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On the Pfaff problem

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The method for the integration of partial differential equations in an arbitrary number of independent variables that Pfaff made known in 1814 in the *Mémoires de l'Académie de Berlin* has been neglected for quite some time. The beautiful discoveries of Jacobi and Cauchy have only attracted the attention of geometers that were occupied with that theory.

Meanwhile, the Pfaff method, which is, moreover, a generalization of one that is due to Lagrange for the case of two independent variables, offers some considerable advantages. It replaces calculations that are often complicated with the use of certain differential identities that give the key to the intuitive solution of difficulties that present themselves in the other methods. The beautiful results that were obtained by Lie in various memoirs that were inserted into the *Mathematischen Annalen* show all of what one can infer from these identities, for example, if one would like to reduce to the smallest possible number the integrations that one must successively perform before arriving at the complete solution of a partial differential equation.

In the work that one is presently reading, I propose to explain the solution to the Pfaff problem without any recourse to the theory of partial differential equations, and above all, I am obliged to exhibit the invariance properties that play a fundamental role in that solution. I am not at all concerned with the integrations that are necessary in order to bring a differential expression into its reduced form, and moreover, from some formulas that I will give, the operations that one must do in order to obtain the solution to that problem can be copied in some fashion onto the ones that refer to the integration of a partial differential equation.

In the first part, I study the reduced forms, and I show that the integration of the first Pfaff system suffices, and immediately gives the reduced form when one is dealing with the differential expression that corresponds to a partial differential equation.

In the second part, I study the relations between the reduced forms, and I prove, in particular, three propositions that serve as the basis for the theory of Lie groups ([1]).

([1]) The first part of this paper was written in 1876 and communicated by Bertrand, who taught the theory of partial differential equations at the Collège de France at the time. Bertrand has kindly explained the method that I have propounded here in his first lecture in January, 1877.

Some time later, a beautiful memoir of Frobenius appeared in the *Journal de Borchardt* that carried a date that was previous to January, 1877, moreover (viz., September, 1876), and in which that geometric scholar followed a path that is very closely analogous to the one that I communicated to Bertrand, in the sense that it rested upon the use of invariants and the bilinear covariants of Lipschitz. Upon returning to my work at that point in time, it seemed to me that my exposition was more free from calculations and, as a consequence, from the importance that Pfaff method is said to take on, so there would be some interest in making it known.

PART ONE

I.

Consider the differential expression:

$$X_1\, dx_1 + \ldots + X_n\, dx_n\,,$$

where $X_1, \ldots, X_n$ are given functions of $x_1, \ldots, x_n$. We denote them by the notation Θ_d, where the index d refers to the system of differentials that is adopted. One will thus have:

$$\Theta_d = X_1\, dx_1 + \ldots + X_n\, dx_n\,, \tag{1}$$

and if one employs other differentials that are denoted by the character δ then one has:

$$\Theta_\delta = X_1\, \delta x_1 + \ldots + X_n\, \delta x_n\,. \tag{2}$$

From the two preceding equalities, one deduces that:

$$\delta\Theta_d = \sum \delta X_i\, dx_i + \ldots + \sum X_i\, \delta dx_i\,,$$

$$d\Theta_\delta = \sum dX_i\, \delta x_i + \ldots + \sum X_i\, d\delta x_i\,,$$

and consequently:

$$\begin{aligned}\delta\Theta_d - d\Theta_\delta &= \sum (\delta X_i\, dx_i - dX_i\, \delta x_i) \\ &= \sum_i \sum_k \left(\frac{\partial X_i}{\partial x_k} - \frac{\partial X_k}{\partial x_i}\right)(dx_i \delta x_k - dx_k \delta x_i)\,,\end{aligned}$$

the summation being extended over all combinations of indices 1, 2, …, n, and consequently consists of $n(n-1)/2$ terms. To abbreviate, we set:

$$a_{ik} = \frac{\partial X_i}{\partial x_k} - \frac{\partial X_k}{\partial x_i}\,, \tag{3}$$

and the preceding equality will become:

$$\delta\Theta_d - d\Theta_\delta = \sum_i \sum_k a_{ik}\,(dx_i \delta x_k - dx_k \delta x_i)\,. \tag{4}$$

By virtue of the identities:

In the same year – viz., 1877 – an important memoir by Lie on the same subject also appeared in the *Archiv for Mathematik* in Christiania (t. II, pp. 338). However, this paper rests upon methods that are completely different from the ones that I will present.

$$a_{ik} + a_{ki} = 0, \quad a_{ii} = 0,$$

which follow from formula (3), one can further write equation (4) in the form:

(4 *cont.*) $$\delta\Theta_d - d\Theta_\delta = \sum_{i=1}^{n}\sum_{k=1}^{n} a_{ik}\,dx_i\,\delta x_k\,.$$

Now suppose that one replaces the variables x_i in the differential expression (1) with some other variables y_i. Upon performing the substitution that is defined by the formulas:

(5) $$x_i = \psi_i\,(y_1, \ldots, y_n),$$

which gives:

$$dx_i = \sum \frac{\partial \psi_i}{\partial y_k}\,dy_k\,,$$

the expression Θ_d will take the form:

(6) $$\Theta_d = \sum Y_i\,dy_k\,.$$

In all of what follows, we shall assume that the n functions y_i are independent. As a result, the new variables y_i can be regarded as independent functions of the old ones x_i . As for the coefficients Y_i , one can always transform them by the use of formulas (5) into functions of the variables y_i .

Having said that, apply formula (4) to the new expression for Θ_d. If we set:

(7) $$b_{ik} = \frac{\partial Y_i}{\partial y_k} - \frac{\partial Y_k}{\partial y_i}$$

then we will have:

$$\delta\Theta_d - d\Theta_\delta = \sum_{i=1}^{n}\sum_{k=1}^{n} b_{ik}\,dy_i\,\delta y_k\,,$$

and consequently:

(8) $$\sum_{i=1}^{n}\sum_{k=1}^{n} a_{ik}\,dx_i\,\delta x_k = \sum_{i=1}^{n}\sum_{k=1}^{n} b_{ik}\,dy_i\,\delta y_k\,.$$

This formula is fundamental to our theory. Furthermore, before continuing, we shall give a direct proof of it without appealing to the property that is expressed by the equation:

$$d\delta x_i = \delta dx_i$$

that we made use of.

From a comparison of expressions (1) and (6) for Θ_d, one deduces the equalities:

$$X_1\frac{\partial x_1}{\partial y_k} + \cdots + X_n\frac{\partial x_n}{\partial y_k} = Y_k\,,$$

which serve to define the quantities Y_k . From them, one deduces that:

$$\frac{\partial Y_k}{\partial y_i} = \sum_\alpha X_\alpha \frac{\partial^2 x_\alpha}{\partial y_k\,\partial y_i} + \sum_\alpha \sum_{\alpha'} \frac{\partial X_\alpha}{\partial x_{\alpha'}} \frac{\partial x_\alpha}{\partial y_k} \frac{\partial x_{\alpha'}}{\partial y_i},$$

and consequently:

$$\frac{\partial Y_k}{\partial y_i} - \frac{\partial Y_i}{\partial y_k} = \sum_\alpha \sum_{\alpha'} \left(\frac{\partial X_\alpha}{\partial x_{\alpha'}} - \frac{\partial X_{\alpha'}}{\partial x_\alpha}\right)\left(\frac{\partial x_\alpha}{\partial y_k}\frac{\partial x_{\alpha'}}{\partial y_i} - \frac{\partial x_\alpha}{\partial y_i}\frac{\partial x_{\alpha'}}{\partial y_i}\right),$$

where the sum on the right-hand side is taken over all systems of different values of α, α', and consequently it consists of $n(n-1)/2$ terms.

If one multiplies the preceding equation by $dy_i\,\delta y_k - dy_k\,\delta y_k$, and one then takes the sum of the $n(n-1)/2$ equation thus obtained then the coefficient of:

$$\frac{\partial X_\alpha}{\partial x_{\alpha'}} - \frac{\partial X_{\alpha'}}{\partial x_\alpha}$$

in the right-hand side will be:

$$\sum_i \sum_k \left(\frac{\partial x_\alpha}{\partial y_k}\frac{\partial x_{\alpha'}}{\partial y_i} - \frac{\partial x_\alpha}{\partial y_i}\frac{\partial x_{\alpha'}}{\partial y_i}\right)(dy_i\,\delta y_k - dy_k\,\delta y_i),$$

i.e.:

$$dx_{\alpha'}\,\delta x_\alpha - dx_\alpha\,\delta x_{\alpha'} .$$

One will then have:

$$(9)\qquad \begin{cases} \displaystyle\sum_i \sum_k \left(\frac{\partial Y_i}{\partial y_k} - \frac{\partial Y_k}{\partial y_i}\right)(dy_i\,\delta y_k - dy_k\,\delta y_i) \\ \displaystyle\qquad = \sum_i \sum_k \left(\frac{\partial X_\alpha}{\partial x_{\alpha'}} - \frac{\partial X_{\alpha'}}{\partial x_\alpha}\right)(dx_\alpha\,\delta x_{\alpha'} - dx_{\alpha'}\,\delta x_\alpha), \end{cases}$$

which is the same thing as equation (8).

II.

Having said this, consider the variables x_i to be functions of one auxiliary variable t that are defined by the differential equations:

$$(10)\qquad \begin{cases} a_{11}dx_1 + \cdots + a_{n1}dx_n = \lambda X_1 dt, \\ a_{12}dx_1 + \cdots + a_{n2}dx_n = \lambda X_2 dt, \\ \qquad\qquad \cdots \\ a_{1n}dx_1 + \cdots + a_{nn}dx_n = \lambda X_n dt, \end{cases}$$

where λ will be a quantity that one can choose arbitrarily to be 0, a constant, or a function of t, depending upon the situation. We remark that equations (10) can be replaced with the single equation:

$$(10)^a \qquad \sum_{i=1}^{n}\sum_{k=1}^{n} a_{ik} dx_i \delta x_k = \lambda\, dt \sum X_i\, \delta x_i$$

that one obtains by adding them, after having multiplied them by $\delta x_1, \ldots, \delta x_n$, respectively, provided that one requires that this equation is verified for all of the values that are attributed to the auxiliary variables δx_i. Therefore, the system (1) can be replaced with the single equation:

$$(10)^b \qquad \delta\Theta_d - d\Theta_\delta = \lambda\, \Theta_\delta\, dt,$$

which must be true for any differentials δ. In the applications, it will always be preferable to directly form the two sides of the latter equation instead of successively calculating the quantities a_{ik} that appear in system (10). From now on, the preceding remarks will lead us to a fundamental property of system (10).

Suppose that one performs a change of variables and one replaces the variables x_i with some other variables y_i that are equal in number and which are independent functions of the former. It is easy to see that the system (10) is transformed into the one that one forms in the same manner by taking new independent variables. This results immediately from the fact that this system, when written in the form $(10)^b$, is obviously independent of any choice of independent variables. However, for the sake of neatness, consider equation $(10)^a$. One knows, by virtue of equality (8), that its left-hand side will become:

$$\sum\sum b_{ik}\, dy_i\, \delta y_k\ .$$

As for the right-hand side, it will obviously transform into the following one:

$$\lambda\, dt \sum Y_k\, \delta y_k\ .$$

Therefore, equation $(10)^a$ will take the form:

$$\sum_i \sum_k b_{ik} dy_i \delta y_k = \lambda\, dt\ \sum_i Y_i \delta y_i\ .$$

Since the functions y_i are independent, their differentials δy_i are arbitrary, like the differentials δx_i. One can then equate the coefficients of the differentials in the two sides, and one will obtain the equations:

$$
(11) \qquad \begin{cases} b_{11}dy_1 + b_{21}dy_1 + \cdots + b_{n1}dy_n = \lambda Y_1 dt, \\ b_{11}dy_2 + \cdots\cdots\cdots\cdots + b_{n2}dy_n = \lambda Y_2 dt, \\ \qquad\qquad\qquad\qquad \cdots\cdots\cdots \\ b_{1n}dy_1 + \cdots\cdots\cdots\cdots + b_{nn}dy_n = \lambda Y_n dt. \end{cases}
$$

Therefore, whenever the functions x_i satisfy equations (10), the functions y_i will satisfy equations (11). The converse is obviously proved in the same manner. One can thus say that systems (10) and (11) are absolutely equivalent, since they are two forms of the same system of differential equations, when written in different variables. As they are composed in the same manner by means of variables that enter into them, we express this property in an abbreviated manner by saying that it amounts to saying that *system* (10) *is invariant*. We shall make use of this proposition in order to indicate the reduced forms into which one can convert the differential expression Θ_d .

III.

First, suppose that n is even. The skew determinant:

$$\sum \pm a_{11}\, a_{22} \, \dots \, a_{nn}$$

will be a perfect square. We begin by assuming that this determinant is non-zero.

One can then solve equations (10) for $dx_1, \dots, dx_n$, and one will obtain a system of the form:

$$\frac{dx_1}{H_1} = \dots = \frac{dx_n}{H_n} = \lambda\, dt$$

that admits $n - 1$ independent integrals of t.

Take these $n - 1$ integrals to be new variables that we denote by $y_1, \dots, y_{n-1}$, and a function y_n that is subject to the single condition that it not be an integral of the system. $y_1, \dots, y_n$ then define a system of n independent functions, and the system (10), when written in the new variables, will take the form (11). One must then express the idea that equations (11) are verified when one assumes that the $n - 1$ functions $y_1, \dots, y_{n-1}$ in them are constants.

One must then have:

$$\left(\frac{\partial Y_1}{\partial y_n} - \frac{\partial Y_n}{\partial y_1}\right) dy_n = -\lambda\, Y_1\, dt,$$

$$\left(\frac{\partial Y_2}{\partial y_n} - \frac{\partial Y_n}{\partial y_2}\right) dy_n = -\lambda\, Y_2\, dt,$$

$$\dots$$

$$0 = \lambda\, Y_n\, dt.$$

From this, one deduces that:

$$Y_n = 0,$$

$$\frac{\partial \log Y_1}{\partial y_n} = \frac{\partial \log Y_2}{\partial y_n} = \dots = \frac{\partial \log Y_{n-1}}{\partial y_n} = \frac{-\lambda\, dt}{dy_n}.$$

The latter equations show that the functions $Y_1, \dots, Y_{n-1}$ depend effectively upon the y_n, but that their mutual ratios are independent. One can thus assume that for $i < n$ one has:

$$Y_i = K Y_i^0,$$

Y_i^0 being independent of the variable y_n, while, on the contrary, K necessarily contains it. One thus comes down to a differential expression of the form:

$$\Theta_d = K(Y_1^0 dy_1 + \dots + Y_{n-1}^0 dy_{n-1}),$$

which has at least one term, but which again enjoys the property of containing the variable y_n only in the factor K. One can further write:

(12) $$\Theta_d = y_n (Y_1^0 dy_1 + \dots + Y_{n-1}^0 dy_{n-1}),$$

upon now denoting the coefficient K by y_n .

Now, suppose that n is odd. The determinant:

$$\Delta = \sum \pm a_{11} \dots a_{nn}$$

will then be zero, since it is skew-symmetric of odd order, and consequently equations (10) will never be impossible if one sets $\lambda = 0$ in them. We first suppose that all of the minors of first order in Δ are non-zero. In this case, equations (10), where one makes $\lambda = 0$, determine the ratios of the differentials completely. They therefore admit $n - 1$ independent integrals that we again denote by $y_1, \dots, y_{n-1}$, and that we take for the new variables, when we add a function y_n to them that will not be an integral, and will consequently form a system of n independent functions with them. Equations (11) must then be verified by the substitution of the equations:

$$\lambda = 0, \quad dy_1 = 0, \quad \dots, dy_{n-1} = 0,$$

which will give:

$$\frac{\partial Y_1}{\partial y_n} - \frac{\partial Y_n}{\partial y_1} = 0,$$

$$\frac{\partial Y_2}{\partial y_n} - \frac{\partial Y_n}{\partial y_2} = 0,$$

$$\dots\dots\dots\dots\dots,$$

$$\frac{\partial Y_{n-1}}{\partial y_n} - \frac{\partial Y_n}{\partial y_{n-1}} = 0.$$

It is easy to find the most general form for the functions that satisfy these equations. Indeed, set:

$$Y_n = \frac{\partial \Psi}{\partial y_n}, \qquad Y_k = \frac{\partial \Psi}{\partial y_k} + Y_k^0 .$$

The equations express the idea that the derivatives of the functions Y_k^0 with respect to y_n are zero. One can thus set:

$$\Theta_d = d\Psi + Y_1^0 dy_1 + \ldots + Y_{n-1}^0 dy_{n-1},$$

in which the functions Y_k^0 do not depend upon y_n .

However, two different cases can present themselves here. In general, Ψ will contain y_n, and consequently Ψ, y_1, …, y_{n-1} will be n independent functions. Upon changing the notation and denoting Ψ by y_n , one will get the first reduced form:

(13) $$\Theta_d = dy_n + Y_1^0 dy_1 + \ldots + Y_{n-1}^0 dy_{n-1} .$$

However, it can also happen that Ψ does not contain y_n . One will then have:

$$\Theta_d = \left(\frac{\partial \Psi}{\partial y_1} + Y_1^0\right) dy_1 + \cdots + \left(\frac{\partial \Psi}{\partial y_{n-1}} + Y_{n-1}^0\right) dy_{n-1},$$

or, more simply:

(14) $$\Theta_d = Y_1^0 dy_1 + \ldots + Y_{n-1}^0 dy_{n-1} .$$

It is, moreover, very easy to distinguish these forms from each other, *a priori*. Indeed, the latter is characterized by the property that Θ_d is annulled when one has:

$$dy_1 = 0, \ldots, dy_{n-1} = 0.$$

One thus sees that one will obtain this form whenever the equation:

$$X_1\, dx_1 + \ldots + X_n\, dx_n = 0$$

is a consequence, a simple linear combination of equations (10) in which one has set λ = 0.

For example, consider the form in three variables:

$$F_d = X\, dx + Y\, dy + Z\, dz = 0.$$

Here, system (10) becomes:

(15) $$\frac{dx}{\frac{\partial Y}{\partial z}-\frac{\partial Z}{\partial y}} = \frac{dy}{\frac{\partial Z}{\partial x}-\frac{\partial X}{\partial z}} = \frac{dz}{\frac{\partial X}{\partial y}-\frac{\partial Y}{\partial x}}.$$

If one replaces dx, dy, dz in the form with quantities that are proportional to them then one obtains the well-known expression:

(16) $$X\left(\frac{\partial Y}{\partial z}-\frac{\partial Z}{\partial y}\right)+Y\left(\frac{\partial Z}{\partial x}-\frac{\partial X}{\partial z}\right)+Z\left(\frac{\partial X}{\partial y}-\frac{\partial Y}{\partial x}\right).$$

If this expression is non-zero then one can convert F_d into the form:

$$d\gamma + M\,d\alpha + N\,d\beta,$$

where α, β are integrals of the system (15), M and N are functions of α and β, and γ is a function that is independent of α, β. On the contrary, if the expression (16) is zero then the term $d\gamma$ will disappear, and what will remain is:

$$F_d = M\,d\alpha + N\,d\beta = \mu\,du,$$

which is in agreement with known results.

IV.

Up to now, we have assumed that the system (10) is determinate. Now, imagine that it is not. Thus, if n is even then the determinant:

$$\sum \pm a_{11} \ldots a_{nn}$$

will be zero, and consequently the same will be true for all of its first-order minors, by virtue of a known property of skew-symmetric determinants. If n is odd then the first-order minors of the same determinant will all be zero.

Equations (10) then reduce to at least n distinct ones, and it no longer suffices to determine the mutual ratios of $dx_1, \ldots, dx_n, dt$. However, I remark that they always form a system that is equivalent to system (11), since the argument that we made in order to establish that equivalence suffers no exception.

To simplify, suppose that one has made $\lambda = 0$. Equations (10) will be indeterminate. Suppose that they reduce to p distinct equations, where p can be equal to zero.

I arbitrarily append $n - p - 1$ differential equations – for example, the following ones:

$$d\varphi_1 = 0, \qquad d\varphi_2 = 0, \qquad \ldots, d\varphi_{n-p-1} = 0,$$

where $\varphi_1, \ldots, \varphi_{n-p-1}$ are arbitrary functions – and I thus obtain a perfectly determined system. I further call the $n - 1$ integrals of the complete system $y_1, \ldots, y_{n-1}$, and upon adjoining to them a function y_n that is not an integral, I again obtain n independent functions y_i that I substitute for the variables x_i. The system (11), in which one sets $\lambda = 0$, will be verified, like the first one, when one sets:

$$dy_1 = 0, \qquad \ldots, \qquad dy_{n-1} = 0.$$

By an argument like the one that we made in the case where n is odd, we are led to the same conclusions, and we find one of the forms (13) or (14). In summary, we can state the following theorem:

A form Θ_d in n variables can always be converted by the integration of the system (10) *into one of the three forms:*

$$(A) \qquad \begin{cases} y_n(Y_1 dy_1 + \cdots + Y_{n-1} dy_{n-1}), \\ \qquad Y_1 dy_1 + \cdots + Y_{n-1} dy_{n-1}, \\ dy_n + Y_1 dy_1 + \cdots + Y_{n-1} dy_{n-1}, \end{cases}$$

where the variables $y_1, \ldots, y_{n-1}$ are independent, and where the functions Y_i depend only upon $y_1, \ldots, y_{n-1}$. Some of the functions Y_i can be zero, moreover. The first of these three forms presents itself only when n is even and the determinant:

$$\sum \pm a_{11} \ldots a_{nn}$$

is non-zero.

One can further state the preceding result in the following manner: Let Θ_d^n denote a differential form in n variables. One can always convert Θ_d^n into one of the three forms:

$$y_n \Theta_d^{n-1}, \qquad \Theta_d^{n-1}, \qquad dy_n + \Theta_d^{n-1},$$

where y_n is a variable that is completely independent of the ones that figure in the new differential expression Θ_d^{n-1}.

V.

We may now prove the following theorem:

A form Θ_d can always be converted into one of the following two types:

$$(17) \qquad \begin{cases} dy - z_1 dy_1 - z_2 dy_2 - \cdots - z_p dy_p, \\ \qquad z_1 dy_1 + z_2 dy_2 + \cdots + z_p dy_p, \end{cases}$$

where the functions $y, y_1, \ldots, z_k$ *constitute a system of independent variables; i.e., they are functions that are independent of all the variables that enter into the form* Θ_d.

The first of these two preceding types will be said to be of *indeterminate* type, while the other one will be said to be of *determinate* type.

We shall prove that this proposition is an almost immediate consequence of the preceding one. Indeed, it is obvious for forms in one and two variables. It will then suffice to show that if it is true for a form in $n - 1$ variables then it is also true for a form that contains one more variable.

In order to do this, we remark that a form in n variables can be converted into one of the three types in A. Neglecting the second one, which depends upon only $n - 1$ variables and for which, consequently, the theorem is allowed, we remark that the other two are composed in a very simple manner with the function in $n - 1$ variables $Y_1\, dy_1 + \ldots + Y_{n-1}\, dy_{n-1}$.

Replacing that form in $n - 1$ variables with one of the two types (17), we obtain one of the following expressions for the form in n variables:

$$y_n\,(du - v_1\, du_1 - v_2\, du_2 - \ldots - v_p\, du_p),$$
$$y_n\,(v_1\, du_1 + v_2\, du_2 - \ldots + v_p\, du_p),$$
$$d(y_n + u) - v_1\, du_1 - v_2\, du_2 - \ldots - v_p\, du_p,$$
$$dy_n + v_1\, du_1 + v_2\, du_2 + \ldots + v_p\, du_p,$$

where u, u_i, v_k are independent functions of $y_1, \ldots, y_{n-1}$ and where, consequently, y_n, u, u_i, v_k are independent functions of the original variables.

The last two expressions obviously fall into the indeterminate type. As for the first two, one converts them into the second type by substituting the following functions for the functions $v_1, \ldots, v_p$:

$$v_1 y_n = \pm\, w_1\,, \ldots, v_p\, y_n = \pm\, w_p\,.$$

The theorem is thus established. The following consequence is an obvious result:

If the reduced form for the expression in n variables Θ_d is:

$$z_1\, dy_1 + \ldots + z_p\, dy_p$$

then the $2p$ functions z_i, y_k of the variables x_i are independent, so one necessarily has $2p \leq n$.

If the reduced form is:

$$dy - z_1\, dy_1 - \ldots - z_p\, dy_p$$

then one must likewise have $2p + 1 \leq n$.

VI.

We shall now solve the following problem:

If one is given a form Θ_d *in n variables then which of the two types* (17) *can it be converted into, and what is the value of the number p then?*

This problem is susceptible to an extremely simple solution. Indeed, suppose that one transforms the expression Θ_d by taking the new variables to be the ones that figure in the reduced form and choosing the other ones in an arbitrary manner so that they would complete the number of n independent functions. Observe that this must become the system (10). This system can be replaced with the single equation:

$$\delta\Theta_d - d\Theta_\delta = \lambda\,\Theta_d\,dt, \tag{18}$$

which must be valid for any differential δ. Suppose, to begin with, that the reduced form of Θ_d is:

$$\Theta_d = dy - z_1\,dy_1 - z_2\,dy_2 - \dots - z_p\,dy_p\,.$$

One will have:

$$\delta\Theta_d - d\Theta_\delta = dz_1\,\delta y_1 - dy_1\,\delta z_1 + \dots + dz_p\,\delta y_p - dy_p\,\delta z_p\,,$$

and the system (10) or equation (18), which is equivalent to it, gives us:

$$\begin{cases} dy_1 = 0, & dz_1 = -\lambda\,z_1 dt, \\ dy_2 = 0, & dz_2 = -\lambda\,z_2 dt, \\ \quad\cdots & \quad\cdots \\ dy_p = 0, & dz_p = -\lambda\,z_p dt, \\ & 0 = \lambda\,dt. \end{cases} \tag{19}$$

One sees that one will necessarily have $\lambda = 0$, and that equations (10) reduce to $2p$, which will be completely integrable.

On the contrary, if the reduced form is:

$$\Theta_d = z_1\,dy_1 + \dots + z_p\,dy_p$$

then system (10) will be equivalent to the following one:

$$\begin{cases} dy_1 = 0, & dz_1 = \lambda\,z_1 dt, \\ dy_2 = 0, & dz_2 = \lambda\,z_2 dt, \\ \quad\cdots & \quad\cdots \\ dy_p = 0, & dz_p = \lambda\,z_p dt. \end{cases} \tag{20}$$

Here, it will not be necessary to make $\lambda = 0$, which distinguishes this case from the first one. Moreover, the equations admit $2p - 1$ independent integrals of t:

$$y_1 = C_1, \qquad \frac{z_2}{z_1} = C_1',$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,$$
$$y_p = C_p, \qquad \frac{z_p}{z_1} = C_{p-1}'.$$

We can therefore state the following theorems:

If equations (10), *when regarded as determining the differentials* dx_i, *are impossible as long as* λ *is non-zero then the form* Θ_d *is reducible to the indeterminate type:*

$$dy - z_1\,dy_1 - z_2\,dy_2 - \ldots - z_p\,dy_p .$$

The number $2p$ *is equal to the number of distinct equations to which equations* (10) *reduce when one sets* $\lambda = 0$, *and consequently, it will be easy to determine,* a priori. *Moreover, the* $2p$ *equations to which equations* (10) *then reduce are completely integrable, and the variables* y_i, z_k *of the reduced form are functions of their* $2p$ *integrals.*

If equations (10) *can be verified by supposing that* λ *is non-zero then the form is reducible to the determinate type:*

$$z_1\,dy_1 + \ldots + z_p\,dy_p .$$

The number $2p$ *is equal to the number of distinct equations to which equations* (10) *then reduce. Moreover, these equations are always completely integrable, and one will have a system of integrals of these equations in terms of the variables of the reduced form that are given by the formulas:*

$$y_1 = \alpha_1, \qquad z_1\,e^{-\int \lambda\,dt} = \beta_1,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,$$
$$y_p = \alpha_p, \qquad z_p\,e^{-\int \lambda\,dt} = \beta_p .$$

In other words, these differential equations admit the functions $y_1, \ldots, y_p$ *and the quotients* $z_2 / z_1, \ldots, z_p / z_1$ *for independent integrals of* t.

As an application, we study the reduced form Θ_d in the most general case.

If n is odd then the determinant:

$$\sum \pm a_{11} \ldots a_{nn}$$

is non-zero, and one can solve equations (10) for the differentials dx_i; λ is non-zero and equations (10) are all distinct. Here, one then has the second type (17), and the reduced form is:

$$z_1\,dy_1 + z_2\,dy_2 + \ldots + z_{n/2}\,dy_{n/2} .$$

On the contrary, if n is odd then the determinant:

$$\sum \pm a_{11} \dots a_{nn}$$

is zero; however, its first-order minors are non-zero, in general. As we have seen, one must then have $\lambda = 0$, apart from an exceptional case, and the equations then reduce to $n - 1$ distinct ones. The reduced form is:

$$dy - z_1\, dy_1 - \dots - z_{(n-1)/2}\, dy_{(n-1)/2} .$$

VII.

We have seen how one recognizes which type is attached to a differential form and how one determines the number p. It remains for us to show the integrations that are necessary in order to convert a given differential expression into its canonical form. The beautiful discoveries of Mayer and Lie greatly diminish the difficulty in this subject. However, in this paper I will occupy myself only with the invariance properties that relate to a differential form. I will thus content myself with explaining the general process of integrations, my sole objective being to show that the Pfaff method, when applied to a partial differential equation, leads to the same results as those of Cauchy.

First, consider a differential expression:

$$\Theta_d^n = X_1\, dx_1 + \dots + X_n\, dx_n ,$$

whose canonical form is:

$$z_1\, dy_1 + \dots + z_p\, dy_p . \tag{21}$$

We know that the Pfaff system:

$$\delta\Theta_d - d\Theta_\delta = \lambda\, \Theta_\delta\, dt$$

is then completely integrable if $2p < n$, and consequently admit $2p - 1$ independent integrals of *t in any case.* There will thus be at least $n - 2p - 1$ variables x_i that are not integrals. Suppose, to fix ideas, that the latter are:

$$x_{2p}, x_{2p+1}, \dots, x_n .$$

When one sets:

$$x_{2p} = x_{2p}^0, \qquad x_{2p+1} = x_{2p+1}^0, \quad \dots, \qquad x_n = x_n^0,$$

$x_{2p}^0, \dots, x_n^0$ being numerical constants, the $2p - 1$ integrals of the Pfaff system reduce to functions of $x_1, \dots, x_{2p-1}$. There will then be one integral that reduces to x_1, another that reduces to x_2, and so on ([1]). We let $[x_i]$ or u_i denote those of these integrals that reduce to

([1]) This classification of integrals of a system of equations is, as one knows, due to Cauchy in the case where there is just one independent variable. As far as completely integrable systems are concerned, it has already been utilized by Lie in the paper that we already cited on the Pfaff problem.

x_i . We know that the functions u_i depend solely upon the variables $y_1, \ldots, y_p$ that appear in the canonical form (21), and the quotients $z_2 / z_1, \ldots, z_p / z_1$. Having said this, we take the new variables to be:

$$u_1, \ldots, \quad u_{2p-1}, \quad x_{2p}, \ldots, \quad x_n ,$$

which are obviously independent functions of the first ones.

The form Θ_d^n becomes:

$$K(U_1\, du_1 + \ldots + U_{2p-1}\, du_{2p-1}), \tag{22}$$

where $U_1, \ldots, U_{2p-1}$ depend upon only the $u_1, \ldots, u_{2p-1}$, while K, by contrast, contains one or more variables $x_{2p}, \ldots, x_n$. This is simple to prove in several ways. For example, if one starts with the canonical form (21):

$$z_1\left(dy_1 + \frac{z_2}{z_1}dy_2 + \cdots + \frac{z_p}{z_1}dy_p\right)$$

then one knows that the z_k / z_1 are functions of the variables u_i . Therefore, if one replaces the y_i , z_k / z_1 with their expressions as functions of the integrals u_i and if one remarks that z_1 is an independent function of the preceding ones, then one indeed finds the expression (22).

I remark that the function K that appears in that expression is not defined completely. Nothing prevents one from dividing it by an arbitrary function $\varphi(u_1, \ldots, u_{2p-1})$, on the condition that one multiplies the quantities u by the same function φ. However, one can determine K completely by the following condition:

Suppose that K reduces to a function:

$$\psi(x_1, x_2, \ldots, x_{2p-1})$$

for $x_{2p} = x_{2p}^0, \ldots, x_n = x_n^0$.

We divide K by $\psi(u_1, u_2, \ldots, u_{2p-1})$, and then the new value of K will be defined completely and will enjoy the property of reducing to 1 when one sets $x_{2p} = x_{2p}^0, \ldots, x_n = x_n^0$.

Having said this, we write down the identity:

$$X_1\, dx_1 + \ldots + X_n\, dx_n = K(U_1\, du_1 + \ldots + U_{2p-1}\, du_{2p-1}),$$

and set $x_{2p} = x_{2p}^0, \ldots, x_n = x_n^0$ on both sides. Let X_p^0 denote what X_p becomes. Since K will then become equal to 1, u_i will become equal to x_i, and one will have:

$$X_1^0\, dx_1 + \ldots + X_{2p-1}^0\, dx_{2p-1} = U_1\, dx_1 + \ldots + U_{2p-1}\, dx_{2p-1} ,$$

and consequently one can write:

$$U_i = X_i^0 ,$$

which leads us to the following theorem:

Suppose that the canonical form of a differential expression:

$$\Theta_d^n = X_1\,dx_1 + \dots + X_n\,dx_n$$

is

$$z_1\,dy_1 + \dots + z_p\,dy_p\,.$$

The former Pfaff system will be completely integrable if $2p < n$*, and will admit* $2p - 1$ *independent integrals in any case. Therefore, there will always be at least* $n - 2p + 1$ *of the variables* x_i *that are not integrals of that system. Let* $x_{2p}, \dots, x_n$ *be* $n - 2p + 1$ *variables that enjoy that property. Consider the* $2p - 1$ *integrals of the Pfaff system that reduces to* $x_1, \dots, x_{2p-1}$ *when one sets* $x_{2p} = x_{2p}^0, \dots, x_n = x_n^0$*, and let* u_i *denote the ones that reduce to* x_i *. If one chooses these integrals to be the new variables then the expression* Θ_d^n *takes the following form:*

$$K(U_1\,du_1 + \dots + U_{2p-1}\,du_{2p-1}),$$

where one deduces U_h *from* X_h *by replacing* $x_1, \dots, x_{2p-1}$ *with* $u_1, \dots, u_{2p-1}$, respectively and $x_{2p}, \dots, x_n$ *with the constants* $x_{2p}^0, \dots, x_n^0$.

Now, consider the case in which the form Θ_d^n is reducible to the type:

$$dy - z_1\,dy_1 - \dots - z_p\,dy_p\,. \tag{23}$$

One knows that the Pfaff system will be possible only if one sets $\lambda = 0$ in it, and that in all cases it will admit $2p$ integrals that will be $z_1, \dots, z_p, y_1, \dots, y_p$. Here, we may argue as in the preceding. Among the n variables x_i, there will be at least $n - 2p$ of them that will not be integrable. Let:

$$x_{2p+1}, \dots, x_n$$

be $n - 2p$ variables that enjoy this property. Let u_i denote those of the integrals that reduce to x_i when one replaces $x_{2p+1}, \dots, x_n$ with numerical constants $x_{2p+1}^0, \dots, x_n^0$. Finally, perform a change of variables that substitutes the following variables:

$$u_1, \dots, u_{2p}, u_{2p+1}, \dots, x_n$$

for the original ones. One will have:

$$dH + U_1\,du_1 + \dots + U_{2p}\,du_{2p} \tag{24}$$

for the new form of the differential expression. Indeed, in the canonical form (23), the variables z_i, y_k that are the integrals of the Pfaff system can be regarded as functions of u_1,

…, u_{2p} . Therefore, if one supposes that they are expressed as functions of $u_1, …, u_{2p}$ then one will indeed obtain a result of the preceding form.

In the expression (24), the function H is not defined, and it is clear that the expression does not change if one replaces H with:

$$H - \varphi(u_1, …, u_{2p}),$$

on the condition that one must add $\partial\varphi / \partial u_i$ to U_i . If H reduces to $\psi(x_1, …, x_{2p})$ for $x_{2p+1} = x_{2p+1}^0, …, x_n = x_n^0$ then we agree to subtract:

$$\psi(u_1, …, u_{2p});$$

the new value of H will then reduce to zero for $x_{2p+1} = x_{2p+1}^0, …, x_n = x_n^0$.

Now, write down the identity:

$$X_1\,dx_1 + … + X_n\,dx_n = dH + U_1\,du_1 + … + U_{2p}\,du_{2p} ,$$

and set $x_{2p+1} = x_{2p+1}^0, …, x_n = x_n^0$ in it. Once more, let X_i^0 be what X_i becomes under that substitution. Since u_i then becomes equal to x_i and H becomes equal to zero, one will have:

$$X_1^0\,dx_1 + … + X_{2p}^0\,dx_{2p} = U_1\,du_1 + … + U_{2p}\,du_{2p} ,$$

and consequently:

$$U_k = X_k^0 .$$

We may thus state the new proposition as follows:

Suppose that the canonical form for a differential expression:

$$\Theta_d^n = X_1\,dx_1 + … + X_n\,dx_n ,$$

is

$$dy - z_1\,dy_1 - … - z_p\,dy_p .$$

The first Pfaff system will be possible only if one set $\lambda = 0$ *in it and will admit* $2p$ *integrals. Let* $x_{2p+1}, …, x_n$ *be a system of variables that do not take part in these integrals, and let* u_i *denote the integral of the Pfaff system that reduces to* x_i *for* $x_{2p+1} = x_{2p+1}^0, …, x_n = x_n^0$. *The expression* Θ_d^n *can be converted into the form:*

$$dH + U_1\,du_1 + … + U_{2p}\,du_{2p} ,$$

where one deduces U_k *from* X_k *by replacing* $x_1, …, x_{2p}$ *with* $u_1, …, u_{2p}$ *and* $x_{2p+1}, …, x_n$, *by constants* $x_{2p+1}^0, …, x_n^0$. *H is a function that reduces to zero for* $x_{2p+1} = x_{2p+1}^0, …, x_n = x_n^0$.

It is good to remark that H will be determined with no difficulty by a quadrature when $u_1, \ldots, u_{2p}$ are known. Because one has:

$$dH = \Theta_d^n - U_1\, du_1 - \ldots - U_{2p}\, du_{2p},$$

and everything will be known in the right-hand side.

The two preceding theorems lead to several consequences. One sees immediately that the various systems of differential equations to which the application of the method leads acquire an independent existence, in a sense. One may write each of them after having integrated the preceding one. Mayer already made analogous remarks relating to the completely integrable systems. Moreover, one sees that by starting with the second system one no longer has indeterminacy, and one no longer finds that the forms belong to the two general types.

One can make an important application of the preceding results to the particular form that one encounters in the theory of partial differential equations.

Let:

(25) $$p_1 = f(z, x_1, \ldots, x_p, p_2, \ldots, p_n)$$

be a partial differential equation, where p_i denotes $\partial z / \partial x_i$. It is clear that the integration of that equation is equivalent to the following problem:

Annul the form:

$$\Theta_d = dz - f\, dx_1 - p_2\, dx_2 - \ldots - p_n\, dx_n$$

in $2n$ *variables* $z, x_1, \ldots, x_n, p_2, \ldots, p_n$ *by establishing* n *relations between these variables.*

One knows that the solution to this problem offers no difficulty as long as Θ_d is reduced to the canonical form. Now, *I say that in order to convert* Θ_d *to the canonical form, it will suffice to integrate the first Pfaff system relative to the form.*

Indeed, write down that system:

$$\delta\Theta_d - d\Theta_\delta = \lambda\, \Theta_\delta\, dt,$$

or

$$\begin{aligned} df\, \delta x_1 - \delta f\, dx_1 + dp_2\, \delta x_2 - \delta p_2\, dx_2 + \ldots + dp_n\, \delta x_n - dx_n\, \delta p_n, \\ = \lambda\, dt(\delta z - f\, \delta r_1 - \ldots - p_n\, \delta x_n), \end{aligned}$$

which gives the equations:

$$\begin{aligned} df - \frac{\partial f}{\partial x_1} dx_1 \qquad &= -\lambda\, f\, dt, \\ -\frac{\partial f}{\partial x_2} dx_1 + dp_2 \quad &= -\lambda\, p_2\, dt, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots &\ldots\ldots, \\ -\frac{\partial f}{\partial x_n} dx_1 + dp_n \quad &= -\lambda\, p_n\, dt, \end{aligned}$$

$$
\begin{aligned}
-\frac{\partial f}{\partial z}dx_1 \qquad\quad &= \lambda\, dt,\\
-\frac{\partial f}{\partial p_2}dx_1 - dp_2 &= 0,\\
\dots\dots\dots\dots\dots\dots\dots\dots&,\\
-\frac{\partial f}{\partial p_n}dx_1 - dx_n &= 0,
\end{aligned}
$$

which one easily puts into the following form:

$$
(26)\qquad \begin{cases} \dfrac{dx_1}{-1} = \dfrac{dx_2}{\dfrac{\partial f}{\partial p_2}} = \cdots = \dfrac{dx_n}{\dfrac{\partial f}{\partial p_n}} = \dfrac{-dp_2}{\dfrac{\partial f}{\partial x_2} + p_2\dfrac{\partial f}{\partial z}} = \cdots = \dfrac{-dp_n}{\dfrac{\partial f}{\partial x_n} + p_n\dfrac{\partial f}{\partial z}}, \\ \qquad\qquad dz = p_1 dx_1 + \cdots + p_n dx_n. \end{cases}
$$

One recognizes the differential equations of the characteristic.

Here, we see that x_1 is never an integral. Let $[z]$, $[p_k]$, $[x_i]$ denote the integrals of that system that z, p_k, x_i reduce to for $x_i = x_1^0$, x_1^0 being an arbitrary constant. *There will be no difficulty in determining these integrals as long as the system* (26) *is completely integrable.* If we now apply the first of the two theorems that we proved then we see that one will have:

$$
(27)\qquad \begin{cases} dz - f\,dx_1 - p_2 dx_2 - \cdots - p_n dx_n \\ \quad = L\{d[z] - [p_2]d[x_2] - [p_3]d[x_3] - \cdots - [p_n]d[x_n]\}, \end{cases}
$$

in which L depends upon x_1. We thus obtain the reduced form that must be the conclusion of our calculations on the first try. The preceding method is encountered in the Cauchy method, and it plays a fundamental role there. It is pointless to return to the well-known propositions and to show how they lead to the integration of the proposed partial differential equation. For us, it suffices that we have established that, by means of a simple supplement, the Pfaff method becomes as perfect as the others. However, it is also justified for us to add that this classification of integrals that allowed us to arrive at our objective constitutes a very essential advance that is once more due to Cauchy.

(*to be continued*)