"Sur la surface des ondes," Ann. sci. de l'E.N.S. (3) **6** (1889), 379-388.

On

On the wave surface

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I.

In volume IX of the *Quarterly Journal of Mathematics*, Niven gave the following remarkable proposition that relates to the wave surface:

The three spheres that pass through the three principal circles and an arbitrary point M of the surface must cut at a second point P that is the foot of the perpendicular that is dropped from the center of the surface onto the tangent plane at M.

Niven remarked that this theorem permits one to construct either the tangent plane at a given point or the point of contact of a given tangent plane. I would like to establish that it leads to a new and simple definition of the wave surface, and the essential character of that definition is that it does not employ an ellipsoid.

Indeed, upon employing the fact that from one part of the preceding proposition, one sees that the spheres that pass through the three principal circles and a point M on the wave surface must cut at a point P such that MPO is a right angle, where O denotes the center of the surface.

The wave surface then appears to us to be a particular case of the following surface: One considers three arbitrary circles in space (A), (B), (C), and an arbitrary point O. One seeks the locus (Σ) of the points M that enjoy the following property: The spheres that pass through the three fixed circles (A), (B), (C), and through an arbitrary point M of the locus must cut at a second point P, such that MPO is a right angle. That locus is obviously a surface. I shall first show that one can construct it from points by employing only the ruler and compass.

Indeed, consider two arbitrary spheres that pass through the circles (A) and (B); they cut along a circle (Γ). I shall seek the points of the locus that are situated on (Γ). In order for that to be true, I remark that any sphere that passes through the circle (C) will cut the

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([1]) Extract of volumes **92** and **97** of the *Comptes rendus de l'Académie des Sciences*. [Translator: More precisely, sections I, II, and III are direct transcriptions of:

"Sur une nouvelle definition de la surface des ondes," Comptes rendus **92** (1881), 446-448.

"Sur les lignes asymptotiques de la surface des ondes," Comptes rendus **97** (1883), 1039-1042.

"Sur les lignes de courbure de la surface des ondes," Comptes rendus **97** (1883), 1133-1135.

circle (Γ) at two points M and P, such that the line MP will meet it at a fixed point H. If M is a point of the locus then the angle MPO, or – what amounts to the same thing – the angle HPO, will be a right angle. The point P will then be found on the sphere that is described with OH as a diameter. There will then be two positions for the point P, and consequently there will also be two positions for the point M. Since that construction is general, it will not require any modification in the case of the wave surface.

There exists a circle (K) that meets each of the circles (A), (B), (C) at two points. We call the radical center of all spheres that pass through those two circles the *radical center of the two circles.* The plane of the circle (K) is the plane of the radical centers of three circles (A), (B), (C), taken two at a time.

The surface (Σ) contains the circle (K).

Each of the spheres that pass through the circle (K) and one of the circles (A), (B), (C) cuts the surface along a new circle. One then obtains three circles (A'), (B'), (C').

The surface (Σ) generally has order five. It admits the circle at infinity as a double line, and, in addition, it cuts the plane at infinity along a line that is in the plane that is perpendicular to the line OH, where H denotes the point at which the planes of the circles (A), (B), (C) meet.

The surface (Σ) reduces to order four when:

1. The planes of the circles (A), (B), (C) intersect along a line.
2. The point O and the point H coincide.

I will examine the latter case especially.

The surface will then admit eight planes that each cut it along a circle and a conic. They are the plane at infinity, which cuts it along a conic and the circle at infinity, the planes of the circles (K), (A), (A'), (B), (B'), (C), (C'). It will then contain sixteen conics, which is all the more remarkable for the fact that it does not generally have any singular points.

In a first study on fourth-order surfaces that admit isolated conics, it seemed to me that there exists a fourth-surface surface that admits eighteen quadruple tangent planes, and consequently thirty-six conics, without having a singular point.

One sees that it results from the preceding study that the wave surface is a simple variety of a fourth-order surface that has no singular point and contains sixteen isolated conics.

I shall conclude by adding a small complement to two of my previous communications. One knows that if three points of an invariable line describe rectangular planes then any point of the line will describe an ellipsoid. I add to this the theorem of Dupin that *the line, in all of its positions, will remain normal to a fixed surface whose lines of curvature are algebraic.* That surface is a variety of surfaces of class four that was considered in my communication on 3 January, and the developable surfaces that are defined by the normals at all points of a line of curvature are tangents to a second-degree surface, since that is true, moreover, for the most general surfaces of that kind.

One sees that one determines the surface on the normals for which the coordinate planes intersect segments of given length. In a general manner, one can always obtain, by simple quadratures, the equation of the surface that is defined by an arbitrary relation between the three lengths of the segments that are included between the foot of the normal and the three coordinate planes, at least, when those three planes are rectangular. Upon studying that equation, one will be led to an interesting theorem:

If there exist two relations between the lengths of three segments of the normal that is included between the foot of that normal and the three coordinate planes then one of those relations will necessarily be the following one: The segments of the normal that are included between the three coordinate planes have invariables ratios.

This theorem is verified, in particular, for the surface that we just considered, and which is normal to all positions of an invariable line, three points of which describe coordinate planes.

II.

Let x, y, z be the rectangular coordinates of an arbitrary point of a surface. Let p, q, r denote quantities that are proportional to the direction cosines of the normal and are required to satisfy the condition:

(1) $$px + qy + rz = 1,$$

in addition.

Finally, let p', q', r' denote the three quantities:

(2) $$p' = qz - ry, \quad q' = rx - pz, \quad r' = py - qx,$$

in such a manner that the six coordinates of the normal will be p, q, r, p', q', r'.

With these notations, the differential equation of the asymptotic lines of the surface will be:

(3) $$dp\,dx + dq\,dy + dr\,dz = 0$$

and that of the lines of curvature will be:

(4) $$dp\,dp' + dq\,dq' + dr\,dr' = 0.$$

I propose to apply those very simple results to the study of the asymptotic lines and lines of curvature of the wave surface.

I shall first examine the matters that concern the asymptotic lines. Since the wave surface is a particular case of the surface with sixteen singular points, one can deduce the determination of those lines from the one that was given by Klein and Lie for the Kummer surface. However, there is some interest to determining them directly, and we shall see, moreover, that the method that is followed in this study will give the asymptotic lines of an infinitude of new surfaces.

The detailed and complete study of the wave surface rests upon the simultaneous use of four variables, which are the following ones: Consider a point M of the surface. The ray that joins the point M to the center O of the surface cuts it at a second point M'. We set:

$$\overline{OM}^2 = \beta, \qquad \overline{OM'}^2 = \alpha'.$$

Similarly, let α and β' denote the squares of the distances from the center to the tangent plane at M and to the parallel tangent plane, resp. Those four variables will be coupled by two relations that are contained in the identity:

$$x(x-\beta)(x-\beta')-(x-a)(x-b)(x-c)=\frac{abc}{\alpha\alpha'}(x-\alpha)(x-\alpha'), \tag{5}$$

which must be true for all values of x.

Having said that, for an arbitrary point of the wave surface, one will have the values of x, y, z, p, q, r that one deduces from the following formulas:

$$\left\{\begin{aligned} x &= C\left(\frac{a-\alpha}{\alpha}\right)^{m_1}\left(\frac{a-\alpha'}{\alpha'}\right)^{m_2}(a-\beta)^{n_1}(a-\beta')^{n_2}, \\ y &= C'\left(\frac{b-\alpha}{\alpha}\right)^{m_1}\left(\frac{b-\alpha'}{\alpha'}\right)^{m_2}(b-\beta)^{n_1}(b-\beta')^{n_2}, \\ z &= C''\left(\frac{c-\alpha}{\alpha}\right)^{m_1}\left(\frac{c-\alpha'}{\alpha'}\right)^{m_2}(c-\beta)^{n_1}(c-\beta')^{n_2}, \end{aligned}\right. \tag{6}$$

when one sets:

$$m_1 = n_2 = 0, \qquad m_2 = n_1 = \tfrac{1}{2}$$

and conveniently assigns the constants C, C', C''.

I shall consider the surfaces that are defined by formulas (6) in a general manner. For them, one has:

$$\left\{\begin{aligned} p &= \frac{1}{C(a-b)(a-c)}\left(\frac{a-\alpha}{\alpha}\right)^{-m_1}\left(\frac{a-\alpha'}{\alpha'}\right)^{-m_2}(a-\beta)^{1-n_1}(a-\beta')^{1-n_2}, \\ q &= \frac{1}{C'(b-a)(b-c)}\left(\frac{b-\alpha}{\alpha}\right)^{-m_1}\left(\frac{b-\alpha'}{\alpha'}\right)^{-m_2}(b-\beta)^{1-n_1}(b-\beta')^{1-n_2}, \\ z &= \frac{1}{C''(c-a)(c-a)}\left(\frac{c-\alpha}{\alpha}\right)^{-m_1}\left(\frac{c-\alpha'}{\alpha'}\right)^{-m_2}(c-\beta)^{1-n_1}(c-\beta')^{1-n_2}. \end{aligned}\right. \tag{7}$$

Here, one can apply formula (3) and write down the differential equation of the asymptotic lines. One will then be led to this very simple result:

Whenever the exponents are linked by the relation:

$$m_1 + n_1 + m_2 + n_2 = 1, \tag{8}$$

the differential equation of the asymptotic lines will be:

$$(9) \qquad \frac{d\beta^2}{(\beta-a)(\beta-b)(\beta-c)} = \frac{d\beta'^2}{(\beta'-a)(\beta'-b)(\beta'-c)},$$

and consequently those lines will be defined by an algebraic relation between β *and* β' *whose form is well-known.*

Since the exponents in the case of the wave surface satisfy the relation (8), the preceding result includes the one that one knows relative to that surface.

The integration of equation (9) leads to the following theorem, which replaces all of the calculations:

Consider each of the Chasles complexes, which are defined by the lines that cut three coordinate planes and the plane at infinity at four points whose anharmonic ratio is constant. The locus of the points of the surface where the cone of the complex is tangent to that surface is an asymptotic line.

When one varies the value of the constant anharmonic ratio, one will get an infinitude of complexes that gives all asymptotic lines.

It seems interesting to me to seek all surfaces that enjoy the property that is expressed by the preceding theorem. One first finds Lamé's tetrahedral surfaces, which are defined by the equation:

$$\left(\frac{x}{a}\right)^m + \left(\frac{y}{b}\right)^m + \left(\frac{z}{c}\right)^m = 1.$$

Their asymptotic lines have been determined already by Lie, and they enjoy the special property that the tangents to each of them all belong to the same Chasles complex (which varies when one passes from one line to the other).

The other surfaces satisfy the partial differential equation:

$$(10) \qquad xyz\,(rt - s^2) + pq\,(s - px - qy) = 0,$$

which one can interpret as follows:

Let N_x, N_y, N_z denote the portions of the normal to the surface that are included between the foot M of that normal and the coordinate planes. Let R, R' be the radii of principal curvature, and let P be the distance from the origin O to the tangent plane at M. Equation (10) is equivalent to the relation:

$$RR' = \frac{N_x N_y N_z}{P},$$

which gives the total curvature, and applies, in particular, to the wave surface.

The following formula, which is just as simple, gives the sum of the radii of curvature. One has:

$$R + R' = N_x + N_y + N_z - \frac{\overline{OM}^2}{P}.$$

III.

The lines of curvature of the wave surface were the subject of various studies as a result of a note by Bertrand that was included in *Comptes rendus* **47** (1858), pp. 817. A geometer had stated that the curve of contact of the developable that circumscribes the surface and a concentric sphere is a line of curvature. Bertrand proved two elegant theorems that showed the incorrectness of that proposition. After that, a very skilled geometer – viz., Combescure – returned to the subject in the *Annali di Tortolini* **2** (1859), pp. 278. A brief note by Brioschi that was placed after the preceding paper contained an interesting transformation of that equation.

I was led to address the lines of curvature of the wave surface upon studying the form that the lines of curvature of an arbitrary surface take in the neighborhood of an umbilic. That interesting question has already been the subject of research by Cayley [*Philosophical Magazine* **26** (4), pps. 373, 441].

The lines of curvature in the neighborhood of an umbilic never resemble a circle, and their form is quite variable. If one lets A, B, C, α, β, γ denote six parameters that depend upon the form of the surface in the neighborhood of the umbilic then the lines of curvature will be defined by the following formulas:

$$(1) \qquad \begin{cases} x = K(p-\alpha)^A (p-\beta)^B (p-\gamma)^C [p^2 - 1 - (\alpha\beta\gamma + \alpha + \beta + \gamma)\, p], \\ y = K(p-\alpha)^A (p-\beta)^B (p-\gamma)^C [\alpha\beta\gamma(1-p^2) - (1 + \alpha\beta + \alpha\beta + \beta\gamma)\, p], \end{cases}$$

in which x, y denote the rectangular coordinates of the projection of the point onto the tangent plane, p is a variable parameter, and K is the arbitrary constant that varies when one passes from one line of curvature to the other one.

The preceding result, to which I will undoubtedly have the opportunity to return and complete, provides a means of recognizing whether the lines of curvature of a surface can be algebraic. A necessary condition for that is that the numbers A, B, C that relate to each umbilic must be commensurable. If that condition is not fulfilled for even one umbilic then one can confirm that the lines of curvature will not be algebraic.

Upon applying that criterion to the wave surface that was always indicated for that type of research, I recognized that A, B, C would be commensurable in that case, and for all umbilics. In the neighborhood of each umbilic, the lines of curvature are similar to algebraic curves of order ten. I was then led to new studies that were communicated to the Congrès de l'Association française in 1878.

Preserve the variables α, β, α', β' that were defined already and set:

$$f(\alpha) = (\alpha - a)\,(\alpha - b)\,(\alpha - c),$$

to abbreviate. The differential equation of the lines of curvature that was given already by Combescure will be:

$$(2)\qquad f(\alpha)\,d\beta^2 + f(\beta)\,d\alpha^2 - d\alpha\,d\beta\left\{2\,f(\alpha) + (\beta-\alpha)\left[f'(\alpha) - \frac{f(\alpha)}{\alpha}\right]\right\} = 0.$$

That equation will keep absolutely the same form when one employs α', β', instead of writing down the variables α, β. I would like to show that one can integrate it *whenever the third-degree function* $f(x)$ *reduces to a second-degree polynomial.*

In order to do that, I remark that if one sets:

$$\varphi(x) = x\,f(x),$$

to abbreviate, and if one replaces β with the variable $v = \alpha(\beta - \alpha)$ then equation (2) will become:

$$(3)\qquad \varphi(\alpha)\left(\frac{dv}{d\alpha}\right)^2 - \varphi'(\alpha)\,v\frac{dv}{d\alpha} + v\,\varphi(\alpha) + \frac{v^2}{2}\varphi''(\alpha) + \frac{v^3\varphi^{\mathrm{IV}}(\alpha)}{6} = 0.$$

If $f(x)$ has degree two, and consequently $\varphi(x)$ has degree three, then the last term in the preceding equation will disappear. I suppose that the coefficient of the third power in $\varphi(x)$ has been reduced to unity and set:

$$w = \frac{\varphi(\alpha)}{v}.$$

The equation in w will be:

$$(4)\qquad \varphi(\alpha)\left(\frac{dw}{d\alpha}\right)^2 - \varphi'(\alpha)\,w\frac{dw}{d\alpha} + w^2 + \frac{w^2}{2}\varphi''(\alpha) = 0.$$

One can then write:

$$\varphi\left(\alpha - w\frac{d\alpha}{dw}\right) + w^2\left(\frac{d\alpha}{dw}\right)^2\left(\frac{d\alpha}{dw} + 1\right) = 0,$$

and if one performs the well-known change of variables that is defined by the formulas:

$$(5)\qquad \left\{\begin{array}{ll} \alpha - w\dfrac{d\alpha}{dw} = y, & -w = \dfrac{dy}{dp}, \\[2ex] \dfrac{d\alpha}{dw} = p, & \alpha = y - p\dfrac{dy}{dp} \end{array}\right.$$

then the equation will become:

$$\varphi(y) - p^2\,(p+1)\left(\frac{dy}{dp}\right)^3 = 0.$$

One only has to separate the variables and integrate, which will give:

$$\int \frac{dp}{p^{2/3}(1+p)^{1/3}} = \int \frac{dy}{[\varphi(y)]^{1/3}}.$$

That first result, which relates to the case in which $f(x)$ has degree two, already proves that *the lines of curvature of the wave surface cannot be algebraic curves of a well-defined degree.* If that fact does not keep us from hoping that the most general integral of equation (2) can be obtained, it will at least show that the integral can only be expressed in a very complicated manner. Finally, there are some geometric applications that I would like to point out in conclusion.

The wave surface is the apsidal of a certain ellipsoid (E). Suppose that the ellipsoid become a cylinder, so one of its axes grows indefinitely. The wave surface will be transformed into a surface whose lines of curvature will be determined by the equation that we just integrated.

When two of the axes of the ellipsoid tend to become equal, one of the sheets of the wave surface will approach a sphere. If the three axes a, b, c tend to a common value r by formulas such as the following ones:

$$a = r + \varepsilon a', \quad b = r + \varepsilon b', \quad c = r + \varepsilon c',$$

in which a', b', c' are fixed quantities, then the two sheets of the surface will approach the sphere of radius r. In one case and the other, the lines of curvature will tend to limiting positions, and the differential equation will reduce to the one that we have integrated.

One can then consider the lines of curvature to be known for all wave surfaces that present themselves in physics, and which are, as one knows, little different from the sphere.