curve at one of its points is super-osculated by a circle at that point.

Later on, we shall return to that property in no. **510**.

Before beginning the detailed study of the lines that are traced on surfaces, we shall give various tables that contain the systems of formulas that we have obtained in this chapter and the preceding one.

TABLE I (Chapter I).

Rotations $p, q, r; p_1, q_1, r_1$; translations $\xi, \eta, 0; \xi_1, \eta_1, 0$:

$$(A)\qquad \begin{cases} \dfrac{\partial p}{\partial v}-\dfrac{\partial p_1}{\partial u}=qr_1-rq_1, & \dfrac{\partial \xi}{\partial v}-\dfrac{\partial \xi_1}{\partial u}=\eta r_1-r\eta_1, \\ \dfrac{\partial q}{\partial v}-\dfrac{\partial q_1}{\partial u}=rp_1-pr_1, & \dfrac{\partial \eta}{\partial v}-\dfrac{\partial \eta_1}{\partial u}=r\xi_1-\xi r_1, \\ \dfrac{\partial r}{\partial v}-\dfrac{\partial r_1}{\partial u}=pq_1-qp_1, & p\eta_1-p_1\eta=q\,\xi_1-q_1\xi; \end{cases}$$

$$(B)\qquad \begin{cases} dx+\xi\,du+\xi_1 dv+(q\,du+q_1 dv)\,z-(r\,du+r_1\,dv)\,y, \\ dy+\eta\,du+\eta_1 dv+(r\,du+r_1\,dv)\,x-(p\,du+p_1 dv)\,z, \\ dz \qquad\qquad +(p\,du+p_1 dv)\,y-(q\,du+q_1 dv)\,x. \end{cases}$$

Curve traced on the surface:

$$(1)\qquad ds\cos\omega=\xi\,du+\xi_1\,dv, \qquad ds\sin\omega=\eta\,du+\eta_1\,dv.$$

Spherical image of the curve:

$$(2)\qquad d\sigma\cos\theta=q\,du+q_1\,dv, \qquad d\sigma\sin\theta=-p\,du-p_1\,dv,$$

$$(3)\qquad d\sigma^2=(p\,du+p_1\,dv)^2+(q\,du+q_1\,dv)^2.$$

Condition for two directions to be conjugate:

$$(4)\qquad (p\,du+p_1\,dv)\,(\eta\,\delta u+\eta_1\,\delta v)-(q\,du+q_1\,dv)\,(\xi\,\delta u+\xi_1\,\delta v)=0.$$

Asymptotic lines:

$$(5)\qquad (p\eta-q\xi)\,du^2+(p_1\eta_1-q_1\,\xi_1)\,dv^2+(p\eta_1+p_1\eta-q\xi_1-q_1\xi)\,du\,dv=0,$$

$$(6)\qquad (p\,du+p_1\,dv)\sin\omega-(q\,du+q_1\,dv)\cos\omega=0.$$

Lines of curvature:

$$(7)\qquad \begin{cases} \xi\,du+\xi_1\,dv+\rho\,(q\,du+q_1 dv)=0, \\ \eta\,du+\eta_1 dv+\rho\,(p\,du+p_1 dv)=0. \end{cases}$$

Equation of the radii of principal curvature:

$$(10)\qquad \rho^2\,(pq_1-qp_1)+\rho\,(p\eta_1+p_1\eta-q\xi_1-q_1\xi)+\xi\eta_1-\eta\xi_1=0.$$

Total curvature:

$$\frac{\xi\eta_1 - \eta\xi_1}{RR'} = pq_1 - qp_1 = \frac{\partial r}{\partial v} - \frac{\partial r_1}{\partial u}. \tag{11}$$

TABLE II (Chapter I)

Curvature and torsion of a line that is traced on the surface: ξ', η', ζ'
Angles with the principal normal: λ', μ', ν'
Angles between the binormal and the axes of the trihedron (T):

$$(1)\qquad \begin{cases} \cos\xi' = -\sin\omega\sin\varpi, & \cos\eta' = \cos\omega\sin\varpi, & \cos\zeta' = \cos\varpi, \\ \cos\lambda' = \sin\omega\cos\varpi, & \cos\mu' = -\cos\omega\cos\varpi, & \cos\nu' = \sin\varpi, \end{cases}$$

$$(2)\qquad \frac{\sin\varpi}{\rho}\, ds = d\omega + r\, du + r_1\, dv,$$

$$(3)\qquad \frac{\cos\varpi}{\rho}\, ds = \sin\omega\,(p\, du + p_1\, dv) - \cos\,\omega\,(q\, du + q_1\, dv),$$

$$(4)\qquad \frac{1}{\tau} - \frac{d\varpi}{ds} = -\frac{p\,du + p_1 dv}{ds}\cos\,\omega - \frac{q\,du + q_1 dv}{ds}\,\sin\,\omega = -\frac{1}{2}\frac{\partial}{\partial\omega}\left(\frac{\cos\varpi}{\rho}\right),$$

$$(5)\qquad \begin{cases} -\dfrac{\cos\varpi}{\rho^2}\dfrac{d\rho}{ds} + \dfrac{\sin\varpi}{\rho}\left(\dfrac{2}{\tau} - 3\dfrac{d\varpi}{ds}\right) = K \\ \qquad = \left[\dfrac{\partial}{\partial u}\left(\dfrac{\cos\varpi}{\rho}\right) - r\dfrac{\partial}{\partial\omega}\left(\dfrac{\cos\varpi}{\rho}\right)\right]\dfrac{du}{ds} + \left[\dfrac{\partial}{\partial v}\left(\dfrac{\cos\varpi}{\rho}\right) - r_1\dfrac{\partial}{\partial\omega}\left(\dfrac{\cos\varpi}{\rho}\right)\right]\dfrac{dv}{ds}. \end{cases}$$

Center of the osculating sphere (x_0, y_0, z_0):

$$(6)\qquad \begin{cases} \dfrac{x_0}{-\sin\omega} = \dfrac{y_0}{\cos\omega} = \rho\sin\varpi + \tau\cos\varpi\dfrac{d\rho}{ds}, \\ z_0 = \rho\cos\varpi - \tau\dfrac{d\rho}{ds}\sin\varpi. \end{cases}$$

Center of the normal curvature (x_1, y_1, z_1):

$$(7)\qquad x_1 = y_1 = 0, \qquad z_1 = \frac{\rho}{\cos\varpi}.$$

Center of geodesic curvature (x_2, y_2, z_2):

$$(8)\qquad x_2 = -\frac{\rho\sin\omega}{\sin\varpi}, \qquad y_2 = \frac{\rho\cos\omega}{\sin\varpi}, \qquad z_2 = 0\,.$$

Center of curvature of the curve (x_3, y_3, z_3):

(9) $$x_3 = -\rho \sin \omega \sin \varpi, \quad x_3 = -\rho \cos \omega \sin \varpi, \quad x_3 = \rho \cos \omega.$$

TABLE III (Chapter II)

(1) $$ds^2 = A^2\,du^2 + C^2\,dv^2 + 2AC\cos\alpha\,du\,dv,$$

(2) $$n - m = \alpha,$$

(3) $$\xi = A\cos m, \quad \eta = A\sin m, \quad \xi_1 = C\cos n, \quad \eta_1 = C\sin n,$$

(A') $$\left\{\begin{aligned} &\frac{\partial p}{\partial v} - \frac{\partial p_1}{\partial v} = qr_1 - rq_1, && r = -\frac{\partial n}{\partial u} - \frac{1}{C\sin\alpha}\left(\frac{\partial A}{\partial v} - \frac{\partial C}{\partial u}\cos\alpha\right), \\ &\frac{\partial q}{\partial v} - \frac{\partial q_1}{\partial v} = rp_1 - pr_1, && r_1 = -\frac{\partial m}{\partial v} + \frac{1}{A\sin\alpha}\left(\frac{\partial C}{\partial u} - \frac{\partial A}{\partial v}\cos\alpha\right) \\ &\frac{\partial r}{\partial v} - \frac{\partial r_1}{\partial v} = pq_1 - qp_1, && A(p_1\sin m - q_1\cos m) = C(p\sin n - q\cos n). \end{aligned}\right.$$

(B') $$\left\{\begin{aligned} &dx + A\cos m\,du + C\cos n\,dv + (q\,du + q_1\,dv)\,z - (r\,du + r_1\,dv)\,y, \\ &dy + A\sin m\,du + C\sin n\,dv + (r\,du + r_1\,dv)\,x - (p\,du + p_1\,dv)\,z, \\ &dz \qquad\qquad\qquad\qquad\qquad\quad + (p\,du + p_1\,dv)\,y - (q\,du + q_1\,dv)\,x, \end{aligned}\right.$$

Line traced on the surface:

(4) $$ds\cos\omega = A\cos m\,du + C\cos n\,dv, \qquad ds\sin\omega = A\sin m\,du + C\sin n\,dv.$$

Angle between two directions:

(5) $$\left\{\begin{aligned} &ds\,\delta s\cos(\omega - \omega') = A^2 du\,\delta u + AC\cos\alpha(du\,\delta v + dv\,\delta u) + C^2 dv\,\delta v, \\ &ds\,\delta s\sin(\omega - \omega') = AC\sin\alpha(dv\,\delta u - du\,\delta v). \end{aligned}\right.$$

Conjugate directions:

(6) $$\left\{\begin{aligned} &A(q\cos m - p\sin m)\,du\,\delta u + C(q_1\cos n - p_1\sin n)\,dv\,\delta v \\ &\quad + A(q_1\cos m - p_1\sin m)\,du\,\delta v + C\,(q\cos m - p\sin m)\,dv\,\delta u = 0. \end{aligned}\right.$$

Asymptotic lines:

(7) $$A(q\cos m - p\sin m)\,du^2 + C(q_1\cos n - p_1\sin n)\,dv^2 + 2A(q_1\cos m - p_1\sin m)\,du\,\delta v = 0.$$

Lines of curvature:

(8) $$\left\{\begin{aligned} &A\cos m\,du + C\cos n\,dv + \rho\,(q\,du + q_1\,dv) = 0, \\ &A\sin m\,du + C\sin n\,dv - \rho\,(p\,du + p_1\,dv) = 0. \end{aligned}\right.$$

Differential equation:

$$(9)\qquad \left\{ \begin{aligned} & A(p\cos m + q\sin m)\,du^2 + C(p_1\cos n + q_1\sin n)\,dv^2 \\ & \quad + [A(p_1\cos m + q_1\sin m) + C\,(p\cos m + q\sin m)\,dv\,du = 0. \end{aligned} \right.$$

Radii of principal curvature:

$$(10)\quad \rho^2\,(pq_1 - qp_1) - \rho\,[A\,(p_1\cos m + q_1\sin m) - C\,(p\cos n + q\sin n)] + AC\sin\alpha = 0,$$

$$(11)\qquad \frac{AC\sin\alpha}{RR''} = -\frac{\partial^2\alpha}{\partial u\,\partial v} - \frac{\partial}{\partial u}\left(\frac{\dfrac{\partial C}{\partial u} - \dfrac{\partial A}{\partial v}\cos\alpha}{A\sin\alpha}\right) - \frac{\partial}{\partial v}\left(\frac{\dfrac{\partial A}{\partial u} - \dfrac{\partial C}{\partial v}\cos\alpha}{C\sin\alpha}\right).$$

TABLE IV (Chapter II)

Arbitrary rectangular coordinates:

(1) $\xi = A, \qquad \eta = 0, \qquad \xi_1 = 0, \qquad \eta_1 = 0, \qquad n = \alpha = \frac{\pi}{2}, \qquad m = 0,$

$$(A) \qquad \left\{ \begin{array}{ll} Aq_2 + Cp = 0, & \frac{\partial p}{\partial v} - \frac{\partial p_1}{\partial v} = qr_1 - rq_1, \\ r = -\frac{1}{C}\frac{\partial A}{\partial v}, & \frac{\partial q}{\partial v} - \frac{\partial q_1}{\partial v} = rp_1 - pr_1, \\ r_1 = \frac{1}{A}\frac{\partial C}{\partial u}, & \frac{\partial r}{\partial v} - \frac{\partial r_1}{\partial v} = pq_1 - qp_1, \end{array} \right.$$

$$(B) \qquad \left\{ \begin{array}{l} dx + A\,du + (q\,du + q_1\,dv)\,z - (r\,du + r_1\,dv)\,y, \\ dy + C\,dv + (r\,du + r_1\,dv)\,x - (p\,du + p_1\,dv)\,z, \\ dz \qquad\quad + (p\,du + p_1\,dv)\,y - (q\,du + q_1\,dv)\,x. \end{array} \right.$$

Line traced on the surface:

(2) $ds \cos \omega = A\,du, \qquad ds \sin \omega = C\,dv.$

Conjugate directions:

(3) $A\,q\,du\,\delta v - C\,p_1\,dv\,\delta v + A\,q_1\,(du\,\delta v + dv\,\delta u) = 0.$

Asymptotic lines:

(4) $A\,q\,du^2 - C\,p_1\,dv^2 + (A\,q_1 - C\,p)\,du\,dv = 0.$

Lines of curvature:

$$(5) \qquad \left\{ \begin{array}{l} A\,du + \rho\,(q\,du + q_1\,dv) = 0, \\ C\,du - \rho\,(p\,du + p_1\,dv) = 0. \end{array} \right.$$

Differential equation:

(6) $A\,p\,du^2 + C\,q_1\,dv^2 + (C\,q + A\,p_1)\,du\,dv = 0.$

Radius of principal curvature:

$$(7) \qquad \rho^2 \left(\frac{\partial r}{\partial v} - \frac{\partial r_1}{\partial u} \right) - \rho\,(p_1 - C\,q) + AC = 0,$$

$$(8) \qquad \frac{AC}{RR'} = -\frac{\partial}{\partial u}\left(\frac{1}{A}\frac{\partial C}{\partial u} \right) - \frac{\partial}{\partial v}\left(\frac{1}{C}\frac{\partial A}{\partial v} \right),$$

$$(9) \qquad AC\left(\frac{1}{R} + \frac{1}{R'} \right) = A\,p_1 - C\,q.$$

TABLE V (Chapter II)

Coordinate system defined by the lines of curvature:

(1) $$\xi = A, \quad \eta = 0, \quad \xi_1 = 0, \quad \eta_1 = 0, \quad p = 0, \quad q_1 = 0,$$

(A) $$\begin{cases} r = -\dfrac{1}{C}\dfrac{\partial A}{\partial v}, & \dfrac{\partial p_1}{\partial u} = -q r_1, \\ r_1 = \dfrac{1}{A}\dfrac{\partial C}{\partial v}, & \dfrac{\partial q}{\partial v} = r p_1, \\ \dfrac{\partial r}{\partial v} - \dfrac{\partial r_1}{\partial u} = -q p_1, & \dfrac{\partial}{\partial u}\left(\dfrac{1}{q}\dfrac{\partial p_1}{\partial u}\right) + \dfrac{\partial}{\partial v}\left(\dfrac{1}{p_1}\dfrac{\partial q}{\partial v}\right) + q p_1 = 0. \end{cases}$$

Conjugate directions:

(2) $$A\, q\, du\, \delta u - C\, p_1\, dv\, \delta v = 0.$$

Asymptotic lines:

(3) $$A\, q\, du^2 - C\, p_1\, dv^2 = 0, \qquad \frac{\cos^2\omega}{R} + \frac{\sin^2\omega}{R'} = 0.$$

Radii of principal curvature:

(4) $$R = -\frac{A}{q}, \qquad R' = \frac{C}{p_1},$$

(5) $$\frac{\partial R}{\partial v} = (R' - R)\,\frac{\partial \log q}{\partial v}, \qquad \frac{\partial R'}{\partial u} = -(R' - R)\,\frac{\partial \log p_1}{\partial u}.$$

Line traced on the surface:

$$\cos\omega = \frac{A\,du}{ds}, \qquad \sin\omega = \frac{C\,dv}{ds},$$

(6) $$\frac{\cos\varpi}{\rho} = \frac{\cos^2\omega}{R} + \frac{\sin^2\omega}{R'},$$

(7) $$\frac{1}{\tau} - \frac{d\varpi}{ds} = \left(\frac{1}{R} - \frac{1}{R'}\right)\sin\omega\cos\omega,$$

(8) $$\begin{cases} -\dfrac{\cos\varpi}{\rho^2}\dfrac{d\rho}{ds} + \dfrac{\sin\varpi}{\rho}\left(\dfrac{2}{\tau} - 3\dfrac{d\varpi}{ds}\right) \\ = -q^2\dfrac{\partial R}{\partial u}\dfrac{du^2}{ds^2} - 3q^2\dfrac{\partial R}{\partial v}\dfrac{du^2}{ds^2}\dfrac{dv}{ds} - 3p_1^2\dfrac{\partial R'}{\partial u}\dfrac{du}{ds}\dfrac{dv^2}{ds^2} - p_1^2\dfrac{\partial R'}{\partial v}\dfrac{dv^3}{ds^3}. \end{cases}$$

TABLE VI (Chapter II)

Symmetric coordinates:

(1) $$ds^2 = 4\lambda^2\, du\, dv,$$

(2) $$\xi = \lambda, \qquad \eta = -i\,\lambda, \qquad \xi_1 = \lambda, \qquad \eta_1 = i\,\lambda,$$

(A) $$\begin{cases} p + p_1 = i(q_1 - q), & \dfrac{\partial p}{\partial v} - \dfrac{\partial p_1}{\partial u} = qr_1 - rq_1, \\ r = -i\dfrac{\partial \log \lambda}{\partial u}, & \dfrac{\partial q}{\partial v} - \dfrac{\partial q_1}{\partial u} = rp_1 - pr_1, \\ r_1 = \;\; i\dfrac{\partial \log \lambda}{\partial v}, & \dfrac{\partial r}{\partial v} - \dfrac{\partial r_1}{\partial u} = pq_1 - qp_1, \end{cases}$$

(B) $$\begin{cases} dx + \lambda(du + dv) + (q\,du + q_1\,dv)\,z - (r\,du + r_1\,dv)\,y, \\ dy + i\lambda(dv - du) + (r\,du + r_1\,dv)\,x - (p\,du + p_1\,dv)\,z, \\ dz \qquad\qquad\qquad + (p\,du + p_1\,dv)\,y - (q\,du + q_1\,dv)\,x. \end{cases}$$

Conjugate directions:

(3) $$(q + ip)\,du\,\delta u + (q_1 - i\,p_1)\,dv\,\delta v + (q - ip)\,(du\,\delta v + dv\,\delta u) = 0.$$

Asymptotic lines:

(4) $$(q + ip)\,du^2 + (q_1 - i\,p_1)\,dv^2 + 2\,(q - ip)\,du\,dv = 0.$$

Lines of curvature:

(5) $$\begin{cases} \lambda(du + dv) + \rho\,(q\,du + q_1 dv) = 0, \\ i\lambda(du - dv) + \rho\,(p\,du + p_1 dv) = 0. \end{cases}$$

Differential equation:

(6) $$(p - i\,q)\,du^2 + (p_1 + i\,q_1)\,dv^2 = 0.$$

Radii of principal curvature:

(7) $$\rho^2\,(qp_1 - qp_1) - \lambda\,\rho\,(p_1 - p - iq - iq_1) + 2i\lambda^2 = 0,$$

(8) $$\frac{1}{RR'} = -\frac{1}{\lambda^2}\frac{\partial^2 \log \lambda}{\partial u\,\partial v}.$$

Curve traced on the surface:

(9) $$e^{i\omega} = \frac{2\lambda\,du}{ds}, \qquad e^{-i\omega} = \frac{2\lambda\,dv}{ds},$$

(10) $$\begin{cases} \dfrac{\sin\varpi}{\rho}\,ds = d\omega + r\,du + r_1 dv = d\omega - i\left(\dfrac{\partial \log \lambda}{\partial u}\,du - \dfrac{\partial \log \lambda}{\partial v}\,dv\right) \\ \qquad = \dfrac{i}{2}\left[d\log\dfrac{dv}{du} - 2\dfrac{\partial \log \lambda}{\partial u}\,du + 2\dfrac{\partial \log \lambda}{\partial v}\,dv\right]. \end{cases}$$

CHAPTER III

NORMAL CURVATURE AND GEODESIC TORSION

Euler's theorem on the curvature of normal sections. – Formula of O. Bonnet. – Theorem of J. Bertrand. – Introduction of geodesic torsion. – Geometric expression for the six rotations that enter into the previously-given formulas. – Relations between the same geometric elements in the case of oblique coordinates. – Joachimsthal's theorem that relates to the lines of curvature that are common to two surfaces. – Laguerre's formula. Its application to the determination of the radius of curvature of a line that is traced on the surface at the points where it is tangent to an asymptotic line. – Beltrami's theorem. – Torsion of an asymptotic line. – Bonnet's formula that relates to the radius of curvature of an asymptotic. – Application to isothermal orthogonal systems.

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505. We begin with the study of the two formulas that give the curvature and torsion:

$$\frac{\cos\varpi}{\rho} = \frac{1}{\rho_n} = \sin\,\omega\left(p\frac{du}{ds} + p_1\frac{dv}{ds}\right) - \cos\,\omega\left(q\frac{du}{ds} + q_1\frac{dv}{ds}\right), \tag{1}$$

$$\frac{d\varpi}{ds} - \frac{1}{\tau} = \cos\,\omega\left(p\frac{du}{ds} + p_1\frac{dv}{ds}\right) + \sin\,\omega\left(q\frac{du}{ds} + q_1\frac{dv}{ds}\right). \tag{2}$$

The first one tells us the variation of the normal curvature for all of the curves that pass through the same point of the surface. It will then contain the celebrated theorem of Euler that relates to the variation of the curvature of normal sections. In fact, if one supposes that the coordinate lines are lines of curvature of the surface, and if one introduces the simplifications that result from that hypothesis then the preceding equations will take the following form:

$$\frac{\cos\varpi}{\rho} = \frac{1}{\rho_n} = \frac{\cos^2\omega}{R} + \frac{\sin^2\omega}{R'}, \tag{3}$$

$$\frac{d\varpi}{ds} - \frac{1}{\tau} = \left(\frac{1}{R'} - \frac{1}{R}\right)\sin\,\omega\cos\,\omega. \tag{4}$$

The first of those formulas immediately gives Euler's theorem. The second one, which was given for the first time by O. Bonnet ([4]), will permit us to present the remarkable laws that Bertrand added to those of Euler ([5]).

If one considers a fixed point M on a surface and an infinitely-close point M' then the direction of the normal at M' can be determined in the following manner:

([4]) O. BONNET, "Mémoire sur la théorie des surfaces," Journal de l'École Polytechnique **32** (1848), 1.
([5]) J. BERTRAND, "Mémoire sur la théorie des surfaces," Journal de Liouville (1) **9** (1844), pp. 133.

1. By the angle that the normal at M makes with the projection of the normal at M' onto the normal plane at M that passes through the point M'.

2. By the angle that the normal at M' makes with its projection onto that normal plane.

The first of those two elements is obviously the angle of contingency of the normal section at M where the plane passes through M'. When the point M' turns around M, the variation of that angle will be known: It is given by Euler's theorem. Before Bertrand, no one had dreamed of determining the magnitude of the second angle and seeking to find how it would vary when the point M' displaced around M. Nonetheless, the study of that element is essential if one would like to know completely the properties of the pinching of normals that are infinitely-close to the normal at M. It is easy to see that such a study can be carried out completely by means of Bonnet's formula.

Indeed, apply it to the normal plane sections that pass through the point M. The torsion is zero for those sections, and formulas (4) will consequently give us:

$$d\varpi = \left(\frac{1}{R'} - \frac{1}{R}\right) ds \sin\omega \cos\omega. \tag{5}$$

Now, in general, ϖ denotes the angle between the osculating plane to the curve and the normal to the surface. In the case that we are dealing with, consider a point M' that is close to M on the plane section considered. ϖ will be the angle between the normal at M' and the plane of the section. Since that angle is zero at the point M, $d\varpi$ will be (upon neglecting second-order infinitesimals) the angle that was considered by Bertrand, which can be established directly from the preceding formula, moreover.

That formula shows us that $d\varpi$ will be zero for the two principal directions. In general, the values of $d\varpi$ will correspond to the same value of ds, and they will be equal and of opposite sign for two rectangular directions.

506. One can substitute the moment of the two normals at M and M' for the angle $d\varpi$. Imagine a force of length equal to unity that is directed along the normal at M'. The moment $\mathcal{M}$ of that force with respect to the normal at M will equal to $\delta\psi$, in which δ denotes the shortest distance, and ψ is the angle between the two lines. Now, if one slides that force along the normal at M' until the point of application arrives at M' then one can decompose it into two other forces, one of which will be directed along the projection of the normal at M' onto the normal plane at M that contains M', and the other of which will be perpendicular to that plane. The moment of the force will be equal to that of the second component, which is equal to $d\varpi$, and whose distance to the normal at M is obviously ds, upon neglecting second-order infinitesimals in the two evaluations.

$$\mathcal{M} = \delta\psi = ds\, dv = ds^2 \left(\frac{1}{R'} - \frac{1}{R}\right) ds \sin\omega \cos\omega.$$

It is easy to verify that this value of $\mathcal{M}$ will not change when one passes to any parallel surface, because upon replacing sin ω, cos ω with their values, one will find that:

$$\mathcal{M} = (R' - R)\, p_1\, q\, du\, dv,$$

and the quantities p_1, q, $R' - R$ will obviously not change (no. **500**) under the passage to the parallel surface.

If one would like to know the angle ψ between the normals at M and M' then one would obviously have:

$$\psi^2 = d\varpi^2 + \left(\frac{ds\cos\varpi}{\rho}\right)^2 = \left(\frac{\cos^2\varpi}{R'^2} + \frac{\sin^2\varpi}{R^2}\right) ds^2.$$

507. Bonnet's formula leads to some other consequences: In particular, it gives rise to the introduction of an element that relates to the curves that are traced on a surface.

The function $\frac{1}{\tau} - \frac{d\varpi}{ds}$ that relates to a point of a curve (C) will remain the same for all the tangent curves to the curve (C) at that point, so consider a tangent geodesic line, in particular. By definition, one will have:

$$\varpi = 0$$

for that line, and the preceding function will reduce to the torsion. Thus:

$$\frac{1}{\tau} - \frac{d\varpi}{ds}$$

represents the torsion of the tangent geodesic line at an arbitrary point of a curve (C); Bonnet gave it the name of *geodesic torsion*. That definition is appropriate, although it has the inconvenience of evoking the idea that there is an analogy with geodesic curvature that does not exist. The geodesic torsion is not preserved when one deforms a surface.

Be that as it may, once one has introduced that new notion, it will be very easy to give geometric expressions for the six rotations p, q, r, p_1, q_1, r_1 .

Let $1 / \rho_{nu}$, $1 / \rho_{gu}$, $1 / t_u$ be the normal and geodesic curvature, and the geodesic torsion, resp., of the arc length $A\,du$, and similarly denote the analogous elements relative to the arc length $C\,dv$ by $1 / \rho_{nv}$, $1 / \rho_{gv}$, $1 / t_v$, resp.

The previously-established formulas will give us:

$$
(6) \qquad \begin{cases}
\dfrac{A}{\rho_{nu}} = p \sin m - q \cos m, & \dfrac{C}{\rho_{nv}} = p_1 \sin n - q_1 \cos n, \\[2ex]
\dfrac{A}{\rho_{gu}} = \dfrac{\partial m}{\partial u} + r, & \dfrac{C}{\rho_{gv}} = \dfrac{\partial n}{\partial v} + r_1, \\[2ex]
\dfrac{A}{t_u} = -p \cos m - q \sin m, & \dfrac{C}{t_v} = -p_1 \cos n - q_1 \sin n.
\end{cases}
$$

If the curvilinear coordinates are rectangular then one will have $n = \pi / 2$, $m = 0$, and the preceding formulas will become:

$$
(7) \qquad \begin{cases}
\dfrac{A}{\rho_{nu}} = -q, & \dfrac{C}{\rho_{nv}} = \quad p_1, \\[2ex]
\dfrac{A}{\rho_{gu}} = \quad r, & \dfrac{C}{\rho_{gv}} = \quad r_1, \\[2ex]
\dfrac{A}{t_u} = -p, & \dfrac{C}{t_v} = -q_1.
\end{cases}
$$

We then have the definition and geometric interpretation of the six rotations. One knows that O. Bonnet, in his beautiful proof of the Codazzi formulas, introduced those six quantities without considering them to be rotations, but by appealing to only their geometric definition that would result from the preceding formulas.

508. In the case of oblique coordinates, one can introduce the expressions for the six absolute curvatures in place of the six rotations, as in the case of rectangular coordinates, and set:

$$
(8) \qquad \begin{cases}
\dfrac{A}{\rho_{nu}} = p \sin m - q \cos m = -Q, \\[2ex]
\dfrac{A}{\rho_{gu}} = \dfrac{\partial m}{\partial u} + r = R - \dfrac{\partial \alpha}{\partial u}, \\[2ex]
\dfrac{A}{t_u} = -p \cos m - q \sin m = -R; \\[2ex]
\dfrac{C}{\rho_{nv}} = p_1 \sin n - q_1 \cos n = P_1, \\[2ex]
\dfrac{C}{\rho_{gv}} = \dfrac{\partial n}{\partial v} + r_1 = R_1 + \dfrac{\partial \alpha}{\partial v}, \\[2ex]
\dfrac{C}{t_v} = -p_1 \cos n - q_1 \sin n = -Q_1.
\end{cases}
$$

One deduces from this that:

$$(9)\qquad \begin{cases} p = R\cos m - Q\sin m, & p_1 = P_1\sin n + Q_1\cos n, \\ q = P\sin m + Q\cos m, & q_1 = -P_1\cos n + Q_1\sin n, \\ r = R - \dfrac{\partial n}{\partial u}, & r_1 = R_1 - \dfrac{\partial m}{\partial v}, \end{cases}$$

and upon substituting those values into formulas (A') [pp. 14], one will obtain the system:

$$(10)\qquad \begin{cases} A(P_1\cos\alpha - Q_1\sin\alpha) = C(P\sin\alpha - Q\cos\alpha), \\ R = -\dfrac{1}{C\sin\alpha}\left(\dfrac{\partial A}{\partial v} - \dfrac{\partial C}{\partial u}\cos\alpha\right),\; R_1 = \dfrac{1}{A\sin\alpha}\left(\dfrac{\partial C}{\partial u} - \dfrac{\partial A}{\partial v}\cos\alpha\right), \\ \dfrac{\partial P}{\partial v} - QR_1 = \sin\alpha\left(\dfrac{\partial P_1}{\partial u} - RQ_1\right) + \cos\alpha\left(\dfrac{\partial Q_1}{\partial u} + RP_1\right), \\ \dfrac{\partial Q}{\partial v} + RP_1 = \sin\alpha\left(\dfrac{\partial Q_1}{\partial u} + RP_1\right) - \cos\alpha\left(\dfrac{\partial P_1}{\partial u} - RQ_1\right), \\ \dfrac{\partial Q}{\partial v} - \dfrac{\partial R_1}{\partial u} = \dfrac{\partial^2\alpha}{\partial u\,\partial v} - (PP_1 + QQ_1)\cos\alpha + (PQ_1 - QP_1)\sin\alpha. \end{cases}$$

Those equations, which coincide (up to notation) with the ones that Codazzi gave at the end of his paper, are obviously more complicated than formulas (A). That fact seems to indicate that the utility of the Codazzi formulas is due, above all, to the fact that the six geometric elements that figure in them can be considered to define a system of rotations, which is true only in the case of oblique coordinates.

509. Bonnet has remarked that his formula immediately exhibits an important theorem. Indeed, from that formula, the only curves that pass through a point of the surface whose geodesic torsion are zero at that point are the ones that admit one of the principal directions at that point for their tangent. The lines of curvature are thus characterized by the property that the geodesic torsion is zero at each of their points. That theorem is sometimes attributed to Lancret, even though that geometer never stated it. It is related most intimately to the beautiful propositions that Joachimsthal gave in regard to planar or spherical lines of curvature, and which we shall rapidly discuss:

When two surfaces intersect at a constant angle, the line of intersection cannot be a line of curvature of one of the surfaces without being a line of curvature of the other one.

Indeed, if ϖ and ϖ' denote the angles that the osculating plane of the line of intersection makes with the normals to the two surfaces then it is clear that the angle between the two normals will be $\varpi - \varpi'$. If that angle is constant then one will have:

$$\frac{1}{\tau} - \frac{d\varpi}{ds} = \frac{1}{\tau} - \frac{d\varpi'}{ds}$$

at each point of the intersection.

That equality shows that the geodesic torsion of the curve will have the same value when one refers to the two surfaces in succession. It cannot be zero for one of the surfaces without being zero for the other one.

Conversely, if the intersection of two surfaces is a line of curvature for the two surfaces then they will cut at a constant angle, because one will then have:

$$\frac{1}{\tau} - \frac{d\varpi}{ds} = 0 = \frac{1}{\tau} - \frac{d\varpi'}{ds}, \qquad \frac{d\varpi}{ds} = \frac{d\varpi'}{ds},$$

and consequently the angle $\varpi - \varpi'$ will be constant.

In the case in which one of the surfaces is a plane or a sphere, those propositions will give the following corollaries:

If a plane or a sphere cuts a surface at a constant angle then the intersection will be a line of curvature of the surface.

If a line of curvature is planar or spherical then the plane or sphere that contains the curve will cut the surface at a constant angle.

In order to attach these propositions to the preceding ones it will suffice to remark that any planar or spherical line is a line of curvature of the plane or sphere on which it is traced.

Moreover, all of those propositions have their true origin in the theory of developments of skew curves. Indeed, we have seen [I, pp. 15] that any normal to a skew curve that envelops a developable will make an angle V with the osculating plane that is defined by the formula:

$$dV = \frac{ds}{\tau}.$$

If one is given a curve that is traced on a surface then in order for it to be a line of curvature – i.e., in order for the normal to the surface to envelop a development of the curve at all points – it will be necessary and sufficient that the preceding relation must be verified when one replaces V with ϖ, which is the theorem that was stated above.

One likewise proves Joachimsthal's theorems. For example, if two surfaces cut along a common line of curvature then the normals to the two surfaces at each point of that curve will envelop two distinct developments, and as a result, they will cut at a constant angle. The converse propositions are established by analogous considerations.

Joachimsthal's theorem leads to a consequence that we have already stated without proof [I, pp. 316]:

Whenever a surface admits one planar line of curvature, the spherical representation of that line of curvature will be a circle whose plane is parallel to the one that contains the line of curvature.

Indeed, the normal to the surface at all points of the line of curvature will then make a constant angle α with the perpendicular to the plane of that line. As a result, the spherical representation of the line of curvature will be the locus of extremities of the radii of the sphere that make an angle α with that perpendicular; viz., it will be a great circle if the plane of the line of curvature is normal to the surface and a minor circle if the angle α is not a right angle. However, in one and the other case, the plane of the circle will obviously be parallel to that of the line of curvature.

Conversely, if the spherical representation of a line of curvature is defined by a circle of the sphere then the tangent at all points of that line will be parallel to the plane of the circle, and consequently the line itself will be located in a plane that is parallel to the plane of the circle.

510. Having studied O. Bonnet's formula, we shall now say a few words about that of Laguerre. If one refers the surface to the coordinate system that is defined by the lines of curvature then it will take the form:

$$-\frac{\cos\varpi}{\rho^2}\frac{d\rho}{ds}+\frac{\sin\varpi}{\rho}\left(\frac{2}{\tau}-3\frac{d\varpi}{ds}\right)=-q^2\frac{\partial R}{\partial u}\frac{du^2}{ds^2}-3q^2\frac{\partial R}{\partial v}\frac{du^2dv}{ds^3}$$
$$-3p_1^2\frac{\partial R'}{\partial u}\frac{du\,dv^2}{ds^3}-p_1^2\frac{\partial R'}{\partial v}\frac{dv^3}{ds^3}.$$

It results from this that the product of the left-hand side with ds^3 will remain constant when one passes from the surface to any parallel surface. Indeed, in order to perform that change, it will suffice to increase R and R' by the same constant without changing q and p_1.

If one applies the formula to a normal section of the surface then one will have:

$$\varpi = 0,$$

and the left-hand side will reduce to $\frac{d(1/\rho)}{ds}$. It follows from this that the first-order, third-degree differential equation:

$$q^2\frac{\partial R}{\partial u}du^3+3q^2\frac{\partial R}{\partial v}du^2dv+3p_1^2\frac{\partial R'}{\partial u}du\,dv^2+p_1^2\frac{\partial R'}{\partial v}dv^3=0 \tag{11}$$

defines the curves that are traced on the surface, and for which:

The normal section of the tangent surface to the curve will be super-osculated by a circle at any point of the curve.

Those lines were considered for the first time by de la Gournerie ([6]). It results from their differential equation that they will be preserved when one passes from one surface to a parallel surface. That remark was made by Ribaucour ([7]). One easily determines them for second-degree surface. They then reduce to two systems of rectilinear generators and the curves on which the total curvature of the surface remains constant. One can attach their theory to that of the contact of a surface with a cylinder of revolution. However, that study will find its place somewhere else.

511. Laguerre's formula permits one to solve a very interesting question to which Bonnet was the first to call attention.

Consider a curve (*C*) that is traced on a surface, and suppose that it is tangent to one of the asymptotic lines that pass through one of its points *M*. We will then have:

$$\frac{\cos\varpi}{\rho} = 0,$$

and consequently if cos ϖ is non-zero – i.e., if the osculating plane does not coincide with the tangent plane to the surface – then ρ will be infinite. That is what happens, for example, for a plane section whose plane passes through one of the asymptotic tangents and does not coincide with the tangent plane.

However, if the osculating plane of the curve does coincide with the tangent plane to the surface then one will have:

$$\cos \varpi = 0,$$

and ρ can have an arbitrary value. The geometric construction that one deduces from Meusnier's theorem likewise breaks down, and also leads to an indeterminacy.

Bonnet, to whom the preceding remark is due, has given a formula for that special case that one easily attach to the one that was proved by Laguerre.

Let $1 / \tau$, $1 / \rho$ denote the torsion and curvature, resp., of the curve considered, and let $1 / \tau_0$, $1 / \rho_0$ denote the same quantities that relate to the tangent asymptotic line. We have seen (nos. **492** and **493**) that the two functions:

$$\frac{1}{\tau} - \frac{d\varpi}{ds} \qquad \text{and} \qquad -\frac{\cos\varpi}{\rho^2}\frac{d\rho}{ds} + \frac{\sin\varpi}{\rho}\left(\frac{2}{\tau} - 3\frac{d\varpi}{ds}\right)$$

have the same values if one calculates them successively for two curves that are tangent to the same point. Here, the angle ϖ is equal to one quadrant for both the curve considered and the asymptotic line. However, $d\varpi / ds$, which is zero at each point of the asymptotic line, is not necessarily zero for the curve considered. One will then have:

([6]) DE LA GOURNERIE, "Étude sur la courbure des surfaces," Journal de Liouville (1) **20** (1855), pp. 155.

([7]) RIBAUCOUR, "Propriétés des lignes tracées sur les surfaces," Comptes rendus **30** (1875), pp. 642.

(12)
$$\begin{cases} \dfrac{1}{\tau} - \dfrac{d\varpi}{ds} = \dfrac{1}{\tau_0}, \\ \dfrac{1}{\rho}\left(\dfrac{2}{\tau} - 3\dfrac{d\varpi}{ds}\right) = \dfrac{2}{\rho_0 \tau_0}, \end{cases}$$

so, upon eliminating $d\varpi / ds$, one will have:

(13)
$$\frac{1}{\tau} - \frac{3}{\tau_0} = -\frac{2\rho}{\tau_0 \rho_0}.$$

That equation will tell one ρ when τ is given.

Suppose, for example, that we would like to determine the radius of curvature of the section by the tangent plane. That section will have two branches that pass through the point of contact. For each of them, one will have:

$$\tau = \infty,$$

and consequently ([8]):

$$\rho = \tfrac{3}{2} \rho_0 .$$

However, in order for the preceding formulas to be truly useful, it is necessary that one must be able to determine ρ_0 and τ_0. Here is how one achieves that:

512. In the case of an asymptotic line, from formula (22) [pp. 9], one has:

$$-\frac{1}{\tau_0} = \frac{p\,du + p_1 dv}{ds} \cos\omega + \frac{q\,du + q_1 dv}{ds} \sin\omega.$$

Moreover, the differential equation of an asymptotic line gives us:

$$\frac{p\,du + p_1 dv}{ds} \sin\omega - \frac{q\,du + q_1 dv}{ds} \cos\omega = 0.$$

These two relations can be replaced with the following one:

(14)
$$\begin{cases} \dfrac{p\,du + p_1 dv}{ds} + \dfrac{\cos\omega}{\tau_0} = 0, \\ \dfrac{q\,du + q_1 dv}{ds} + \dfrac{\sin\omega}{\tau_0} = 0. \end{cases}$$

([8]) This elegant relation is due to Beltrami, who gave it in an article "Sur la courbure de quelques lignes tracées sur une surface," that was included in Nouvelle Annales de Mathématiques (2) **4** (1865), pp. 258.

Upon replacing ds sin ω, ds cos ω with their expressions (5) [pp. 14] as functions of du, dv, one will find that:

(15)
$$\begin{cases} \left(p+\dfrac{A\cos m}{\tau_0}\right)du+\left(p_1+\dfrac{C\cos n}{\tau_0}\right)dv=0, \\ \left(p+\dfrac{A\sin m}{\tau_0}\right)du+\left(q_1+\dfrac{C\sin n}{\tau_0}\right)dv=0, \end{cases}$$

or, upon eliminating du / dv:

$$(pq_1 - qp_1)\,\tau_0^2 + AC\sin\alpha = 0.$$

Upon introducing the product of the radii of principal curvature using formula (14) [pp. 16], one will obtain:

(16)
$$\tau_0 = \pm\sqrt{-RR'},$$

which is a remarkable expression for torsion that is due to Enneper. One deduces from it, in particular, that the asymptotic lines of surfaces with constant total curvature are skew surfaces whose torsion is invariable.

If one substitutes the expression for τ_0 in formula (13) then it will become:

(17)
$$\frac{1}{2\rho} = \frac{\dfrac{1}{\rho}-\dfrac{1}{\rho_0}}{\dfrac{\sqrt{-RR'}}{\tau}-1}.$$

That formula, which is due to O. Bonnet ([9]), implicitly subsumes that of Enneper, because it suffices to make $\rho = \rho_0$ in order to recover the expression for the torsion of an asymptotic line.

513. It remains for us to determine ρ_0. Here again, the formula was given by Bonnet.

Take the coordinate system that is defined by the lines of curvature. The direction of the asymptotic lines will be defined by the equation:

$$\frac{\sin^2\omega}{R'}+\frac{\cos^2\omega}{R} = 0,$$

which will give:

(18)
$$\cos\omega = \frac{\sqrt{R}}{\sqrt{R-R'}}, \qquad \sin\omega = \frac{\sqrt{-R}}{\sqrt{R-R'}}, \qquad \tan\omega = \frac{\sqrt{R}}{\sqrt{-R'}},$$

([9]) Nouvelles Annales de Mathématiques (2) **4** (1865), pp. 267.

in which radicals are always given the same sign. Formula (18) [pp. 7] will give us:

$$\frac{ds}{\rho_0} = d\omega + r\,du + r_1\,dv,$$

and consequently:

$$\frac{1}{\rho_0} = \frac{\partial\omega}{\partial u}\frac{du}{ds} + \frac{\partial\omega}{\partial v}\frac{dv}{ds} + r\frac{du}{ds} + r_1\frac{dv}{ds},$$

or upon replacing du / ds, dv / ds with their expressions as functions of ω:

$$\frac{1}{\rho_0} = \frac{\cos\omega}{A}\frac{\partial\omega}{\partial u} + \frac{\sin\omega}{C}\frac{\partial\omega}{\partial v} + \frac{r}{A}\cos\omega + \frac{r_1}{C}\sin\omega.$$

Now, one deduces from the formulas of no. **500** that:

$$\frac{r}{A} = \frac{R'}{CR}\frac{\frac{\partial R}{\partial v}}{R-R'}, \qquad \frac{r_1}{C} = \frac{R}{AR'}\frac{\frac{\partial R'}{\partial u}}{R-R'},$$

and upon replacing r, r_1 with those values in the expression for ρ_0, one will obtain:

$$\frac{1}{\rho_0} = \frac{\cos\omega}{A}\frac{\partial\omega}{\partial u} + \frac{\cos^2\omega\sin\omega}{A}\frac{\partial\log R'}{\partial u} + \frac{\sin\omega}{C}\frac{\partial\omega}{\partial v} - \frac{\sin^2\omega\cos\omega}{C}\frac{\partial\log R}{\partial v},$$

$$\frac{1}{\rho_0} = \frac{\cos^2\omega\sin\omega}{A}\frac{\partial\log(R'\tan\omega)}{\partial u} - \frac{\sin^2\omega\cos\omega}{C}\frac{\partial\log(R\cot\omega)}{\partial v}.$$

Now, replace sin ω and cos ω with their values, and one will find that:

$$(19) \qquad \frac{(R-R')^{3/2}}{\rho_0} = \frac{R^2}{2(-R)^{5/2}}\frac{\partial}{A\partial u}\left(\frac{-R'^3}{R}\right) - \frac{R'^2}{2R^{5/2}}\frac{\partial}{C\,\partial v}\left(\frac{R^3}{-R'}\right).$$

Upon taking the auxiliary variables to be $\frac{R'^3}{R}$ and $\frac{R^3}{R'}$, one will easily arrive at the transform of that formula, and put it into the elegant form:

$$(20) \qquad \frac{1}{\rho_0} = \frac{4(-RR')^{7/8}}{(R-R')^{3/2}}\left[-\frac{\partial}{A\partial u}\left(\frac{R}{-R'^3}\right)^{1/3} + \frac{\partial}{C\,\partial v}\left(\frac{-R'}{R^3}\right)^{1/3}\right],$$

which was given by Bonnet, but which presented more difficulties than the preceding one in terms of observing the signs.

One can give an entirely geometric form to the expressions that were found by remarking that:

$$\frac{\partial}{A\partial u}, \quad \frac{\partial}{C\partial v}$$

represent the derivatives that relate to the displacements that are performed along the lines of curvature. Upon denoting those displacements by ds_1, ds_2, one will find that the absolute values of the two radii of curvature satisfy:

$$\frac{1}{\rho_0} = \frac{4(-RR')^{7/8}}{(R-R')^{3/2}}\left[\frac{\partial}{\partial s_1}\left(\frac{R}{-R'^3}\right)^{1/3} \mp \frac{\partial}{\partial s_2}\left(\frac{-R'}{R^3}\right)^{1/3}\right]. \tag{21}$$

An important application will show all of the interest that can be presented by research of the same nature as what we just discussed. In his studies of mathematical physics, the illustrious Lamé proposed to determine all triple systems that were both orthogonal and isothermal, and the solution that he gave to that important and difficult question did not fail to go on at some length and even raise some difficulties. In a paper that is already classical ([10]), Bonnet showed that all surfaces that belong to a system that is both orthogonal and isothermal must enjoy the following property: The principal radius of each line of curvature that corresponds to that line is proportional to the cube of the other radius. In other words, one must have:

$$\frac{\partial}{\partial s_1}\left(\frac{R}{-R'^3}\right)^{1/3} = 0, \qquad \frac{\partial}{\partial s_2}\left(\frac{-R'}{R^3}\right)^{1/3} = 0, \tag{22}$$

and consequently O. Bonnet could then deduce from his formula that the asymptotic lines of each of those surface must have an infinite radius of curvature; i.e., they must reduce to lines. The surfaces that the system is comprised of must then be necessarily of degree two.

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([10]) O. BONNET, "Mémoire sur la théorie des surfaces isothermes orthogonales," Journal de l'École Polytechnique **30** (1845), pp. 141.

CHAPTER IV

GEODESIC LINES

Various forms for the differential equation of geodesic lines. – Null-length lines for a surface that satisfy that differential equation. – Geodesic line that passes through two sufficiently-close points. – Gauss's theorem that relates to the geodesic lines that pass through a point of the surface. – Shortest path between two sufficiently-close points. – Geodesics that are normal to an arbitrary curve. – Gauss's second theorem. Extension to the definition of parallel curves in the plane. – Orthogonal trajectories of an arbitrary family of geodesics. They are determined by a simple quadrature. – Variation of length of a segment of a geodesic line. – Orthogonal system that is defined by two families of geodesic ellipses and hyperbolas. – Weingarten's theorem. – Bipolar coordinates in the plane and on the sphere. – Liouville's theorem that relates to the two families of geodesic lines that mutually cut at constant angles.

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514. It remains for us to undertake the study of the formula:

$$\frac{ds}{\rho_g} = d\omega + r\,du + r_1\,dv, \tag{1}$$

which gives the radius of geodesic curvature ρ_g of an arbitrary curve that is traced on the surface. That formula is distinguished from the preceding one by an essential property that we have already pointed out: The quantities that figure in it depend exclusively upon the form of the line element, and in turn, the geodesic curvature will remain invariant when one deforms the surface in an arbitrary manner. We begin by studying the geodesic lines. Their differential equation:

$$d\omega + r\,du + r_1\,dv = 0 \tag{2}$$

has order two. One knows how to integrate it only in a small number of cases. Nevertheless, in his celebrated paper ([11]), Gauss, and the geometers who followed him have enriched the theory of geodesic lines with a great number of interesting propositions that we shall develop first of all.

In the first place, we shall indicate some different forms for the differential equation. If we preserve all of the notations of Chapter I [pp. 2] then we will have:

$$\cos\omega\,ds = \xi\,du + \xi_1\,dv, \qquad \sin\omega\,ds = \eta\,du + \eta_1\,dv, \tag{3}$$

and as a result:

([11]) GAUSS, "Disquisitiones generales circa superficies curvas," Mémoires de la Société Royale des Sciences de Goettingue **6** (1828), and *Gesammelte Werke*, Bd. IV, pp. 217. The paper by Gauss has often been reproduced. In particular, one can find it, as edited by Liouville in 1850, in MONGE's *Application de l'Analyse à la Géométrie*.

$$\omega = \arctan \frac{\eta\, du + \eta_1\, dv}{\xi\, du + \xi_1\, dv}. \tag{4}$$

Formulas (*A*) (no. **481**) give us:

$$\left\{\begin{aligned} \Delta r &= -\frac{1}{2}\frac{\partial E}{\partial v} + \eta \frac{\partial \eta_1}{\partial u} + \xi \frac{\partial \xi_1}{\partial u} = \frac{\partial F}{\partial v} - \frac{1}{2}\frac{\partial E}{\partial v} - \eta_1 \frac{\partial \eta}{\partial u} - \xi_1 \frac{\partial \xi}{\partial u}, \\ \Delta r_1 &= \quad \frac{1}{2}\frac{\partial G}{\partial u} - \eta_1 \frac{\partial \eta}{\partial u} - \xi_1 \frac{\partial \xi}{\partial u} = \frac{1}{2}\frac{\partial G}{\partial u} - \frac{\partial F}{\partial v} + \eta \frac{\partial \eta_1}{\partial v} + \xi \frac{\partial \xi_1}{\partial v}, \end{aligned}\right. \tag{5}$$

in which Δ always denotes the determinant:

$$\Delta = \xi\, \eta_1 - \eta\, \xi_1 . \tag{6}$$

If one substitutes the values of ω, r, r_1 in equation (2), and if one develops the calculations by taking into account the relations:

$$\xi^2 + \eta^2 = E, \qquad \xi\, \xi_1 + \eta\, \eta_1 = F, \qquad \xi_1^2 + \eta_1^2 = G, \tag{7}$$

then one will obtain the following equation:

$$\left\{\begin{aligned} &2(EG - F^2)(du\, d^2v - dv\, d^2u) \\ &\quad = +\left(E\frac{\partial E}{\partial v} + F\frac{\partial E}{\partial u} - 2E\frac{\partial F}{\partial u}\right) du^2 \\ &\qquad + \left(3F\frac{\partial E}{\partial u} + G\frac{\partial E}{\partial u} - 2F\frac{\partial F}{\partial u} - 2E\frac{\partial G}{\partial u}\right) du^2 dv \\ &\qquad - \left(3F\frac{\partial G}{\partial u} + E\frac{\partial G}{\partial v} - 2F\frac{\partial F}{\partial v} - 2G\frac{\partial E}{\partial v}\right) du\, dv^2 \\ &\qquad - \left(G\frac{\partial G}{\partial u} + F\frac{\partial G}{\partial v} - 2G\frac{\partial F}{\partial v}\right) dv^2, \end{aligned}\right. \tag{8}$$

which characterizes the geodesic lines and contains only the quantities E, F, G.

515. If one adopts the arc length s of the geodesic line as the independent variable then one can replace the preceding equation with two differential equations that define u and v as functions of s. For example, one deduces from the first formula in (3) that:

$$\omega = \arccos \frac{\xi\, du + \xi_1 dv}{ds}.$$

Choose the system of translations for which one has:

(9) $$\xi_1 = 0,$$

so one can take:

(10) $$\eta_1 = \sqrt{G}\,, \qquad \eta = \frac{F}{\sqrt{G}}\,, \qquad \xi = \frac{\sqrt{EG - F^2}}{\sqrt{G}}.$$

The substitution of the values of r, r_1, ω in formula (2) will permit one to calculate $\frac{d^2u}{ds^2}$ and will give one the first of the following two equations:

(11) $$\begin{cases} 2(EG - F^2)\dfrac{d^2u}{ds^2} \\ \quad = \left(2F\dfrac{\partial F}{\partial u} - G\dfrac{\partial E}{\partial u} - F\dfrac{\partial E}{\partial v}\right)\dfrac{du^2}{ds^2} \\ \qquad + 2\left(F\dfrac{\partial G}{\partial u} - G\dfrac{\partial E}{\partial v}\right)\dfrac{du\,dv}{ds^2} + \left(F\dfrac{\partial G}{\partial v} + G\dfrac{\partial G}{\partial u} - 2G\dfrac{\partial F}{\partial v}\right)\dfrac{du^2}{ds^2}, \\ \\ 2(EG - F^2)\dfrac{d^2v}{ds^2} \\ \quad = \left(F\dfrac{\partial E}{\partial u} + E\dfrac{\partial E}{\partial v} - 2E\dfrac{\partial F}{\partial u}\right)\dfrac{du^2}{ds^2} \\ \qquad + 2\left(F\dfrac{\partial E}{\partial v} - E\dfrac{\partial G}{\partial u}\right)\dfrac{du\,dv}{ds^2} + \left(2F\dfrac{\partial F}{\partial v} - E\dfrac{\partial G}{\partial v} - F\dfrac{\partial G}{\partial u}\right)\dfrac{dv^2}{ds^2}, \end{cases}$$

which are deduced from each other by the exchange of u and v, as well as E and G. In order to determine u and v as functions of s completely, one must append the following relation to them:

(12) $$E\frac{du^2}{ds^2} + 2F\frac{du\,dv}{ds^2} + G\frac{dv^2}{ds^2} = 1,$$

which will serve to define the auxiliary variable s.

One can replace the two equations (11) with the following ones:

(13) $$\begin{cases} 2\dfrac{d}{ds}\left(\dfrac{E\,du + F\,dv}{ds}\right) = \dfrac{\partial E}{\partial u}\dfrac{du^2}{ds^2} + 2\dfrac{\partial F}{\partial u}\dfrac{du\,dv}{ds^2} + \dfrac{\partial G}{\partial u}\dfrac{dv^2}{ds^2}, \\ 2\dfrac{d}{ds}\left(\dfrac{F\,du + G\,dv}{ds}\right) = \dfrac{\partial E}{\partial v}\dfrac{du^2}{ds^2} + 2\dfrac{\partial F}{\partial v}\dfrac{du\,dv}{ds^2} + \dfrac{\partial G}{\partial v}\dfrac{dv^2}{ds^2}, \end{cases}$$

which have a more elegant form, but are not solved with respect to the second derivatives.

516. The various equations that we just obtained exhibit a fundamental property of geodesic lines that one can state as follows:

If one is given a geodesic line and two arbitrary points A, B that are taken on that line then the first variation of the arc length of the geodesic that is found between those two points will be zero when one passes from that geodesic to any other infinitely-close line that has the same extremities, and conversely any line that enjoys that property will be a geodesic.

Indeed, consider an arbitrary geodesic line that is found between the points A and B and is defined by the equation:

$$v = \varphi(u) .$$

Its arc length will be given by the formula:

$$\int_A^B \sqrt{E + 2Fv' + Gv'^2}\, du ,$$

in which v' denotes the derivative of v with respect to u. If one desires that the first variation of the arc length should be zero then upon applying the principles of the calculus of variations, one will get the differential equation:

$$\frac{d}{du}\left(\frac{Gv' + F}{\sqrt{E + 2Fv' + Gv'^2}}\right) - \frac{\dfrac{\partial E}{\partial v} + 2\dfrac{\partial F}{\partial v}v' + \dfrac{\partial G}{\partial v}v'^2}{2\sqrt{E + 2Fv' + Gv'^2}} = 0. \tag{14}$$

If one develops the calculations then one will recover equation (8). The proposition is then established.

However, one can also choose the arc length s to be the independent variable; equation (14) will then immediately (and with no calculation) take the form of the second equation (13). The first of those two equations is deduced from the second one by the exchange of u and v, so one can consider the system (13) to have been proved, which is a system that one can, in turn, solve with respect to the second derivatives $\dfrac{d^2u}{ds^2}$, $\dfrac{d^2v}{ds^2}$, which will give equations (11).

517. The calculations in the verification that we just pointed out lead to an interesting consequence. For example, recall equation (14), in which we regard u as an independent variable. Upon developing it, one will first have the following form:

$$\left[2\,d(Gv'+F)-du\left(\frac{\partial E}{\partial v}+2\frac{\partial F}{\partial v}v'+\frac{\partial G}{\partial v}v'^2\right)\right](E + 2Fv' + Gv'^2)$$
$$- (Gv' + F)\, d\,(E + 2F\,v' + G\,v'^2) = 0.$$

One then recognizes immediately that:

The null-length lines on the surface, which are defined by the equation:

$$E + 2F\,v' + G\,v'^2 = 0,$$

satisfy the differential equation of the geodesic lines.

Indeed, it is easy to establish directly that these lines are true geodesic lines, and that their osculating plane is normal to the surface at each point. It suffices to remark that the osculating plane to any null-length line is tangent to the circle at infinity and consequently normal to the tangent. Hence, if a null-length line is traced on a surface then since its osculating plane at each point is normal to the tangent at that point, it will necessarily contain the normal to the surface.

One can, moreover, verify the property that we just established in another way. If one supposes that the surface is referred to its null-length lines then one will have:

$$E = 0, \quad G = 0.$$

Equation (8) will then take the particularly simple form:

$$F\,(du\,d^2v - dv\,d^2u) - dv\,du\left(\frac{\partial F}{\partial u}du-\frac{\partial F}{\partial v}dv\right) = 0. \tag{15}$$

One then recognizes that all of the coordinate lines, which are defined by one or the other of the two equations:

$$dv = 0 \qquad \text{or} \qquad du = 0,$$

satisfy the differential equation of the geodesic lines ([12]).

([12]) The null-length lines are nonetheless distinguished from the other geodesics by a property that is good to point out. The first variation of the arc length will be presented in an indeterminate form when one passes from one such line to an infinitely-close curve. That amounts to saying that the arc length of any infinitely-close line to a null-length line is an infinitesimal of order 1 / 2. Indeed, let:

$$v = 0$$

be a null-length line. For any infinitely-close line that is defined by the equation:

$$v = \varepsilon\,\varphi(u),$$

in which ε an infinitely-small quantity, one will have:

518. The equations of various forms that we obtained for the geodesic lines exhibit an essential fact to which we shall appeal at every point along the way in the theory, namely:

Only one geodesic line will pass through any point of the surface that admits a well-defined tangent to the surface for its tangent at that point.

In other words:

A geodesic line is uniquely determined by the condition that it passes through a point of the surface and admits a given tangent.

On the contrary, if one would like to subject a geodesic line to the condition that it must pass through two points then it would be easy to see that the problem could have an infinitude of solutions, even when the two points are very close to each other. For example, suppose that the given surface is a cylinder of revolution; the geodesic lines will be helices. One will easily recognize that if one takes two points M and M' on the cylinder, no matter how close they are, then there will be an infinitude of geodesic lines that pass through those two points. Those helices are distinguished from each other by the number of circuits that a point on them will make when it starts at M before arriving at M', and by the sense in which one performs that motion.

However, no matter what the surface considered, one can determine a magnitude l such that if one takes a length λ that is less than or equal to l along each geodesic line that passes through M and starts at M then no other geodesic line whose length is less than l will pass through the point M and the extremity of that length. That proposition is not absolutely obvious, but one can establish it rigorously in the following manner:

We have seen that the coordinates u and v of a point of the geodesic line are defined as functions of s by equations (11), which have the following form:

$$(16)\qquad \left\{\begin{aligned} \frac{d^2u}{ds^2} &= a\left(\frac{du}{ds}\right)^2 + 2b\frac{du}{ds}\frac{dv}{ds} + c\left(\frac{dv}{ds}\right)^2, \\ \frac{d^2v}{ds^2} &= a'\left(\frac{du}{ds}\right)^2 + 2b'\frac{du}{ds}\frac{dv}{ds} + c'\left(\frac{dv}{ds}\right)^2, \end{aligned}\right.$$

in which a, b, a', … are given functions of u and v. If one would like to study the set of geodesic lines that pass through a point M whose coordinates are u_0, v_0 then differential equations will give u and v as functions of s, which is measured by starting at M, and the initial values u_0, v_0, $\left(\frac{du}{ds}\right)_0$, $\left(\frac{dv}{ds}\right)_0$ that relate to that point. Moreover, we remark that the preceding differential equations will not change form when one replaces s with αs, in which α denotes an arbitrary constant. It will then be necessary that the values of u and v

---

$$s = \sqrt{\varepsilon}\int\sqrt{2F\,\varphi'(u)}\,du\,.$$

must not change when one replaces, s, $\left(\frac{du}{ds}\right)_0$, $\left(\frac{dv}{ds}\right)_0$ with αs, $\frac{1}{\alpha}\left(\frac{du}{ds}\right)_0$, $\frac{1}{\alpha}\left(\frac{dv}{ds}\right)_0$, resp. That will be true only if u, v depend upon not only u_0, v_0, but also the variables:

$$u' = s\left(\frac{du}{ds}\right)_0, \qquad v' = s\left(\frac{dv}{ds}\right)_0.$$

One will then have:

$$u = f(u, v, u_0, v_0), \qquad v = \varphi(u', v', u_0, v_0).$$

If the coefficients E, F, G, and in turn, the functions a, b, a', …, are developable in integer powers of $u - u_0$, $v - v_0$ then the functions f and φ will be developable in powers of u' and v', and one will have two series of the following form:

$$u - u_0 = u' + \alpha u'^2 + 2\alpha' u' v' + \alpha'' v'^2 + \ldots,$$
$$v - v_0 = v' + \beta u'^2 + 2\beta' u' v' + \beta'' v'^2 + \ldots,$$

in which the coefficients α, β, … will be functions of u_0, v_0, and which will converge for all values of u', v' whose modulus is less than a fixed quantity.

Since the functional determinant:

$$\frac{\partial(u, v)}{\partial(u', v')}$$

is equal to 1 for $u' = v' = 0$, the preceding equations can be solved for u', v' and will give those quantities as series that are ordered in powers of $u - u_0$, $v - v_0$, which are series that will remain convergent as long as the moduli of those two differences remain less than a fixed quantity. In other words, only one geodesic line will pass through the point M and a sufficiently-close point M' for which u' and v' are less than a fixed quantity – i.e., the length of the line is less than a given quantity ([13]). That is the proposition that we would like to establish. One can also state it by saying that one can delimit a region around the point M such that only one geodesic line will pass through an arbitrary point of that region and the point M that is contained within the region completely ([14]).

([13]) Since one has:

$$E_0\left(\frac{du}{ds}\right)_0^2 + 2F_0\left(\frac{du}{ds}\right)_0\left(\frac{dv}{ds}\right)_0 + G_0\left(\frac{dv}{ds}\right)_0^2 = 1,$$

in which E_0, F_0, G_0 denote the values of E, F, G at the point M, upon multiplying by s^2, one will obtain the relation:

$$s^2 = E_0\, u'^2 + 2F_0\, u'\, v' + G_0\, v'^2,$$

which shows that if u', v' are less than a fixed quantity then the same thing will be true for s.

([14]) The variables u', v' are the ones to which Lipschitz gave the name of *normal variables* in some more general research. In particular, see Bulletin des Sciences mathématiques (1) **4**, pp. 97-110.

519. It results from that proposition that if one determines each point of the preceding region by the length of the geodesic that joins that point to the point M and by the angle v that the geodesic line makes at M with one of the tangents at that point on the surface then one will have constituted a coordinate system that is entirely analogous to the polar coordinate system in the plane, and in which just one system of values for u and v will correspond to each point of the surface if one agrees to takes, for example, u to be positive and v to be between 0 and 2π. The line element of the surface will be given by the formula:

$$ds^2 = du^2 + 2F\,du\,dv + G\,dv^2,$$

in which E is equal to 1. One will obviously have:

$$F = 0, \quad G = 0$$

for $u = 0$.

Having said that, express the idea that the lines v = const. are geodesics. If one employs equation (8), for example, then upon annulling dv and d^2v, one will get the condition:

$$\frac{\partial F}{\partial u} = 0.$$

F can only depend upon just the variable v, and since one has $F = 0$ for $u = 0$, F will be identically zero. As a result, the line element of the surface will take the simple form:

$$ds^2 = du^2 + G\,dv^2. \tag{17}$$

520. One can further establish the same result by adopting the form of the line element that was studied in Chapter II [pp. 13]. Here, one will have:

$$ds^2 = du^2 + 2C\cos\alpha\,du\,dv + G\,dv^2.$$

We remark, moreover, that since the arc length $C\,dv$ that is found between two infinitely-close geodesic lines must diminish indefinitely when u tends to zero, it is necessary that one must have $C = 0$ for $u = 0$ for any v.

Having said that, we express the idea that the lines v = const. are geodesics. Upon applying formula (2) and remarking that ω is equal to m here, one will have:

$$\frac{\partial m}{\partial u} + r = 0,$$

or, upon replacing r with its value that is deduced from formulas (A') (no. **493**):

$$-\frac{\partial \alpha}{\partial u} + \frac{\cot\alpha}{C}\frac{\partial C}{\partial u} = 0, \qquad \frac{\partial (C\log\alpha)}{\partial u} = 0.$$

Hence, C must be a function of v:

$$C \cos \alpha = \varphi(v). \tag{18}$$

However, since C is zero for $u = 0$ and any v, it is necessary that one must have:

$$\varphi(v) = 0.$$

The preceding equation gives us:

$$C \cos \alpha = 0$$

for an arbitrary value of u, and in turn:

$$\cos \alpha = 0.$$

We then recover the formula that was given already:

$$ds^2 = du^2 + C^2 dv^2 \tag{19}$$

for the line element of the surface.

521. That formula is of paramount importance. It permits us to prove that the shortest path between two sufficiently-close points on a surface is always a geodesic line.

Indeed, take an arbitrary point M' whose coordinates are u_0, v_0 on a portion of the surface that we defined above, and which can be regarded as being generated by a geodesic line of length l that turns around its extremity M. u_0 will have the length of the geodesic line that passes through M and M'. If we consider any other path that connects those two points and is included entirely within a portion of the surface considered then the length of that path will be expressed by the integral:

$$\int_0^{u_0} \sqrt{du^2 + C^2 dv^2} \, .$$

Now, that integral is obviously greater than:

$$\int_0^{u_0} du \, .$$

Hence, the path length is greater than u_0.

If the path leaves the portion of the surface that we just defined then it will be necessary that it first go from M to a point μ on the limit. Since the path length $M\mu$ is equal to at least l, from the preceding proof, it will already be greater than u_0 ; the same thing will then be true *a fortiori* for the total path.

One can also present the preceding argument in a geometric form. Construct curves u = const around the point M that present the general appearance of a series of concentric circles around that point as their center in the plane. Consider two infinitely-close curves; the arc length of any line that is found between the two curves will be:

$$\sqrt{du^2 + C^2 dv^2} \, .$$

Its minimum value will then be du, and it will correspond to the case in which one follows a normal geodesic in order to go from one curve to another. The shortest path from M to M' will then necessarily be the geodesic that passes through those two points.

522. One can generalize the proposition that was obtained in no. **520** as follows:

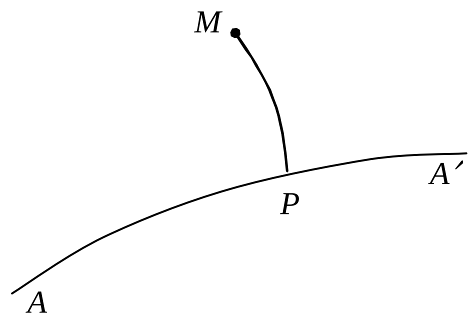

Figure 32.

If one is given (Fig. 32) an arbitrary curve AA' then construct the geodesics that are normal to that curve. We define an arbitrary point of the surface in the neighborhood of AA' by the arc $v = AP$ that determines the foot of the geodesic that passes through the point M and by the length $u = MP$, when it is measured by starting with P on that surface. As long as u is less than a fixed limit, a point will have only one pair of coordinates ([15]).

([15]) One can prove that proposition rigorously; it suffices to appeal to the results that were obtained in no. **518**.

We have seen that the values of u and v that relate to any point of a geodesic line that passes through a point M with coordinates u_0, v_0 are functions of four variables:

$$u_0,\ v_0,\ s\left(\frac{du}{ds}\right)_0,\ s\left(\frac{dv}{ds}\right)_0 .$$

If the point M is taken on a curve (C), and if, moreover, the geodesic line must be normal to (C) then u_0, v_0, $s\left(\frac{du}{ds}\right)_0$, $s\left(\frac{dv}{ds}\right)_0$ will become functions of the variable that fixes the position of that point on the curve; denote that variable by σ. u and v will be functions of s and σ. The functional determinant:

$$\frac{d(u,v)}{d(s,\sigma)}$$

will obviously have the initial value:

$$\begin{vmatrix} \left(\frac{du}{ds}\right)_0 & \frac{du_0}{d\sigma} \\ \left(\frac{dv}{ds}\right)_0 & \frac{dv_0}{d\sigma} \end{vmatrix} .$$

Since $\left(\frac{du}{ds}\right)_0$, $\left(\frac{dv}{ds}\right)_0$ determine the tangent to the geodesic line, they cannot be proportional to $\frac{du_0}{d\sigma}$, $\frac{dv_0}{d\sigma}$, which define the tangent to the curve (C).

Indeed, it suffices to remark that the geodesic lines that are normal to AA' will not criss-cross as long as u is less than a limit that one can determine.

Upon taking u and v to be variables, one will further have:

$$ds^2 = du^2 + 2C \cos \alpha \, du \, dv + C^2 \, dv^2,$$

with the condition:

$$C \cos \alpha = \varphi(v),$$

which expresses the idea that the parameter lines of v are geodesics.

Furthermore, for $u = 0$, one will have:

$$\cos \alpha = 0$$

for any v. One will then have:

$$\varphi(v) = 0,$$

and consequently one will recover the form that was obtained already for the line element:

$$ds^2 = du^2 + C^2 \, dv^2.$$

Thus:

When one moves along the geodesic lines that are normal to a curve of constant length, the locus of the extremities of those lengths will be a curve that is likewise normal to the geodesic lines.

That is the generalization of a well-known result that relates to parallel curves in the plane. As one knows, the two preceding theorems are due to Gauss.

523. One can also prove the last theorem in the following manner: Consider a family of geodesics on a portion of the surface such that only one of those lines passes through each point of the region considered, and associate those lines with another family of arbitrary curves that will permit one to define a coordinate system that is suitable to determine all points of the region when it is combined with the first one. The line element of the surface will be represented by a formula such as the following one:

$$ds^2 = E \, du^2 + 2F \, du \, dv + G \, dv^2,$$

in which we suppose that the geodesic lines are curves with parameter v. If one refers to equation (8), and if one expresses the idea that it is verified when one introduces the hypothesis $dv = 0$ then one will be led to the equation of condition:

Since the initial value of the functional determinant is not zero, that determinant will remain non-zero for sufficiently-small values of s. Consequently, u and v will be independent functions of s and σ in the region that neighbors the curve (C), and conversely, s and σ will be independent functions of u and v that admit only one determination in the neighborhood of the curve (C).

$$E\frac{\partial E}{\partial v} + F\frac{\partial E}{\partial u} - 2E\frac{\partial F}{\partial u} = 0,$$

to which one can give the following form:

$$\frac{\partial \sqrt{E}}{\partial v} = \frac{\partial}{\partial u}\frac{F}{\sqrt{E}}. \tag{20}$$

One can then set:

$$\sqrt{E} = \frac{\partial \theta}{\partial u}, \qquad \frac{F}{\sqrt{E}} = \frac{\partial \theta}{\partial v}, \qquad F = \frac{\partial \theta}{\partial u}\frac{\partial \theta}{\partial v}. \tag{21}$$

Upon substituting the values of E and F in the line element, one will give it the following form:

$$ds^2 = d\theta^2 + \frac{EG - F^2}{E}\, dv^2.$$

One then sees that the curves that are defined by the equation:

$$\theta = \int \left(\sqrt{E}\, du + \frac{F}{\sqrt{E}}\, dv \right) = \text{const.} \tag{22}$$

are the orthogonal trajectories of the geodesics considered. Therefore:

One can always determine the orthogonal trajectories of an arbitrary family of geodesics by a simple quadrature, and if one refers the points of the surface to the coordinate system that is defined by geodesics (v = const.) *and their orthogonal trajectories* (θ = const.) *then the line element will take the form:*

$$ds^2 = d\theta^2 + G\, dv^2. \tag{23}$$

The geometric interpretation is immediate.

Two arbitrary orthogonal trajectories will intercept the same arc length on all geodesics considered.

Those results are in perfect accord with the ones that we have obtained already.

In the case where the geodesic lines all pass through a point, there will obviously be orthogonal trajectories that remain infinitely-close to that point during a portion of their traversal, as seen from a point at a finite angle, and consequently the point itself can be included in an orthogonal trajectory, which proves Gauss's first theorem.

524. We see that the consideration of the geodesic lines leads us to new systems for which one must make:

$$A = 1$$

in the formulas of no. **499**. We point out the exceptionally simple form that the expression for total curvature will take in those systems. From formula (22) (no. **499**), one will have:

$$\frac{1}{RR'} = -\frac{1}{C}\frac{\partial^2 C}{\partial u^2}, \tag{24}$$

which is an expression that is due to Gauss, and which we will often have to make use of.

By analogy, we give the name of *parallel curves* to the orthogonal trajectories of a family of geodesics.

525. One can deduce a fundamental formula that relates to the variation of length of a segment of a geodesic line from the preceding results.

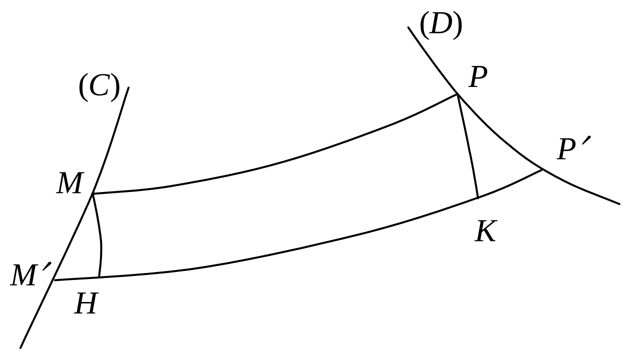

Figure 33.

Let (Fig. 33) *MP* be a segment of the geodesic line whose extremities *M* and *P* will describe two given curves (*C*) and (*D*). Employ the curvilinear coordinate system that is defined by successive positions *MP*, *M′ P′*, … of the segment, and their orthogonal trajectories. The line element will take the form (23) in that system; if u, u_0 denote the values of u at the points *M* and *P* then one will have:

$$\text{arc } MP = u - u_0 .$$

Similarly, if $u + du$, $u_0 + du_0$ are the values of u that correspond to the points *M′*, *P′* then one will have:

$$\text{arc } MP = u + du - u_0 - du_0 ,$$

which will give:

$$d \text{ arc } MP = du - du_0 .$$

Now, in the infinitely-small triangles *MM′H*, *PP′K* that are defined by the orthogonal trajectories *MH* and *PK* then one will have:

$$M'H = \quad du \;= -\,MM'\cos\,\widehat{M'MP}\,,$$
$$KP' \;= -\,du_0 = -\;PP'\cos\,\widehat{P'PM}\,,$$

and consequently the substitution of those values for du, du_0 leads to the following result:

$$d\text{ arc }MP = -\,MM'\cos\,\widehat{M'MP} - PP'\cos\,\widehat{P'PM}\,. \tag{25}$$

That formula is identical to the one that gives the differential of a line segment. Since it is easy to obtain directly by the calculus of variations, it can lead to Gauss's propositions along a path that is inverse to the one that we have followed.

526. Formula (25) permits one to extend a great number of propositions to geodesic lines that apply to systems of straight lines in the geometry of the plane. For example, one can formulate a theory on any surface that is analogous to that of the developables and developments of a plane curve. We shall leave to the reader the task of pursuing those generalizations, and we prefer to attach them to the following consequence of the fundamental formula.

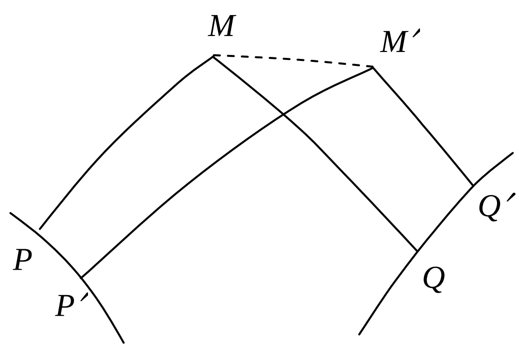

Figure 34.

Consider (Fig. 34) two curves (C), (C'), and look for the locus of points such that the sum or difference of their *geodesic distances* to those two curves is constant. If one drops *geodesic normals MP*, *MQ* from a point M of that locus to the two curves then one must have:

$$MP \pm MQ = \text{const.},$$

and as a result, when one passes from a point M of the locus to an infinitely-close point M', one will get:

$$d\,MP \pm d\,MQ = 0.$$

Formula (25) gives us:

$$d\,MP = -\,MM'\cos\,\widehat{M'MP}\,,$$
$$d\,MQ = -\,MM'\cos\,\widehat{M'MQ}\,.$$

Upon substituting those values for the differentials in the preceding relation, one will find that:

$$\cos \widehat{M'MP} \pm \cos \widehat{M'MQ} = 0.$$

In the case in which one takes the + sign, and in which consequently the sum of the distances is constant, the equation will express the idea that the tangent to the locus is the bisector of the angle that is defined by a geodesic line and prolongation of the other one. When one takes the – sign (i.e., when the difference between the distances is constant), the tangent will be the bisector of the angle that is defined by the two geodesic normals.

Upon comparing these two results, we will obtain the following theorem:

If one constructs all of the curves on an arbitrary surface that are loci of points for which the sum or difference of the geodesic distances to two given curves remains constant then one will obtain two families of curves that cut at a right angle in any case.

In what follows, we shall give the name of *geodesic ellipses* and *hyperbolas* to the curves that comprise those families. Their definition will not change if one substitutes arbitrary parallel curves for the two basic curves (C) and (C'). Nonetheless, one must remark that this change can transform the *ellipses* into *hyperbolas*, and *vice versa*.

527. We shall look for the form that the line element of the surface will take when one adopts the curvilinear coordinate system that we just defined. However, we shall take an oblique coordinate system that is defined by two series of parallel curves as an intermediary.

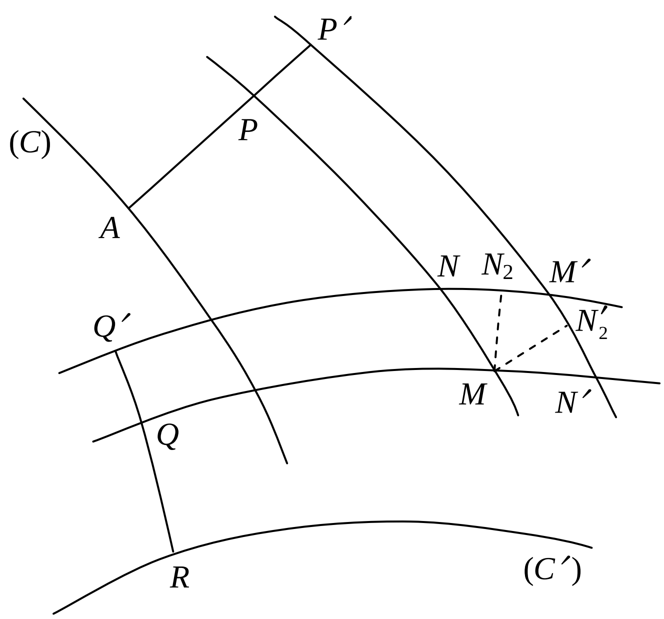

Figure 35.

Consider a first family of parallel curves that we shall define by their distance $u = AP$ to one of them (C), which is a distance that is measured along a normal geodesic.

Similarly, let there be a second family of parallel curves that we define by their distances $v = BQ$ to one of them (C').

Construct the four curves with parameters u, $u + du$, v, $v + dv$ that define a curvilinear parallelogram $MNM'N'$ whose angle M will be denoted by α and whose edges will have the values:

$$MN' = A\,du, \quad MN = C\,dv.$$

A and C are the quantities that figure in the expression:

$$ds^2 = A^2\,du^2 + C^2\,dv^2 + 2AC\cos\alpha\,du\,dv$$

for the line element. If one draws geodesics MN_1, MN_1' through the point M that are normal to the opposite edges of the parallelogram then the lengths of those geodesics will be:

$$MN_1 = dv, \qquad MN_1' = du.$$

In the triangles MNN_1, MN_1', which one can regard as rectilinear triangles, one will have:

$$MN_1 = MN\sin\alpha, \qquad MN_1' = MN'\sin\alpha,$$

i.e.:

$$du = A\,du\sin\alpha, \qquad dv = C\,dv\sin\alpha,$$

and as a result:

$$A = C = \frac{1}{\sin\alpha}.$$

The expression for the line element will then be:

$$ds^2 = \frac{du^2 + dv^2 + 2du\,dv\cos\alpha}{\sin^2\alpha}. \tag{26}$$

If one now takes:

$$u + v = 2u', \qquad u - v = 2v' \tag{27}$$

then the curves with parameters u', v' will be the geodesic ellipses and hyperbolas that were defined above, and the expression for the line element will take the form:

$$ds^2 = \frac{du'^2}{\sin^2\frac{\alpha}{2}} + \frac{dv'^2}{\cos^2\frac{\alpha}{2}}, \tag{28}$$

which exhibits the orthogonality that was proved already.

528. Weingarten, to whom the preceding result is due ([16]), established it by a different method, which we shall point out. Let:

$$ds^2 = E\,du^2 + 2F\,du\,dv + G\,dv^2$$

be the expression for the line element. Since u denotes the geodesic distance to a curve (C), the line element can be put into the form:

$$ds^2 = du^2 + \sigma^2\,du'^{\,2}.$$

It is then necessary that the difference:

$$ds^2 - du^2 = (E - 1)\,du^2 + 2F\,du\,dv + G\,dv^2$$

must be a perfect square. That will give us the condition:

$$F^2 = G\,(E - 1).$$

Upon likewise expressing the idea that v is the geodesic distance to a second curve, one will obtain the condition:

$$F^2 = E\,(G - 1).$$

Those two relations, when employed simultaneously, will give us:

$$E = G, \qquad F = \sqrt{E(E-1)}\,,$$

and the line element will take the form:

$$ds^2 = E\,(du^2 + dv^2) + 2\sqrt{E(E-1)}\;du\,dv. \tag{29}$$

One will recover the formula that we proved directly with geometry by replacing E with $1 / \sin^2 \alpha$.

529. The function α that figures in that formula will depend upon the nature of the surface, as well as the basic curves (C), (C'), and cannot be determined, in general. We shall point out two applications in which one will obtain the expression for α with no difficulty.

First, consider the bipolar coordinates in a plane. If one calls the distances from a point in the plane to two fixed points r, r' then formula (26) will give:

([16]) WEINGARTEN (J.), "Ueber die Oberflächen für welche einer der beiden Hauptkrümmungshalbmesser eine Function der anderen ist," Crelle's Journal **66** (1862), 160-173.

(30) $$ds^2 = \frac{dr^2 + dr'^2 + 2dr\,dr'\cos\alpha}{\sin^2\alpha}.$$

Let O, O' be two poles, and let M be the point considered. Let $2c$ denote the distance between the two poles. The triangle $OO'M$ will give us:

(31) $$4c^2 = r^2 + r'^2 + 2rr'\cos\alpha,$$

which is an equation from which we can infer α. However, we remark beforehand that if one sets:

(32) $$\begin{cases} r + r' = 2\mu, \\ r - r' = 2\nu \end{cases}$$

then the expression for the line element will become:

(33) $$ds^2 = \frac{d\mu^2}{\sin^2\frac{\alpha}{2}} + \frac{d\nu^2}{\cos^2\frac{\alpha}{2}},$$

and one will deduce the following equation from formula (31):

(34) $$c^2 = \mu^2\cos^2\frac{\alpha}{2} + \nu^2\sin^2\frac{\alpha}{2},$$

which will tell one the values of $\sin\frac{\alpha}{2}$, $\cos\frac{\alpha}{2}$. Upon substituting them in formula (33), one will have:

(35) $$ds^2 = (\mu^2 - \nu^2)\left(\frac{d\mu^2}{\mu^2 - c^2} + \frac{d\nu^2}{c^2 - \nu^2}\right).$$

The curves μ = const. are homofocal ellipses, and the curves ν = const. are hyperbolas with those same foci. One then sees that the elliptic coordinate system is only a modification – and in fact an advantageous one – of the bipolar coordinate system. That explains why the latter system is rarely employed.

If one similarly takes the bipolar coordinate system on the sphere, while always denoting the distance between the poles by $2c$, then one must substitute the following equation for equation (31):

(36) $$\cos 2c = \cos r\cos r' - \sin r\sin r'\cos\alpha,$$

which will give the spherical triangle $OO'M$ immediately. One can write:

$$\cos 2c = \cos 2\mu\cos^2\frac{\alpha}{2} + \cos 2\nu\sin^2\frac{\alpha}{2},$$

and consequently one will have:

$$(37) \qquad ds^2 = (\cos 2\mu - \cos 2\nu) \left(\frac{d\mu^2}{\cos 2\mu - \cos 2c} - \frac{dv^2}{\cos 2c - \cos 2v} \right)$$

for the sphere.

The coordinate curves are the homofocal ellipses and hyperbolas.

530. After those particular applications, we point out this proposition, which is due to Liouville, and which is a general consequence of formula (8) that relates to an arbitrary surface:

If one has two systems of geodesic lines on a surface that cut at a constant angle then the surface will be a plane or a developable.

Indeed, if we construct the parallel curves that are orthogonal trajectories to those geodesics then they as well will cut at a constant angle, and consequently the line element of the surface can be put into the form (26), or even in the form (28), in which the angle α is constant. Then set:

$$u' = x \sin \frac{\alpha}{2},$$

$$v' = y \cos \frac{\alpha}{2}.$$

What will remain is:

$$ds^2 = dx^2 + dy^2,$$

which indeed shows that the surface can be mapped to the plane.

CHAPTER V

FAMILIES OF PARALLEL CURVES

General method of searching for geodesic lines. – Definition of the differential parameter $\Delta\theta$. – Any function whose parameter is equal to 1 will define a family of parallel curves. – When that function contains an arbitrary constant, one can determine the geodesic lines of the surface. – Converse proposition: When one knows the geodesic lines, one can integrate the equation $\Delta\theta = 1$ by a simple quadrature. – Jacobi's theorem: When one has obtained a first integral of the second-order differential equation of the geodesic lines, one can always determine a factor of that integral. – Various consequences. – Expression for the line element in terms of the function θ and its derivatives with respect to the arbitrary constant *a*. – Third-order equation that the function θ must satisfy. – Description of another method that permits one to establish the preceding results. – Geodesic distance between two points. – Propositions relating to that distance.

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531. The propositions that were established in the preceding chapter lead to an elegant method for finding geodesic lines, which we shall present in all of its necessary detail.

Consider an arbitrary geodesic on the surface. One can obviously associate it with an infinitude of other geodesics – for example, all of the ones that pass through one of its points and constitute a system of curvilinear coordinates along with the orthogonal trajectories to those lines for which the line element will take the form:

$$ds^2 = d\theta^2 + \sigma^2 \, d\theta_1^2 .$$

One will then be assured of obtaining all of the geodesic lines if one knows how to solve the following problem of analysis in full generality:

Given the line element of a surface in its most general form:

$$ds^2 = E\, du^2 + 2F\, du\, dv + G\, dv^2,$$

determine three functions θ, σ, θ_1 of u and v such that one will have:

$$E\, du^2 + 2F\, du\, dv + G\, dv^2 = d\theta^2 + \sigma^2\, d\theta_1^2 \tag{1}$$

identically.

That equation obviously decomposes into the following three:

$$(2)\qquad \begin{cases} E = \left(\dfrac{\partial\theta}{\partial u}\right)^2 + \sigma^2\left(\dfrac{\partial\theta_1}{\partial u}\right)^2, \\ F = \dfrac{\partial\theta}{\partial u}\dfrac{\partial\theta}{\partial v} + \sigma^2 \dfrac{\partial\theta_1}{\partial u}\dfrac{\partial\theta_1}{\partial v}, \\ G = \left(\dfrac{\partial\theta}{\partial v}\right)^2 + \sigma^2\left(\dfrac{\partial\theta_1}{\partial v}\right)^2, \end{cases}$$

between which one can eliminate $\sigma\dfrac{\partial\theta_1}{\partial u}$ and $\sigma\dfrac{\partial\theta_1}{\partial v}$. One will then be led to the relation:

$$(3)\qquad G\left(\frac{\partial\theta}{\partial u}\right)^2 - 2F\,\frac{\partial\theta}{\partial u}\frac{\partial\theta}{\partial v} + E\left(\frac{\partial\theta}{\partial v}\right)^2 = EG - F^2,$$

which one can obtain immediately, moreover, by writing down the idea that the homogeneous polynomial in du, dv:

$$ds^2 - d\theta^2$$

is a perfect square. If one sets, to abbreviate:

$$(4)\qquad \Delta\theta = \frac{G\left(\dfrac{\partial\theta}{\partial u}\right)^2 - 2F\dfrac{\partial\theta}{\partial u}\dfrac{\partial\theta}{\partial v} + E\left(\dfrac{\partial\theta}{\partial v}\right)^2}{EG - F^2},$$

then equation (3) can also be written in the form:

$$(5)\qquad \Delta\theta = 1.$$

We frequently encounter the function $\Delta\theta$ in what follows, to which we – with Beltrami – will give the name of *first-order differential parameter of* θ.

Conversely, it is easy to prove that any solution of equation (5) will correspond to a family of parallel curves; i.e., curves whose orthogonal trajectories are geodesic lines.

Indeed, as we have seen, equation (5) will express the idea that $ds^2 - d\theta^2$ is a perfect square; one will then have:

$$ds^2 - d\theta^2 = (m\,du + n\,dv)^2$$

for any differentials du, dv, in which m and n are functions of u and v. Now, one can always convert the linear function $m\,du + n\,dv$ to the form $\sigma\,d\theta_1$. One will then have:

$$ds^2 = d\theta^2 + \sigma^2\,d\theta_1^2,$$

θ_1 will be a distinct function of θ, moreover, because otherwise ds^2 would be a perfect square. Our converse proposition is then established, and any solution of equation (5) will give us a family of parallel curves.

532. As one knows, equation (5) will have various types of solutions. One can find some that contain no arbitrary constant, while others contain one or more arbitrary constants, or even an arbitrary function. From the standpoint of the question that we are addressing, it is essential to consider these various solutions in succession.

If one has obtained a solution to equation (5) that contains no arbitrary constant then an application of the preceding method, which prescribes that one must put the differential $m\,du + n\,dv$ into the form $\sigma\,d\theta_1$, will demand the integration of the equation:

$$m\,du + n\,dv = 0, \tag{6}$$

which is that of the geodesic lines that are orthogonal trajectories of the curves θ = const. One does not know of any proposition that permits one to perform the integration of that equation or render it the simplest one.

On the contrary, suppose that one has obtained a solution to the partial differential equation (5) that contains a constant besides the one that can always be combined with θ by addition, *which is a constant that must consequently figure in at most one of the two derivatives* $\frac{\partial \theta}{\partial u}$, $\frac{\partial \theta}{\partial v}$. We shall see that in this case (to which one can convert all of the ones in which the function contains several constants or an arbitrary function), one can obtain σ and θ_1 by simple derivations, and in turn, the finite equations of geodesic lines that are the orthogonal trajectories of the curves θ = const.

Indeed, recall the identity:

$$ds^2 = d\theta^2 + \sigma^2\,d\theta_1^2\,.$$

That equation is true for five variables: namely, u, v, du, dv, and the arbitrary constant that enters into θ, which we shall denote by a. Differentiate with respect to a while treating the other *four* variables as constants. The differential of θ:

$$d\theta = \frac{\partial \theta}{\partial u}du + \frac{\partial \theta}{\partial v}dv$$

will become:

$$\frac{\partial^2 \theta}{\partial a\,\partial u}du + \frac{\partial^2 \theta}{\partial a\,\partial v}dv = d\,\frac{\partial \theta}{\partial a},$$

and one will have an analogous result for θ_1. Since ds^2 does not contain a, we will then have:

$$0 = d\theta\,d\left(\frac{\partial \theta}{\partial a}\right) + \sigma\,d\theta_1\left[\frac{\partial \sigma}{\partial a}d\theta_1 + \sigma d\left(\frac{\partial \theta_1}{\partial a}\right)\right]. \tag{7}$$

The preceding equation shows us that $d\theta_1$, which is a linear function of du, dv, must divide either $d\theta$ or $d\left(\frac{\partial\theta}{\partial a}\right)$. Now, $d\theta_1$ cannot divide $d\theta$, because θ_1 would then be a function of θ. It is then necessary that $d\theta_1$ must divide $d\frac{\partial\theta}{\partial a}$. θ_1 will then be a function of $\frac{\partial\theta}{\partial a}$, and consequently one can take:

$$\theta_1 = \frac{\partial\theta}{\partial a}.$$

Before deducing some other consequences of equation (7), we shall stop with that first result. We see that the geodesics lines that cut the curves θ = const. at a right angle have the equation:

$$\frac{\partial\theta}{\partial a} = \text{const.} = a', \tag{8}$$

and that their arc length is equal to θ precisely when one measures it from one of their trajectories.

Equation (8) contains two arbitrary constants that one can arrange in such a manner that it makes the geodesic line pass through an arbitrary point and give it an arbitrary tangent at that point. In order to establish that essential point in full rigor, we show that one can make one of the curves:

$$\theta = \text{const.}$$

pass through an arbitrary point (u_0, v_0) and give it a well-defined tangent at that point.

We first remark that the ratio:

$$\frac{\partial\theta}{\partial u} : \frac{\partial\theta}{\partial v}$$

cannot be independent of a. Indeed, if that were true then if one had:

$$\frac{\partial\theta}{\partial u} = \frac{\partial\theta}{\partial v} f(u, v),$$

then upon appending the equation:

$$\Delta\theta = 1$$

to that equation, one could determine the values of $\frac{\partial\theta}{\partial u}$, $\frac{\partial\theta}{\partial v}$, which would both be independent of a, which would be contrary to hypothesis.

Having said that, consider the curve (θ) that is defined by the equation:

$$\theta(u, v, a) = \theta(u_0, v_0, a) = \theta_0 .$$

It obviously passes through the point (u_0, v_0), and the direction of its tangent at that point will depend upon the ratio $\frac{\partial\theta}{\partial u} : \frac{\partial\theta}{\partial v}$. Since that ratio is not independent of a, it can take on all possible values. Hence, the curves θ = const. can pass through an arbitrary point of the surface and admit an arbitrary tangent there. The same thing will then be true for the geodesic lines that are represented by equation (8), which are their orthogonal trajectories. Since a geodesic line is determined by the condition that it must pass through a point and admit a given tangent, we can say that equation (8) represents all of the geodesic lines and state the following theorem:

In order to determine the geodesic lines, one considers the partial differential equation:

$$\Delta\theta = 1.$$

Any solution to that equation will determine a family of parallel curves when it is equated to a constant.

If one has a solution that contains an arbitrary constant a then the equation of the most general geodesic line will be:

$$\frac{\partial\theta}{\partial a} = a',$$

and the arc length that is included along that geodesic line between two points will be equal to the difference of the values of θ that relate to those points.

Conversely, suppose that one has determined the geodesic lines by any procedure. We shall show that one will have to integrate the equation:

$$\Delta\theta = 1.$$

For example, look for the solution θ to that equation that is equal to zero at all points of a curve (C) that is given in advance. One will construct all of the geodesic lines that are normal to (C). The arc length of one of those lines, when measured by starting on (C), will be a function of the coordinates of its extremity that one will obtain by a quadrature, and which will be the desired solution. That remark, when suitably extended, is very important in the theory of partial differential equations. Here, at least, it permits us to recognize that the method of searching for geodesic lines that is established by the preceding theorem will introduce no difficulty that is foreign to the problem.

533. To begin with, we point out the following consequence of the general theory that we just developed:

Imagine that one knows a differential equation:

$$\frac{dv}{du} = v' = \varphi(u, v) \tag{9}$$

such that all of its particular integrals are geodesic lines and look for the differential equation of their orthogonal trajectories. Upon applying the formula:

$$E\,du\,\delta v + F\,(du\,\delta v + dv\,\delta u) + G\,dv\,\delta v = 0,$$

which expresses the orthogonality of the two directions, one will obtain the desired solution in the form:

(10) $$(E + F\,v')\,du + (F + G\,v')\,dv = 0,$$

in which v' is replaced by its value that one infers from equation (9).

Now, one knows that one can find a factor λ such that the product:

$$\lambda\,[(E + F\,v')\,du + (F + G\,v')\,dv]$$

is the differential of a function θ that satisfies the equation:

$$\Delta\theta = 1.$$

If one then sets:

$$\frac{\partial\theta}{\partial u} = \lambda\,(E + F\,v'), \qquad \frac{\partial\theta}{\partial v} = \lambda\,(F + G\,v'),$$

and if one expresses the idea that the preceding partial differential equation is verified then one will obtain the value of λ, which will be:

$$\lambda = \frac{1}{\sqrt{E + 2Fv' + Gv'^2}}.$$

One can then state the following theorem, which was established in a different form in no. **523**, moreover.

If the differential equation:

$$\frac{dv}{du} = v' = \varphi(u, v)$$

represents geodesic lines then the expression:

$$\frac{(E + Fv')\,du + (F + Gv')\,dv}{\sqrt{E + 2Fv' + Gv'^2}}$$

will be the exact differential of a function θ*; the equation:*

$$\theta = \text{const.}$$

will represent the orthogonal trajectories of geodesic lines that satisfy the proposed differential equation, and θ will denote the geodesic distance from an arbitrary point of the surface along one of its orthogonal trajectories.

That proposition will lead us to a beautiful theorem by Jacobi:

If one knows a first integral of the differential equation of the geodesic lines then one can obtain the equation in finite terms for those lines by a simple quadrature.

Indeed, let:

$$v' = \varphi(u, v, a) \tag{11}$$

be the first integral, which contains the constant a. From what we just proved, the function:

$$\theta = \int \frac{(E + Fv')\,du + (F + Gv')\,dv}{\sqrt{E + 2Fv' + Gv'^2}} \tag{12}$$

will satisfy the equation $\Delta\theta = 1$, which contains the arbitrary constant a. Therefore, the equation of the geodesic lines will be:

$$\frac{\partial\theta}{\partial a} = a'.$$

Upon taking the derivative with respect to a under the integral sign, one will find that:

$$\frac{\partial\theta}{\partial a} = \int \frac{(EG - F^2)\dfrac{\partial v'}{\partial a}}{(E + 2Fv' + Gv'^2)^{3/2}}(dv - v\,du), \tag{13}$$

which will permit one to state the following theorem:

When one has obtained a first integral of the equation of geodesic lines

$$v' = \varphi(u, v, a)$$

by any means, one can immediately determine a factor from it in such a way that:

$$\frac{(EG - F^2)\dfrac{\partial v'}{\partial a}}{(E + 2Fv' + Gv'^2)^{3/2}}(dv - v\,du)$$

will become an exact differential after one has replaced v' with its value $\varphi(u, v, a)$.

534. We shall now point out some less important consequences of the results that were obtained, and in particular, equation (7). If one replaces θ_1 with $\frac{\partial \theta}{\partial a}$ in it, and one divides $d\theta_1$ then it will take the form:

$$d\theta_1 + \sigma \frac{\partial \sigma}{\partial a} d \frac{\partial \theta}{\partial a} + \sigma^2 \; d \frac{\partial^2 \theta}{\partial a^2} = 0.$$

That relation must be true for all values of du and dv, so one replaces du, dv in it with $\frac{\partial^2 \theta}{\partial a \, \partial v}$ and $-\frac{\partial^2 \theta}{\partial a \, \partial u}$, respectively. If one denotes the functional determinant:

$$\frac{\partial \alpha}{\partial u}\frac{\partial \beta}{\partial v} - \frac{\partial \alpha}{\partial v}\frac{\partial \beta}{\partial u}$$

by (α, β), to abbreviate, then one will have:

$$\left(\theta, \frac{\partial \theta}{\partial a}\right) + \sigma^2 \left(\frac{\partial^2 \theta}{\partial a^2}, \frac{\partial \theta}{\partial a}\right) = 0,$$

and in turn:

(14) $$\sigma^2 = \frac{\left(\theta, \frac{\partial \theta}{\partial a}\right)}{\left(\frac{\partial \theta}{\partial a}, \frac{\partial^2 \theta}{\partial a^2}\right)}.$$

The expression for the line element will then become:

(15) $$ds^2 = d\theta^2 + \frac{\left(\theta, \frac{\partial \theta}{\partial a}\right)}{\left(\frac{\partial \theta}{\partial a}, \frac{\partial^2 \theta}{\partial a^2}\right)} \left(d \frac{\partial \theta}{\partial a}\right)^2,$$

and:

In that new form, no trace will remain of the original expression for that element. The formula will contain only θ and its derivatives.

We likewise point out the formulas:

(16) $$ds^2 = d\theta^2 + \frac{1}{\Delta \frac{\partial \theta}{\partial a}} \left(d \frac{\partial \theta}{\partial a}\right)^2,$$

(17) $$\sigma^2\left(\theta, \frac{\partial\theta}{\partial a}\right)^2 = EG - F^2,$$

which one will easily deduce from the relations (2). However, the latter one is distinguished from the preceding one by the fact that it contains both the coefficients *E*, *F*, *G*, and the derivatives of θ.

The combination of formulas (14) and (17) gives us the new relation:

(18) $$\frac{\left(\theta, \frac{\partial\theta}{\partial a}\right)}{\left(\frac{\partial\theta}{\partial a}, \frac{\partial^2\theta}{\partial a^2}\right)} = EG - F^2,$$

and if one takes the logarithmic derivative of the two sides with respect to *a* then one will obtain the equation:

(19) $$3\left(\frac{\partial\theta}{\partial a}, \frac{\partial^2\theta}{\partial a^2}\right)\left(\frac{\partial\theta}{\partial a}, \frac{\partial^2\theta}{\partial a^2}\right) - \left(\theta, \frac{\partial\theta}{\partial a}\right)\left(\frac{\partial\theta}{\partial a}, \frac{\partial^2\theta}{\partial a^2}\right) = 0,$$

which no longer contains *E*, *F*, *G*. That relation, which one can arrive at in various ways, must be regarded as a third-order partial differential equation that θ must satisfy when it is considered to be a function of the variables *u*, *v*, and *a*. Its complete integration will then tell one all of the surfaces on which one has to determine the geodesic lines.

535. Suppose that the line element of the surface is expressed as a function of the variables θ and $\theta_1 = \frac{\partial\theta}{\partial a}$. We have seen (no. **524**) that the expression for the total curvature will be given by Gauss's formula:

$$\frac{\sigma}{RR'} = -\frac{\partial^2\sigma}{\partial\theta^2}.$$

In the deeper study of the shortest path between two points on a surface, we have to consider the second-order differential equation:

(20) $$\frac{\partial^2\omega}{\partial\theta^2} + \frac{\omega}{RR'} = 0.$$

The preceding formula will show a particular first integral of that equation:

$$\omega = \sigma.$$

Another integral will then be given by the formula:

$$\sigma \int \frac{d\theta}{\sigma^2},$$

in which one performs the quadrature while supposing that θ_1 is constant. One will then have:

$$\frac{\partial \theta_1}{\partial u} du + \frac{\partial \theta_1}{\partial v} dv = 0.$$

If one replaces $\frac{\partial \theta_1}{\partial u}, \frac{\partial \theta_1}{\partial v}$ with the proportional quantities – dv, du in the expression (14) then that will give:

$$\sigma^2 = -\frac{d\theta}{d\dfrac{\partial^2 \theta}{\partial a^2}},$$

and in turn:

$$-\sigma \int \frac{d\theta}{\sigma^2} = \sigma \int d\frac{\partial^2 \theta}{\partial a^2} = \sigma \frac{\partial^2 \theta}{\partial a^2}.$$

The second integral of the linear equation (20) will then be:

$$\omega = \sigma \frac{\partial^2 \theta}{\partial a^2}, \tag{21}$$

as one can verify directly.

536. The proposition that we just established was obtained by considering orthogonal systems that are defined by a family of geodesic lines. Upon concluding this chapter, we shall rapidly describe a very different method that rests upon the calculus of variations, and which offers the advantage of exhibiting a very important element in the theory of geodesic lines.

Consider a segment of the geodesic line that is terminated by two points M, M_0. If the coordinates u and v of an arbitrary point of that segment are expressed as functions of another variable t then the length θ of that segment will be given by the formula:

$$\theta = \int_{M_0}^{M} \sqrt{E\,u'^2 + 2F\,u'v' + G\,v'^2}\; dt,$$

in which u' and v' denote the derivatives of u and v.

If the points M, M_0 are displaced while describing arbitrary curves then the application of the methods of the calculus of variations will immediately give us the variation of θ by the formula:

(22)
$$\delta\theta = \left[\frac{(E\,u' + F\,v')\,\delta u + (F\,u' + G\,v')\,\delta v}{\sqrt{E\,u'^2 + 2F\,u'v' + G\,v'^2}}\right]_{M_0}^{M},$$

in which the preceding notation indicates that one must take the difference of the values of the expression for the points M and M_0. From the relation (10) that was given already [I, pp. 136], one can write the value of θ in the form:

(23)
$$\delta\theta = [\delta s \cos(ds, \delta s)]_{M_0}^{M},$$

and one will then recover the relation that was established already in no. **525** along an entirely analytical path. However, we shall imagine some other consequences of equation (22).

Let u, v; u_0, v_0 be the coordinates of the points M, M_0. The value of θ can obviously be expressed as a function of u, v; u_0, v_0. From the results of no. **518**, it will even be a perfectly well-defined function of those four variables as long as the points M and M_0 are sufficiently close if one agrees to take the shortest geodesic line that connects the two points. In what follows, we shall refer to that function θ by the name of the *geodesic distance between the two points M, M_0*.

Now, if one denotes the values of E, F, G at the point M_0 (i.e., for $u = u_0$, $v = v_0$) by E_0, F_0, G_0 then formula (22) can be written as follows:

(24)
$$\delta\theta = \frac{(E\,u' + F\,v')\,\delta u + (F\,u' + G\,v')\,\delta v}{\sqrt{E\,u'^2 + 2F\,u'\,v' + G\,v'^2}} - \frac{(E\,u'_0 + F_0\,v'_0)\,\delta u_0 + (F_0\,u'_0 + G_0\,v'_0)\,\delta v_0}{\sqrt{E_0\,u'^2_0 + 2F_0\,u'_0\,v'_0 + G_0\,v'^2_0}},$$

and consequently it will give us the four equations:

(25)
$$\begin{cases} \dfrac{\partial\theta}{\partial u} = \dfrac{E\,u' + F\,v'}{\sqrt{E\,u'^2 + 2F\,u'\,v' + G\,v'^2}} = E\dfrac{du}{ds} + F\dfrac{dv}{ds}, \\[2ex] \dfrac{\partial\theta}{\partial v} = \dfrac{F\,u' + G\,v'}{\sqrt{E\,u'^2 + 2F\,u'\,v' + G\,v'^2}} = F\dfrac{du}{ds} + G\dfrac{dv}{ds}, \end{cases}$$

(26)
$$\begin{cases} \dfrac{\partial\theta}{\partial u_0} = \dfrac{E_0\,u'_0 + F_0\,v'_0}{\sqrt{E_0\,u'^2_0 + 2F_0\,u'_0\,v'_0 + G_0\,v'^2_0}} = -E_0\left(\dfrac{du}{ds}\right)_0 - F_0\left(\dfrac{dv}{ds}\right)_0, \\[2ex] \dfrac{\partial\theta}{\partial v_0} = \dfrac{F_0\,u'_0 + G_0\,v'_0}{\sqrt{E_0\,u'^2_0 + 2F_0\,u'_0\,v'_0 + G_0\,v'^2_0}} = -F_0\left(\dfrac{du}{ds}\right)_0 - G_0\left(\dfrac{dv}{ds}\right)_0, \end{cases}$$

from which, one will immediately deduce, upon eliminating u', v', and u'_0, v'_0, the two equations:

$$(27) \qquad \begin{cases} \Delta\theta = 1, \\ \Delta_0\theta = 1, \end{cases}$$

in which Δ_0 denotes the symbol Δ where one has replaced u, v with u_0, v_0, resp., and $\frac{\partial\theta}{\partial u}$, $\frac{\partial\theta}{\partial v}$ with $\frac{\partial\theta}{\partial u_0}$, $\frac{\partial\theta}{\partial v_0}$, resp.

Those are the properties of the *geodesic distance* θ. When one knows that function, the two equations (26), which reduce to just one by virtue of the second of formulas (27), will give the equation of the geodesic line that passes through the point (u_0, v_0) and admits a well-defined tangent in the most elegant form. The equation:

$$\theta = \text{const.}$$

will represent the orthogonal trajectories of all of the geodesic lines that pass through the point (u_0, v_0).

537. Once one has obtained the equation:

$$(28) \qquad \Delta\theta = 1,$$

one can treat the problem of geodesic lines as another problem in mechanics and apply the methods of Hamilton and Jacobi without modification. One will then recover all of the preceding results. In the following chapters, we shall study the relationships between the theory of geodesic lines and the methods of analytical mechanics that are presented here in a deeper manner, and we shall now content ourselves by showing how one determines the geodesic distance when one knows a complete integral (which is arbitrary, moreover) of the partial differential equation (28).

Let:

$$\theta = f(u, v, a)$$

be that solution. The geodesic lines of the surface that passes through the points (u_0, v_0) will be determined by the equation:

$$(29) \qquad \frac{\partial}{\partial a} f(u, v, a) = \frac{\partial}{\partial a} f(u_0, v_0, a),$$

and the arc length that is included between the points (u_0, v_0), (u, v) will have the expression (no. **532**):

$$(30) \qquad \theta = f(u, v, a) - f(u_0, v_0, a).$$

If one wishes to obtain the desired geodesic distance then it will suffice to substitute the value of a that is deduced from equation (29) in that expression. Therefore:

When one knows an integral with constant:

$$\theta = f(u, v, a)$$

of the equation:

$$\Delta\theta = 1,$$

the geodesic distance between the two points (u, v), (u_0, v_0) *will be obtained by eliminating a from the equation:*

$$\theta = f(u, v, a) - f(u_0, v_0, a)$$

and its derivative with respect to a.

That proposition can also be established by geometry, because the rule that it points to amounts to taking the envelope of all parallel curves:

$$f(u, v, a) = \text{const.}$$

that pass at the same distance θ from the point (u_0, v_0).

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CHAPTER VI

ANALOGIES BETWEEN THE DYNAMICS OF MOTIONS IN THE PLANE AND THE THEORY OF GEODESIC LINES

Equations of motion in the plane. – Definition of a family of trajectories. – Jacobi's partial differential equation. – Use that one can make of a particular solution and a complete solution. – Jacobi's fundamental theorems. – Determination of the solutions to the partial differential equations by various initial conditions. – Application to the motion of ponderous bodies. – Thomson and Tait's theorem. – Principle of least action for the case of planar motions. – Hamilton's principle. – Correspondence that is established between the plane and a surface in such a manner that the trajectories of things that move in the plane correspond to geodesic lines on the surface. – The solution to any problem in mechanics exhibits an infinitude of orthogonal systems in the plane. – Brachistochrones. – Some general results that relate to the case in which one associates trajectories that do not correspond to the same value of the *vis viva* const. – Generalization of those results and application to the theory of minimal surfaces.

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538. In the preceding two chapters, we established a set of properties of geodesic lines. We first defined them by the property of their osculating plane, which amounts to considering them to be the trajectories of a point that moves on the surface without being subject to the action of any force. Then, by entirely elementary considerations, we attached the properties of orthogonality and minimization to that definition. It now seems interesting to apply the same method to the study of all problems in mechanics in which there exists a force function. In order to exhibit the simplicity of those arguments, we begin with the motions that are performed in the plane.

One then has the equations:

$$\frac{d^2x}{dt^2} = \frac{\partial U}{\partial x}, \qquad \frac{d^2y}{dt^2} = \frac{\partial U}{\partial y}, \tag{1}$$

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = 2\,(U + h), \tag{2}$$

the latter of which is the *vis viva* integral. *If we regard the vis viva constant as having been given* then the integrals of the preceding equations will admit only two arbitrary constants besides the one that one can add to time. In other words, the trajectory of a material point will be determined by the condition that it must pass through a point and that it must have a given tangent. Indeed, if one measures time by starting at the moment when the moving body passes through that point then the stated condition will determine the initial values of x, y, dy / dx, and since the equation of *vis viva* gives the values of dx / dt, dy / dt as functions of dy / dx, one can calculate the initial values dx / dt, dy / dt. The motion is then determined completely. We remark that one has two systems of values for dx / dt, dy / dt that are equal and opposite in sign and correspond to the same trajectory when it is traversed in the two senses.

Moreover, one can obtain the differential equation of the trajectories by an easy calculation. Indeed, upon combining equations (1), one will find that:

$$dx\, d^2y - dy\, d^2x = \left(\frac{\partial U}{\partial y} dx - \frac{\partial U}{\partial x} dy\right) dt^2,$$

and upon replacing dt^2 with its values that one infers from the *vis viva* equation:

$$(3) \qquad dx\, d^2y - dy\, d^2x = \left(\frac{\partial U}{\partial y} dx - \frac{\partial U}{\partial x} dy\right) \frac{dx^2 + dy^2}{2(U+h)}.$$

That relation will not change form when the time ceases to be the independent variable, as one easily recognizes. It will then constitute the differential equation of the trajectories that correspond to a given value of the *vis viva* constant. Since it has order two, one will see that the trajectories depend upon only two constants. However, it is, moreover, linear with respect to the second-order differentials, and consequently a trajectory will clearly be determined by the condition that it must pass through a point and have a given tangent.

Among all of the motions that correspond to the same value of h, consider all of the ones whose trajectories satisfy a condition – for example, that they pass through a point, they are normal to a curve, etc. Those trajectories will define a family of curves that will depend upon just one parameter; a limited number of them will pass through each point in the plane. Let:

$$M\, dx + N\, dy = 0$$

be the differential equation of that family of curves. Upon taking the *vis viva* equation into account, one can express dx / dt, dy / dt as functions of x and y. One will have:

$$\frac{dx}{N} = \frac{dy}{-M} = \frac{dt\sqrt{2(U+h)}}{\sqrt{N^2 + M^2}}.$$

One can then consider dx / dt, dy / dt to be functions of x' and y'. Upon substituting them in equations (1) and (2), and denoting them by x and y, to abbreviate, one will have:

$$(4) \qquad x'^2 + y'^2 = 2\,(U + h),$$

$$\frac{dx'}{dt} = \frac{\partial U}{\partial x}, \qquad \frac{dy'}{dt} = \frac{\partial U}{\partial y},$$

or, upon remarking that x' and y' are expressed as functions of x and y:

(5)
$$\begin{cases} \dfrac{\partial x'}{\partial x}x' + \dfrac{\partial x'}{\partial y}y' = \dfrac{\partial U}{\partial x}, \\ \dfrac{\partial y'}{\partial x}x' + \dfrac{\partial y'}{\partial y}y' = \dfrac{\partial U}{\partial y}. \end{cases}$$

Upon differentiation, equation (4) will give us the following values for $\dfrac{\partial U}{\partial x}$, $\dfrac{\partial U}{\partial y}$:

$$\frac{\partial U}{\partial x} = x'\frac{\partial x'}{\partial x} + y'\frac{\partial x'}{\partial y}, \qquad \frac{\partial U}{\partial y} = x'\frac{\partial y'}{\partial x} + y'\frac{\partial y'}{\partial y}.$$

If we substitute those values in the equations (5) then we will have:

$$y'\left(\frac{\partial x'}{\partial y} - \frac{\partial y'}{\partial x}\right) = 0, \qquad x'\left(\frac{\partial x'}{\partial y} - \frac{\partial y'}{\partial x}\right) = 0.$$

Those two equations, which reduce to each other, express the idea that x', y', when considered to be functions of x and y, are the derivatives of the same function. One can then set:

(6)
$$x' = \frac{\partial \theta}{\partial x}, \qquad y' = \frac{\partial \theta}{\partial y},$$

and θ must satisfy the *single* equation:

(7)
$$\left(\frac{\partial \theta}{\partial x}\right)^2 + \left(\frac{\partial \theta}{\partial y}\right)^2 = 2\,(U + h).$$

539. If one has only a particular solution with no arbitrary constant of the partial differential equation then one will obtain only a family of trajectories. In order to find all of the trajectories of the moving body, one must then know a solution θ that contains at least one arbitrary constant. We shall further show here that if one is given such a solution then one will not have to integrate in order to obtain all of the trajectories.

Indeed, let:

$$\theta = f(x, y, a)$$

be a solution to equation (7) that contains an arbitrary constant a and figures in at least one of the two derivatives $\dfrac{\partial \theta}{\partial x}$, $\dfrac{\partial \theta}{\partial y}$. Differentiating equation (7) with respect to a will give:

(9) $$\frac{\partial\theta}{\partial x}\frac{\partial^2\theta}{\partial a\,\partial x}+\frac{\partial\theta}{\partial y}\frac{\partial^2\theta}{\partial a\,\partial y}=0.$$

That equation expresses the idea that the curves:

$$\theta = \text{const.},\qquad \frac{\partial\theta}{\partial a} = \text{const.}$$

cut at a right angle. Thus, the trajectories of the moving body will have the equation:

(10) $$\frac{\partial\theta}{\partial a}=a'.$$

One will also see that if one remarks that the identity (9) can also be written:

$$\frac{\partial^2\theta}{\partial a\,\partial x}\frac{dx}{dt}+\frac{\partial^2\theta}{\partial a\,\partial y}\frac{dy}{dt} = 0 = \frac{d}{dt}\left(\frac{\partial\theta}{\partial a}\right).$$

If we now differentiate equation (7) with respect to h then we will get:

$$\frac{\partial^2\theta}{\partial h\,\partial x}\frac{\partial\theta}{\partial x}+\frac{\partial^2\theta}{\partial h\,\partial y}\frac{\partial\theta}{\partial y} = 1,$$

or even, by virtue of equations (6):

$$\frac{\partial^2\theta}{\partial h\,\partial x}\frac{dx}{dt}+\frac{\partial^2\theta}{\partial h\,\partial y}\frac{dy}{dt} = 1.$$

The integration of the two sides gives us:

(11) $$\frac{\partial\theta}{\partial x} = t + \tau,$$

in which τ denotes an arbitrary constant. We recognize the fundamental propositions of Jacobi.

In summary, if one would like to determine the motion that is defined by the equations:

$$\frac{d^2x}{dt^2}=\frac{\partial U}{\partial x},\qquad \frac{d^2y}{dt^2}=\frac{\partial U}{\partial y}$$

then one should consider the partial differential equation:

$$\left(\frac{\partial\theta}{\partial x}\right)^2 + \left(\frac{\partial\theta}{\partial y}\right)^2 = 2U + 2h.$$

Any integral of that equation, when equated to a constant, will give a family of curves whose orthogonal trajectories will be trajectories of the moving body that correspond to the value h of the *vis viva* constant, and that one will obtain by integrating the two equations:

$$\frac{dx}{dt} = \frac{\partial\theta}{\partial x}, \qquad \frac{dy}{dt} = \frac{\partial\theta}{\partial y}.$$

However, if one knows an integral to the partial differential equation that contains a constant a then one will get the finite equations of the trajectory and time by the formulas:

$$\frac{\partial\theta}{\partial a} = a', \qquad \frac{\partial\theta}{\partial h} = t + \tau.$$

One will then obtain the geometric interpretation of the Jacobi method. It consists of defining orthogonal systems such that one of the families is composed of different trajectories of the moving body that all correspond to the same value of the *vis viva* constant.

540. From the foregoing, we see that when one has found a solution that contains an arbitrary constant of the partial differential equation in θ, one can obtain the complete solution to the problem of mechanics that is being considered. Conversely, if one has obtained the equations in finite terms of all of the trajectories that correspond to a well-defined value of h by whatever means then one can show that all of the solutions of the partial differential equation are obtained by a simple quadrature. For example, one can look for the solutions that are annulled along a curve (C) that is given in advance. We determine all of the trajectories of the moving body that are normal to the curve (C) and we express x', y' as functions of x and y. As we have seen, the expression:

$$x'\,dx + y'\,dy$$

will be the exact differential of a function of two variables, and the function:

$$\theta = \int_{x_0, y_0}^{x, y} (x'\,dx + y'\,dy), \tag{12}$$

in which x_0, y_0 denote the coordinates of an arbitrary point of the curve (C), will obviously be the desired solution. Since one can take x', y' to be two systems of values that are equal and opposite in sign, one will have two values for θ that differ by only their sign.

One knows that one can convert the integration of a differential in several variables to that of an ordinary differential in an infinitude of ways. We apply that remark here.

Suppose that one displaces along the trajectory that is normal to the curve (C) that passes through the point (x, y). In that case, x_0, y_0 will be the coordinates of the point of departure of that trajectory. One will have:

$$x' \, dx + y' dy = (x'^2 + y'^2) \, dt = 2 \, (U + h) \, dt.$$

Calculate the integral:

$$\int_{x_0, y_0}^{x, y} 2 \, (U + h) \, dt,$$

after replacing x and y with functions of time. The result will be a function of t and the parameter that fixes the position of the point (x_0, y_0) along the curve (C). It will suffice to express it as a function of only x and y in order to obtain the function θ. If one supposes that the curve (C) diminishes indefinitely and reduces to a point then that second method will coincide with the one that was given by Jacobi, because all of the trajectories that are normal to (C) will then be transformed into the trajectories that pass through a fixed point in the plane.

541. The orthogonal systems that we just defined, and for which one of the families is composed of a series of trajectories of the moving body, play an important role in the study of certain questions, as we shall show. However, before we do that, we must show how one can obtain all of them with no new integration when one knows a complete solution to the partial differential equation (7).

Let:

$$\theta = f(x, y, a) + b \tag{13}$$

be one such solution. Here is the method that was prescribed by Lagrange for obtaining the most general solution: One sets:

$$b = \varphi(a),$$

in which $\varphi(a)$ denotes an arbitrary function of a. The result of the elimination of a from the equations:

$$\begin{cases} \theta = f(x, y, a) + \varphi(a), \\ 0 = \dfrac{\partial f}{\partial a} + \varphi'(a) \end{cases} \tag{14}$$

will provide the required solution. We can add the following remark here, which is easily confirmed: Let:

$$\theta = F(x, y)$$

be the solution thus-obtained. The orthogonal trajectories of the curves:

$$\theta = \text{const.}$$

will be defined by the second of equations (14):

$$\frac{\partial f}{\partial a} + \varphi'(a) = 0, \tag{15}$$

in which one gives all possible values to the constant a.

542. Once one has accepted those conditions, suppose that one would like to determine the orthogonal system such that one of its families is composed of trajectories that are normal to a given curve (C).

That problem is obviously equivalent to the following one:

Find a solution θ to the partial differential equation that takes a given constant value (zero, for example) at all points of the curve (C).

Let:

$$y = \lambda(x)$$

be the equation of that curve. We propose to determine the function θ that will reduce to $\mu(x)$ when one has:

$$y = \lambda(x)$$

in a more general manner.

Upon substituting the values of θ and y in equations (14), one will find the equations of condition:

$$\left\{\begin{aligned} \mu(x) &= f(x, \lambda, a) + \varphi(a), \\ 0 &= \frac{\partial f}{\partial a} + \varphi'(a), \end{aligned}\right. \tag{16}$$

which will exhibit the function $\varphi(a)$. On first glance, it seems that in order to solve the question that was posed, one must integrate a differential equation, because if one eliminates x from the two equations (16) then one will be led to an equation of the form:

$$\mathcal{F}(a, \varphi(a), \varphi'(a)) = 0.$$

However, if one differentiates the first of equations (16) then upon taking the second one into account, one will find that:

$$\frac{\partial}{\partial x} f(x, \lambda, a) + \frac{\partial f}{\partial \lambda} \lambda'(x) = \mu'(x).$$

It is easy to show that one can substitute the following system for the system (16):

$$(17)\qquad \begin{cases} f(x, \lambda, a) + \varphi(a) = \mu(x), \\ \dfrac{\partial f}{\partial x} + \dfrac{\partial f}{\partial \lambda}\lambda'(x) = \mu'(x). \end{cases}$$

Indeed, those two systems have an equation in common, and the total differentiation of that equation will show us that the second equation of each of them is always a consequence of the second equation of the other one.

Now, upon eliminating x, the two equations (17) will give us a relation that exhibits $\varphi(a)$ as a function of a. The question that was posed is then resolved.

It would not be pointless for what follows to remark that there are two, and only two, distinct integrals that take on values that are given in advance at all points of a curve (C), because if one would like to determine the derivatives with respect to x of the desired integral θ at every point of the curve (C) then one must append to the partial differential equation:

$$(18)\qquad \left(\frac{\partial \theta}{\partial x}\right)^2 + \left(\frac{\partial \theta}{\partial y}\right)^2 = 2(U + h),$$

the relation:

$$(19)\qquad \frac{\partial \theta}{\partial x} + \frac{\partial \theta}{\partial y}\lambda'(x) = \mu'(x),$$

which must be true for all points of the curve (C). Now, the preceding two equations determine two different systems of values for the derivatives $\dfrac{\partial \theta}{\partial x}$, $\dfrac{\partial \theta}{\partial y}$ when they are taken at an arbitrary point (C). Since an integral is defined entirely when one gives its values, along with that of its first derivatives at all points of a curve, one sees that the question that was posed will indeed admit two and only two solutions.

In the case that we have in mind, in which the function θ must have a constant value (zero, for example) at all points of the desired curve, one will have:

$$\mu(x) = 0,$$

so the two solutions that are obtained will be equal, up to sign, and cannot be regarded as truly distinct.

543. As consequences of the preceding propositions, we can state the following theorem:

Whenever one knows a complete integral of the partial differential equation (7), *one can always determine, without integration, an orthogonal system such that one of the families will contain an arbitrary curve* (C) *that is given in advance. The other family will be defined by the trajectories of the moving body that cut that curve* (C) *at a right angle and correspond to the same value of the vis viva constant.*

In the case where the curve (C) becomes infinitely small and reduces to a point, one will have the orthogonal system such that one of its families is composed of trajectories of the moving body that pass through that point. If one remarks that in that case equation (15), which represents all of those trajectories, must be verified when one replaces x, y with the coordinates x_0, y_0 of the point considered then one will see that one must have:

$$\varphi'(a) + \frac{\partial}{\partial a} f(x_0, y_0, a) = 0,$$

and as a result, one can take:

$$\varphi(a) = -f(x_0, y_0, a).$$

One will then have:

$$\theta = f(x, y, a) - f(x_0, y_0, a),$$

and from the rule that was given in no. **542**, one must eliminate a from that equation, along with the derivative of that equation with respect to a.

In order to give an application, consider the motion of ponderable bodies, in which the force function is:

$$U = g(y + h).$$

The equation in θ becomes:

$$\left(\frac{\partial\theta}{\partial x}\right)^2 + \left(\frac{\partial\theta}{\partial y}\right)^2 = 2g(y + h)$$

here.

It admits the following solution:

$$\frac{\theta}{\sqrt{2g}} = ax + \int \sqrt{y + h - a^2}\, dy = ax + \tfrac{2}{3}(y + h - a^2)^{3/2} + b. \tag{20}$$

In order to find the curves that cut all of the parabolic trajectories that pass through a fixed point (the origin, for example) at a right angle, from the preceding rule, one must determine b from the condition that θ must be annulled at that point, which gives:

$$b = -\tfrac{2}{3}(h - a^2)^{3/2},$$

and then eliminate a from equation (20), in which one replaces b with its preceding value, and the derivative of that equation with respect to a. After a calculation that we shall omit, we will then find that:

$$\frac{3\theta}{\sqrt{2g}} = [2h + y + \sqrt{x^2 - y^2}\,]^{3/2} - [2h + y - \sqrt{x^2 - y^2}\,]^{3/2}. \tag{21}$$

That is the equation for the orthogonal trajectories for all parabolas that pass through the same point.

544. In a general manner, consider the orthogonal systems that we just defined and for which one of the families is composed of trajectories of the moving body. The line element of the plane will take the form:

(22) $$ds^2 = H^2\, d\theta^2 + H_1^2\, d\theta_1^2 .$$

If one displaces along a trajectory θ_1 = const. then one will have:

(23) $$ds^2 = H^2\, d\theta^2 = 2\,(U + h)\, dt^2.$$

Furthermore, the equation:

$$2\,(U + h) = \left(\frac{\partial\theta}{\partial x}\right)^2 + \left(\frac{\partial\theta}{\partial y}\right)^2$$

can be written:

(24) $$2\,(U + h) = \frac{\partial\theta}{\partial x}\frac{dx}{dt} + \frac{\partial\theta}{\partial y}\frac{dy}{dt} = \frac{d\theta}{dt}.$$

Upon substituting the value of $d\theta / dt$ in the relation (23), we will have:

$$2H^2\,(U + h) = 1,$$

$$\frac{1}{H^2} = 2U + 2h.$$

Formula (22) will then take the form:

(25) $$2\,(U + h)\, ds^2 = d\theta^2 + \sigma^2 d\theta_1^2 ,$$

from which we shall deduce several consequences.

We shall first see that if one considers two curves with parameter θ:

$$\theta = \alpha, \qquad \theta = \beta$$

and the portion of any of the trajectories of the moving body that is found between those two curves then the integral:

$$\int \sqrt{2(U+h)}\; ds = \int d\theta ,$$

which is taken from the beginning to the end of that arc, will be constant and equal to the difference $\beta - \alpha$ of the values of θ. We give the name of *action* to the preceding integral. Since the curve $\theta = \alpha$ can be chosen arbitrarily, we can state the following theorem:

If one is given an arbitrary curve (C) *and the trajectories of the moving body that are normal to that curve then if one measures lengths along those trajectories that start from* *their point of incidence such that the action has a value that is given in advance, but still arbitrary, the locus of the extremities of all of those lengths will define a curve that will again be normal to all of the trajectories.*

That remarkable proposition, which is due to Thomson and Tait ([17]), is analogous to the one that we gave in no. **522** for the geodesic lines. Here again, one can prove directly by the calculus of variations and deduce all of the preceding results. One will then recover the method that was followed by Hamilton and Jacobi.

In particular, if one considers all of the trajectories that pass through a point *A*, and if one determines a point *M* on each of them such that action, when extended along the arc *AM*, has a given constant value then the locus of points *M* will be a curve that is normal to all of the trajectories.

545. If one refers the points of the plane to the coordinate system that is defined by the trajectories that pass through *A* and the curves that cut them at a right angle then the line element of the plane will be given by formula (25), in which θ denotes the action when it is measured by starting at *A*. We shall deduce a direct proof of the *principle of least action* from that remark.

That principle can be stated as follows:

Among all of the motions that take the moving body from a point A to a point M, with the velocity on each trajectory being dictated by the equation:

$$v^2 = 2\,(U + h),$$

the natural motion is the one for which the action – i.e., the integral:

$$\int_A^M \sqrt{2(U+h)}\;ds = \int_A^M v\;ds,$$

is a minimum.

The proof is identical to the one that we developed in the case of geodesic lines. Construct all of the trajectories of the moving body that correspond to the given value of *h* and pass through the point *A*. They give rise to an orthogonal system for which one has:

$$2\,(U + h)\,ds^2 = d\theta^2 + \sigma^2\,d\theta_1^2\,.$$

Having said that, it is clear that minimum of the integral:

$$\int \sqrt{2U+2h}\;ds = \int \sqrt{d\theta^2+\sigma^2 d\theta_1^2}\,,$$

([17]) SIR WILLIAM THOMSON and TAIT, *Treatise on natural Philosophy*, vol. I, Part I, pp. 353 in the second edition (1879).

when taken between the points A and M and path that is followed is the trajectory that connects those points, will correspond to the case in which $d\theta_1$ is zero. I shall not insist upon all of the conditions that must be true in order for the proof to be valid; they are identical to the ones that were enumerated in the case of geodesic lines.

546. Hamilton's principle refers to some hypotheses that are completely different from the ones that are involved with the principle of least action. It concerns the integral:

$$\int \left(\frac{ds^2}{2\,dt^2} + U \right) dt\,.$$

The motion in nature is the one for which that integral is a maximum or minimum. *However, the motion here is compared to all of the ones that can exist between the same points* and at the same time, *and no law is imposed upon the velocity, moreover.* We shall show that there is actually a *minimum.*

If A and M again denote the extreme positions, and if one preserves the orthogonal system such that one of the families is composed of trajectories that pass through A then the preceding integral will become:

$$\int \left[\frac{d\theta^2 + \sigma^2 d\theta_1^2}{4(U+h)\,dt^2} + U \right] dt.$$

We shall compare that to the one that corresponds to the natural motion, for which θ_1 remains constant.

Let θ_0, U_0 be the values of θ and U for the natural motion, in which θ, U, θ_0, U_0 are assumed to correspond to the same value of time; set:

$$\theta = \theta_0 + \omega, \qquad U = U_0 + U_1\,.$$

As we have seen, one will have:

$$(26) \qquad \frac{d\theta_0}{dt} = 2\,(U_0 + h).$$

The increment in Hamilton's integral when one passes from the natural motion to another one is:

$$\int \left[\frac{d\theta^2 + \sigma^2 d\theta_1^2}{4(U+h)\,dt^2} + U + U_0 - \frac{d\theta_0^2}{4(U_0+h)\,dt^2} \right] dt.$$

Replace θ with its value $\theta_0 + \omega$ and then substitute the value of $d\theta_0 / dt$ that is deduced from formula (26). The increment in the integral will become:

$$\int \left\{ \frac{\sigma^2 \dfrac{d\theta_1^2}{dt^2}}{4(U+h)} + \frac{\dfrac{d\omega^2}{dt^2}}{4(U+h)} + \frac{U_0+h}{U+h}\frac{d\omega}{dt} + U_1 - \frac{U_1(U_0+h)}{U+h} \right\} dt$$

or, after some reductions:

$$\int \left\{ \frac{\sigma^2 \dfrac{d\theta_1^2}{dt^2} + \dfrac{d\omega^2}{dt^2}}{4(U+h)} + \frac{U_1^2}{U+h} + \frac{d\omega}{dt} - \frac{U_1}{U+h}\frac{d\omega}{dt} \right\} dt .$$

Since the two motions take place between the same point and in the same amount of time, ω will be zero at the two limits. One can then suppress the term $d\omega / dt$, and what will remain for the increment in the integral is the expression:

(27)
$$\int \left\{ \frac{\sigma^2 \dfrac{d\theta_1^2}{dt^2}}{4(U+h)} + \frac{\left(\dfrac{d\omega}{dt} - 2U_1\right)^2}{4(U+h)} \right\} dt .$$

In that form, one sees clearly that Hamilton's integral has increased. In order for the preceding integral to be zero, it is necessary that one must have:

$$\frac{d\theta_1}{dt} = 0, \qquad \frac{d\omega}{dt} = 2U_1$$

at each instant, or:

$$\frac{d\theta_1}{dt} = 0, \qquad \frac{d\theta}{dt} = 2\,(U+h), \qquad \frac{ds}{dt} = \sqrt{2(U+h)}\,,$$

and those equations characterize the natural motion.

547. We shall not dwell any further upon the preceding principles, and in conclusion we remark only that the proof of the least-action principle can be attached directly to the theory of geodesic lines in the following manner:

If x and y are the rectangular coordinates of a point in the plane, U is the force function, and h is the *vis viva* constant then consider the surface whose line element is given by the formula:

$$ds^2 = 2\,(U+h)\,(dx^2 + dy^2).$$

That surface will be represented on the plane *with conservation of angles*. However, the correspondence is such that *any trajectory of the moving body in the plane will correspond to a geodesic line on the surface*, and vice versa.

That proposition has already been presented several times in the preceding arguments. We could establish it by either comparing the differential equation (3) for the trajectories and that (8) of the geodesic lines (no. **514**) or by comparing the partial differential equation (7) with equation (5) (no. **531**) upon which the search for geodesic lines depends. We can now prove that immediately, because if one refers the points in the plane to a coordinate system such that one of the families is composed of trajectories of the moving body then the line element of the plane will be given by the formula (25); that of the corresponding surface will then have the expression:

$$ds^2 = d\theta^2 + \sigma^2\, d\theta_1^2 .$$

As a result, the lines θ_1 = const. – i.e., the trajectories of the moving body in the plane – necessarily correspond to geodesics on the surface, and *vice versa*.

As an application, consider the motion of a point that is attracted to a fixed center in inverse proportion to the square of the distance. If r denotes the distance to the fixed center then one will have:

$$U + h = \frac{2\mu}{r} - \frac{\mu}{a} .$$

The surface whose geodesic lines correspond to the trajectories of the moving body will have the line element:

$$ds^2 = \left(\frac{2\mu}{r} - \frac{\mu}{a}\right)(dx^2 + dy^2),$$

or, upon passing to polar coordinates r, v:

$$ds^2 = \left(\frac{2\mu}{r} - \frac{\mu}{a}\right)(dr^2 + r^2\, dv^2) . \tag{28}$$

The surfaces of revolution that admit that line element are defined by the formulas:

$$\left\{\begin{aligned} x &= m\sqrt{\frac{\mu r(2a-r)}{a}}\cos\frac{v}{m}, \\ y &= m\sqrt{\frac{\mu r(2a-r)}{a}}\sin\frac{v}{m}, \\ z &= \sqrt{\frac{\mu}{a}}\int\sqrt{\frac{(2a-r)^2 - m^2(a-r)^2}{r(2a-r)}}\,dr. \end{aligned}\right. \tag{29}$$

Whenever m is commensurable, a point in the plane will correspond to a limited number of points of the surface, and as a result, all of the geodesic lines that do not meet the boundary of the surface will be closed, like the ellipses in the plane to which they correspond.

548. That correspondence, which is established between a plane and a surface in such a manner that the trajectories of the plane correspond to the geodesic lines of the surface, immediately exhibits the principle of least action, which is nothing but the translation of the minimum property of geodesic lines into the plane. However, it will lead to a large number of other propositions with no calculation. For example, we have seen that that the curves on a surface that are loci of the points such that the sum or difference of their geodesic distances to two fixed curves (C), (C') is constant define an orthogonal system. That system can obviously be determined without integration whenever one knows two curves (C), (C') and one has an expression for the geodesic distance between two points of the surface. One can even add that if one of the two curves (C) is given then one can determine the other one (C') in such a manner that one of the families of the orthogonal system contains a curve (D) that is given in advance. Upon referring that result to the plane, we will obtain the following proposition:

Whenever one has the complete solution to a problem in mechanics in the plane and the function θ that relates to that problem, one can determine, with no new integrations, an infinitude of orthogonal systems in the plane that contain a curve (D) *that is given in advance. The equations that define those systems will contain an arbitrary function of one variable.*

Moreover, that proposition can be proved directly in the simplest manner. Indeed, let θ and σ be two arbitrary solutions of the partial differential equation (7). One will have:

$$\left(\frac{\partial\theta}{\partial x}\right)^2 + \left(\frac{\partial\theta}{\partial y}\right)^2 = \left(\frac{\partial\sigma}{\partial x}\right)^2 + \left(\frac{\partial\sigma}{\partial y}\right)^2,$$

and consequently:

$$\frac{\partial(\theta-\sigma)}{\partial x}\frac{\partial(\theta+\sigma)}{\partial x} + \frac{\partial(\theta-\sigma)}{\partial y}\frac{\partial(\theta+\sigma)}{\partial y} = 0.$$

That equation expresses the idea that the curves:

$$\theta - \sigma = \text{const.}, \qquad \theta + \sigma = \text{const.}$$

cut at a right angle and define the two families of an orthogonal system. If one desires that a certain curve (D) must belong to one of those families then it will suffice to determine two solutions θ, σ of the partial differential equation that have the same value at each point of the curve (D). One takes σ arbitrarily, which will introduce an arbitrary function. θ will, in turn, be determined by the condition that it must have the same value as σ at all points of the curve (D). We know (no. **542**) that θ will be a distinct function of σ.

549. The preceding general propositions permit one to establish that one can determine an infinitude of algebraic orthogonal systems that an arbitrary algebraic curve will belong to when it has been given in advance. We first remark that there is an infinitude of problems in mechanics for which the *action* is an algebraic function; i.e., for which, the equation:

$$\left(\frac{\partial\theta}{\partial x}\right)^2 + \left(\frac{\partial\theta}{\partial y}\right)^2 = 2\,(U + h) \tag{30}$$

admits a complete algebraic integral. Without discussing the case in which the force function is zero, take:

$$U = A\,x^{1/m} + B\,y^{1/m}, \tag{31}$$

for example, in which A and B are arbitrary constants, and m, n are two integers. One will have the complete solution:

$$\theta = \int \sqrt{2A^{1/m} + h + a}\,dx + \int \sqrt{2B^{1/m} + h - a}\,dy\,, \tag{32}$$

which is obviously algebraic. If one applies the preceding methods upon employing that value of θ then one will see that all of the solutions to equation (30) that are required to take on an algebraic value at all points of an algebraic curve will be algebraic. One can then obtain an infinitude of algebraic orthogonal systems to which a given algebraic curve will belong. There are two different kinds of those systems. One of them, in which one of the families is composed of the trajectories of the moving body that cut the given curve at a right angle, are analogous to the orthogonal systems that are composed of a family of parallel curves and their common normals. The other ones are analogous to the orthogonal system that is composed of the two families of curves that are loci of points such that the sum of or the difference between their geodesic distances to two fixed curves (C), (C') is constant. Their definition contains an arbitrary algebraic function, even though one has required a curve that is given in advance to belong to one of the two families of the orthogonal system.

550. It is easy to see that the preceding method can be extended to the study of the motion of a point on a surface, and in general, to all problems of mechanics in which there is a force function, since the position of the moving system depends upon only *two* variables. We shall not develop the calculations, which are quite tedious, when we treat the most general problem in mechanics, and we shall be content to point out here some other questions of mechanics in which one recovers the properties that we just studied.

One must thank various geometers ([18]) for some properties of the brachistochrone that are analogous to the ones that Gauss brought to light for geodesic lines. The explanation of that fact rests upon the following remark:

([18]) For example, see ROGER, "Thése sur les brachistochrones," Journal de Liouville (1) **13** (1848), pp. 41.

We propose to determine the brachistochrones on a surface (Σ). If the velocity of the moving body is given by the *vis viva* equation:

$$v^2 = U + h$$

then the brachistochrones will be the curves for which the integral:

$$\int \frac{ds}{v} = \int \frac{ds}{\sqrt{U+h}},$$

when taken between two arbitrary points of the curve, be a minimum. Now, if one considers the surface (Σ') for which the line element ds' is determined by the formula:

$$ds'^2 = \frac{ds^2}{U+h} \tag{33}$$

then it will correspond to the surface (Σ) with conservation of angles, and the brachistochrones of (Σ) will correspond to the geodesic lines on (Σ'), such that the arc length of each geodesic is equal to the time during which the corresponding portion of the brachistochrone is traversed. That simple remark permits one to extend all of the properties of geodesic lines to brachistochrones. In particular, one then recognizes that the brachistochrones actually satisfy their definition and that the time in which an arbitrary arc of those curves is traversed is actually a minimum, provided that the arc is not extended too far.

One can then associate the brachistochrones with trajectories in a planar motion, and that comparison offers the advantage of being extensible to brachistochrones in space.

Suppose that the element of the surface (Σ) reduces to the form:

$$ds^2 = \lambda\,(dx^2 + dy^2). \tag{34}$$

The integral that must be a minimum is:

$$\int \frac{\sqrt{\lambda}\sqrt{dx^2+dy^2}}{\sqrt{U+h}}.$$

By virtue of the principle of least action, one will recognize immediately that brachistochrones correspond to the trajectories of a planar motion in which the force function U' has the value:

$$U' = \frac{\lambda}{U+h},$$

ANDOYER, "Sur la réduction du problème des brachistochones aux équations canonique," Comptes rendus **100** (1885), 1577.

while the velocity of the moving body is given by the formula:

$$v^2 = 2U', \tag{35}$$

in which the constant vis viva has the particular value of zero.

Some analogous remarks can also be made in regard to the figures of equilibrium of a flexible and inextensible string. However, we leave it to the reader to examine that point.

551. In the preceding developments, we have associated only those trajectories for which the *vis viva* constant has the same value. That restriction is indeed in accord with the spirit of modern mechanics, which attaches less importance to forces than to *energy*, and which permits one to regard two problems as distinct when the force function is the same, but the total energy is different. Be that as it may, upon grouping together the trajectories for which the *vis viva* constant takes on different values, one will get the following results, which we shall rapidly discuss:

Consider some arbitrary trajectories that constitute a family that is analogous to the ones that we defined in no. **538**. x' and y' will again be functions of x and y, but the constant h, which varies when one passes from one trajectory to the other, must be considered to be a function of x and y here. One will again have the equations:

$$\left\{\begin{aligned} & x'\frac{\partial x'}{\partial x} + y'\frac{\partial x'}{\partial y} = \frac{\partial U}{\partial x}, \\ & x'\frac{\partial y'}{\partial x} + y'\frac{\partial y'}{\partial y} = \frac{\partial U}{\partial y}, \\ & x'^2 + y'^2 = 2h + 2U. \end{aligned}\right. \tag{36}$$

However, differentiating the equation of *vis viva* will give different results. One must not regard h in it as a constant that is independent of x and y. Differentiation will then give the equations:

$$\left\{\begin{aligned} & x'\frac{\partial x'}{\partial x} + y'\frac{\partial y'}{\partial x} = \frac{\partial h}{\partial x} + \frac{\partial U}{\partial x}, \\ & x'\frac{\partial x'}{\partial y} + y'\frac{\partial y'}{\partial y} = \frac{\partial h}{\partial y} + \frac{\partial U}{\partial y}. \end{aligned}\right. \tag{37}$$

If one eliminates $\frac{\partial U}{\partial x}, \frac{\partial U}{\partial y}$ from these equations, along with the preceding ones, then one will find that:

$$y'\left(\frac{\partial y'}{\partial x} - \frac{\partial x'}{\partial y}\right) = \frac{\partial h}{\partial x},$$

$$x'\left(\frac{\partial y'}{\partial x}-\frac{\partial x'}{\partial y}\right) = -\frac{\partial h}{\partial y}.$$

Now set:

(38) $$\frac{\partial y'}{\partial x}-\frac{\partial x'}{\partial y} = \lambda\,;$$

one will first have:

(39) $$y' = \frac{1}{\lambda}\frac{\partial h}{\partial x}, \qquad x' = -\frac{1}{\lambda}\frac{\partial h}{\partial y}.$$

The substitution of those values for x', y' in the *vis viva* equation will give the relation:

$$\Delta h = 2\lambda^2 (h + U),$$

in which Δh denotes Lamé's differential parameter:

(40) $$\Delta h = \left(\frac{\partial h}{\partial x}\right)^2 + \left(\frac{\partial h}{\partial y}\right)^2,$$

and from which one can find λ. One will then obtain:

(41) $$y' = \frac{\sqrt{2(h+U)}}{\sqrt{\Delta h}}\frac{\partial h}{\partial x}, \qquad x' = -\frac{\sqrt{2(h+U)}}{\sqrt{\Delta h}}\frac{\partial h}{\partial y}.$$

Upon substituting those values in formulas (38), which serves to define λ, one will find the second-order partial differential equation:

(42) $$\frac{\partial}{\partial x}\left(\frac{\sqrt{U+h}}{\sqrt{\Delta h}}\frac{\partial h}{\partial x}\right)+\frac{\partial}{\partial y}\left(\frac{\sqrt{U+h}}{\sqrt{\Delta h}}\frac{\partial h}{\partial y}\right) = \frac{1}{2}\frac{\sqrt{\Delta h}}{\sqrt{U+h}},$$

which defines the function h. When one has an arbitrary solution of that equation, the curves:

$$h = \text{const.}$$

will be the trajectories of a corresponding family, and equations (41) will exhibit the components of the velocity of the moving body at each of their points. Conversely, if one knows how to determine the trajectories then one will also know how to integrate the partial differential equation (42). When one has obtained the general equation of the trajectory:

$$y = \varphi(x, a, b, h),$$

with arbitrary constants a and b, it will suffice to replace a and b with arbitrary functions of h in order to obtain the general integral of equation (42).

For example, suppose that the force function is zero. The trajectories will be straight lines that are represented by the equation:

$$y = ax + b.$$

The integral of the corresponding partial differential equation will be given by the formula:

$$y = x\,\varphi(h) + \psi(h),$$

which is easy to verify.

A particular situation will show the interest in the preceding remarks. The partial differential equation (42) shows up in the study of a question of the minimum of a double integral:

$$\iint \sqrt{\left(\frac{\partial h}{\partial x}\right)^2 + \left(\frac{\partial h}{\partial y}\right)^2}\sqrt{U+h}\;dx\,dy, \tag{43}$$

which is a form that is analogous to the one that Riemann considered in Dirichlet's principle.

Imagine that the function h is given for all points of a closed contour that bounds a planar area A. If one expresses the idea that the preceding double integral, when taken over all points of that area, is a minimum then one will be led to a partial differential equation that is precisely equation (42) when one equates its first variation to zero.

Therefore, *any problem in mechanics in the plane* (and more generally, in two independent variables) *can be attached to a property of giving a minimum to a double integral.*

Some considerations of geometry that the reader can easily supply will permit one to deduce that minimum property from the principle of least action, moreover.

552. In the following two chapters, we will associate only trajectories for which the *vis viva* constant has the same value. We shall then point out here, without proof, the extension that one can give to the preceding properties. For better clarity, we shall content ourselves with considering the motions in space in the statement of the generalized properties.

If one seeks to determine the functions λ and μ of x and y in such a manner as to render a minimum to the triple integral:

$$\left\{\begin{aligned}&\iiint \sqrt{\left(\frac{\partial\lambda}{\partial y}\frac{\partial\mu}{\partial z}-\frac{\partial\lambda}{\partial z}\frac{\partial\mu}{\partial y}\right)^2+\left(\frac{\partial\lambda}{\partial z}\frac{\partial\mu}{\partial x}-\frac{\partial\lambda}{\partial x}\frac{\partial\mu}{\partial z}\right)^2+\left(\frac{\partial\lambda}{\partial x}\frac{\partial\mu}{\partial y}-\frac{\partial\lambda}{\partial x}\frac{\partial\mu}{\partial y}\right)^2}\\ &\quad\times\varphi(x,y,z,\lambda,\mu)\,dx\,dy\,dz,\end{aligned}\right. \tag{44}$$

when extended over a closed volume, and the functions λ and μ are required to take on given values at all points of the surface or surfaces that bound that volume, it will suffice to integrate the equations of motion that relate to a problem in mechanics in which the *force function is $\varphi(x, y, z, \lambda, \mu)$, the vis viva constant is zero, and λ and μ are treated as constants, and then to replace the arbitrary constants in the general equations of the trajectory with arbitrary functions of λ and μ. One will then obtain two equations that will give one λ and μ.*

If one seeks the function λ that assures the minimum of the triple integral:

$$\iiint \sqrt{\left(\frac{\partial\lambda}{\partial x}\right)^2 + \left(\frac{\partial\lambda}{\partial y}\right)^2 + \left(\frac{\partial\lambda}{\partial z}\right)^2}\,\varphi(x, y, z, \lambda, \mu)\,dx\,dy\,dz\,, \tag{45}$$

when extended over a closed volume, and λ is required to take on given values at all points of the surface that bounds that volume then the surfaces:

$$\lambda = \text{const.}$$

must be the ones for which the following double integral is a minimum:

$$\iint \varphi(x, y, z, \lambda)\;d\sigma,$$

in which $d\sigma$ denotes the area of an element of the surface, and the integral is extended over the portion of the surface that is found inside an arbitrary contour.

If one considers, for example, the integral:

$$\iiint \sqrt{\left(\frac{\partial\lambda}{\partial x}\right)^2 + \left(\frac{\partial\lambda}{\partial y}\right)^2 + \left(\frac{\partial\lambda}{\partial z}\right)^2}\,dx\,dy\,dz\,, \tag{46}$$

which corresponds to the hypothesis that $\varphi = 1$, one will recognize that the surfaces λ = const. must be the minimal surfaces. One is then led to the following result:

If the equation:

$$\lambda = \text{const.}$$

represents a family of minimal surfaces then λ must satisfy the partial differential equation:

$$\frac{\partial}{\partial x}\left(\frac{\dfrac{\partial\lambda}{\partial x}}{\sqrt{\Delta\lambda}}\right) + \frac{\partial}{\partial y}\left(\frac{\dfrac{\partial\lambda}{\partial y}}{\sqrt{\Delta\lambda}}\right) + \frac{\partial}{\partial z}\left(\frac{\dfrac{\partial\lambda}{\partial z}}{\sqrt{\Delta\lambda}}\right) = 0, \tag{47}$$

in which $\Delta\lambda$ is the first-order differential parameter:

(48) $$\Delta\lambda = \left(\frac{\partial\lambda}{\partial x}\right)^2 + \left(\frac{\partial\lambda}{\partial y}\right)^2 + \left(\frac{\partial\lambda}{\partial z}\right)^2 .$$

That result is due to Riemann ([19]), who even proved, as one can easily verify by direct calculation, that if one has just one surface that is represented by the equation:

$$\lambda = 0$$

then in order for the surface to be minimal, it will suffice that the preceding partial differential equation must be satisfied only by virtue of the equation of the surface, rather than identically.

The preceding form (47) for the partial differential equation for minimal surfaces is attached directly to that of Lagrange (I, no. **175**), which one can recover immediately, moreover, upon supposing that the equation of the surface has been put into the form:

$$z = \varphi(x, y).$$

The remarks by which we obtained it show that one can immediately write down the partial differential equation for minimal surfaces in curvilinear coordinates, because if the line element of space is given by the formula:

(49) $$ds^2 = H^2\, d\rho^2 + H_1^2 d\rho_1^2 + H_2^2 d\rho_2^2$$

then the integral (46) will take the form:

(50) $$\iiint \sqrt{\Delta\lambda}\; H\, H_1\, H_2\, d\rho\, d\rho_1\, d\rho_2 ,$$

and the minimum property, which we have pointed out without calculation, will lead to the equation:

(51) $$\frac{\partial}{\partial\rho}\left(\frac{H_1 H_2}{H}\frac{\dfrac{\partial\lambda}{\partial\rho}}{\sqrt{\Delta\lambda}}\right) + \frac{\partial}{\partial\rho_1}\left(\frac{H H_2}{H_1}\frac{\dfrac{\partial\lambda}{\partial\rho_1}}{\sqrt{\Delta\lambda}}\right) + \frac{\partial}{\partial\rho_2}\left(\frac{H H_1}{H_2}\frac{\dfrac{\partial\lambda}{\partial\rho_2}}{\sqrt{\Delta\lambda}}\right) = 0,$$

which replaces equation (47). One can follow the same method that one employed for oblique curvilinear coordinates.

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([19]) *Riemann's Gesammelte Werke*, pp. 311.

CHAPTER VII

APPLICATION OF THE PRECEDING METHODS TO THE STUDY OF MOTIONS IN SPACE

Differential equations of motion. – All of the trajectories that correspond to the same value of the *vis viva* constant and are normal to a surface are, by that fact itself, normal to a family of surfaces. – Partial differential equation of Hamilton and Jacobi. – Use that one can make of a complete integral. – Conditions that the integral must satisfy. – Definition of *action*. – Consideration of certain orthogonal systems. – Formulas that relate to the variation of the action. – Generalization of the Malus-Dupin theorem.

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553. Now consider the motions in space. If U denotes the force function then the equations of motion will be:

(1) $$\frac{d^2x}{dt^2} = \frac{\partial U}{\partial x}, \quad \frac{d^2y}{dt^2} = \frac{\partial U}{\partial y}, \quad \frac{d^2z}{dt^2} = \frac{\partial U}{\partial z},$$

(2) $$x'^2 + y'^2 + z'^2 = 2\,(U + h),$$

in which x', y', z' denote the components of the velocity.

Among all of the motions that correspond to a given value of the *vis viva* constant, we shall study, in particular, the ones whose trajectories pass through a point or are normal to a surface or, in general, satisfy any condition that leaves two arbitrary constants in the equations of the trajectory. We will then have a congruence of curves that is represented by equations such as the following ones:

(3) $$\begin{cases} f(x, y, z, a, b) = 0, \\ \varphi(x, y, z, a, b,) = 0. \end{cases}$$

Moreover, the components of the velocity at each point must satisfy two equations:

(4) $$\begin{cases} \dfrac{\partial f}{\partial x}x' + \dfrac{\partial f}{\partial y}y' + \dfrac{\partial f}{\partial z}z' = 0, \\[2ex] \dfrac{\partial \varphi}{\partial x}x' + \dfrac{\partial \varphi}{\partial y}y' + \dfrac{\partial \varphi}{\partial z}z' = 0, \end{cases}$$

which, when combined with the *vis viva* equation, will obviously permit one to determine them, and then express them by substituting the values for a and b that are deduced from equations (3) uniquely as functions of x, y, z.

Suppose that one has obtained those expressions. The equations of motion take the following form:

$$(5)\qquad \begin{cases} \dfrac{\partial x'}{\partial x}x' + \dfrac{\partial x'}{\partial y}y' + \dfrac{\partial x'}{\partial z}z' = \dfrac{\partial U}{\partial x}, \\[2mm] \dfrac{\partial y'}{\partial x}x' + \dfrac{\partial y'}{\partial y}y' + \dfrac{\partial y'}{\partial z}z' = \dfrac{\partial U}{\partial y}, \\[2mm] \dfrac{\partial z'}{\partial x}x' + \dfrac{\partial z'}{\partial y}y' + \dfrac{\partial z'}{\partial z}z' = \dfrac{\partial U}{\partial z}, \end{cases}$$

which is analogous to the one that one encounters in the study of the permanent motion of fluids; x', y', z' must also satisfy the *vis viva* equation.

One can deduce the values of $\dfrac{\partial U}{\partial x}$, $\dfrac{\partial U}{\partial y}$, $\dfrac{\partial U}{\partial z}$ from the last equation. For example, one will have:

$$\frac{\partial U}{\partial x} = x'\frac{\partial x'}{\partial x} + y'\frac{\partial y'}{\partial x} + z'\frac{\partial z'}{\partial x}.$$

Upon substituting that value of $\dfrac{\partial U}{\partial x}$ in the first of equations (5), one will obtain the relation:

$$x'\frac{\partial x'}{\partial x} + y'\frac{\partial x'}{\partial y} + z'\frac{\partial x'}{\partial z} = x'\frac{\partial x'}{\partial x} + y'\frac{\partial y'}{\partial x} + z'\frac{\partial z'}{\partial x},$$

which one can write in the following manner:

$$y'\left(\frac{\partial x'}{\partial y} - \frac{\partial y'}{\partial x}\right) = z'\left(\frac{\partial z'}{\partial x} - \frac{\partial x'}{\partial z}\right).$$

The second and third equations in (5) give analogous formulas, from which one will deduce the following system:

$$(6)\qquad \frac{\dfrac{\partial y'}{\partial z} - \dfrac{\partial z'}{\partial y}}{x'} = \frac{\dfrac{\partial z'}{\partial x} - \dfrac{\partial x'}{\partial z}}{y'} = \frac{\dfrac{\partial x'}{\partial y} - \dfrac{\partial y'}{\partial x}}{z'},$$

which contains all of the relations between x', y', z' that are independent of the force function.

One recognizes immediately that one can satisfy those equations by annulling the numerators; i.e., upon supposing that x', y', z' are the derivatives of the same function θ. Then set:

$$(7)\qquad x' = \frac{\partial \theta}{\partial x}, \qquad y' = \frac{\partial \theta}{\partial y}, \qquad z' = \frac{\partial \theta}{\partial z}.$$

If one substitutes those values for x', y', z' in equations (2) and (5) then equation (2) will take the form:

$$\left(\frac{\partial\theta}{\partial x}\right)^2 + \left(\frac{\partial\theta}{\partial y}\right)^2 + \left(\frac{\partial\theta}{\partial z}\right)^2 = 2U + 2h, \tag{8}$$

and the system (5) will be composed of equations that one deduces from the preceding ones by differentiating with respect to x, y, or z. *It will then suffice that θ must merely satisfy the partial differential equation* (8).

Equations (7) show us immediately what the geometric significance of the function θ is. If one considers the family of surfaces that are represented by the equation θ = const., in which θ is an arbitrary integral of equation (6), then the curves that are orthogonal trajectories of the family of surfaces will also be trajectories of the moving body, and the velocity of that moving body will be equal to the derivative $\frac{\partial\theta}{\partial n}$ of θ along the normal. In other words, *θ is a velocity potential*.

The preceding method rests upon the consideration of certain particular congruences that are composed of the moving trajectory, so it is natural to demand that they should all give solutions to the problem of mechanics; i.e., all of the possible trajectories. Let (C) be one of those trajectories that passes through the point M_0 (x_0, y_0, z_0), and let x_0', y_0', z_0' be the components of the velocity of the moving body at that point, which are components that necessarily verify the *vis viva* equation:

$$x_0'^2 + y_0'^2 + z_0'^2 = 2U_0 + 2h.$$

There is obviously an infinitude of solutions θ to equation (8) whose partial derivatives $\frac{\partial\theta}{\partial x}$, $\frac{\partial\theta}{\partial y}$, $\frac{\partial\theta}{\partial z}$ take on the values x_0', y_0', z_0' at the point M_0. Consider any of those solutions θ'. The orthogonal trajectories of the surfaces θ' = const. will be the trajectories of the moving body. Those of the trajectories that pass through the point M_0 will obviously coincide with the curve (C), since the initial conditions of the motion are the same on one and the other of those trajectories.

554. One is again led to the consideration of particular congruences for which there is a velocity potential by the following reasoning, which will once more exhibit the preceding result.

Let λ denote the common value of the ratios (6). One has:

$$(9) \qquad \begin{cases} \dfrac{\partial y'}{\partial z} - \dfrac{\partial z'}{\partial y} = \lambda x', \\ \dfrac{\partial z'}{\partial x} - \dfrac{\partial x'}{\partial z} = \lambda y', \\ \dfrac{\partial x'}{\partial y} - \dfrac{\partial y'}{\partial x} = \lambda z', \end{cases}$$

and in turn, upon making use of the a well-known identity:

$$(10) \qquad \frac{\partial(\lambda x')}{\partial x} + \frac{\partial(\lambda y')}{\partial y} + \frac{\partial(\lambda z')}{\partial z} = 0.$$

That relation, which recalls the *continuity equation* of hydrodynamics, confirms the analogy that we have already pointed out above, and upon which we shall not insist, moreover. If one performs the differentiations then it will take the form:

$$\frac{\partial \lambda}{\partial x} x' + \frac{\partial \lambda}{\partial y} y' + \frac{\partial \lambda}{\partial z} z' = -\lambda \left(\frac{\partial x'}{\partial x} + \frac{\partial y'}{\partial y} + \frac{\partial z'}{\partial z} \right).$$

The left-hand side, which one can also write:

$$\frac{\partial \lambda}{\partial x} \frac{dx}{dt} + \frac{\partial \lambda}{\partial y} \frac{dy}{dt} + \frac{\partial \lambda}{\partial z} \frac{dz}{dt},$$

has a very simple significance: *It expresses the derivative $d\lambda / dt$ of λ when one displaces along a trajectory of the moving body.* If one then sets:

$$(11) \qquad \Omega = -\frac{\partial x'}{\partial x} - \frac{\partial y'}{\partial y} - \frac{\partial z'}{\partial z}$$

to abbreviate, then one will have:

$$\frac{d\lambda}{dt} = \lambda \, \Omega,$$

from which one will deduce that:

$$(12) \qquad \lambda = \lambda_0 \, e^{\int_{t_0}^{t} \Omega \, dt},$$

in which λ_0 denotes the value of λ for $t = t_0$. Hence:

If λ is zero for an arbitrary point on a trajectory then it will be zero for all of the other points of the same trajectory.

From that, among the trajectories of the moving body (which always correspond to the same value of the *vis viva* constant), consider the ones that are normal to a surface (Σ) and remark that as a result of the definition of λ and the *vis viva* equation, one will have:

$$(13) \qquad \lambda = \frac{x'\left(\frac{\partial y'}{\partial z}-\frac{\partial z'}{\partial y}\right)+y'\left(\frac{\partial z'}{\partial x}-\frac{\partial x'}{\partial z}\right)+z'\left(\frac{\partial x'}{\partial y}-\frac{\partial y'}{\partial x}\right)}{2(U+h)}.$$

It results from that expression that λ will be zero for the point where each trajectory meets the surface (Σ) normally. In order to see that immediately, it will suffice to remark that the differential equations of the curves of the congruence are:

$$(14) \qquad \frac{dx}{x'} = \frac{dy}{y'} = \frac{dz}{z'},$$

in which x', y', z' play the role of the quantities X, Y, Z in no. **438** here.

Since λ is non-zero for a point on each trajectory, it will be zero by that fact itself on all of the trajectories, which will, in turn, be normal to a family of surfaces, from the theorem in the cited number. We then recover the proposition of Thomson and Tait, which we will establish, moreover, in another manner:

All of the trajectories of the moving body that correspond to the same value of the vis viva constant and are normal to just one surface will be, by that fact itself, normal to all surfaces of a family.

It results from the preceding argument that in order to obtain all of those families of surfaces that are normal to the trajectories, one must integrate the partial differential equation (8). We shall examine various solutions to that equation.

555. We have only to repeat here what we said in the case of planar motions. If the solution θ contains no constant then in order to get the corresponding trajectories, one must integrate the three equations (5) or equations (14). However, I would like to show that if the solution θ contains two other arbitrary constants a and b, in addition to the constant that one can always add to it, then one can obtain the complete solution to the problem of mechanics with no integration.

Indeed, substitute θ in equation (8) and take the derivative with respect to a; we will have:

$$\frac{\partial\theta}{\partial x}\frac{\partial^2\theta}{\partial a\,\partial x}+\frac{\partial\theta}{\partial y}\frac{\partial^2\theta}{\partial a\,\partial y}+\frac{\partial\theta}{\partial z}\frac{\partial^2\theta}{\partial a\,\partial z} = 0.$$

If one replaces $\frac{\partial\theta}{\partial x}$, $\frac{\partial\theta}{\partial y}$, $\frac{\partial\theta}{\partial z}$ with x', y', z', respectively, then the preceding equation will take the form:

$$\frac{d}{dt}\left(\frac{\partial\theta}{\partial a}\right) = 0.$$

Hence, $\dfrac{\partial\theta}{\partial a}$ is constant on each trajectory of the moving body. Upon applying the same argument to b, one will see that the equations:

$$(15) \qquad \frac{\partial\theta}{\partial a} = a', \qquad \frac{\partial\theta}{\partial a} = b',$$

in which a', b' denote two new constants, define a trajectory of the moving body. Moreover, one verifies immediately that the two surfaces that are represented by each of the preceding equations cut all of the surfaces θ = const. at a right angle.

Upon likewise differentiating equation (8) with respect to h, one will find:

$$\frac{d}{dt}\left(\frac{\partial\theta}{\partial h}\right) = 1,$$

and upon integration, one will get, in turn:

$$(16) \qquad \frac{\partial\theta}{\partial h} = t + \tau,$$

in which τ denotes a new constant.

Equations (15) and (16), which contain six arbitrary constants a, b, h, a', b', τ, indeed define the most general solution of the problem that was posed. Upon attempting to prove that rigorously, one will recognize the conditions that the solution θ, which contains the constants a, b, must satisfy. Indeed, if one desires to determine the trajectory of the moving body that passes through the point M (x, y, z) when the moving body admits the velocities x', y', z', which are necessarily coupled by the *vis viva* equation, then one will have three equations:

$$x' = \frac{\partial\theta}{\partial x}, \qquad y' = \frac{\partial\theta}{\partial y}, \qquad z' = \frac{\partial\theta}{\partial z},$$

which reduce to two, by virtue of equations (2), (8), and which must determine a and b as functions of the six given functions x, y, z, x', y', z'.

For example, take the first two. In order for one to be able to deduce the values of a and b, in general, it will be necessary and sufficient that $\dfrac{\partial\theta}{\partial x}$, $\dfrac{\partial\theta}{\partial y}$ must be functions that are independent of one of the variables a and b. It will then be necessary that the determinant:

$$\frac{\partial\left(\dfrac{\partial\theta}{\partial x},\dfrac{\partial\theta}{\partial y}\right)}{\partial(a,b)}$$

must be non-zero. Upon reasoning similarly with $\dfrac{\partial\theta}{\partial y}$, $\dfrac{\partial\theta}{\partial z}$, one will be led to the following conclusion:

The solution θ must be such that the two equations:

$$\text{(17)}\qquad \frac{\dfrac{\partial^2\theta}{\partial a\,\partial x}}{\dfrac{\partial^2\theta}{\partial b\,\partial x}} = \frac{\dfrac{\partial^2\theta}{\partial a\,\partial y}}{\dfrac{\partial^2\theta}{\partial b\,\partial y}} = \frac{\dfrac{\partial^2\theta}{\partial a\,\partial z}}{\dfrac{\partial^2\theta}{\partial b\,\partial z}}$$

(which will reduce to just one of them, moreover) are not verified identically.

One can also state that condition in one or the other of the following forms:

If θ is considered to be a function of a and b then it cannot satisfy a first-order equation:

$$F\left(\frac{\partial\theta}{\partial a},\frac{\partial\theta}{\partial b},a,b\right) = 0$$

that is independent of x, y, z.

One can also say that if θ is considered to be a function of x, y, z then it cannot satisfy a first-order equation:

$$\varphi\left(x, y, z,\frac{\partial\theta}{\partial x},\frac{\partial\theta}{\partial y},\frac{\partial\theta}{\partial z}\right) = 0$$

that is distinct from equation (8) and does not depend upon either a or b.

For example, suppose that the force function is zero. Equation (8) will be:

$$\left(\frac{\partial\theta}{\partial x}\right)^2 + \left(\frac{\partial\theta}{\partial y}\right)^2 + \left(\frac{\partial\theta}{\partial z}\right)^2 = 2h,$$

and it will admit the solution:

$$\theta = z\sqrt{2h-1} + \sqrt{(x-a)^2 + (y-b)^2}\,.$$

That solution is not suitable, even though it contains two constants. One will recognize that immediately upon applying any of the three criteria that we just discussed.

556. Here, we see a new fact presenting itself: In the plane, all possible families of trajectories of the moving body must belong to an orthogonal system that corresponds to a certain solution θ of Jacobi's partial differential equation. The same thing will no longer be true in space: One can certainly associate the trajectories of a moving body with congruences that admit surfaces that cut them at a right angle, as we just proved, but there also exist families of trajectories that do not possess that important property.

That result could have been predicted. Indeed, consider the case in which there is no force. The trajectories that correspond to the same value of h are the lines in space that are all traversed with the same velocity. Now, one certainly knows that a system of rectilinear rays is not always composed of the normals to a surface. However, one also knows that if the lines are normal to a surface then there will be an infinitude of other surfaces. As one sees, that property is only a particular case of the one that belongs to the trajectories of a moving body, and that we proved in no. **554**.

If one leaves aside the results that were established in this number then one can also prove the theorem of Thomson and Tait as follows:

If one is given a surface (Σ) then one will always know how to determine a solution θ of Jacobi's equations that is zero for all points of that surface. The trajectories of the moving bodies, which are normal to all of the surfaces θ = const., will be normal to (Σ), in particular. Since the set of all of them is determined by the latter condition, the proposition will then be proved.

In particular, if the surface (Σ) becomes infinitely small and reduces to a point then one will have all of the trajectories that pass through that point. One will then see that:

All of the trajectories of the moving body that pass through an arbitrary point are normal to a family of surfaces.

Here again, one can introduce an integral that is analogous to the one that we defined in no. **544**. One has:

$$\left(\frac{\partial\theta}{\partial x}\right)^2 + \left(\frac{\partial\theta}{\partial y}\right)^2 + \left(\frac{\partial\theta}{\partial z}\right)^2 = \frac{\partial\theta}{\partial x}x' + \frac{\partial\theta}{\partial y}y' + \frac{\partial\theta}{\partial z}z' = 2\,(U + h).$$

When one displaces along a trajectory, the preceding equation will take the form:

$$\frac{d\theta}{dt} = 2\,(U + h), \tag{18}$$

$$d\theta = 2\,(U + h)\,dt = \sqrt{2U + 2h}\;ds. \tag{19}$$

It follows from this that the difference between the values θ_M, $\theta_{M'}$ of θ that relate to the points M, M', resp., of the same trajectory is expressed by the formula:

$$\theta_M - \theta_{M'} = \int_{M'}^{M} \sqrt{2(U + h)}\;ds\,. \tag{20}$$

Once more, the integral that figures in the left-hand side will be called the *action* from M' to M. The developments that were given by Thomson and Tait show all of the importance of that element, which must be considered to have the same status as work in the study of problems in mechanics. The preceding formula gives the following theorems, in particular, which are analogous to the ones in no. **544**.

If one considers the arcs of the trajectories that pass through a point M_0 or are normal to an arbitrary surface, when those arcs are measured from their point of incidence, and for which the action has a given value, then the locus of extremities of all of those arcs will be normal to all of the trajectories.

557. We shall now point out how one can employ a complete integral:

$$\theta = f(x, y, z, a, b) \tag{21}$$

of the Jacobi partial differential equation to solve two of the problems that we just encountered.

From Lagrange's rule, the most general solution to the partial differential equation is provided by the relations:

$$\left\{ \begin{aligned} &\theta = f(x, y, z, a, b) + \varphi(a, b), \\ &0 = \frac{\partial f}{\partial a} + \frac{\partial \varphi}{\partial a}, \\ &0 = \frac{\partial f}{\partial b} + \frac{\partial \varphi}{\partial b}, \end{aligned} \right. \tag{22}$$

between which, one must eliminate a and b.

If one desires that the solution θ should be zero, or more generally, that it should have a given value $\mu(x, y)$ at each point of a surface (Σ) that is given by its equation:

$$z = \lambda(x, y) \tag{23}$$

then one must replace θ with μ and z with λ in the preceding equations, which would give the system:

$$\left\{ \begin{aligned} &\mu = f(x, y, z, a, b) + \varphi(a, b), \\ &0 = \frac{\partial f}{\partial a} + \frac{\partial \varphi}{\partial a}, \qquad 0 = \frac{\partial f}{\partial b} + \frac{\partial \varphi}{\partial b}. \end{aligned} \right. \tag{24}$$

Upon differentiating the first relation with respect to x and y successively and taking the other two into account, one will have:

$$(25) \qquad \begin{cases} \dfrac{\partial f}{\partial x} + \dfrac{\partial f}{\partial \lambda}\dfrac{\partial \lambda}{\partial x} - \dfrac{\partial \mu}{\partial x} = 0, \\ \dfrac{\partial f}{\partial y} + \dfrac{\partial f}{\partial \lambda}\dfrac{\partial \lambda}{\partial y} - \dfrac{\partial \mu}{\partial y} = 0. \end{cases}$$

The elimination of x and y from those two equations and the first of equations (24) will give φ as a function of a and b, and will, in turn, determine the desired solution.

If one would like to have the solution θ that corresponds to all of the trajectories that pass through a point (x_0, y_0, z_0) then one must take:

$$(26) \qquad \varphi = -f(x_0, y_0, z_0, a, b) .$$

I shall be content to point out those propositions, which belong to the theory of partial differential equations and are analogous to the ones that were given above (nos. **541** and **542**).

558. The preceding results lead us to imagine curvilinear coordinate systems in which one defines a point in space by the values of three quantities:

$$\theta, \quad \theta_1 = \frac{\partial \theta}{\partial a}, \quad \theta_2 = \frac{\partial \theta}{\partial b}.$$

A point is then determined by the intersection of three surfaces that belong to different families. The surfaces of the families:

$$\theta_1 = \text{const.}, \qquad \theta_2 = \text{const.}$$

are generated by trajectories of the moving body that are normal to the surfaces:

$$\theta = \text{const.}$$

Upon considering x, y, z to be functions of θ, θ_1, θ_2, one will then have:

$$(27) \qquad \begin{cases} \dfrac{\partial x}{\partial \theta}\dfrac{\partial x}{\partial \theta_1} + \dfrac{\partial y}{\partial \theta}\dfrac{\partial y}{\partial \theta_1} + \dfrac{\partial z}{\partial \theta}\dfrac{\partial z}{\partial \theta_1} = 0, \\ \dfrac{\partial x}{\partial \theta}\dfrac{\partial x}{\partial \theta_2} + \dfrac{\partial y}{\partial \theta}\dfrac{\partial y}{\partial \theta_2} + \dfrac{\partial z}{\partial \theta}\dfrac{\partial z}{\partial \theta_2} = 0, \end{cases}$$

and as a result, the line element of space will be given by an equation of the form:

$$ds^2 = H^2\, d\theta^2 + M\, d\theta_1^2 + 2N\, d\theta_1\, d\theta_2 + P\, d\theta_2^2 .$$

As before (no. **541**), the value of H will satisfy:

$$H^2 = \frac{1}{2U + 2h},$$

and, in turn, the value of ds^2 can be written:

$$(28) \qquad (2U + 2h)\, ds^2 = d\theta^2 + m\; d\theta_1^2 + 2n\, d\theta_1\, d\theta_2 + p\; d\theta_2^2 ,$$

in which the quantities m, p, $mp - n^2$ are essentially positive. One can deduce the principle of least action, as well as Hamilton's principle, from that formula by arguments that are analogous to the ones that we developed in the case of two variables. Instead of insisting upon that subject, which will be reprised in a general manner in the following chapter, we shall, in conclusion, point out an important formula that relates to *action.*

559. Recall the relation:

$$(29) \qquad \theta_M - \theta_{M'} = \int_{M'}^{M} \sqrt{2(U+h)}\; ds ,$$

which gives the action $\overline{M'M}$ along the arc $M'M$ of a trajectory. Let:

$$\theta = f(x, y, z, a, b)$$

be a complete solution of the Jacobi equation, and let:

$$(30) \qquad \frac{\partial \theta}{\partial a} = a', \qquad \frac{\partial \theta}{\partial b} = b'$$

be the equation of the trajectory that one considers. Let x, y, z denote the coordinates of M, and let x_0, y_0, z_0 be those of M'. By virtue of equations (30), one will have:

$$(31) \qquad \begin{cases} \dfrac{\partial}{\partial a} f(x,y,z,a,b) = \dfrac{\partial}{\partial a} f(x_0,y_0,z_0,a,b), \\[2ex] \dfrac{\partial}{\partial b} f(x,y,z,a,b) = \dfrac{\partial}{\partial b} f(x_0,y_0,z_0,a,b). \end{cases}$$

Those two equations give a and b as functions of x, y, z, x_0, y_0, z_0, and will, in turn, express the action $\overline{M'M}$ by a formula:

$$(32) \qquad \overline{M'M} = \Theta\,(x, y, z;\, x_0, y_0, z_0)$$

that contains only the coordinates of the points M, M'. It is important to calculate the derivatives of that function. Now, one has:

$$\overline{M'M} = f(x, y, z, a, b) - f(x_0, y_0, z_0, a, b),$$

and in turn, upon totally differentiating:

$$\delta \overline{M'M} = \frac{\partial f}{\partial x}\delta x + \frac{\partial f}{\partial y}\delta y + \frac{\partial f}{\partial z}\delta z - \frac{\partial f_0}{\partial x_0}\delta x_0 - \frac{\partial f_0}{\partial y_0}\delta y_0 - \frac{\partial f_0}{\partial z_0}\delta z_0$$
$$+ \frac{\partial (f - f_0)}{\partial a}\delta a + \frac{\partial (f - f_0)}{\partial b}\delta b .$$

Since the coefficients of δa, δb are zero, by virtue of equations (31), what will remain is simply:

$$\delta \overline{M'M} = \frac{\partial f}{\partial x}\delta x + \frac{\partial f}{\partial y}\delta y + \frac{\partial f}{\partial z}\delta z - \frac{\partial f_0}{\partial x_0}\delta x_0 - \frac{\partial f_0}{\partial y_0}\delta y_0 - \frac{\partial f_0}{\partial z_0}\delta z_0 ,$$

or, upon replacing the derivatives of f and f_0 with the velocities:

(33) $$\delta \overline{M'M} = x' \,\delta x + y' \,\delta y + z' \,\delta z - x'_0 \,\delta x_0 - y'_0 \,\delta y_0 - z'_0 \,\delta z_0 .$$

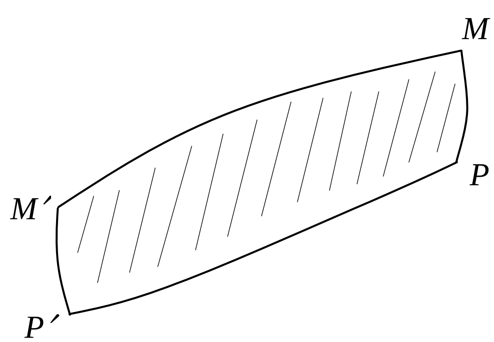

Figure 36.

That relation, in which the function f has disappeared completely, and that one can also establish by the calculus of variations, is analogous to the ones that we proved in nos. **525** and **540**. It gives the variation of the action along a segment of the trajectory $M'M$ (Fig. 36) when one passes to an infinitely-close segment $P'P$. One can give it an entirely geometric form. If $M'MP$, $MM'P'$ denote the angles that the trajectory makes at M, M', resp., with the infinitesimal displacements MP, $M'P'$, resp., then one will obviously have:

$$x' \,\delta x + y' \,\delta y + z' \,\delta z = - MP \frac{ds}{dt} \cos \widehat{M'MP} ,$$
$$x'_0 \,\delta x_0 + y'_0 \,\delta y_0 + z'_0 \,\delta z_0 = MP \left(\frac{ds}{dt}\right)_0 \cos \widehat{MM'P'} .$$

Upon replacing the velocities $\dfrac{ds}{dt}$, $\left(\dfrac{ds}{dt}\right)_0$ with their values that are deduced from the *vis viva* equation, one will have:

(34) $$\delta \overline{M'M} = - MP\sqrt{2U_M + 2h}\cos\widehat{M'MP} - M'P'\sqrt{2U_{M'} + 2h}\cos\widehat{MM'P'}.$$

That formula includes the particular case that relates to the differential of a line segment, and it gives rise to analogous consequences. We shall point out only the following one, which the reader can establish by extending the method that was given in no. **450** for the proof of Malus's theorem.

560. If one is given an arbitrary trajectory and a surface (D) then one imagines that the trajectory is reflected or refracted according to the sine law at the point where it meets the surface (D) in the same manner as a light ray. The law of reflection or refraction determines the tangent to the reflected or refracted trajectory, since a trajectory that corresponds to a given value of the *vis viva* constant will clearly be defined when one knows one of its points and the tangent at that point, so one will see that one can always determine what one calls the reflected or refracted trajectory by a geometric construction. Once that definition has been assumed, the arguments in no. **450** and the use of formulas (34) will lead us to the following theorem:

Consider all of the trajectories of the moving body that are normal to a surface (Σ), *and suppose that they are reflected or refracted on a surface* (D). *The reflected or refracted trajectories will also be normal to a surface* (Σ_1) *that one can construct in the following manner: If M is the point where the trajectory is normal to* (Σ), *and P is the one where it meets* (D) *then one will take an arc PM' on the refracted trajectory such that the action along the arc PM' is equal to the product of the action along the arc MP with the constant* $-1/n$, *where n is the index of refraction.*

561. One can imagine some dynamical conditions that obligate the trajectories to be reflected or refracted according to the laws that we just discussed. Indeed, suppose that the force function varies abruptly in the neighborhood of the surface (D), in such a manner that it is replaced by:

$$n^2 (U + h) - h,$$

in which n is the index of refraction. The law of the trajectories will be the same after crossing the surface (D). Granted, the equations of motion (1) and (2) will change, but in order to convert them to their original form, it will suffice to replace dt with dt / n. One will then have the same trajectories, but traversed with velocities that are augmented by the ratio of the index n to unity. In order to know how the trajectories are converted in the neighborhood of (D), it will suffice to apply the Thomson-Tait theorem. If the incident trajectories are normal to a surface (Σ), they will remain normal to all of the surfaces that one obtains by starting with their point of incidence on (Σ) and moving along an arc such that the action along that arc has a given value. Let M be the point of incidence for one of the trajectories, let P be the point where it meets D, and let M' be a point of the refracted trajectory. One will have:

$$\overline{MP} = \int_M^P \sqrt{2U + 2h}\ ds,$$

$$\overline{PM'} = n\int_P^{M'} \sqrt{2U + 2h}\ ds.$$

Hence, if one determines the point M' by the equation:

$$\int_M^P \sqrt{2U + 2h}\ ds + n\int_P^{M'} \sqrt{2U + 2h}\ ds = \text{const.}$$

then one will obtain a surface that is normal to the refracted trajectories. As one easily sees, that condition, when combined with formula (34), will determine the law of refraction, and one will recover precisely the law of Descartes that we assumed *a priori* in the preceding number.

CHAPTER VIII

THE GENERAL PROBLEM IN DYNAMICS

Lagrange equations. – Hamilton's transformation. – Definition of a *family* of solutions. – Partial differential equations that define a family. – Orthogonal families. – Jacobi's partial differential equation. – Use that one can make of its various solutions. – Expression for the *vis viva* that is due to Lipschitz. – Principle of least action. – Liouville's formula. – Definition of *action*. – Expression for the elementary action by means of a complete integral of the Jacobi equation and its derivatives with respect to the constant. – Another method of presenting the preceding results. Elimination of time with the aid of the *vis viva* principle. – Definition of the angles with respect to a quadratic form. Work of Beltrami. – Definition and invariance properties of the differential parameters $\Delta\theta$, $\Delta(\theta, \theta_1)$. – Remarkable transformations of the quadratic form. – Geodesic lines of the form, extension of Gauss's theorems. – Application of the general problem of dynamics.

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502. The methods that we have applied in the preceding two chapters can be extended to the study of the general problem in mechanics. It would not be pointless to develop this new way of presenting the fundamental results that are due to Hamilton and Jacobi here, because we would then be led to certain general properties of the quadratic forms that clarify the preceding results and will be useful to us in what follows.

Imagine a problem in mechanics in which there exists a force function, which we assume to be independent of time. Let $q_1, q_2, \ldots, q_n$ be the independent variables upon which the position of the moving body depends, let $q_1', \ldots, q_n'$ be their derivatives with respect to time, and let $2T$ be the *vis viva*, which is defined by the formula:

(1) $$2T = a_{11}\, q_1'^2 + 2a_{12}\, q_1' q_2' + \ldots = \sum\sum a_{ik}\, q_i' q_k' ,$$

in which the coefficients a_{ik} are given functions of $q_1, q_2, \ldots, q_n$. The motion will be defined by the Lagrange equations:

(2) $$\frac{d}{dt}\frac{\partial T}{\partial q_i'} - \frac{\partial T}{\partial q_i} - \frac{\partial U}{\partial q_i} = 0 \qquad (i = 1, 2, \ldots, n).$$

Hamilton showed that if one introduces the auxiliary variables:

(3) $$p_i = \frac{\partial T}{\partial q_i'} = \sum a_{ik}\, q_k' \qquad (i = 1, 2, \ldots, n)$$

then one can transform those equations in the following manner:

Set:

(4) $$H = T - U.$$

One expresses H as a function of the variables p_i, q_k. In order to do that, it will suffice to deduce the values of $q'_1, \ldots, q'_n$ from (3) and substitute them in the preceding expression. If one sets:

$$D = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n1} & \cdots & a_{nn} \end{vmatrix}, \tag{5}$$

or, upon adopting a notation that is due to Kronecker:

$$D = |\, a_{ik} \,| \qquad (i, k = 1, 2, \ldots, n),$$

and denoting the coefficient of a_{ik} in the preceding determinant by A_{ik}, then formulas (3) will give us:

$$q'_i = \frac{A_{i1}}{D} p_1 + \ldots + \frac{A_{in}}{D} p_n . \tag{6}$$

Since one has:

$$2T = p_1 q'_1 + \ldots + p_n q'_n , \tag{7}$$

from the theorem of homogeneous functions, one will obtain, with no difficulty, the following value of T:

$$2T = \frac{1}{D} \sum \sum A_{ik}\, p_i\, p_k , \tag{8}$$

which one can further write as follows:

$$2T = -\frac{1}{D} \begin{vmatrix} a_{11} & \cdots & a_{1n} & p_1 \\ \cdots & \cdots & \cdots & \cdots \\ a_{1n} & \cdots & a_{nn} & p_n \\ p_n & \cdots & p_n & 0 \end{vmatrix}, \tag{9}$$

and one deduces from this that:

$$H = T - U = \frac{1}{2D} \sum \sum A_{ik}\, p_i\, p_k - U. \tag{10}$$

Once one knows that value for H, the equations of motion will be presented in the *canonical* form:

$$\frac{dq_i}{dt} = \frac{\partial H}{\partial p_i}, \qquad \frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i} \qquad (i = 1, 2, \ldots, n), \tag{11}$$

and the equation of *vis viva*:

$$T = U + h \tag{12}$$

can then be written:

$$H = h. \tag{13}$$

That is the first result that Hamilton established.

503. Now consider all of the solutions of the problem for which the *vis viva* constant has a given value h, and let:

$$q_i = f_i\,(c_1, c_2, \ldots, c_{2n-2}, h, t - t_0) \qquad (i = 1, 2, \ldots, n) \tag{14}$$

be the equation that gives the values of the variables q_i as functions of time.

The values of the variables p_i that are defined by formulas (3) will be:

$$p_i = a_{i1}\frac{\partial f_1}{\partial t} + a_{i2}\frac{\partial f_2}{\partial t} + \cdots a_{in}\frac{\partial f_n}{\partial t} \qquad (i = 1, 2, \ldots, n). \tag{15}$$

Instead of preserving the most general solutions, imagine that one establishes $n - 1$ relations between the $2n - 1$ constants c_i and h, which are arbitrary, moreover. For example, one annuls $n - 1$ constants, or perhaps one considers the set of solutions that correspond to the same given initial position of the system, etc. One will then obtain some formulas that contain only $n - 1$ constants:

$$q_i = \varphi_i\,(c_1, c_2, \ldots, c_{n-1}, h, t - t_0) \qquad (i = 1, 2, \ldots, n), \tag{16}$$

which define what we call a *family* of solutions.

One can eliminate $t - t_0$ and the constants c_i from the preceding equations and their derivatives:

$$q_1' = \frac{d\varphi_1}{dt}, \qquad \ldots, \qquad q_n' = \frac{d\varphi_n}{dt}.$$

One will then be led to a system of differential equations:

$$q_i' = \frac{dq_i}{dt} = \Phi_i\,(q_1, q_2, \ldots, q_n, h) \qquad (i = 1, 2, \ldots, n), \tag{17}$$

whose integration will permit one to recover equations (6), and which can be considered to define the family of solutions by the same right as system (16). If one substitutes the preceding values in formulas (3) then one will deduce some expressions for $p_1, \ldots, p_n$ as functions of $q_1, \ldots, q_n$:

$$p_i = \Psi_i\,(q_1, q_2, \ldots, q_n, h) \tag{18}$$

that can take the place of system (17). We shall now give the equations that determine the functions Ψ_i .

First, consider the following n equations:

$$\frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i}$$

of the system (11). When the p_i are expressed as functions of the variables q_i, those equations will take the form:

$$\frac{\partial p_i}{\partial q_1} q'_1 + \frac{\partial p_i}{\partial q_2} q'_2 + \cdots + \frac{\partial p_i}{\partial q_n} q'_n = -\frac{\partial H}{\partial q_i},$$

and if one replaces q'_i with its value $\frac{\partial H}{\partial p_i}$ then one will find that:

$$\sum_k \frac{\partial p_i}{\partial q_k} \frac{\partial H}{\partial p_k} + \frac{\partial H}{\partial q_i} = 0. \tag{19}$$

On the other hand, if one substitutes the expressions for $p_1, \ldots, p_n$ in the *vis viva* equation:

$$H = h \tag{20}$$

then one must obtain an identity. It is then necessary that the derivative of the left-hand side with respect to q_i must become zero, which will give the relation:

$$\frac{\partial H}{\partial q_i} + \sum_k \frac{\partial H}{\partial p_k} \frac{\partial p_k}{\partial q_i} = 0.$$

Upon subtracting that from equation (19), one will find that:

$$\sum_k \frac{\partial H}{\partial p_k} \left(\frac{\partial p_k}{\partial q_i} - \frac{\partial p_i}{\partial q_k} \right) = 0 \qquad (i = 1, 2, \ldots, n). \tag{21}$$

Hence, the variables p_i, when considered to be functions of $q_1, \ldots, q_n$, must satisfy equations (20) and (21).

Now consider the second group of differential equations (11):

$$\frac{dq_i}{dt} = \frac{\partial H}{\partial p_i}. \tag{22}$$

When one has replaced the p_i with their values, one will have defined a system of n first-order differential equations that is equivalent to the system (17), and whose integration will give the values of $q_1, q_2, \ldots, q_n$ as functions of time and $n - 1$ arbitrary constants that must be adjoined to the *vis viva* constant. As one sees, all of the difficulty is then reduced to first determining the expressions for $p_1, p_2, \ldots, p_n$ that satisfy equations (20) and (21).

564. Once the problem has been transformed in that way, one does not at all believe that one has made a step towards solving it: On the surface of things, the integration of equations (20) and (21) constitutes a question that is much more difficult than the one that one is supposed to solve.

It is only that problem that exhibits some particular solutions. Indeed, one recognizes immediately that equations (21) will be verified if one takes $p_1, \ldots, p_n$ to be the derivatives of the same arbitrary function θ:

$$p_1 = \frac{\partial \theta}{\partial q_1}, \qquad p_2 = \frac{\partial \theta}{\partial q_2}, \qquad \ldots, \qquad p_n = \frac{\partial \theta}{\partial q_n}. \tag{23}$$

As for equation (20), if one substitutes the preceding values of the variables p_i then it will be transformed into a partial differential equation that defines the function θ. If one sets:

$$\Delta \theta = \sum \sum \frac{A_{ik}}{D} \frac{\partial \theta}{\partial q_i} \frac{\partial \theta}{\partial q_k}, \tag{24}$$

to abbreviate, then one will then find that:

$$\Delta \theta = 2\,(U + h), \tag{25}$$

and one can then state the following theorem:

Each integral of the partial differential equation (25) *corresponds to a family of solutions of the problem that was posed for which h is the vis viva constant and that one can determine completely upon performing the integration of the system of differential equations:*

$$p_i = \sum_k a_{ik} \frac{dq_i}{dt} = \frac{\partial \theta}{\partial q_i} \qquad (i = 1, 2, \ldots, n). \tag{26}$$

From the arguments that we just presented, it is clear that the preceding theorem will provide only some particular families of solutions. However, we see (and we can prove this at the moment) that those particular families comprise all of the possible solutions to the proposed problem. Indeed, let (γ) be one such solution. It is defined entirely by the initial values p_i^0, q_k^0 of the variables p_i, q_k, which are values that must satisfy the *vis viva* equation:

$$H = h,$$

moreover.

Now, there exists an infinitude of solutions θ to the partial differential equations (25) such that one has:

$$\frac{\partial \theta}{\partial q_i} = p_i^0 \qquad \text{for} \qquad q_1 = q_1^0, \ldots, q_n = q_n^0.$$

In each of the corresponding families, the solution (γ') that is defined by the initial values q_i^0 of the variables q_i will coincide with the solution (γ), because for the two solutions, the initial values p_i^0, q_k^0 of all the variables p_i, q_k will be the same.

565. We give the name of *orthogonal families* to all of the ones that are defined by equation (25) and the system (26). When one knows the solution θ, the complete determination of the corresponding family will require the integration of the system (26).

The latter integration will be facilitated, and can even be obviated, when the solution θ contains a certain number of arbitrary constants. That is what Jacobi's fundamental theorem amounts to, which we shall first prove.

Let:

$$\theta = f(q_1, \ldots, q_n, c_1, \ldots, c_\lambda, h)$$

be a solution to equation (25) that contains the arbitrary constants $c_1, \ldots, c_\lambda$. Upon differentiating both sides of that equation with respect to any one c_p of the preceding constants and remarking that $U + h$ does not contain c_p, one will find that:

$$\sum\sum \frac{A_{ik}}{D} \frac{\partial\theta}{\partial q_i} \frac{\partial^2\theta}{\partial q_k\, \partial c_p} = 0.$$

From formula (6), the set of coefficients of the $\dfrac{\partial^2\theta}{\partial q_k\, \partial c_p}$ is precisely q_k'. One will then have:

$$\sum \frac{\partial^2\theta}{\partial q_k\, \partial c_p} q_k' = 0,$$

or more simply:

$$\frac{d}{dt}\left(\frac{\partial\theta}{\partial c_p}\right) = 0,$$

and upon integrating:

$$\frac{\partial\theta}{\partial c_p} = \text{const.} = c_p'.$$

One proves in the same manner that the equation:

$$\frac{d}{dt}\left(\frac{\partial\theta}{\partial h}\right) = 1 \quad \text{will give} \qquad \frac{\partial\theta}{\partial h} = t + \tau.$$

Hence:

If the function θ contains the arbitrary constants $c_1, c_2, \ldots, c_\lambda$ then the equations:

(27) $$\frac{\partial\theta}{\partial c_1} = c_1', \qquad \ldots, \qquad \frac{\partial\theta}{\partial c_\lambda} = c_\lambda'$$

will be as good as integrals of the system (26). *Moreover, if one has not attributed a numerical value to the vis viva constant then the equation:*

(28) $$\frac{\partial\theta}{\partial h} = t + \tau$$

will give the time.

If the integral θ is complete – i.e., if it contains n – 1 *arbitrary contains c_p – then the equations:*

(29) $$\frac{\partial\theta}{\partial c_1} = c_1', \qquad \ldots, \qquad \frac{\partial\theta}{\partial c_{n-1}} = c_{n-1}', \quad \frac{\partial\theta}{\partial h} = t + \tau$$

will give the complete integration of the system (26).

Jacobi's proposition is then found to be established.

Here again, by arguments that are analogous to the ones in no. **555**, one will recognize the conditions that the complete integral must satisfy. If one considers it to be a function of the constants c_p then it will be necessary that it must not satisfy any equation of the form:

$$F\left(\frac{\partial\theta}{\partial c_1}, \ldots, \frac{\partial\theta}{\partial c_{n-1}}, c_1, \ldots, c_{n-1}, h\right) = 0.$$

That condition is, moreover, equivalent to the following one: θ, when considered to be a function of $q_1, \ldots, q_n$, must not verify any partial differential equation that is independent of the constants c_p and distinct from equation (24).

566. Equations (29), to which, one must adjoin the following ones:

(30) $$p_1 = \frac{\partial\theta}{\partial q_1}, \qquad \ldots, \qquad p_n = \frac{\partial\theta}{\partial q_n},$$

which will give one the velocities, define the most general solution to the problem that was posed. Among all of the solutions, consider the ones that correspond to the same initial position of the moving system. Let $q_1^0, \ldots, q_n^0$ be the values of the variables q_i that define that position, and let:

$$\theta = f(q_1, \ldots, q_n, c_1, \ldots, c_{n-1}, h)$$

be the solution that figures in formulas (30). We set:

$$f_0 = f(q_1^0, \ldots, q_n^0, c_1, \ldots, c_{n-1}, h).$$

Equations (29) must be verified when one sets $q_i = q_i^0$, so one will have:

$$c_i' = \frac{\partial f_0}{\partial c_i},$$

and as a result, those equations will take the form:

$$(31) \qquad \frac{\partial}{\partial c_i}(f - f_0) = 0 \qquad (i = 1, 2, \ldots, n - 1).$$

Upon attributing all possible values to the constants c_i in those equations, one will obtain what we have called a family of solutions. *That family is orthogonal.* One can prove that in the following manner:

From the definition itself of the orthogonal families, everything amounts to establishing that the expression:

$$(32) \qquad \sum p_i \, dq_i = \sum \frac{\partial f}{\partial q_i} dq_i$$

will become an exact differential after one has replaced the constants c_i with their expressions as functions of $q_1, \ldots, q_n$ that are deduced from equations (31). Now, if one considers the functions:

$$\sigma = f - f_0 ,$$

in which one has replaced the constants c_λ with their values that are deduced from equations (31), and if one totally differentiates them then one will find that:

$$d\sigma = \sum \frac{\partial f}{\partial q_i} dq_i + \sum \frac{\partial (f - f_0)}{\partial c_p} dc_p .$$

Since the coefficients of the differentials dc_p are zero by virtue of equations (31), $d\sigma$ must be equal to the expression (32). The proposition that have in mind is then established. We shall recover it later on along an entirely different path.

The particular solution σ that we just obtained, which is a solution that one can examine as a function of $q_1, \ldots, q_n$, $q_1^0, \ldots, q_n^0$, plays a fundamental role in Hamilton's theory, as one knows.

The preceding argument persists without modification when one replaces f_0 with an arbitrary function:

$$\varphi(c_1, c_2, \ldots, c_{n-1}, h),$$

in such a way that the equations:

$$\frac{\partial}{\partial c_i}(f - \varphi) = 0 \qquad (p = 1, 2, \ldots, n - 1)$$

always define an orthogonal family. That family corresponds to the solution θ of the partial differential equation that one obtains from Lagrange's rule by eliminating c_1, c_2, …, c_{n-1} from the equation:

$$\theta = f - \varphi$$

and its derivatives with respect to the arbitrary constants.

567. The preceding remarks apply to all hypotheses in which the solution θ contains arbitrary constants. When that is not true, if one is to determine the family of solutions that correspond to the solution θ then it will be necessary to integrate the system (26). Any integral:

$$F = \text{const.}$$

of that system must satisfy the linear equation:

$$\sum \frac{\partial F}{\partial q_k} q_k' = \sum \sum \frac{A_{ik}}{D} \frac{\partial \theta}{\partial q_i} \frac{\partial F}{\partial q_k} = 0.$$

We let $\Delta(\theta, F)$ denote the expression:

$$\Delta(\theta, F) = \sum \sum \frac{A_{ik}}{D} \frac{\partial \theta}{\partial q_i} \frac{\partial F}{\partial q_k}, \tag{33}$$

to abbreviate.

The integration of the system (26) is then equivalent to that of the linear equation:

$$\Delta(\theta, F) = 0. \tag{34}$$

We remark that the preceding symbol will reduce to $\Delta\theta$ when one supposes that $F = \theta$ in it.

The consideration of orthogonal families will lead us to a remarkable expression for the *vis viva* of the moving system that was pointed out by Lipschitz ([20]). Let θ be an arbitrary solution of equation (25), and let $\theta_1, \theta_2, \ldots, \theta_{n-1}$ be the $n - 1$ *distinct* integrals of the linear equation that corresponds to (34).

$$\theta_1, \theta_2, \ldots, \theta_{n-1}$$

will then define a system of n independent functions, because if θ can be expressed as a function of $\theta_1, \ldots, \theta_{n-1}$ then one will have:

$$\Delta(\theta, \theta) = \Delta\theta = 0,$$

([20]) R. LIPSCHITZ, "Untersuchung eines Problems der Variations-rechunung in welchem das Problem of Mechanik enhalten ist," Crelle's Journal **74** (1871). One can also consult an analysis of that paper that was edited by the author in the Bulletin des Sciences mathématiques (1) **4** (1873), pp. 212.

which is impossible, since $\Delta\theta$ is equal to $U + h$. We can then introduce the variables θ, θ_i in place of the variables q_i in the quadratic form:

$$\sum\sum a_{ik} dq_i dq_k ,$$

which will give the *vis viva* when it is divided by dt^2. One will then obtain an expression:

$$\sum\sum a_{ik} dq_i dq_k = B\, d\theta^2 + 2\sum B_i\, d\theta\, d\theta_i + \sum\sum a_{ik}\, d\theta_i\, d\theta_k ; \tag{35}$$

we shall first look for the values of the coefficients B, B_i .

From the preceding equation, one will have:

$$\left\{\begin{aligned} B &= \sum\sum a_{ik} \frac{\partial q_i}{\partial\theta}\frac{\partial q_k}{\partial\theta}, \\ B_p &= \sum\sum a_{ik} \frac{\partial q_i}{\partial\theta}\frac{\partial q_k}{\partial\theta_p}. \end{aligned}\right. \tag{36}$$

Now, when θ alone varies, the equations of motion will be verified; that will result from the fact that $\theta_1, \ldots, \theta_{n-1}$ are the integrals of the system (26). One will then have:

$$\frac{\partial q_i}{\partial\theta} = q_i' \frac{dt}{d\theta}, \tag{37}$$

in which $d\theta / dt$ denotes the derivative of θ with respect to time in the natural motion. One then deduces from this that:

$$\sum_k a_{ik} \frac{\partial q_k}{\partial\theta} = \left(\sum_k a_{ik} q_i'\right)\frac{dt}{d\theta} = p_i \frac{dt}{d\theta},$$

or furthermore:

$$\sum_k a_{ik} \frac{\partial q_k}{\partial\theta} = \frac{\partial\theta}{\partial q_i}\frac{dt}{d\theta}. \tag{38}$$

If one multiplies that equation by $\dfrac{\partial q_i}{\partial\theta}$, and if one adds all of the similar equations then one will have:

$$\sum\sum a_{ik} \frac{\partial q_i}{\partial\theta}\frac{\partial q_k}{\partial\theta} = \left(\sum \frac{\partial\theta}{\partial q_i}\frac{\partial q_i}{\partial\theta}\right)\frac{dt}{d\theta},$$

or upon remarking that from the formulas that relate to changing variables, the coefficient of $dt / d\theta$ is unity, one will have:

$$\sum\sum a_{ik} \frac{\partial q_i}{\partial\theta}\frac{\partial q_k}{\partial\theta} = \frac{dt}{d\theta}.$$

The use of formula (37) will permit us to eliminate the derivatives $\frac{\partial q_i}{\partial \theta}$, and that will give us:

$$\sum \sum a_{ik} q_i' q_k' = \frac{d\theta}{dt},$$

or, upon taking the *vis viva* equation into account:

$$\frac{d\theta}{dt} = 2\,(U + h). \tag{39}$$

That is the formula that gives the derivative of θ for the natural motion. One deduces the following value for B from this:

$$B = 2\,(U + h)\,\frac{dt^2}{d\theta^2} = \frac{1}{2(U+h)}.$$

Now, calculate the value of B_p. If one multiplies the two sides of equation (38) by $\frac{\partial q_i}{\partial \theta}$ then upon adding all similar equations, one will get:

$$B_p = \sum \sum a_{ik} \frac{\partial q_i}{\partial \theta} \frac{\partial q_k}{\partial \theta_p} = \frac{dt}{d\theta} \left[\sum_i \frac{\partial \theta}{\partial q_i} \frac{\partial q_i}{\partial \theta} \right],$$

and since the left-hand side is obviously zero, from the formulas that relate to a change of variables, one will have:

$$B_p = 0.$$

Upon substituting the values of B and B_p in equation (35), one will then be led to the fundamental identity:

$$(2U + 2h) \sum \sum a_{ik}\, dq_i\, dq_k = d\theta^2 + f\,(d\theta_1, \ldots, d\theta_{n-1}), \tag{40}$$

in which f denotes a quadratic form of $n - 1$ differentials $d\theta_1, \ldots, d\theta_{n-1}$ that will necessarily be *positive-definite.*

588. That is the formula that was established by Lipschitz. One can deduce a neat and precise proof of the least-action principle from it. In the form that Jacobi gave it ([21]), that principle can be stated as follows:

([21]) *Vorlesungen über Dynamik*, sixth lecture.

If one is given two positions (P_0) *and* (P_1) *of a moving system then imagine all continuous displacements that take the system from the first position to the second one, while the velocities satisfy the vis viva equation:*

$$2T = \sum\sum a_{ik} q_i' q_k' = 2(U + h)$$

at each instant.

If one considers the integral:

$$\int_{(P_0)}^{(P_1)} \sum mv^2 dt = \int_{(P_0)}^{(P_1)} \sqrt{2U+2h}\sqrt{\sum\sum a_{ik} dq_i dq_k} \tag{41}$$

for each of those displacements then it will be less for the natural motion than for all other displacements.

Indeed, we shall see later on (no. **571**) that the first variation of the preceding integral will always be zero when one passes from the natural motion to any other motion from (P_0) to (P_1) that is infinitesimally different from it. Here, we shall prove a more precise proposition and prove that the integral will actually become a minimum when the position (P_1) is sufficiently close to (P_0).

Indeed, let (γ) be one of the natural motions. Consider an orthogonal family of solutions (F) to which the motion (γ) belongs. For example, one can choose any of the solutions that correspond to an initial position (P') that is one of them that the system takes in the natural motion. Define a continuous domain of positions that are characterized, for example, by certain inequalities that the $\theta_1, \theta_2, \ldots, \theta_n$ must satisfy when they are subject to the single condition that the solution θ to equation (25), which characterizes the orthogonal family, will remain finite and uniform, along with its first derivatives, except possibly for a certain well-defined position (P''). Moreover, suppose that this domain includes a subset of the positions that the moving system will occupy under the motion (γ) in its interior. If (P_0) and (P_1) denote two of those positions then we shall show that the integral:

$$\mathfrak{A} = \int_{(P_0)}^{(P_1)} \sqrt{2U+2h}\sqrt{\sum\sum a_{ik} dq_i dq_k}\ , \tag{42}$$

to which we shall give the name of *action*, will be smaller for the natural motion than for any other motion that takes one between the same position and belongs to the interior of the domain that was defined above. Indeed, it is impossible that two different positions (P_0), (P_1) that are included in the interior of the domain that was defined will belong to two distinct solutions of the orthogonal family that correspond to the determination of θ that we have chosen. If that were the case then one of the two positions (P_0), (P_1) would be distinct from (P'), and since the velocities that relate to those positions are determined by equations (26), in which, by hypothesis, the derivatives $\frac{\partial \theta}{\partial q_i}$ are neither infinite nor indeterminate, it will result that two distinct solutions will correspond to the same initial position and the same initial velocity, which is obviously impossible.

From that, we evaluate the action $\mathfrak{A}$ by appealing to formula (40). We will have:

$$\mathfrak{A} = \int_{(P_0)}^{(P_1)} \sqrt{d\theta^2 + f(d\theta_1, d\theta_2, \ldots, d\theta_{n-1}}\ . \tag{43}$$

For the natural motion, one has:

$$d\theta_1 = d\theta_2 = \ldots = d\theta_{n-1} = 0,$$

and on the other hand, since $\dfrac{\partial\theta}{\partial t}$ is always positive from formula (39), θ will be an increasing function. One will then have:

$$\mathfrak{A} = \int_{(P_0)}^{(P_1)} d\theta = \theta_{(P_1)} - \theta_{(P_0)}\ .$$

If one now considers any other motion that can be performed in that domain that was defined then, from the preceding proof, one cannot connect the two positions (P_0), (P_1) and constitute a solution of the orthogonal family that corresponds to the determination of θ that have chosen. As a result, the differentials $d\theta_1, \ldots, d\theta_{n-1}$ will not always be zero under the second motion, and since f is a positive-definite form, the integral $\mathfrak{A}$, when evaluated for the new motion, will be greater than:

$$\int_{(P_0)}^{(P_1)} \sqrt{d\theta^2}\ ,$$

and *a fortiori*, it will be greater than:

$$\int_{(P_0)}^{(P_1)} d\theta\ .$$

Now, since the function θ is well-defined inside the domain, the latter integral will always be $\theta_{(P_1)} - \theta_{(P_0)}$. The proposition that we have in mind is then established.

In particular, suppose that the orthogonal family considered is the one that is defined by all of the solutions that have the initial position (P_0) in common. Let θ be the integral that corresponds to the partial differential equation, to which one can always add a constant in such a manner that it is annulled for the position (P_0). The continuous domain can be characterized here by the inequality ([22]):

$$\theta < A,$$

in which A is a positive constant that is chosen by the single condition that θ and its derivatives will not become either infinite or indeterminate in the interior of that domain. The action in the natural motion that is performed in the interior of that domain between

([22]) The complete and precise definition of that domain will demand some developments that are analogous to the ones that we made in nos. **518** and **521** in regard to geodesic lines.

the position (P_0) and another position (P_1) that are included in the domain will be an absolute minimum, because from the preceding proof, it will be smaller than the one that relates to any other motion that is performed in the domain. However, it is also less than the one that refers to any motion that goes beyond the limits, since the action for that motion, when it is extended to just the first position for which it leaves the domain, will already be equal to A, and as a result, greater than $\theta_{(P_1)}$. That argument differs from the one that we presented in no. **521** only by the number of variables.

569. The preceding proof then establishes the principle of least action without the intervention of the calculus of variations and by purely algebraic methods. From that standpoint, it must be compared with the one that Liouville described in an article that was included in *Comptes rendus* in 1856 ([23]). We shall rapidly discuss a new method that leads to the results that were obtained by the illustrious geometer.

Let ϖ_i denote the expressions:

$$\varpi_i = a_{i1}\,dq_1 + a_{i2}\,dq_2 + \ldots + a_{in}\,dq_n\,, \tag{41}$$

which are equal to the quantities p_i, multiplied by dt, and consider the quadratic form:

$$K = \begin{vmatrix} a_{11} & \cdots & a_{1n} & \dfrac{\partial\theta}{\partial q_1} & \varpi_1 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n1} & \cdots & a_{nn} & \dfrac{\partial\theta}{\partial q_n} & \varpi_n \\ \dfrac{\partial\theta}{\partial q_1} & \cdots & \dfrac{\partial\theta}{\partial q_n} & 0 & 0 \\ \varpi_1 & \cdots & \varpi_n & 0 & 0 \end{vmatrix}. \tag{45}$$

One easily recognizes that it is always positive or zero, and that it can be annulled only if one has:

$$\frac{\varpi_1}{\dfrac{\partial\theta}{\partial q_1}} = \frac{\varpi_2}{\dfrac{\partial\theta}{\partial q_2}} = \ldots = \frac{\varpi_n}{\dfrac{\partial\theta}{\partial q_n}}.$$

Having said that, let the notations b_{11}, b_{12}, b_{21}, b_{22} denote the four elements that are zero and belong to the last two rows and the last two columns. A formula that we have recalled already (pp. 124 in original) will give us the relation:

([23]) J. LIOUVILLE, "Expression remarquable de la quantité qui, dans le movement d'un système de points matériels à liaisons quelconques, est un minimum en vertu du principe de la moindre action," Comptes rendus **42** (1856), pp. 1146, and Journal de Liouville (2) **1** (1856), pp. 297.

(46)
$$K\,\frac{\partial^2 K}{\partial b_{11}\,\partial b_{22}} = \frac{\partial K}{\partial b_{11}}\frac{\partial K}{\partial b_{22}} - \left(\frac{\partial K}{\partial b_{12}}\right)^2 .$$

One establishes immediately, or by a simple combination of columns, that one has:

$$\frac{\partial^2 K}{\partial b_{11}\,\partial b_{22}} = D, \qquad \frac{\partial K}{\partial b_{11}} = -\,D\sum\sum a_{ik}dq_i\,dq_k\,,$$

$$\frac{\partial K}{\partial b_{12}} = -\,D\,d\theta, \qquad \frac{\partial K}{\partial b_{12}} = -\,D\,\Delta\theta,$$

in which D and $\Delta\theta$ are the expressions that were defined already by formulas (5) and (24). Upon substituting those values in equation (46), one will find that:

$$\Delta\theta\sum\sum a_{ik}dq_i\,dq_k = d\theta^2 + \frac{K}{D}.$$

If one now supposes that θ satisfies the equation:

$$\Delta\theta = 2\,(U + h)$$

then one will have:

(47)
$$2\,(U + h)\sum\sum a_{ik}dq_i\,dq_k = d\theta^2 + \frac{K}{D}.$$

That equation, which gives an expression for the elementary action, like formula (40), can replace that identity and plays the same role in the proof that we gave of the principle of least action. Moreover, one can very easily deduce the first formula from the second one.

Indeed, from the expression (45), one recognizes immediately that K is a quadratic function of the binomials:

$$\varpi_i\,\frac{\partial\theta}{\partial q_k} - \varpi_k\,\frac{\partial\theta}{\partial q_i}\,.$$

All of those quantities, which are annulled when one displaces on a trajectory by virtue of the differential equations (26), will necessarily have the form:

$$P_1\,d\theta_1 + \ldots + P_{n-1}\,d\theta_{n-1}\,.$$

$K\,/\,D$ will then be a quadratic form in the differentials $d\theta_1$, $d\theta_2$, …, $d\theta_{n-1}$. That remark will suffice to show that equation (40) is a consequence of formula (47).

570. Here, we shall neglect everything that is concerned with Hamilton's principle. In order to establish that principle and see that there is actually a minimum of the corresponding integral, it will suffice to repeat the proof in no. **546**, while substituting the function $f(d\theta_1, \ldots, d\theta_{n-1})$ that figures in formula (40) for the term $\sigma^2 d\theta_1^2$ everywhere. On the contrary, we insist upon the following generalization of a result that was established in no. **534**.

When one knows a complete integral θ to equation (25), one can take the functions $\theta_1, \ldots, \theta_{n-1}$ to be the derivatives of θ with respect to the constants c_i. Set:

$$\theta_i = \frac{\partial \theta}{\partial c_i},$$

$$\theta_{ik} = \frac{\partial^2 \theta}{\partial c_i \, \partial c_k}.$$

We shall see that one can express the right-hand side of formula (40) entirely as a function of θ and its derivatives.

Replace the characteristic d with $d + \lambda\,\delta$ in that formula everywhere, and equate the coefficients of λ in the two sides. We will have the relation:

$$2\,(U + h) \sum\sum a_{ik}\, dq_i\, \delta q_k = d\theta\, \delta\theta + \frac{1}{2} \sum \frac{\partial f}{\partial\, d\theta_i} \delta\theta_i\,, \tag{48}$$

which is equivalent to the equation from which one has deduced it, but which contains two systems of differentials. The two sides can be considered to depend upon the $4n - 1$ variables q_i, dq_k, δq_k, c_λ. Differentiate with respect to c_λ, while keeping all of the other variables constant. Upon remarking that the left-hand side does not contain c_λ and that one has:

$$\frac{\partial}{\partial c_\lambda} du = d\, \frac{\partial u}{\partial c_\lambda},$$

in general, we will find the following result:

$$0 = d\theta_\lambda\, \delta\theta + d\theta\, \delta\theta_\lambda + \frac{1}{2} \sum \frac{\partial f}{\partial\, d\theta_i} \delta\theta_{i\lambda} + \frac{1}{2} \sum \frac{\partial f}{\partial\, \delta\theta_i} d\theta_{i\lambda} + \frac{1}{2} \sum \frac{\partial^2 f}{\partial c_\lambda\, \partial\, d\theta_i} \delta\theta_i\,.$$

Having said that, choose values for the differentials $\delta q_1, \ldots, \delta q_n$ that annul $\delta\theta_1$, $\delta\theta_2$, …, $\delta\theta_{n-1}$. If one sets:

(49) $$(0) = \begin{vmatrix} \dfrac{\partial\theta}{\partial q_1} & \cdots & \dfrac{\partial\theta}{\partial q_n} \\ \dfrac{\partial\theta_1}{\partial q_1} & \cdots & \dfrac{\partial\theta_1}{\partial q_n} \\ \cdots & \cdots & \cdots \\ \dfrac{\partial\theta_{n-1}}{\partial q_1} & \cdots & \dfrac{\partial\theta_{n-1}}{\partial q_n} \end{vmatrix}$$

then the values of $\delta q_1, \ldots, \delta q_n$ will be proportional to the coefficients of $\dfrac{\partial\theta}{\partial q_1}, \ldots, \dfrac{\partial\theta}{\partial q_n}$ in the preceding determinant. If one denotes the expression that the determinant will become when one replaces θ with θ_{ik} by (ik) then the preceding equation will give us:

(50) $$0 = (0)\, d\theta_\lambda + \frac{1}{2}\sum \frac{\partial f}{\partial\, d\theta_i}(i\lambda) \qquad (\lambda = 1, 2, \ldots, n-1),$$

while all of the other terms will disappear by virtue of the hypothesis that was made. If one appends the identity:

$$f = \frac{1}{2}\sum \frac{\partial f}{\partial\, d\theta_i}\, d\theta_i$$

to the preceding equations then one can eliminate all of the derivatives $\dfrac{1}{2}\dfrac{\partial f}{\partial\, d\theta_i}$ and obtain the equation:

$$\begin{vmatrix} (1,1) & \cdots & (1, n-1) & d\theta_1 \\ (2,1) & \cdots & (2, n-1) & d\theta_2 \\ \cdots & \cdots & \cdots & \cdots \\ (n-1,1) & \cdots & (n-1, n-1) & d\theta_{n-1} \\ d\theta_1 & \cdots & d\theta_{n-1} & -\dfrac{f}{(0)} \end{vmatrix} = 0,$$

which will give one f. One will then find that:

(51) $$f = \frac{(0)}{D'}\begin{vmatrix} (1,1) & \cdots & (1, n-1) & d\theta_1 \\ (2,1) & \cdots & (2, n-1) & d\theta_2 \\ \cdots & \cdots & \cdots & \cdots \\ (n-1,1) & \cdots & (n-1, n-1) & d\theta_{n-1} \\ d\theta_1 & \cdots & d\theta_{n-1} & 0 \end{vmatrix},$$

in which D' denotes the determinant:

$$| (i\,k) | \qquad\qquad (i, k = 1, 2, \ldots, n-1).$$

Hence, the quadratic form $2\,(U + h) \sum\sum a_{ik}\,dq_i\,dq_k$, which represents what one can call the square of the *elementary action*, will be expressed entirely as a function of θ and its derivatives by a formula in which no unknowns will still remain. As we have stated, that result will include what we proved in no. **531**.

571. The variable t plays a very self-effacing role and disappears almost completely in the preceding arguments. One can eliminate it from the outset, and thus recover the results that we just established in a different way. Those results are very important, so we shall rapidly describe that new mode of exposition.

One can make time disappear from the Lagrange equations by employing the principle of *vis viva*. Indeed, if one sets:

$$2\,(T) = \sum\sum a_{ik}\,dq_i\,dq_k$$

then the Lagrange equations can be written as follows:

$$d\left[\frac{\partial (T)}{\partial\, dq_i}\frac{1}{dt}\right] - \frac{1}{dt}\frac{\partial (T)}{\partial\, dq_i} - dt\,\frac{\partial U}{\partial\, dq_i} = 0.$$

Now, from the principle of *vis viva*, one will have:

$$dt = \sqrt{\frac{(T)}{U+h}}\,.$$

If one substitutes that value for dt in the Lagrange equations then they will take the form:

$$d\left[\frac{\partial (T)}{\partial\, dq_i}\frac{\sqrt{U+h}}{\sqrt{(T)}}\right] - \frac{\partial (T)}{\partial\, dq_i}\frac{\sqrt{U+h}}{\sqrt{(T)}} - \frac{\sqrt{(T)}}{U+h}\frac{\partial U}{\partial\, dq_i} = 0,$$

or, more simply:

(52) $$d\,\frac{\partial}{\partial\, dq_i}\sqrt{(U+h)(T)} - \frac{\partial}{\partial q_i}\sqrt{(U+h)(T)} = 0.$$

Those are the ones to which one will be led by equating the first variation of the integral:

(53) $$\int_{(P_0)}^{(P_1)}\sqrt{(U+h)(T)} = \tfrac{1}{2}\int_{(P_0)}^{(P_1)}\sqrt{2(U+h)\sum\sum a_{ik}\,dq_i\,dq_k}$$

to zero.

Set

(54) $$ds^2 = (2U + 2h) \sum\sum a_{ik} dq_i\, dq_k .$$

ds denotes what we have called the *elementary action*, and one sees that the solution to the general problem in mechanics is thus reduced to the search for the maximum or minimum of the integral:

$$\int_{(P_0)}^{(P_1)} ds ,$$

in which ds^2 denotes a quadratic form that is subject to only the condition that it must be *positive-definite.* That is what the principle of least action consists of, and thanks to that principle, one will see immediately that the general problem of mechanics is only an extension of the problem of the search for geodesic lines to an arbitrary number of variables. That is the viewpoint that we shall now assume, while taking a beautiful paper by Beltrami ([24]) as a guide.

572. If one is given the quadratic form:

(55) $$ds^2 = \sum\sum a_{ik} dq_i\, dq_k$$

then if one makes a change of variables that gives:

$$\sum\sum a_{ik} dq_i\, dq_k = \sum\sum b_{ik} dr_i\, dr_k ,$$

one will also have, upon introducing two systems of differentials

$$\sum\sum a_{ik} dq_i\, \delta q_k = \sum\sum b_{ik} dr_i\, \delta r_k .$$

As a result, the angle $(ds, \delta s)$, which is defined by the equation:

(56) $$ds\ \delta s \cos (ds, \delta s) = \sum\sum a_{ik} dq_i\, \delta q_k ,$$

will be an invariant. In the case where the form is definite, one will be easily assured that $\cos (ds, \delta s)$ has an absolute value that is less than unity, with the use of an identity of Lagrange. We say that the element $(ds, \delta s)$ is the *angle* between the two directions that are defined by the two systems of differentials d and δ. Two directions will be *perpendicular* when one has:

(57) $$\sum\sum a_{ik} dq_i\, \delta q_k = 0.$$

If one is given an arbitrary relation:

([24]) E. BELTRAMI, "Sulla teorica generale dei parametri differenziale," Memorie dell' Accademia delle Scienze dell' Istituto di Bologna (2) **8** (1869), pp. 549.

(58) $$\varphi(q_1, q_2, \ldots, q_n) = 0$$

then it will define what we call a *surface*. Suppose that $q_1, \ldots, q_n$ varies without ceasing to satisfy that equation. At each instant, their differentials will verify the relation:

(59) $$\sum \frac{\partial \varphi}{\partial q_i} \delta q_i = 0.$$

We reserve the name of *line* for the set of values for $q_1, \ldots, q_n$ that are given functions of a variable parameter t. We shall consider only lines and surfaces in this chapter. In general, a line and a surface have a limited number of common elements. The orthogonality condition for two lines that have a common element is expressed by formula (57), in which the characteristics d and δ refer to the displacements that were performed on the two lines, respectively.

If a line and a surface have a common element then we will say that the line is *normal to the surface* when it is normal to all of the lines in the surface that contain the common element. In order for that to be true, it is necessary that one must have:

(60) $$\sum\sum a_{ik} dq_i \, \delta q_k = 0,$$

in which the differential d refers to a displacement on the line, and the differential δ refers to a displacement on the surface. If the surface is represented by equation (58) then the differentials δq_i will satisfy the single condition (59). It will then be necessary that the coefficients of the differentials δq_i in the two equations (57) and (59) must be proportional. One will then be led to the system:

(61) $$\sum_k a_{ik} \frac{dq_k}{ds} = \lambda \frac{\partial \varphi}{\partial q_i} \qquad (i = 1, 2, \ldots, n),$$

in which λ is a factor of proportionality.

If one solves this for the derivatives dq_k / dt then one will obtain the following values:

(62) $$\frac{dq_i}{ds} = \lambda \sum_k \frac{A_{ik}}{D} \frac{\partial \varphi}{\partial q_k},$$

in which the symbols D and A_{ik} denote the quantities that were defined before; i.e., the determinant $| a_{ik} |$ and its first-order minors, respectively.

Since one has:

(63) $$\sum a_{ik} \frac{dq_i}{ds} \frac{dq_k}{ds} = 1,$$

the multiplication of equations (61) and (62) will give:

$$\lambda^2 \sum\sum \frac{A_{ik}}{D}\frac{\partial\varphi}{\partial q_i}\frac{\partial\varphi}{\partial q_k} = 1.$$

Here, upon setting:

(64) $$\Delta\theta = \sum\sum \frac{A_{ik}}{D}\frac{\partial\theta}{\partial q_i}\frac{\partial\theta}{\partial q_k},$$

one will find that:

(65) $$\lambda^2 = \frac{1}{\Delta\varphi}.$$

If one now remarks that, from formula (61), the set of terms that multiply dq_i / ds in equation (63) is equal to $\lambda\dfrac{\partial\varphi}{\partial q_i}$ then one can further write that equation in the following form:

$$\lambda \sum \frac{\partial\varphi}{\partial q_i}\frac{dq_i}{ds} = 1,$$

which will give:

(66) $$\frac{1}{\lambda} = \frac{d\varphi}{ds},$$

in which the differential $d\varphi$ refers to a displacement that is performed on the curve that is normal to the surface. A comparison of formulas (65) and (66) will then give us:

(67) $$\Delta\varphi = \left(\frac{d\varphi}{ds}\right)^2,$$

and that expression shows immediately that $\Delta\varphi$ is an invariant ([25]).

573. That essential fact can also be established in the following manner: Consider the quadratic form ds^2 and look for the function m such that the difference:

$$ds^2 - \frac{d\theta^2}{m},$$

when considered to be a function of $dq_1, \ldots, dq_n$, reduces to a sum of $n - 1$ squares. If one takes the derivatives of the preceding difference with respect to $dq_1, dq_2, \ldots, dq_n$ then one will obtain n equations:

$$\sum_k a_{ik}\frac{dq_k}{ds} - \frac{1}{m}\frac{\partial\theta}{\partial q_i}d\theta = 0 \qquad (i = 1, 2, \ldots, n),$$

([25]) E. BELTRAMI, *loc. cit.*

in which the determinant must be zero. One can append the equation:

$$\sum_k \frac{\partial\theta}{\partial q_i} dq_i - d\theta = 0$$

to them, which defines $d\theta$. Upon eliminating $d\theta$, dq_1, …, dq_n, one will be led to an equation that gives:

$$m = \Delta\theta$$

precisely.

Since the function m is, from its very definition, an invariant, the same thing will be true for $\Delta\theta$.

574. Since the difference:

$$ds^2 - \frac{d\theta^2}{\Delta\theta}$$

reduces to a sum of $n - 1$ squares, which are all positive when the form is positive-definite, one will be led to introduce the element (θ, ds) that is defined by the equation:

$$\frac{d\theta}{\sqrt{\Delta\theta}} = ds \sin(\theta, ds). \tag{68}$$

The element (θ, ds) will be called the *angle* between the surface (θ) and the curve to which the differentials dq_i refer. That definition is in accord with the one that we gave for orthogonality above, since, from the relation (67), one will then have:

$$\sin^2(\theta, ds) = 1.$$

From the invariant $\Delta\theta$, one will immediately deduce the following ([26]):

$$\Delta(\theta, \theta_1) = \sum\sum \frac{A_{ik}}{D} \frac{\partial\theta}{\partial q_i} \frac{\partial\theta}{\partial q_k}, \tag{69}$$

which is the coefficient of 2λ in the development of $\Delta(\theta + \lambda\theta_1)$, when it is ordered in powers of the constant λ.

That leads us to further introduce the element (θ, θ_1) that is defined by the formula:

([26]) One can give a formula for $\Delta(\theta, \theta_1)$ that is analogous to equation (67). Indeed, one has:

$$\Delta(\theta, \theta_1) = \sqrt{\Delta\theta}\,\frac{d\theta_1}{ds},$$

in which the differential d refers to a displacement that is normal to the surface (θ).

$$\cos(\theta, \theta_1) = \frac{\Delta(\theta, \theta_1)}{\sqrt{\Delta\theta}\sqrt{\Delta\theta_1}}, \tag{70}$$

which will be the *angle between the two surfaces* (θ), (θ_1).

In summary, we have defined the angle between two lines, two surfaces, and a line and a surface. One will easily verify that *those angles do not change when the form is multiplied by an arbitrary function of the independent variables.* If one considers lines and surfaces that have a common element then the angle between a line and a surface is the complement of the angle that the line makes with the normal direction to the surface. The angle between two surfaces is equal to the angle between lines that are normal to them. Some elementary calculations will establish those propositions, which is easy to predict and which the reader can verify with no effort.

575. We shall now point out an interesting consequence of one of the preceding results. We have seen that the difference:

$$ds^2 - \frac{d\theta^2}{\Delta\theta} \tag{71}$$

is always reducible to a sum of $n - 1$ squares:

$$P_1^2 + P_2^2 + \cdots + P_{n-1}^2.$$

Equate each of these squares to zero. We will then have a system of differential equations:

$$B_{i1}\, dq_1 + \ldots + B_{in}\, dq_n = 0,$$

which are $n - 1$ in number. Let $\theta_1, \theta_2, \ldots, \theta_{n-1}$ be the $n - 1$ integrals of that system – i.e., the functions that will give the most general relations between the values of $q_1, \ldots, q_n$ that satisfy these equations when they are equated to zero. We will obviously have $n - 1$ identities of the following form:

$$P_i = C_{i1}\, d\theta_1 + C_{i2}\, d\theta_2 + \ldots + C_{i,\,n-1}\, d\theta_{n-1} \qquad (i = 1, 2, \ldots, n-1),$$

and in turn, the difference (71) will take the form:

$$ds^2 - \frac{d\theta^2}{\Delta\theta} = f_1(d\theta_1, \ldots, d\theta_{n-1}),$$

in which f_1 denotes a quadratic form in the $n - 1$ differentials $d\theta_i$.

That inequality can exist only if the functions $\theta, \theta_1, \ldots, \theta_{n-1}$, are mutually independent, and as a result, upon substituting them for the original variables, one can express $\Delta\theta$ and the coefficients of f_1 as functions of $\theta, \theta_1, \ldots, \theta_{n-1}$.

Therefore:

By a change of variables that requires only the integration of $n - 1$ ordinary differential equations, one can always reduce the quadratic function ds^2 to the following form:

$$ds^2 = \frac{d\theta^2}{\Delta\theta} + f_1(d\theta_1, \ldots, d\theta_{n-1}), \tag{72}$$

in which θ is an arbitrarily-chosen function that is subject to the single condition that $\Delta\theta$ must not be zero.

In particular, if one has:

$$\Delta\theta = 1$$

then one will find that:

$$ds^2 = d\theta^2 + f_1(d\theta_1, \ldots, d\theta_{n-1}). \tag{73}$$

That remarkable proposition, to which one can adjoin the preceding one, is due to Beltrami. In the theory of forms in n variables, it plays the same role as Gauss's proposition that relates to geodesic lines (nos. **519** and **531**).

One can define, in an elegant manner, the system of differential equations whose integration gives the $n - 1$ functions θ_i when one has chosen the function θ. For that, it suffices to appeal to the invariance properties of the symbol $\Delta(\theta, \theta_1)$.

Indeed, if one looks for the functions u that satisfy the equation:

$$\Delta(\theta, u) = 0$$

then one will easily find, upon calculating $\Delta(\theta, u)$ with the left-hand side of equation (73), that the preceding equation reduces to the simple form:

$$\frac{\partial u}{\partial \theta} = 0.$$

As a result, it admits the $n - 1$ independent integrals $\theta_1, \theta_2, \ldots, \theta_{n-1}$. It will then suffice to write the linear equation in the original variables:

$$\Delta(\theta, u) = \frac{1}{D}\sum\sum A_{ik}\frac{\partial\theta}{\partial q_i}\frac{\partial\theta}{\partial q_k} = 0. \tag{74}$$

The $n - 1$ independent integrals of that linear equation will be functions that one can associate with θ in order to obtain the new system of variables.

If the quadratic function ds^2 is such that there exists a partial differential equation:

$$\Delta\theta = U \tag{75}$$

that one can integrate completely, in which U is a function of $q_1, \ldots, q_n$, moreover, that is chosen at will then one can give it the form:

$$ds^2 = \frac{d\theta^2}{U} + f_1(d\theta_1, \ldots, d\theta_{n-1}) \tag{76}$$

without integration and in an infinitude of ways.

To show that, let θ be a complete integral of equation (75). Upon differentiating the two sides of that equation with respect to any of the constants c_i that enter into θ, one will find that:

$$\Delta\left(\theta, \frac{\partial\theta}{\partial c_i}\right) = 0,$$

and as a result, $\frac{\partial\theta}{\partial c_1}, \ldots, \frac{\partial\theta}{\partial c_{n-1}}$ will be the various integrals of the linear equation (74). These are Jacobi's fundamental results, but in a slightly different form.

576. The transformations that we just pointed out permit us to treat the problem that was posed above of finding the minimum of the integral:

$$\int ds,$$

when it is taken between two given systems of extreme values. In order to preserve the analogy, we give the name of *geodesics* to all of the lines that give us a solution to that problem. It is clear that these solutions are invariant, moreover – i.e., that they persist when one changes the independent variables.

From that, we first suppose that one has chosen those variables in the following manner: $n - 1$ of the variables $y_1, \ldots, y_{n-1}$ are such that the equations:

$$y_1 = C_1, \qquad \ldots, \qquad y_{n-1} = C_{n-1} \tag{77}$$

define a solution to the problem *no matter what the constants C_i are* – i.e., a geodesic line. The last variable θ will be simply a function that is independent of the preceding ones. For more simplicity, one can suppose that θ is the value of the integral $\int ds$ when measured along each of the geodesic lines (77) upon starting at a fixed, but arbitrary, origin. ds^2 will then take the form:

$$ds^2 = d\theta^2 + 2\,(a_1\, dy_1 + \ldots + a_{n-1}\, dy_{n-1})\, d\theta + \sum\sum b_{ik}\, dy_i\, dy_k\,. \tag{78}$$

Take θ to be the independent variable, and set:

$$s' = \frac{ds}{d\theta},$$

$$y_i' = \frac{dy_i}{d\theta}.$$

Upon equating the first variation of the integral:

$$\int ds = \int s' d\theta$$

to zero, we will have the equations:

$$\frac{d}{d\theta}\frac{\partial s'}{\partial y_i'} - \frac{\partial s'}{\partial y_i} = 0 \qquad (i = 1, 2, \ldots, n - 1).$$

If we write that they are verified for the hypothesis that:

$$dy_1 = dy_2 = \ldots = dy_{n-1} = 0$$

then we will find the equations:

$$\frac{\partial a_i}{\partial \theta} = 0 \qquad (i = 1, 2, \ldots, n - 1),$$

which are integrated immediately and give:

(79) $$a_i = \varphi_i (y_1, y_2, \ldots, y_{n-1}) \qquad (i = 1, 2, \ldots, n - 1).$$

Therefore:

It is necessary and sufficient that all of the coefficients a_i should be independent of θ.

From that, consider an arbitrary curve on the locus that is defined by the equation $\theta = 0$ and contains the starting points of all the geodesics:

$$y_i = \text{const.} \qquad (i = 1, 2, \ldots, n - 1),$$

and look for the angle between that curve and the geodesic that passes through one of its points. One will have:

$$d\theta = ds,$$
$$dy_1 = dy_2 = \ldots = dy_{n-1} = 0$$

for the geodesic and:

$$\delta\theta = 0$$

for the curve, in which δy_1, δy_2, …, δy_{n-1} are arbitrary. The angle ω between those two lines, which is defined by formula (56), in general, will then have the value here that is given by the equation:

$$\cos\omega = \frac{a_1\delta y_1 + a_2\delta y_2 + \cdots + a_{n-1}\delta y_{n-1}}{\sqrt{\sum\sum b_{ik}\delta y_i\,\delta y_k}}. \tag{80}$$

In order for the geodesic lines to be normal to the geometric locus ($\theta = 0$) of their starting points, it is necessary and sufficient that the preceding expression for $\cos\omega$ must be zero for all values of the differentials δy_i – i.e., that one must have:

$$a_1 = a_1 = \ldots = a_{n-1} = 0$$

for $\theta = 0$.

However, since those coefficients do not contain θ*, they will be identically zero.*

Hence:

If the geodesic lines are normal to an arbitrary surface ($\theta = 0$) *then all of the coefficients a_i will disappear in the expression* (78); ds^2 *will take the following form:*

$$ds^2 = d\theta^2 + f_1(dy_1, \ldots, dy_{n-1}), \tag{81}$$

and as a result the geodesic lines will also be normal to all of the surfaces:

$$\theta = \text{const.}$$

that one obtains by measuring a given length along each of them upon starting at its point of incidence.

That is the proposition that was proved by Beltrami. It also applies to the case in which one considers all of the geodesic lines that pass through the same point, because if one takes that point to be the origin of the arc then, from the definition itself of θ, one must have:

$$ds = d\theta \qquad \text{for} \qquad \theta = 0,$$

no matter what the variables y_i are. As a result, the coefficients a_i and b_{ik} must be annulled for $\theta = 0$. However, since the first a_i do not contain θ, they will be identically zero. Hence:

If one measures out equal arcs along the geodesics that pass through a point when starting with that point then the geometric locus of the extremities of those lengths will be normal to all of the geodesics.

577. The preceding theorems, which constitute a generalization of Gauss's theorem, immediately exhibit the following result, which was established already in nos. **531** and **532** for the case of two variables:

The determination of the geodesic lines of the quadratic form and the integration of the partial differential equation:

$$\Delta\theta = 1$$

constitute two equivalent problems. The solution of one of them will necessarily imply the solution of the other one.

All that remains for us to do is to add a word in regard to the most general problem in mechanics. As we have remarked already, it amounts to the determination of the geodesic lines of the form:

$$dS^2 = (U + h)\sum\sum a_{ik}dq_i dq_k = (U + h)\, ds^2. \tag{82}$$

If one now remarks that the differential parameter $\Delta\theta$, when evaluated for dS^2, is equal to the quotient of the same differential parameter that relates to ds^2 by $U + h$ then one will immediately see that the solution to the problem in mechanics comes down to the integration of the equation:

$$\Delta\theta = U + h, \tag{83}$$

in which Δ is the differential parameter that relates to ds^2.

From the remark that we made above (no. **574**), the angles between the lines and surface are the same with respect to the two forms dS^2 and ds^2. Upon taking that result into account, one can extend the two propositions that relate to orthogonality that we proved in the preceding number like Beltrami, to the general problem of mechanics. One then recovers two theorems that Lipschitz stated in the paper that we pointed out above, and to which we shall refer the reader.

END OF PART TWO