"Sur les systèmes non holonomes," Bull. Soc. Math. France **37** (1909), 120-132.

On non-holonomic systems

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Since the work of Appell, one knows that the equations of motion of a non-holonomic equation can be put into a form that is analogous to that of the Lagrange equations, but the right-hand sides contain correction terms. One will then have the *corrected Lagrange equations*. If one proposes to reduce those equations to first order then one will get *corrected canonical equations*, which differ from the canonical equations by corrective terms that are added to the right-hand sides. Certain theorems in the dynamics of holonomic systems will persist for non-holonomic systems; other must be modified. In particular, Poisson's theorem does not apply to non-holonomic systems in the form that is suitable to holonomic systems. When one knows two integrals of the corrected canonical equations, one can also form a third. However, in order to do that, one must find a particular system of a system of linear differential equations. In addition, the integral that is formed will no longer have the simple form that is suited to holonomic systems. We propose to direct our attention to those various points.

I. – CORRECTED CANONICAL EQUATIONS.

For more simplicity, we shall consider a non-holonomic system whose constraints are frictionless and independent of time and is subject to forces that are derived from a force function U that does not contain time explicitly. First recall Appell's results ([1]).

Let q_1 , q_2 , …, q_n be the independent parameters whose expression as a function of time determines the position of the system at each instant.

Set:

$$q_i' = \frac{dq_i}{dt}, \qquad q_i'' = \frac{d^2 q_i}{dt^2} \qquad (i = 1, 2, \dots, n) . \tag{1}$$

If one calls the *vis viva* $2T$ and the energy of acceleration $2S$ then one will have:

$$\left\{ \begin{aligned} T &= \varphi_1(q_1', q_2', \dots, q_n'), \\ S &= \varphi_2(q_1'', q_2'', \dots, q_n'') + \psi_1 \, q_1'' + \dots + \psi_n \, q_n'' + \chi, \end{aligned} \right. \tag{2}$$

([1]) "Remarques d'orde analytique sur une nouvelle forme des équations de la Dynamique," J. Math. pures et appl. (5) **7** (1901), pp. 5.

in which φ_1 is a quadratic form in q'; φ_2 has the same form as φ_1 when one replaces the q' with q''. The ψ are quadratic forms in q'; χ is a form of order four in q'. The coefficients of those various forms are functions of $q_1, q_2, \ldots, q_n$ that do not include time explicitly. However, those coefficients are not mutually independent. Indeed, one has the identity:

$$E = \psi_1 q_1' + \ldots + \psi_n q_n' = \frac{\partial \varphi_1}{\partial q_1} q_1' + \ldots + \frac{\partial \varphi_1}{\partial q_n} q_n'. \tag{3}$$

The motion of the system is then given by the corrected Lagrange equations:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial q_i'}\right) - \frac{\partial T}{\partial q_i} = \frac{\partial U}{\partial q_i} + \Delta_i' \qquad (i = 1, 2, \ldots, n), \tag{4}$$

in which:

$$\Delta_i' = \frac{\partial E}{\partial q_i'} - 2\frac{\partial \varphi_1}{\partial q_i} - \psi_i. \tag{5}$$

Equations (4) admit the *vis viva* integral. Indeed, if one multiplies those equations by $q_1', q_2', \ldots, q_n'$, and adds them then one will get:

$$dT = dU + (\Delta_1' q_1' + \cdots + \Delta_n' q_n')\, dt. \tag{6}$$

However, one has:

$$\sum_i \Delta_i' q_i' = \sum_i \frac{\partial E}{\partial q_i'} q_i' - 2\sum_i \frac{\partial \varphi_1}{\partial q_i} q_i' - \sum \psi_i q_i'.$$

We remark that E is a form of order three in q', and when one takes the identity (3) into account, one will get:

$$\sum_i \Delta_i' q_i' = 3E - 2E - E = 0.$$

Equation (6) then gives the *vis viva* integral:

$$T = U + h.$$

We point out the identity:

$$\sum_i \Delta_i' q_i' = 0. \tag{7}$$

Those are the results that Appell gave.

Apply the Poisson-Hamilton transformations to equations (4). Set:

$$p_i = \frac{\partial T}{\partial q_i'} \qquad (i = 1, 2, \ldots, n),$$

$$H = T - U.$$

The argument that applies to holonomic systems can be repeated in the case that we are dealing with, and it will lead to the equations:

$$\frac{dq_i}{dt} = \frac{\partial H}{\partial p_i}, \quad \frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i} + \Delta_i \qquad (i = 1, 2, \ldots, n), \tag{8}$$

in which Δ_i denotes the function into which the preceding substitution transformed Δ_i'. The identity (7) is now written:

$$\sum_i \Delta_i' \frac{\partial H}{\partial p_i} = 0. \tag{9}$$

Equations (8) are the *corrected canonical equations*.

We seek the condition for the equation:

$$f(t, q_1, \ldots, q_n, p_1, \ldots, p_n) = \text{const.}$$

to be an integral of (8). One has:

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \sum_i \frac{\partial f}{\partial q_i}\frac{dq_i}{dt} + \sum_i \frac{\partial f}{\partial p_i}\frac{dp_i}{dt}$$

$$= \frac{\partial f}{\partial t} + \sum_i \left(\frac{\partial f}{\partial q_i}\frac{\partial H}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial H}{\partial q_i}\right) + \sum_i \frac{\partial f}{\partial p_i}\Delta_i$$

$$= \frac{\partial f}{\partial t} + (f, H) + \sum_i \frac{\partial f}{\partial p_i}\Delta_i .$$

The desired condition is then:

$$\frac{\partial f}{\partial t} + (f, H) + \sum_i \frac{\partial f}{\partial p_i}\Delta_i = 0. \tag{10}$$

One deduces from this that if f = const. is an integral then $\partial f / \partial t$ = const. will be another.

Indeed, if the constraints are independent of time then none of the functions φ_i and ψ contain time explicitly. The same thing is true for the Δ and for E, by virtue of (3) and (5). Since the functions T and U do not contain time explicitly, the same thing will be true for H. Upon taking the partial derivative with respect to t of the left-hand side of (10)m one will get:

$$\frac{\partial}{\partial t}\left(\frac{\partial f}{\partial t}\right)+\left(\frac{\partial f}{\partial t}, H\right)+\sum_i \frac{\partial}{\partial p_i}\left(\frac{\partial f}{\partial t}\right)\Delta_t = 0,$$

which is a relation that shows that $\partial f / \partial t$ = const. is an integral.

One likewise sees that f = const. is an integral, and if H and the Δ are independent of q_k or p_k then $\partial f / \partial q_k$ = const. or $\partial f / \partial p_k$ = const. will also be an integral.

One will then have some theorems that relate to holonomic system that are preserved for the non-holonomic systems that we shall consider. However, Poisson's theorem no longer applies, at least in the form that is suitable for holonomic systems.

Let us follow the argument to which one is accustomed in establishing that theorem. Let f = const. and φ = const. be two integrals of the corrected canonical equations. One has:

$$[(f, \varphi), H] + [(\varphi, H), f] + [(H, f), \varphi] = 0$$

identically.

However, since (10) applies to the functions f and φ, one will have:

$$(\varphi, H) = -\frac{\partial \varphi}{\partial t} - \sum_i \Delta_i \frac{\partial \varphi}{\partial p_i},$$

$$(H, f) = \frac{\partial f}{\partial t} + \sum_i \Delta_i \frac{\partial f}{\partial p_i},$$

and as a result, when one uses the properties of the Poisson brackets:

$$[(\varphi, H), f] = -\left(\frac{\partial \varphi}{\partial t}, f\right) - \left(\sum_i \Delta_i \frac{\partial \varphi}{\partial p_i}, f\right),$$

$$[(\varphi, H), f] = \left(f, \frac{\partial \varphi}{\partial t}\right) + \sum_i \Delta_i \left(f, \frac{\partial \varphi}{\partial p_i}\right) - \sum_i \frac{\partial \varphi}{\partial p_i}(\Delta_i, f).$$

One likewise finds that:

$$[(H, f), \varphi] = \left(\frac{\partial f}{\partial t}, \varphi\right) + \sum_i \Delta_i \left(\frac{\partial f}{\partial p_i}, \varphi\right) + \sum_i \frac{\partial f}{\partial p_i}(\Delta_i, \varphi).$$

Upon substituting that in (10), one will get:

$$[(f, \varphi), H] + \left(f, \frac{\partial \varphi}{\partial t}\right) + \left(\frac{\partial f}{\partial t}, \varphi\right)$$

$$+\sum_i \Delta_i \left[\left(f, \frac{\partial \varphi}{\partial p_i}\right) + \left(\frac{\partial f}{\partial p_i}, \varphi\right)\right] + \sum_i \left[\frac{\partial f}{\partial p_i}(\Delta_i, \varphi) - \frac{\partial \varphi}{\partial p_i}(\Delta_i, f)\right] = 0,$$

which one can write:

$$\frac{\partial}{\partial t}(f, \varphi) + [(f, \varphi), H] + \sum_i \Delta_i \frac{\partial}{\partial p_i}(f, \varphi) + \sum_i \left[\frac{\partial f}{\partial p_i}(\Delta_i, \varphi) - \frac{\partial \varphi}{\partial p_i}(\Delta_i, f)\right] = 0.$$

Upon comparing that identity with the condition (10), one will see that (f, φ) = const. will not be an integral, in general; Poisson's theorem does not apply. It will persist if, when one is given the integral f = const., one can choose the second integral φ = const. in such a manner that one has:

$$\sum_i \left[\frac{\partial f}{\partial p_i}(\Delta_i, \varphi) - \frac{\partial \varphi}{\partial p_i}(\Delta_i, f)\right] = 0,$$

identically.

We shall pass over the examination of that particular case.

II. – CORRECTED POISSON THEOREM.

In his paper on the three-body problem (Acta Mathematica, v. XIII, pp. 46), Poincaré exhibited the equations that he called the *equations of variation* of canonical equations. He showed that two solutions of the equations of variations are linked by a certain relation and that if the solutions are known then that relation will constitute an integral of the canonical equations. One can find a solution to the equations of variation when one knows an integral, and conversely. Finally, if one knows an integral of the canonical equations then one can find a solution to the equations of variation, and Poisson's theorem will be an immediate consequence of the latter remark. Upon following Poincaré's method, one will get a generalization of Poisson's theorem.

Recall the corrected canonical equations:

(1) $$\frac{dq_i}{dt} = \frac{\partial H}{\partial p_i}, \quad \frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i} + \Delta_i \qquad (i = 1, 2, \ldots, n).$$

Consider two infinitely-close solutions to those equations q_i, p_i, and $q_i + \xi_i$, $p_i + \eta_i$, in which the ξ, η are small enough that one can neglect their squares. The ξ, η will then satisfy the linear differential equations:

(2) $$\begin{cases} \dfrac{d\xi_i}{dt} = \displaystyle\sum_k \frac{\partial^2 H}{\partial p_i\,\partial q_k}\xi_k + \sum_k \frac{\partial^2 H}{\partial q_i\,\partial p_k}\eta_k, \\ \dfrac{d\eta_i}{dt} = -\displaystyle\sum_k \frac{\partial^2 H}{\partial q_i\,\partial q_k}\xi_k - \sum_k \frac{\partial^2 H}{\partial p_i\,\partial p_k}\eta_k + \sum_k \frac{\partial \Delta_i}{\partial q_k}\xi_k + \sum_k \frac{\partial \Delta_i}{\partial p_k}\eta_k, \end{cases}$$

which are the equations of variation of equations (1). Let ξ'_i, η'_i be another solution of equations (2), in such a way that:

$$(2')\qquad \begin{cases} \dfrac{d\xi'_i}{dt} = \displaystyle\sum_k \frac{\partial^2 H}{\partial p_i\,\partial q_k}\xi'_k + \sum_k \frac{\partial^2 H}{\partial q_i\,\partial p_k}\eta'_k, \\ \dfrac{d\eta'_i}{dt} = -\displaystyle\sum_k \frac{\partial^2 H}{\partial q_i\,\partial q_k}\xi'_k - \sum_k \frac{\partial^2 H}{\partial p_i\,\partial p_k}\eta'_k + \sum_k \frac{\partial \Delta_i}{\partial q_k}\xi'_k + \sum_k \frac{\partial \Delta_i}{\partial p_k}\eta'_k. \end{cases}$$

Multiply equations (2) and (2) by η'_i, $-\xi'_i$, $-\eta_i$, ξ_i, and sum over all of the results that are obtained; one will get:

$$\frac{d}{dt}\sum_i (\eta'_i\,\xi_i - \xi'_i\,\eta_i) + \sum_{i,k}\xi'_i\left(\xi_k\frac{\partial \Delta_i}{\partial q_k} + \eta_k\frac{\partial \Delta_i}{\partial p_k}\right) - \sum_{i,k}\xi_i\left(\xi'_k\frac{\partial \Delta_i}{\partial q_k} + \eta'_k\frac{\partial \Delta_i}{\partial p_k}\right) = 0,$$

which one can write as:

$$\frac{d}{dt}\sum_i (\eta'_i\,\xi_i - \xi'_i\,\eta_i) + \sum_{i,k}\xi'_k\left(\xi_i\frac{\partial \Delta_k}{\partial q_i} + \eta_i\frac{\partial \Delta_k}{\partial p_i}\right) - \sum_{i,k}\xi_i\left(\xi'_k\frac{\partial \Delta_i}{\partial q_k} + \eta'_k\frac{\partial \Delta_i}{\partial p_k}\right) = 0,$$

or rather:

$$\frac{d}{dt}\sum_i (\eta'_i\,\xi_i - \xi'_i\,\eta_i) + \sum_i\left\{\xi_i\sum_k\left[\xi'_k\left(\frac{\partial \Delta_k}{\partial q_i} - \frac{\partial \Delta_i}{\partial q_k}\right) - \eta'_k\frac{\partial \Delta_i}{\partial p_k}\right] + \eta'_i\sum_k \xi'_k\frac{\partial \Delta_k}{\partial p_i}\right\} = 0\,.$$

One will deduce form this that:

$$(3)\qquad \sum_i (\eta'_i\,\xi_i - \xi'_i\,\eta_i) + \int \sum_i\left\{\xi_i\sum_k\left[\xi'_k\left(\frac{\partial \Delta_k}{\partial q_i} - \frac{\partial \Delta_i}{\partial q_k}\right) - \eta'_k\frac{\partial \Delta_i}{\partial p_k}\right] + \eta'_i\sum_k \xi'_k\frac{\partial \Delta_k}{\partial p_i}\right\}dt = \text{const}\,.$$

That is the relation that exists between two solutions of the equations of variation. That relation is no longer algebraic, as it would be for holonomic systems. If all of the Δ are zero then one will indeed recover the relation:

$$\sum_i (\eta'_i\,\xi_i - \xi'_i\,\eta_i) = \text{const.}$$

that Poincaré gave.

In the case where ξ_i, η_i, ξ'_i, η' are known, the relation (3) will give an integral of the canonical equations.

If one supposes that ξ'_i, η' denote a known particular solution of equations (2), and ξ_i, η_i denote their general solution then the relation (3) will become an integral of the equations of variation. Hence, knowing a particular solution of those equations will provide another integral.

Conversely, suppose that we know an integral of equations (2):

$$(4)\qquad \sum_i (A_i\,\xi_i + B_i\,\eta_i) + \int \sum_i \left\{ \xi_i \left[B_k \left(\frac{\partial \Delta_i}{\partial q_k} - \frac{\partial \Delta_k}{\partial q_i} \right) - A_k \frac{\partial \Delta_i}{\partial q_k} \right] - \eta_i \sum_k B_k \frac{\partial \Delta_k}{\partial p_i} \right\} dt = \text{const.},$$

in which the A, B are given. Upon differentiating and remarking that ξ_i, η_i verify (2), one will have:

$$\sum_i \left(\frac{dA_i}{dt}\xi_i + \frac{dB_i}{dt}\eta_i \right) + \sum_i A_i \left(\sum_k \frac{\partial^2 H}{\partial p_i\,\partial q_k}\xi_k + \sum_k \frac{\partial^2 H}{\partial q_i\,\partial p_k}\eta_k \right)$$

$$+ \sum_i B_i \left(-\sum_k \frac{\partial^2 H}{\partial q_i\,\partial q_k}\xi_k - \sum_k \frac{\partial^2 H}{\partial q_i\,\partial p_k}\eta_k + \sum_k \frac{\partial \Delta_i}{\partial q_k}\xi_k + \sum_k \frac{\partial \Delta_i}{\partial p_k}\eta_k \right)$$

$$+ \sum_i \xi_i \sum_k \left[B_k \left(\frac{\partial \Delta_i}{\partial q_k} - \frac{\partial \Delta_k}{\partial q_i} \right) \right] - \sum_i \eta_i \sum_k B_k \frac{\partial \Delta_k}{\partial p_i} = 0.$$

One can write that relation as:

$$\sum_i \xi_i \left\{ \frac{dA_i}{dt} + \sum_k \left[B_k \left(\frac{\partial \Delta_i}{\partial q_k} - \frac{\partial \Delta_i}{\partial q_k} \right) - A_k \frac{\partial \Delta_i}{\partial p_k} \right] \right\}$$

$$+ \sum_i \eta_i \left(\frac{dB_i}{dt} - \sum_k \frac{\partial \Delta_k}{\partial p_i} \right) + \sum_{i,k} A_k \left(\frac{\partial^2 H}{\partial q_i\,\partial p_k}\xi_i + \frac{\partial^2 H}{\partial p_i\,\partial p_k}\eta_i \right)$$

$$+ \sum_{i,k} B_k \left(-\xi_i \frac{\partial^2 H}{\partial q_i\,\partial q_k} - \eta_i \frac{\partial^2 H}{\partial p_i\,\partial q_k} + \xi_i \frac{\partial \Delta_k}{\partial q_i} + \eta_i \frac{\partial \Delta_k}{\partial p_i} \right) = 0,$$

or rather:

$$\sum_i \xi_i \left(\frac{dA_i}{dt} + \sum_k A_k \frac{\partial^2 H}{\partial p_i\,\partial q_k} - \sum_k B_k \frac{\partial^2 H}{\partial q_i\,\partial q_k} + \sum_k B_k \frac{\partial \Delta_i}{\partial q_k} - \sum_k A_k \frac{\partial \Delta_i}{\partial p_k} \right)$$

$$+ \sum_i \eta_i \left(\frac{dB_i}{dt} - \sum_k B_k \frac{\partial^2 H}{\partial p_i\,\partial q_k} + \sum_k A_k \frac{\partial^2 H}{\partial p_i\,\partial p_k} \right) = 0.$$

Upon identifying corresponding terms, one then deduces that:

$$\frac{dA_i}{dt} = -\sum_k \frac{\partial^2 H}{\partial p_i\,\partial p_k} + \sum_k B_k \frac{\partial^2 H}{\partial p_i\,\partial q_k} - \sum_k B_k \frac{\partial \Delta_i}{\partial q_k} + \sum_k A_k \frac{\partial \Delta_i}{\partial p_k},$$

$$\frac{dB_i}{dt} = -\sum_k B_k \frac{\partial^2 H}{\partial p_i\,\partial q_k} - \sum_k A_k \frac{\partial^2 H}{\partial p_i\,\partial p_k}.$$

If one sets:

$$\alpha_i = B_i\,, \qquad \beta_i = -A_i$$

then one will find that:

$$\frac{d\alpha_i}{dt} = \sum_k \frac{\partial^2 H}{\partial p_i\,\partial q_k}\alpha_k + \sum_k \frac{\partial^2 H}{\partial p_i\,\partial p_k}\beta_k\,,$$

$$\frac{d\beta_i}{dt} = -\sum_k \frac{\partial^2 H}{\partial q_i\,\partial q_k}\alpha_k + \sum_k \frac{\partial^2 H}{\partial q_i\,\partial p_k}\beta_k + \sum_k \frac{\partial \Delta_i}{\partial q_k}\alpha_k + \sum_k \frac{\partial \Delta_i}{\partial p_k}\beta_k\,,$$

and one will see that α_i , β_i indeed constitute a particular solution of the equations of variations.

It remains for us to examine whether an integral of the canonical equations might lead to a solution of the equations of variation. Let:

$$f(q, p, t) = \text{const.}$$

be an integral of the canonical equations.

That relation will persist if one replaces q, p with $q + \xi$, $p + \eta$, resp., where ξ, η are an arbitrary solution to the equations of variation. One will then have the integral of the equations of variation:

$$\sum_i \left(\frac{\partial f}{\partial q_i}\xi_i + \frac{\partial f}{\partial p_i}\eta_i \right) = 0.$$

If one can put that integral into the form (4) then one can deduce a solution to the equations of variation from the preceding. Upon denoting some auxiliary unknowns by u_i, v_i, the preceding integral can be written:

$$\sum_i \left[\left(\frac{\partial f}{\partial q_i} + u_i \right) \xi_i + \left(\frac{\partial f}{\partial p_i} + v_i \right) \eta_i \right] - \sum_i (u_i\,\xi_i + v_i\,\eta_i) = 0.$$

However, one has:

$$u_i\,\xi_i = \int (u_i\,d\xi_i + \xi_i\,du_i)$$

$$= \int \left[u_i \sum_k \left(\frac{\partial^2 H}{\partial p_i \, \partial q_k} \xi_k + \frac{\partial^2 H}{\partial p_i \, \partial p_k} \eta_k \right) + \xi_i \frac{du_i}{dt} \right] dt ,$$

and

$$v_i \; \eta_i = \int (v_i \, d\eta_i + \eta_i \, dv_i)$$

$$= \int \left[v_i \sum_k \left(-\frac{\partial^2 H}{\partial q_i \, \partial q_k} \xi_k - \frac{\partial^2 H}{\partial q_i \, \partial p_k} \eta_k + \frac{\partial \Delta_i}{\partial q_k} \xi_k + \frac{\partial \Delta_i}{\partial p_k} \eta_k \right) + \eta_i \frac{dv_i}{dt} \right] dt .$$

One will then have the integral:

$$\sum_i \left[\left(\frac{\partial f}{\partial q_i} + u_i \right) \xi_i + \left(\frac{\partial f}{\partial p_i} + v_i \right) \eta_i \right]$$

$$- \int \; \sum_i \left[u_i \left(\sum_k \frac{\partial^2 H}{\partial p_i \, \partial q_k} \xi_k - \sum_k \frac{\partial^2 H}{\partial p_i \, \partial p_k} \eta_k \right) + \xi_i \frac{du_i}{dt} \right.$$

$$\left. + \; v_i \sum_k \left(\frac{\partial^2 H}{\partial q_i \, \partial p_k} \xi_k - \frac{\partial^2 H}{\partial q_i \, \partial p_k} \eta_k + \frac{\partial \Delta_i}{\partial q_k} \xi_k + \frac{\partial \Delta_i}{p_k} \eta_k \right) + \eta_i \frac{dv_i}{dt} \right] = 0.$$

The coefficient of dt under the ∫ sign can be written:

$$\sum_i \left(\xi_i \frac{du_i}{dy} + \eta_i \frac{dv_i}{dy} \right)$$

$$+ \sum_{i,k} \left(u_i \, \xi_k \frac{\partial^2 H}{\partial p_i \, \partial q_k} + u_i \, \eta_k \frac{\partial^2 H}{\partial p_i \, \partial p_k} - v_i \, \xi_k \frac{\partial^2 H}{\partial q_i \, \partial q_k} - v_i \, \eta_k \frac{\partial^2 H}{\partial q_i \, \partial p_k} + v_i \, \xi_k \frac{\partial \Delta_i}{\partial q_k} + v_i \, \eta_k \frac{\partial \Delta_i}{\partial p_k} \right),$$

or rather, when one permutes the indices i, k in the double sum and then puts ξ_i and η_i in as factors:

$$\sum_i \left\{ \xi_i \left[\frac{du_i}{dt} + \sum_k \left(u_k \frac{\partial^2 H}{\partial q_i \, \partial p_k} - v_k \frac{\partial^2 H}{\partial q_i \, \partial q_k} + v_k \frac{\partial \Delta_k}{\partial q_i} \right) \right] \right.$$

$$\left. + \; \eta_i \left[\frac{dv_i}{dt} + \sum_k \left(u_k \frac{\partial^2 H}{\partial p_i \, \partial p_k} - v_k \frac{\partial^2 H}{\partial p_i \, \partial q_k} + v_k \frac{\partial \Delta_k}{\partial p_i} \right) \right] \right\} ,$$

and the integral can be written in the form:

$$\sum_i \left[\left(\frac{\partial f}{\partial q_i} + u_i \right) \xi_i + \left(\frac{\partial f}{\partial p_i} + v_i \right) \eta_i \right]$$

$$- \int \sum_i \left\{ \xi_k \left[\frac{du_i}{dt} + \sum_k \left(u_i \frac{\partial^2 H}{\partial p_i \, \partial q_k} - v_k \frac{\partial^2 H}{\partial p_i \, \partial p_k} + v_k \frac{\partial \Delta_k}{\partial q_i} \right) \right] \right.$$

$$\left. + \left[\frac{dv_i}{dt} + \sum_k \left(u_k \frac{\partial^2 H}{\partial p_i \, \partial p_k} - v_k \frac{\partial^2 H}{\partial p_i \, \partial q_k} + v_k \frac{\partial \Delta_k}{\partial p_i} \right) \right] \right\} = 0.$$

If one takes u_i, v_i to be solutions of the differential equations:

$$\frac{du_i}{dt} + \sum_k \left(u_i \frac{\partial^2 H}{\partial p_i \, \partial q_k} - v_k \frac{\partial^2 H}{\partial p_i \, \partial p_k} + v_k \frac{\partial \Delta_k}{\partial q_i} \right)$$

$$= - \sum_k \left[\left(\frac{\partial f}{\partial p_k} + v_k \right) \left(\frac{\partial \Delta_i}{\partial q_k} - \frac{\partial \Delta_k}{\partial q_i} \right) - \left(\frac{\partial f}{\partial q_i} + u_i \right) \frac{\partial \Delta_i}{\partial p_k} \right],$$

$$\frac{dv_i}{dt} + \sum_k \left(u_k \frac{\partial^2 H}{\partial p_i \, \partial p_k} - v_k \frac{\partial^2 H}{\partial p_i \, \partial q_k} + v_k \frac{\partial \Delta_k}{\partial p_i} \right) = \sum_k \left(\frac{\partial f}{\partial p_k} + v_k \right) \frac{\partial \Delta_k}{\partial p_i},$$

which can be written:

$$(5) \qquad \begin{cases} \dfrac{du_i}{dt} + \displaystyle\sum_k \left[u_k \left(\frac{\partial^2 H}{\partial p_i \, \partial q_k} - \frac{\partial \Delta_i}{\partial p_k} \right) - v_k \left(\frac{\partial^2 H}{\partial q_i \, \partial q_k} - \frac{\partial \Delta_i}{\partial q_k} \right) \right] = (f, \Delta_i) + \displaystyle\sum_k \frac{\partial f}{\partial p_k} \frac{\partial \Delta_k}{\partial q_i}, \\ \dfrac{dv_i}{dt} + \displaystyle\sum_k \left(u_k \frac{\partial^2 H}{\partial p_i \, \partial p_k} - v_k \frac{\partial^2 H}{\partial p_i \, \partial q_k} \right) = \displaystyle\sum_k \frac{\partial f}{\partial p_k} \frac{\partial \Delta_k}{\partial p_i}, \end{cases}$$

upon simplifying, then the integral can be presented in the form (4), in which one has:

$$A_i = \frac{\partial f}{\partial q_i} + u_i \,, \qquad B_i = \frac{\partial f}{\partial p_i} + v_i \,.$$

It results from this that when one is given the integral $f(q, p) = \text{const.}$ of the corrected canonical equations, if one can find a solutions to equations (5) then one will have a solution to the equations of variation, namely:

$$\xi_i = \frac{\partial f}{\partial p_i} + v_i \,, \qquad \eta_i = - \frac{\partial f}{\partial q_i} - u_i \,.$$

For the holonomic systems, since all Δ are zero, equations (5) will admit the obvious solution $u_i = v_i = 0$, and one will recover the known solution to the equations of variation:

$$\xi_i = \frac{\partial f}{\partial p_i}, \qquad \eta_i = -\frac{\partial f}{\partial q_i}.$$

For non-holonomic systems, one cannot find a solution to equations (5), *a priori*.

Leaving aside the determination of a particular solution to equations (5), suppose that we know two integrals f = const. and φ = const. of the corrected canonical equations; one of them is a solution u_i, v_i of (5) and the other is a solution of equations (5) when f is replaced with φ. The relation (3) will then give an integral of the corrected canonical equations.

After some easy reductions, that integral can be written:

$$(f, \varphi) + \sum_i (u_i v_i' - v_i u_i') + \sum_i \left(u_i \frac{\partial \varphi}{\partial p_i} - v_i \frac{\partial \varphi}{\partial q_i} \right) - \sum_i \left(u_i' \frac{\partial f}{\partial p_i} - v_i' \frac{\partial f}{\partial q_i} \right)$$

$$+ \int \left\{ \sum_i \left[\frac{\partial f}{\partial p_i} (\varphi, \Delta_i) - \frac{\partial \varphi}{\partial p_i} (f, \Delta_i) \right] + \sum_i [v_i (\varphi, \Delta_i) - v_i' (f, \Delta_i)] \right.$$

$$+ \sum_{i,k} \left[\frac{\partial \varphi}{\partial p_k} \left(v_i \frac{\partial \Delta_k}{\partial q_i} - u_i \frac{\partial \Delta_k}{\partial p_i} \right) - \frac{\partial f}{\partial p_k} \left(v_i' \frac{\partial \Delta_k}{\partial q_i} - u_i' \frac{\partial \Delta_k}{\partial p_i} \right) \right.$$

$$\left. \left. + v_i v_i' \left(\frac{\partial \Delta_k}{\partial q_i} - \frac{\partial \Delta_i}{\partial q_k} \right) + v_i u_i' \frac{\partial \Delta_i}{\partial p_k} - u_i v_i' \frac{\partial \Delta_k}{\partial p_i} \right] \right\} dt = \text{const.}$$

That is the form that Poisson's theorem presents for non-holonomic system. If the Δ are zero then one can take:

$$u_i = v_i = u_i' = v_i' = 0,$$

and one will indeed recover the integral (f, φ) = const.

One can simplify the preceding results by appealing to a theorem of de Donder ([1]). If one is given the system:

$$\frac{dx_i}{dt} = X_i, \tag{6}$$

whose equations of variation are:

$$\frac{d\xi_i}{dt} = \sum_k \frac{\partial X_i}{\partial x_k} \xi_k, \tag{7}$$

([1]) "Étude sur les invariants intégraux," Rend. Circ. Mat. Palermo **15** (1901) and **16** (1902).

if one knows an integral f = const. of the equations (6), and if ξ is a solution of the equations of variation (7) then equations (6) will admit the new integral:

$$\sum_k \frac{\partial f}{\partial x_k} \xi_k = \text{const.} \tag{8}$$

Moreover, if we suppose that two integrals of the corrected canonical equations f = const. and φ = const. are known, along with a solution u_i , v_i of equations (5) then we will have the new integral:

$$\sum_k \left[\frac{\partial \varphi}{\partial p_k} \left(\frac{\partial f}{\partial p_k} + v_k \right) - \frac{\partial \varphi}{\partial q_k} \left(\frac{\partial f}{\partial q_k} + u_k \right) \right] = \text{const.},$$

or rather:

$$(f, \varphi) + \sum_k \left(\frac{\partial \varphi}{\partial q_k} u_k - \frac{\partial \varphi}{\partial p_k} v_k \right) = \text{const.},$$

which gives a simpler form for Poisson's theorem.

One can further write the new integral:

$$(\varphi, f) + \sum_k \left(\frac{\partial f}{\partial q_k} u'_k - \frac{\partial f}{\partial p_k} v'_k \right) = \text{const.},$$

in which u', v' are solutions of equations (5) in which φ replaces f.

We finally remark that when one associates the integral f = const. with the solution $\xi_i = \partial f / \partial p_i + v_i$, $\eta_i = - \partial f / \partial q_i + u_i$, one will get the integral:

$$\sum_k \left[\frac{\partial f}{\partial q_k} \left(\frac{\partial f}{\partial p_k} + v_k \right) - \frac{\partial f}{\partial p_k} \left(\frac{\partial f}{\partial q_k} + u_k \right) \right] = \text{const.}$$

or

$$\sum_k \left(\frac{\partial f}{\partial q_k} v_k - \frac{\partial f}{\partial p_k} u_k \right) = \text{const.}$$

For a holonomic system, the latter integral is an identity.

It remains for us to examine whether the new integrals are distinct from the ones that served to define them.
