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CHAPTER XIII

NORMAL LINES TO A SURFACE.

Direct theory for rectilinear congruences. – Condition for lines that start at different points of a surface to be normal to another surface. – Remark by Hamilton. – Partial differential equation for a family of parallel surfaces. – Applications. – Malus's theorem. – Propositions of Dupin relating to the case in which the developables formed by the incident rays are not destroyed by reflection. Definition of the optical axes of a surface. – In order for incident rays that are normal to a surface to have their developables preserved by reflection, it is necessary and sufficient that those developables should cut out a conjugate net from the reflecting surfaces. – Dupin's catoptric umbilics. – Particular examples. – Case in which the incident rays emanate from a unique point. – Case in which the reflecting surface has degree two.

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417. In the preceding chapter, we attached the theory of rectilinear congruences to some propositions that apply to more general congruences. One can also treat them directly in the following manner:

Trace out an arbitrary surface (*S*) in space that is subject to only the condition that it is not composed of lines of the congruence. The direction cosines *u*, *v*, *w* of any of the lines of that congruence are well-defined functions of the rectangular coordinates *x*, *y*, *z* of the point where that line meets the surface (*S*). We shall show that the necessary and sufficient condition for the lines to be normal to the same surfaces is that *the expression:*

$$u\,dx + v\,dy + w\,dz$$

must be an exact differential expression for all of the displacements that are performed on the surface (*S*).

That condition is necessary, since it is satisfied whenever the lines are normal to a surface (Σ). Indeed, let *X*, *Y*, *Z* be the coordinates of the point where the line of the congruence is normal to (Σ). One will have:

$$(1) \qquad u\,dX + v\,dY + w\,dZ = 0$$

for all displacements considered.

Moreover, one can write:

$$(2) \qquad X = x - u\,\rho, \quad Y = y - v\,\rho, \quad Z = z - w\,\rho,$$

in which ρ denotes the distance between two points (*x*, *y*, *z*), (*X*, *Y*, *Z*). If one replaces *X*, *Y*¸ *Z* with these values in equation (1) then one will find:

(3) $$d\rho = u\,dx + v\,dy + w\,dz\ ;$$

the right-hand side is therefore the differential of the function ρ.

Conversely, if the right-hand side is the exact differential of a certain function ρ then the point defined by equations (2) will verify the relation (1), and all of the lines of the congruence will be normal to the surface that is the locus of points (X, Y, Z). The proposition that just stated is then found to be established.

448. The preceding proof leads naturally to the following remarks, which are due to Hamilton.

Imagine that one draws a line through each point (x, y, z) of space. The direction cosines of that line are given, but arbitrary, functions of x, y, z, and the line will depend upon three parameters, in general. In a paper that we shall soon cite, Malus considered such assemblages of lines for the first time, which had been known since the work of Plücker and which have been given the name of *complexes.* Having accepted those definitions, here is what Hamilton's proposition consists of:

The necessary and sufficient condition for the lines to form a congruence (instead of a complex) and to be normal to a surface is that the expression:

$$u\,dx + v\,dy + w\,dz$$

must be an exact differential for all of the possible displacements of the point (x, y, z).

The condition is necessary. In order to see that, it will suffice to repeat the proof that we just made. All that will then remain is to prove that it is sufficient.

Since one has, by hypothesis:

(4) $$u\,dx + v\,dy + w\,dz = d\theta,$$

one can write:

(5) $$u = \frac{\partial\theta}{\partial x}, \qquad v = \frac{\partial\theta}{\partial y}, \qquad w = \frac{\partial\theta}{\partial z},$$

which will then give:

(6) $$\left(\frac{\partial\theta}{\partial x}\right)^2 + \left(\frac{\partial\theta}{\partial y}\right)^2 + \left(\frac{\partial\theta}{\partial z}\right)^2 = 1.$$

Hamilton's proposition then amounts to the following one, which is well-known:

If one is given a function θ that satisfies the preceding equation then the surfaces:

(7) $$\theta = \text{const.}$$

will all be parallel to each other.

Furthermore, one can prove that proposition directly in the following way: Draw some parallel tangent planes to the surfaces that are represented by equation (7). The contact points of those planes will be distributed along the curves that are represented by the equations:

(8) $$u = \frac{\partial \theta}{\partial x} = \text{const.}, \qquad v = \frac{\partial \theta}{\partial y} = \text{const.}$$

As one sees immediately, these curves are orthogonal trajectories to the family of surfaces (7). Indeed, one has:

$$\frac{\partial \theta}{\partial x}\frac{\partial u}{\partial x} + \frac{\partial \theta}{\partial y}\frac{\partial u}{\partial y} + \frac{\partial \theta}{\partial z}\frac{\partial u}{\partial z} = \frac{\partial \theta}{\partial x}\frac{\partial^2 \theta}{\partial x^2} + \frac{\partial \theta}{\partial y}\frac{\partial^2 \theta}{\partial x \partial y} + \frac{\partial \theta}{\partial z}\frac{\partial^2 \theta}{\partial x \partial z} = 0,$$

since the right-hand side is nothing but the derivative with respect to x of the left-hand side of equation (6).

Like the plane that is tangent to an invariant direction at the points where all the surfaces (7) are cut by the lines (8), these orthogonal trajectories will necessarily reduce to lines, and as a result, equation (7) will represent a family of parallel surfaces. Moreover, the line that is drawn through an arbitrary point in space and is defined by the direction cosines u, v, w will obviously be the normal to the particular parallel surface that passes through that point, and it will consequently be the common normal to all of the surfaces. Hamilton's remark is then found to be justified completely.

449. The preceding propositions are very convenient to employ in applications. For example, suppose that the equations of a line are written in the form:

(9) $$\begin{cases} x = az + p, \\ y = bz + q, \end{cases}$$

in which a, b, p, q are functions of two parameters. If one considers the point of the line that is found in the xy-plane and which has the coordinates p, q, 0 then one will have:

(10) $$u\,dx + v\,dy + w\,dz = \frac{a\,dp + b\,dq}{\sqrt{a^2 + b^2 + 1}}$$

here, and that expression must be an exact differential for all normal congruences.

If the line is defined in the most general manner by the equations:

(11) $$\begin{cases} bz - cy + a' = 0, \\ cx - az + b' = 0, \\ ay - bx + c' = 0, \end{cases}$$

which were given in no. **130** [I, pp. 194], then one will have:

$$\frac{u}{a} = \frac{v}{b} = \frac{w}{c} = \frac{1}{\sqrt{a^2+b^2+c^2}}.$$

Upon replacing the expression:

$$u\,dx + v\,dy + w\,dz$$

with the following one:

$$x\,du + y\,dw + z\,dw,$$

which is, like the former, an exact differential, one will be led to the expression:

$$(12) \qquad (a^2 + b^2 + c^2)^{-3/2} \begin{vmatrix} da & db & dc \\ a & b & c \\ a' & b' & c' \end{vmatrix},$$

which must be an exact differential for all normal congruences.

450. One can deduce a celebrated theorem from the preceding results whose original idea goes back to Malus, but which was established completely only by the combined efforts of Dupin, Gergonne, and Quetelet. Here is its statement:

If light rays are normal to a surface then they will not cease to possess that property after an arbitrary number of reflections and refractions.

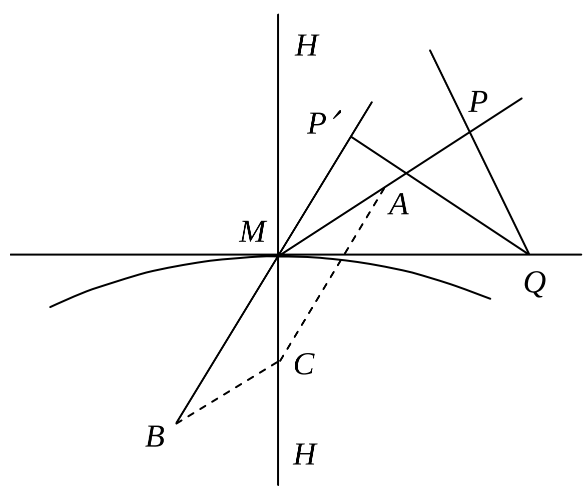

Figure. 30.

Since reflection can be regarded as refraction with index – 1, it is obviously sufficient to prove the theorem for the case of refraction. Here is how one can state Descarte's law:

Give the incident ray (Fig. 30) a length of MA = 1 and its refracted ray a length of MB = n, where n denotes the index of refraction. Compose MA, MB with the parallelogram law, so the resultant MC will be normal to the interface. Indeed, for the triangle MBC, one will have:

$$\frac{BC}{\sin BMC} = \frac{MB}{\sin MCB},$$

or, what amounts to the same thing:

$$\sin AMK = n \sin BMH. \tag{13}$$

This is the known law of refraction.

Let α, β, γ, u, v, w, u', v', w' be the direction cosines of the normal to the surface, the incident ray, and the refracted ray, resp. Upon equating the projection of MC to the sum of the projections MA, MB, one will have:

$$\left\{ \begin{aligned} n u' + u &= \lambda \alpha, \\ n v' + v &= \lambda \beta, \\ n w' + w &= \lambda \gamma, \end{aligned} \right. \tag{14}$$

in which λ denotes the length of $\overline{MC}$. Let x, y, z be the rectangular coordinates of M. One has:

$$\alpha\, dx + \beta\, dy + \gamma\, dz = 0 \tag{15}$$

for any displacement of M on the interface, or, upon eliminating α, β, γ by means of the preceding equations:

$$u\, dx + v\, dy + w\, dz = -n\,(u'\, dx + v'\, dy + w'\, dz). \tag{16}$$

Having said that, suppose that the incident rays are normal to a surface (Σ). The left-hand side will be the differential of a function ρ that is, as we say above, the distance from the point M to the point P where the ray cuts the surface (Σ) normally. By virtue of the preceding formula:

$$u'\, dx + v'\, dy + w'\, dz$$

will also be an exact differential $-\, d(\rho / n)$, and as a result, *the refracted rays will also be normal to surface* (Σ'). One will obtain the point P' where the refracted ray cuts (Σ') normally by moving along the refracted ray through a distance of $-\rho / n$ in the sense that is determined by its sign. One easily verifies that the planes that are normal to the two rays – viz., incident and refracted – at P and P', resp., must intersect along a line that is situated in the plane that is tangent to the interface at M. That relation between the tangent planes to the three surfaces is, moreover, obvious in the system of oscillation. One then deduces that if p, p', and δ denote the terms that are all known in the equations of the tangent planes at P, P', and M to the surfaces (Σ),(Σ'), and the interface then one will have:

$$n\, p' + p = \lambda\, \delta, \tag{14$'$}$$

in which λ has the same value as in formulas (14).

The normal surfaces to the refracted rays have been given the name of *anti-caustics*; the preceding results can then be stated thus:

If the incident rays are normal to a surface (Σ) *then consider that surface to be the envelope of spheres that have their centers on the interface. In order to obtain the anti-caustic that relates to the refracted rays, one must take the envelope of all the spheres that one obtains by reducing the radius of the preceding in the ratio of unity to the index of refraction* ([1]).

It results from this construction that it will, in general, be impossible to analytically separate two systems of refracted rays that correspond to equal values and opposite signs of the index of refraction.

451. In his study of the preceding theorem, which he proved only for the case of reflection, Dupin posed the following question, which gave rise to some interesting research:

([1]) Here are some facts on the subject of the discovery of that beautiful proposition: In a paper that bore the simple title "Optique" and was inserted into the XIV$^{\text{th}}$ letter of the *Journal de l'École Polytechnique* in 1808, Malus proved for the first time an interesting property of these assemblages of lines to which we give the name of *complexes.* Imagine that each point of space corresponds to a line that passes through that point according to some law, and let M be an arbitrary point through which a line (d) of the system passes. If one seeks the points M' that are infinitely close to M and for which the corresponding line meets (d) then one will find that the locus of directions MM' is a second-degree cone. After establishing that proposition, the illustrious physicist studied the assemblages of lines that depend upon two parameters – i.e., *congruences* – and he showed that one can, in general, distribute the lines of the congruence into two families of developable surfaces. He sought the condition for those two families of developables to cut at a right angle, and he recognized that it will be satisfied when the lines are normal to a surface.

Upon then applying these general principles to optics, Malus proved that when the incident rays that emanate from a fixed point are reflected by an arbitrary surface, they will remain normal to a surface after reflection.

In the second part of his paper, which was inserted into page 84 of the same letter, Malus extended that proposition to the case of a unique refraction, and attempted (pp. 101) to extend it to the case of several refractions; however, deluded by an error in calculation, he believed that light rays generally cease to be normal to a surface after a second refraction.

It was Dupin who deserved the credit for having stated the general theorem for the first time in a "Mémoire sur les routes de la lumiére" that we shall cite later on and for having given a very simple geometric proof of it, but only for the case of reflection. Moreover, as Dupin pointed out, Cauchy reprised and corrected Malus's calculations in such a way that no doubt remained in regard to the importance and generality of the theorem. Dupin was content to regard it, in the case of refraction, as a simple corollary to some propositions that he gave in his theory of cutting and filling (*déblais et remblais*). That view is exact, but the theorems to which Dupin appealed have been proved only incompletely.

Several years later, Quételet introduced a new idea into that theory by replacing the *caustics*, whose determination is very arduous, with *secondary caustics*, or better, the *anti-caustics*, which are normal to the reflected or refracted rays. It is thanks to his efforts and those of Gergonne that the theorem has finally been proved simply and completely. (See volume I of *Correspondence mathématique et physique*, 1825, volume XVI of *Annales de Gergonne* and *Nouveaux Mémoires de l'Académie de Bruxelles*, t. III and IV, 1826 and 1827, resp.)

In the study that we cited above [pp. 237], Lévistal extended the Malus-Dupin theorem to the case of double refraction.

If the light rays are normal to a surface then one can assemble them into two families of orthogonal developables. The reflection on a given surface generally transforms these developables into skew surfaces. Dupin gave some beautiful propositions that related to the case in which the incident rays that form a developable are reflected along rays that likewise form a developable, and he recognized that the traces of the two series of developables on the reflecting surface must form a conjugate system. Dupin's proofs rest upon the properties of the indicatrix and upon those of second-degree surfaces of revolutions ([1]). One can replace them with the following ones, which will, moreover, give us some new results:

We begin by considering some light rays that form just one developable and seek the condition for the reflected rays to likewise generate a developable surface.

Let AA', … be a line of curvature of the developable that is formed by the incident rays. The generators that pass through the points A, A', … meet the reflecting surface (Σ) at M, M', … and reflect along the rays MB, $M'B'$, … Take $MB = MA$, $MB' = MA'$, … The curve BB'… will be an orthogonal trajectory of the reflected rays, and as a result, it will necessarily be a line of curvature if those reflected rays also form a developable. The spheres with centers M, M', …, and radii MA, MA', … envelop a surface (S) with circular lines of curvature), and the two curves AA', .., BB', … must be non-circular lines of curvature on that surface. As a result, the tangents to those two lines at the corresponding points A and B will be two generators of a cone of revolution that is circumscribed by the surface (S) along a circle, and it will necessarily meet it. Now, consider the ruled surface (Δ) that is generated by the line AB. From the property that we just proved, it will be a developable surface, since it will admit the same tangent plane to the two distinct points A and B. Let AB, $A'B'$ be two consecutive positions of AB. Since they are perpendicular to the two tangent planes to the reflecting surface (Σ) at M and M', respectively, they will also be perpendicular to the intersection Mt of those two planes, which is the conjugate tangent to MM'. Thus:

The developable (Δ) that is generated by the line AB has its tangent plane perpendicular to the tangent Mt at every instant.

As a result, the normal plane to the developable of the incident ray that is drawn through MA (which is a plane that is obviously perpendicular to the tangent at A to the line of curvature AA', … of that developable) will cut the tangent plane to the reflecting surface at M along the line that is perpendicular to the tangent at A – i.e., along the tangent Mt. In other words: *The tangent plane to the developable along the incident rays and the normal plane that contains that ray must cut the tangent plane to the reflecting surface along two conjugate rays.*

That condition, which is necessary, is also sufficient: In order to see that, it will suffice to recall the preceding argument in the opposite order. One will then deduce the following consequence:

The tangent to the curve of incidence and its conjugate tangent are found in the two rectangular planes that contain the incident ray. Those two planes are obviously the conjugate diametral planes to any cylinder of revolution that has the incident ray for its

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([1]) Consult the *Quatrième Mémoire* in *Applications de Géométrie et de Méchanique*, which was entitled: "Sur les routes suivies par la lumière et par les corps élastiques, en générale, dans les phénomèmes de la réflexion et de la refraction." (Presented to the Academy of Sciences in 22 January 1816.)

axis. As a result, the lines considered will be conjugate diameters to the section of the cylinder that goes through the tangent plane at M.

From that, if one is given an arbitrary incident ray that cuts the reflecting surface at M then there will be only two possible directions for the tangent to the curve of incidence at M: They will the directions of the two conjugate diameters that are common to the indicatrix of the surface and the section of the tangent plane by the cylinder of revolution whose axis is the incident ray.

One must remark that that there is an analogy between that theory and the theory of lines of curvature; moreover, the two theories will coincide if one supposes that the incident ray is normal to the reflecting surface ([1]).

452. The preceding construction leads us to consider lines that possess some remarkable optical properties. Suppose that the conics that were employed in that construction are similar; i.e., that the incident ray is the axis of one of the four cylinders of revolution that contain the indicatrix. The directions of the curve of incidence will no longer be determined then. Thus:

If one considers the four lines at each point of a surface that are axes of cylinders of revolution that cut the tangent plane along the indicatrix, or loci of points where one sees two arbitrary conjugate tangents define a right angle, then those lines will form four systems of rectilinear rays whose developables reflect along developables.

Those lines will be imaginary if the indicatrix is hyperbolic; in the case of an elliptic indicatrix, there will be two of them that are real: They will be the asymptotes of the focal hyperbola. They will be in the principal plane, which contains the two foci of the indicatrix and will be placed symmetrically with respect to the normal. They will form two different systems, and each of them will be obtained by reflecting the other one on the surface. We call them *optical axes*, to abbreviate. One can also construct them as follows:

Draw a plane tangent to the circle at infinity through the two asymptotic tangents to the reflecting surface. Those four planes intersect along four lines that are placed pair-wise symmetric with respect to the tangent plane, and which will be the four optical axes relative to the point considered.

It follows from this that in the case of a second-degree surface, the optical axes of the surface will be the rectilinear generators of the second-degree surface that is homofocal to the proposed one. One can determine the developables that are formed by these lines with no integration, since one knows that they are the double tangents to the developable in which all of the homofocal surfaces are inscribed. Now, when some lines of a congruence are the double tangents to a developable, it will be obvious that there will be

([1]) The results that were established by our method include that ones that Dupin relied upon, because, if the reflecting surface is a second-order surface of revolution with two foci F, F' then the light rays that emanate from F will be reflected towards F'. Construct the cone that is formed from the incident rays that meet the section of the surface through a plane (P). The normal plane along each generator must contain the conjugate tangent to the tangent to the plane section, and as a result, it must pass through the pole to the plane (P). Now, a cone is obviously one of revolution when its normal plane passes through a fixed point. Therefore, *all plane sections will appear to be circles when viewed from a focus.*

an infinitude of them in each tangent plane to the developable. Indeed, that plane touches the developable along a line (d) and cuts it along a curve (C). All of the tangents to (C) will be double tangents to the developable and must be considered to form a developable surface, in their own right. Therefore, in the case of second-degree surfaces, we will have to distribute the optical axes into two series of developables. Moreover, we remark that these two series of developables will be imaginary.

453. In any case, the optical axes enjoy a remarkable property: Imagine that light rays emanate from a point of one of these lines and form an infinitely small pencil around the line. From the property of optical axes, the reflected rays – no matter how one assembles them – must be considered as forming a developable surface, and consequently, the reflected pencil will also seem to emanate from a unique point.

One can establish that conclusion in a more rigorous manner by giving the complement according to our first proposition. Imagine a ray AM that meets the reflected surface at M and reflects from it along a ray MB, and suppose that AM belongs to a developable that reflects along a developable. When one passes from AM to the infinitely-close ray $A'M'$, the plane AMB will be replaced with a plane $A'M'B'$ that cuts the first one along a line $\alpha\mu\beta$, where α, μ, β are the points of that line that are situated on AM, the normal to M, and on BM, respectively. Since the planes AMB, $A'M'B'$ pass through two infinitely-close generators of the developable that is generated by AM, the limiting position of α will be the contact point of the ray AM with the curve that it envelops, and similarly β will be the contact point of the reflected ray BM with the edge of regression of the developable that is generated by that ray. As for μ, it is obviously the point where the normal to the point M of the curve of incidence that is infinitely close to M cuts the plane AMB. We thus have the following theorem:

When a developable is reflected along a developable, the line that joins the points of contact of the incident and reflected rays with the curves that they envelop will cut the normal to the reflecting surface at the point where the skew surface that is formed by the normals at all points of the curve of incidence is tangent to the plane of the incident and reflected rays.

In particular, when one is dealing with an optical axis (which is necessarily situated in a principal plane), all of the normals that are infinitely close to the normal will cut that plane at a point whose limiting position coincides with that of the two principal centers where the principal plane is tangent to the surface of the centers. If γ denotes that center then one will see that β must be on the line $\alpha\gamma$, and that construction will always be the same, no matter what the developables that are formed by the incident rays are. Thus, if they emanate from α then the reflected rays will seem to emanate from β.

The properties that were established give rise to a certain number of consequences, and it is not futile to state them explicitly.

If the incident rays are formed by a system of optical axes of the surface then the incident developables will reflect along other developables, such that the incident rays will or will not be normal to a surface. Except for that exceptional case, one can say that if the reflection does not destroy the two series of developables (which are assumed to be distinct) that are formed from the incident rays then they will necessarily be normal to a surface. Indeed, we have seen that if the incident ray is given then the only two lines that must be the tangent at M to the trace of a developable on the reflecting surface will be in two rectangular planes that pass through the incident ray. Thus, if the two distinct series of developables that are formed from the incident rays are to persist after the reflection then it will be necessary for them to cut at a right angle and for them to cut out a conjugate system on the reflecting surface. These two conditions are sufficient, moreover. Therefore:

Whenever the two series of developables that are formed by the incident rays are all reflected along developables, the incident rays will be normal to a surface, unless it constitutes one of the four systems of optical axes of the surface.

In order for the incident rays that are normal to a surface to have their developables preserved by the reflection, it is necessary and sufficient that these developables cut out a conjugate system on the reflecting surface.

454. For example, consider a surface (Σ), which we assume to be arbitrary, and some light rays that emanate from a point O. These light rays, which form a system (I), reflect on (Σ) and give a system (R) a reflected rays that are normal to a surface (Σ_1) that is enveloped by spheres that pass through the point O and have their center on (Σ). That surface (Σ_1) is obviously homothetic to the (*podaire*) of the point O with respect to (Σ), because it is the locus of the symmetric images of the point O with respect to all the planes that are tangent to (Σ). Suppose that the eye is placed at a point O' on the direction of the reflected ray. It will receive a pencil of rays that emanate from O and reflect into the region of (Σ) that neighbors the point M. That pencil of reflected rays, which is formed from normals to (Σ_1), will generally have two focal lines that are mutually-perpendicular and are placed at the two centers of principal curvature of (Σ_1) that are on the normal $O'M$. The image of the luminous point for an observer that is placed at O' will be more or less indistinct. That observer will refer it to a point that cannot be determined by any rule. However, in the particular case where the line OM is one of the optical axes of the point M, all of the reflected rays will appear to emanate from a unique point that we have learned how to construct. The image of the light point will become clear, and the reflected rays will have a focus that can be *real* or *virtual*. Dupin gave the point M the name of *catoptric umbilic*, by the analogy that it presented with ordinary umbilics, which they will coincide with, moreover, when the incident ray is normal to the surface. If the surface (Σ) has degree two then there will be twelve catoptric umbilics for each point O, four of which will be real and situated at the intersection of that surface with two rectilinear generators of the homofocal hyperboloid that passes through the point O.

Here, the incident rays (no matter how one assembles them) will always form cones, and as a result, developables. We propose to determine the cones that are formed by the incident rays that correspond to reflected rays that form a developable. From the preceding propositions, these cones must cut out two systems of conjugate lines from the reflecting surface that seem to cut at a right angle when one regards them from the point

O. Moreover, the determination of these cones is equivalent to that of the lines of curvature of the surface (Σ_1), or – what amounts to the same thing – to the podaire of (Σ) relative to the point *O*. Thus:

If one is given a surface (Σ) *and a point O then the lines of curvature of the podaire of* (Σ) *relative to the point O will correspond to two systems of conjugate lines that are traced on* (Σ) *and which seem to cut at a right angle for an observer that is placed at O.*

That proposition, which the reader can establish in a very simple manner, permits one to determine the developables that are formed by the reflected rays when the surface (Σ) has degree two. One then obtains the following construction, which we will be content to state:

The curves of incidence of the cones that reflect along the developables are, at the intersection of (Σ) *and the cones of degree two, with their summit at O, homofocal to the cone with the same summit that is circumscribed by* (Σ)*; they are also the curves of contact of the developables that are circumscribed by* (Σ) *and any of the second-degree surfaces that pass through the intersection of* (Σ) *and the sphere of radius zero that has its center at the point O.*

455. In two notes that were published in 1872 ([1]), Ribaucour pointed out some very interesting applications of Dupin's theorem. They rest essentially upon the following remark, which is almost obvious:

In order for the developables of a congruence to cut out a conjugate net from a second-degree surface, it is necessary and sufficient that the two focal planes of each line of the congruence should be conjugate with respect to that surface; i.e., that each of them must contain the pole of the other ([2]).

Since the preceding condition is independent of the point where each line of the congruence cuts the surface, one will first of all deduce the following proposition:

If the developables that are formed by lines of a congruence cut out a conjugate net on a second-degree surface when they enter it then they will also cut our a second conjugate net when they leave it.

Now, consider the case in which the developables intersect at a right angle. One will then have a congruence of normals (*I*) whose developables will, by hypothesis, cut out a

([1]) RIBAUCOUR, "Sur la théorie des lignes de courbure," Comptes rendus, t. LXXIV, pp. 1489 and 1570, 1st semester 1872.

([2]) Indeed, let *M* be a point of the surface, let (*d*) be the line of the congruence that passes through that point, and let *Mt*, *Mt′* be the traces of the two focal planes of that line on the plane that is tangent at *M*. As one knows, the focal plane that passes through *Mt* has its pole on the conjugate tangent to *Mt*. In order for the two focal planes to be conjugate, it is therefore necessary and sufficient that *Mt* and *Mt′* must be conjugate tangents.

conjugate net from the second-degree surface (S). Since they are both normal and conjugate with respect to (S), the two focal planes of each line (d) of the congruence will also be conjugate with respect to all of the surfaces (S_i) that are homofocal to (S). Consequently, the developables of (I) will cut out conjugate nets from all of the surfaces (S_i), either when they enter or when they leave. If one imagines that the lines of the congruence form a system of incident rays (I) then one can make those rays reflect from any one (S_1) of the surfaces (S_i). One will then have a first system of reflected rays (I1) whose developables correspond to those of (I), and since they cut out the same conjugate net on (S_1) as the developables of (I), they will again cut out conjugate nets on all of the homofocal surfaces. Now, one can make the pencil (I_1) reflect on another homofocal surface (S_2), and so on. Upon continuing in that manner, one will obtain a sequence of congruences of normals (I), (I_1), (I_2), … that is generally unlimited and whose developables mutually correspond and cut out conjugate nets on all of the homofocal surfaces. If one follows one of the incident rays then one will easily recognize that the two homofocal surfaces that are tangent to that ray will also remain tangents at all of their successive positions.

We felt that these elegant properties seemed worthy of being pointed out to us. In the following chapter, we shall pursue the study that will permit us to complete the results that were given above on the geodesic lines of second-degree surfaces, moreover.