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On a transformation of motion

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In a recent article ([1]), Appell showed that homographic transformations can be advantageously applied to various questions in mechanics. At the end of his paper, Appell proposed the following generalization of the results that he had obtained.

Suppose that one has the Lagrange equations:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial q_i'}\right) - \frac{\partial T}{\partial q_i} = Q_i\,, \qquad q_i' = \frac{dq_i}{dt} \qquad (i = 1, 2, \ldots, k),$$

in which T is a quadratic form in $q_1', \ldots, q_k'$ with coefficients that are functions of $q_1, \ldots, q_k$, and in which $Q_1, \ldots, Q_k$ depend upon only $q_1, \ldots, q_k$. Find transformations of the form:

$$r_i = f_i(q_1, \ldots, q_k) \qquad (i = 1, 2, \ldots, k),$$
$$dt_1 = \lambda(q_1, \ldots, q_k)\, dt$$

that transform those equations into other ones of the form:

$$\frac{d}{dt_1}\left(\frac{\partial S}{\partial r_i'}\right) - \frac{\partial S}{\partial r_i} = R_i\,, \qquad r_i' = \frac{dr_i}{dt_1} \qquad (i = 1, 2, \ldots, k),$$

in which S denotes a quadratic form in $r_1', \ldots, r_k'$ with coefficients that are functions of $r_1, \ldots, r_k$, and in which $R_1, \ldots, R_k$ depend upon only $r_1, \ldots, r_k$.

We propose to consider the case of motion of a point on a surface ($K = 2$) and to show that the desired transformations are the ones that preserve the geodesic lines, as Appell had predicted.

([1]) P. Appell, "De l'homographie en Mécanique," Am. J. Math. **12** (1889), pp. 103.

I.

Consider two surfaces S and S_1, and make a real point of the first one corresponds to a real point of the second. From a theorem of Tissot, one knows that there exists an orthogonal system on the first one that corresponds to an orthogonal system on the second one. Refer those two surfaces to those two orthogonal systems. Let:

$$ds^2 = E\,du^2 + G\,dv^2, \qquad ds_1^2 = E_1\,du^2 + G_1\,dv^2$$

be the expressions for the line elements on S and S_1 . Consider a material point of mass equal to unity whose motion on the surface S is determined by the equations:

$$(1) \qquad \begin{cases} \dfrac{d}{dt}\left(\dfrac{\partial T}{\partial u'}\right) - \dfrac{\partial T}{\partial u} = P, \quad u' = \dfrac{du}{dt}, \\[2ex] \dfrac{d}{dt}\left(\dfrac{\partial T}{\partial v'}\right) - \dfrac{\partial T}{\partial v} = Q, \quad v' = \dfrac{dv}{dt}, \end{cases}$$

in which

$$2\,T = E\,u'^2 + G\,v'^2,$$

and in which P, Q depend upon only u, v.

Now consider a second point of mass equal to unity that moves on the surface S_1, and imagine that the coordinates of that point are functions of a new variable t_1 that is coupled with t by the equation:

$$(2) \qquad dt_1 = \lambda\,(u, v)\,dt\,.$$

The motion of the second point will be determined by the equations:

$$(1) \qquad \begin{cases} \dfrac{d}{dt_1}\left(\dfrac{\partial T_1}{\partial u_1'}\right) - \dfrac{\partial T_1}{\partial u} = P_1, \quad u_1' = \dfrac{du}{dt_1}, \\[2ex] \dfrac{d}{dt_1}\left(\dfrac{\partial T}{\partial v_1'}\right) - \dfrac{\partial T_1}{\partial v} = Q_1, \quad v_1' = \dfrac{dv}{dt_1}, \end{cases}$$

in which

$$2\,T_1 = E_1\,u_1'^2 + G_1\,v_1'^2.$$

The question that one proposes to solve reduces to this one: Is it possible to determine the function λ in such a manner that P_1 and Q_1 are independent of u_1' and v_1' ?

One deduces from equations (1) and (3) that:

(4)
$$\begin{cases} P = E\dfrac{du'}{dt} + \dfrac{1}{2}\dfrac{\partial E}{\partial u}u'^2 + \dfrac{\partial E}{\partial v}u'v' - \dfrac{1}{2}\dfrac{\partial G}{\partial u}v'^2, \\ Q = G\dfrac{dv'}{dt} - \dfrac{1}{2}\dfrac{\partial E}{\partial v}u'^2 + \dfrac{\partial G}{\partial v}u'v' + \dfrac{1}{2}\dfrac{\partial G}{\partial v}v'^2, \end{cases}$$

and

(5)
$$\begin{cases} P_1 = E_1\dfrac{du_1'}{dt_1} + \dfrac{1}{2}\dfrac{\partial E_1}{\partial u}u_1'^2 + \dfrac{\partial E_1}{\partial v}u_1'v_1' - \dfrac{1}{2}\dfrac{\partial G_1}{\partial u}v_1'^2, \\ Q_1 = G_1\dfrac{dv_1'}{dt_1} - \dfrac{1}{2}\dfrac{\partial E_1}{\partial v}u_1'^2 + \dfrac{\partial G_1}{\partial v}u_1'v_1' + \dfrac{1}{2}\dfrac{\partial G_1}{\partial v}v_1'^2. \end{cases}$$

One now has:

$$u' = u_1'\lambda(u,v), \qquad v' = v_1'\lambda(u,v),$$

from which one infers that:

$$\frac{du_1'}{dt_1} = \frac{1}{\lambda^2}\frac{du'}{dt} - \frac{1}{\lambda}\left(\frac{\partial\lambda}{\partial u}u_1'^2 + \frac{\partial\lambda}{\partial v}u_1'v_1'\right),$$
$$\frac{dv_1'}{dt_1} = \frac{1}{\lambda^2}\frac{dv'}{dt} - \frac{1}{\lambda}\left(\frac{\partial\lambda}{\partial u}u_1'v_1' + \frac{\partial\lambda}{\partial v}v_1'^2\right).$$

If one replaces $\dfrac{du'}{dt}$, $\dfrac{dv'}{dt}$ in those equations with the values that are deduced from (4) and substitutes the values thus-obtained in (5) for $\dfrac{du_1'}{dt_1}$, $\dfrac{dv_1'}{dt_1}$ then one will ultimately find that:

$$\begin{aligned} P_1 = \frac{E_1 P}{E\lambda^2} &- \left(\frac{E_1}{2E}\frac{\partial E}{\partial u} - \frac{1}{2}\frac{\partial E_1}{\partial u} + \frac{E_1}{\lambda}\frac{\partial\lambda}{\partial u}\right)u_1'^2 \\ &- \left(\frac{E_1}{E}\frac{\partial E}{\partial v} - \frac{\partial E_1}{\partial v} + \frac{E_1}{\lambda}\frac{\partial\lambda}{\partial v}\right)u_1'v_1' \\ &+ \left(\frac{E_1}{2E}\frac{\partial G}{\partial u} - \frac{1}{2}\frac{\partial G_1}{\partial u}\right)v_1'^2, \end{aligned}$$

$$\begin{aligned} Q_1 = \frac{G_1 Q}{G\lambda^2} &+ \left(\frac{G_1}{2G}\frac{\partial E}{\partial v} - \frac{1}{2}\frac{\partial E_1}{\partial v}\right)u_1'^2 \\ &- \left(\frac{G_1}{G}\frac{\partial G}{\partial u} - \frac{\partial G_1}{\partial u} + \frac{G_1}{\lambda}\frac{\partial\lambda}{\partial v}\right)u_1'v_1' \\ &- \left(\frac{G_1}{2G}\frac{\partial G}{\partial v} - \frac{1}{2}\frac{\partial G_1}{\partial v} + \frac{G_1}{\lambda}\frac{\partial G_1}{\partial v}\right)v_1'^2. \end{aligned}$$

Upon expressing the idea that P_1, Q_1 are independent of u_1', v_1', one will have the equations:

$$\frac{E_1}{2E}\frac{\partial E}{\partial u}-\frac{1}{2}\frac{\partial E_1}{\partial u}+\frac{E_1}{\lambda}\frac{\partial \lambda}{\partial u}=0\,,$$
$$\frac{E_1}{E}\frac{\partial E}{\partial v}-\frac{\partial E_1}{\partial v}+\frac{E_1}{\lambda}\frac{\partial \lambda}{\partial v}=0\,,$$
$$\frac{E_1}{E}\frac{\partial G}{\partial u}-\frac{\partial G_1}{\partial u}=0\,,$$
$$\frac{G_1}{G}\frac{\partial E}{\partial v}-\frac{\partial E_1}{\partial v}=0\,,$$
$$\frac{G_1}{G}\frac{\partial G}{\partial u}-\frac{\partial G_1}{\partial u}+\frac{G_1}{\lambda}\frac{\partial \lambda}{\partial v}=0\,,$$
$$\frac{G_1}{2G}\frac{\partial G}{\partial v}-\frac{1}{2}\frac{\partial G_1}{\partial v}+\frac{G_1}{\lambda}\frac{\partial G_1}{\partial v}=0\,.$$

Those equations are equivalent to the following ones:

$$(6)\qquad\left\{\begin{aligned}
&\frac{1}{\lambda}\frac{\partial \lambda}{\partial u}=-\frac{1}{2}\left(\frac{1}{E}\frac{\partial E}{\partial u}-\frac{1}{E_1}\frac{\partial E_1}{\partial u}\right),\\
&\frac{1}{\lambda}\frac{\partial \lambda}{\partial v}=-\frac{1}{2}\left(\frac{1}{E}\frac{\partial E}{\partial v}-\frac{1}{E_1}\frac{\partial E_1}{\partial v}\right),\\
&\frac{1}{E}\frac{\partial G}{\partial u}=\frac{1}{E_1}\frac{\partial G_1}{\partial u},\\
&\frac{1}{G}\frac{\partial E}{\partial u}=\frac{1}{G_1}\frac{\partial E_1}{\partial u},\\
&\frac{2}{G}\frac{\partial G}{\partial u}-\frac{1}{E}\frac{\partial E}{\partial u}=\frac{2}{G_1}\frac{\partial G_1}{\partial u}-\frac{1}{E_1}\frac{\partial E_1}{\partial u},\\
&\frac{2}{E}\frac{\partial E}{\partial v}-\frac{1}{G}\frac{\partial G}{\partial v}=\frac{2}{E_1}\frac{\partial E_1}{\partial v}-\frac{1}{G_1}\frac{\partial G_1}{\partial v}.
\end{aligned}\right.$$

Since the last four of them are independent of λ, they represent necessary and sufficient conditions for the problem that was posed to admit a solution. One recognizes that those conditions express the idea that the geodesic lines correspond on the two surfaces considered ([1]).

If one supposes that those equations are verified identically then the first two will give λ. Indeed, one then deduces that:

$$\frac{\partial}{\partial u}\log\frac{E\lambda^2}{E_1}=0\,,\qquad \frac{\partial}{\partial v}\log\frac{E\lambda}{E_1}=0\,,$$

or rather:

([1]) G. DARBOUX, *Leçons sur la théorie générale des surfaces*, Part III, pp. 49.

$$\frac{E\lambda^2}{E_1} = V_2\,, \qquad \frac{E\lambda}{E_1} = U_2\,,$$

in which V_2 denote a function of v, and U_2 denotes a function of u. We adopt the notations that Darboux employed in his book *Sur la théorie générale des surfaces* ([1]). The integration of the last four equations (6) will give:

$$\frac{E}{E_1} = V U^2,$$

in which V is a function of v, and U is a function of u. We will then have:

$$\lambda^2 = \frac{V_2}{VU^2}, \quad \lambda = \frac{U_2}{VU^2}.$$

From that, we will have:

$$V_2\, V = \left(\frac{U_2}{U}\right)^2 = K^2,$$

in which K is a constant.

As a result, we will have:

$$V_2 = \frac{K^2}{V}, \qquad U_2 = K\,U\,, \qquad \lambda = \frac{K}{VU}.$$

The solution of the problem is thus achieved, and one sees that the desired transformations are precisely the ones that permit one to represent one of the surfaces in question on the other one geodesically.

II.

In particular, consider the case in which one of the surfaces is a plane. We can perform the calculations completely and obtain the transformations that we have in mind explicitly.

Let a planar motion be defined by the equations:

(1) $$\frac{d^2x}{dt^2} = X\,, \qquad \frac{d^2y}{dt^2} = Y\,,$$

in which x, y denote the rectangular Cartesian coordinates, and in which one supposes that X, Y are functions of only x, y. Consider a surface that is referred to the curvilinear coordinate system that is defined by a family of geodesics and their orthogonal trajectories. The expression for the line element is:

([1]) *Loc. cit.*

$$ds^2 = du^2 + C^2\,dv^2\,.$$

The motion of a point of mass equal to unity on the surface is determined by the equation:

$$(2)\qquad \begin{cases} \dfrac{d}{dt_1}\left(\dfrac{\partial T}{\partial u'}\right) - \dfrac{\partial T}{\partial u} = P, \quad u' = \dfrac{du}{dt_1}, \\ \dfrac{d}{dt_1}\left(\dfrac{\partial T}{\partial v'}\right) - \dfrac{\partial T}{\partial v} = Q, \quad v' = \dfrac{dv}{dt_1}, \\ 2T = u'^2 + C^2\,v'^2, \end{cases}$$

in which t_1 denotes time.

We propose to find the transformations of the form:

$$(3)\qquad x = f(u, v)\,, \quad y = \varphi(u, v)\,, \quad dt_1 = \lambda(u, v)\,dt$$

that transform equations (1) into equations (2) with the condition that P and Q must be independent of u', v'.

One deduces from (2) that:

$$(4)\qquad \begin{cases} P = \dfrac{du'}{dt_1} - C\,\dfrac{\partial C}{\partial u}\,v'^2, \\ Q = C^2\,\dfrac{dv'}{dt_1} + 2C\,\dfrac{\partial C}{\partial u}\,u'\,v' + C\,\dfrac{\partial C}{\partial u}\,v'^2. \end{cases}$$

Upon differentiating (3) and taking (1) into account, one will have:

$$X = \lambda^2\,\frac{\partial f}{\partial u}\,\frac{\partial u'}{\partial t_1} + \lambda^2\,\frac{\partial f}{\partial v}\,\frac{\partial v'}{\partial t_1} + \left(\lambda^2\,\frac{\partial^2 f}{\partial u^2} + \lambda\,\frac{\partial \lambda}{\partial u}\,\frac{\partial f}{\partial u}\right)u'^2$$
$$+ \left(2\lambda^2\,\frac{\partial^2 f}{\partial u\,\partial v} + \lambda\,\frac{\partial \lambda}{\partial v}\,\frac{\partial f}{\partial u} + \lambda\,\frac{\partial \lambda}{\partial u}\,\frac{\partial f}{\partial u}\right)u'\,v' + \left(\lambda^2\,\frac{\partial^2 f}{\partial v^2} + \lambda\,\frac{\partial \lambda}{\partial v}\,\frac{\partial f}{\partial v}\right)v'^2,$$

$$Y = \lambda^2\,\frac{\partial \varphi}{\partial u}\,\frac{\partial u'}{\partial t_1} + \lambda^2\,\frac{\partial \varphi}{\partial v}\,\frac{\partial v'}{\partial t_1} + \left(\lambda^2\,\frac{\partial^2 \varphi}{\partial u^2} + \lambda\,\frac{\partial \lambda}{\partial u}\,\frac{\partial \varphi}{\partial u}\right)u'^2$$
$$+ \left(2\lambda^2\,\frac{\partial^2 \varphi}{\partial u\,\partial v} + \lambda\,\frac{\partial \lambda}{\partial v}\,\frac{\partial \varphi}{\partial u} + \lambda\,\frac{\partial \lambda}{\partial u}\,\frac{\partial \varphi}{\partial u}\right)u'\,v' + \left(\lambda^2\,\frac{\partial^2 \varphi}{\partial v^2} + \lambda\,\frac{\partial \lambda}{\partial v}\,\frac{\partial \varphi}{\partial v}\right)v'^2.$$

If one substitutes the values of $\dfrac{du'}{dt_1}$, $\dfrac{dv'}{dt_1}$ that are provided by those equations in (4) then one will obtain the values of P, Q as functions of u, v, u', v'. Upon equating the coefficients of u'^2,

$u'v'$, v'^2 in *P* and *Q* to zero, one will have six equations for determining *f*, *φ*, and *λ*. Upon setting $\Delta = \frac{\partial f}{\partial u}\frac{\partial \varphi}{\partial v} - \frac{\partial \varphi}{\partial u}\frac{\partial f}{\partial v}$, to abbreviate the notations, one will then find that:

$$
(5) \qquad \left\{
\begin{aligned}
&\frac{1}{\Delta}\left(\frac{\partial^2 f}{\partial u^2}\frac{\partial \varphi}{\partial v} - \frac{\partial^2 \varphi}{\partial u^2}\frac{\partial f}{\partial v}\right) + \frac{1}{\lambda}\frac{\partial \lambda}{\partial u} = 0, \\
&\frac{1}{\Delta}\left(\frac{\partial^2 f}{\partial u \,\partial v}\frac{\partial \varphi}{\partial v} - \frac{\partial^2 \varphi}{\partial u\,\partial v}\frac{\partial f}{\partial v}\right) + \frac{1}{\lambda}\frac{\partial \lambda}{\partial v} = 0, \\
&\frac{1}{\Delta}\left(\frac{\partial^2 f}{\partial v^2}\frac{\partial \varphi}{\partial v} - \frac{\partial^2 \varphi}{\partial v^2}\frac{\partial f}{\partial v}\right) + C\,\frac{\partial C}{\partial v} = 0, \\
&\frac{\partial f}{\partial u}\frac{\partial^2 \varphi}{\partial u^2} - \frac{\partial \varphi}{\partial u}\frac{\partial^2 f}{\partial u^2} = 0, \\
&\frac{1}{\Delta}\left(\frac{\partial f}{\partial u}\frac{\partial^2 \varphi}{\partial u\,\partial v} - \frac{\partial \varphi}{\partial u}\frac{\partial^2 f}{\partial u\,\partial v}\right) + \frac{1}{\lambda}\frac{\partial \lambda}{\partial u} - \frac{2}{C}\frac{\partial C}{\partial u} = 0, \\
&\frac{1}{\Delta}\left(\frac{\partial f}{\partial u}\frac{\partial^2 \varphi}{\partial v^2} - \frac{\partial \varphi}{\partial u}\frac{\partial^2 f}{\partial v^2}\right) + \frac{1}{\lambda}\frac{\partial \lambda}{\partial v} - \frac{1}{C}\frac{\partial C}{\partial v} = 0.
\end{aligned}
\right.
$$

The fourth equation can be integrated immediately and gives:

$$f = V\,\varphi + V_1 ,$$

in which *V* and V_1 denote two functions of *v*.

Imagine that the family of geodesics that is involved with the curvilinear coordinate system to which one refers the surface is composed of geodesics that pass through the point of the surface that corresponds to the origin of the coordinates in the plane. Each value of *v* will then correspond to a value of *u* for which one will have $f = \varphi = 0$ identically. It will then result that V_1 is identically zero, and that one will have $f = V\,\varphi$. Having said that, by means of some simple reductions, the system (5) will become:

$$
(6) \qquad \begin{cases}
f = V\varphi, \\[4pt]
\dfrac{2\left(\dfrac{\partial\varphi}{\partial v}\right)^2}{\varphi\dfrac{\partial\varphi}{\partial u}} + \dfrac{V''\dfrac{\partial\varphi}{\partial v}}{V'\dfrac{\partial\varphi}{\partial u}} - \dfrac{\dfrac{\partial^2\varphi}{\partial v^2}}{\dfrac{\partial\varphi}{\partial u}} = C\,\dfrac{\partial C}{\partial u}, \\[4pt]
\dfrac{\partial}{\partial u}\log\left(\lambda\dfrac{\partial\varphi}{\partial u}\right) = 0, \\[4pt]
\dfrac{\partial}{\partial v}\log\left[\dfrac{\lambda\left(\dfrac{\partial\varphi}{\partial u}\right)^2}{\varphi^2}\right] = 0, \\[4pt]
\dfrac{\partial}{\partial u}\log\left(\dfrac{\lambda\varphi^2}{C^2}\right) = 0, \\[4pt]
\dfrac{\partial}{\partial v}\log\left(\dfrac{V'\lambda\varphi^2}{C}\right) = 0,
\end{cases}
$$

in which V' and V'' are the first and second derivatives of the function V, resp.

One integrates the last four equations immediately, and one will find that:

$$
\lambda\frac{\partial\varphi}{\partial u} = V_1, \qquad \frac{\lambda}{\varphi^2}\left(\frac{\partial\varphi}{\partial u}\right)^2 = U_1, \qquad \frac{\lambda\varphi^2}{C^2} = V_2, \qquad \frac{V'\lambda\varphi^2}{C} = U_2,
$$

in which V_1, V_2 denote functions of v, and U_1, U_2 denote functions of u. Those equations are written:

$$
\lambda\frac{\partial\varphi}{\partial u} = V_1, \qquad \frac{1}{\varphi^2}\frac{\partial\varphi}{\partial u} = \frac{U_1}{V_1}, \qquad C^2 = \frac{V_1^2}{U_1 V_1}, \qquad V'C = \frac{U_2}{V_2}.
$$

The last two show that C must be the product of a function of u with a function of v. Therefore, set:

$$
C = \alpha\,\beta,
$$

in which α denotes a given function of u, and β denotes a given function of v. One will have:

$$
\frac{\alpha}{U_2} = \frac{1}{\beta V' V_2} = \frac{B}{A}, \qquad \alpha^2 U_1 = \frac{V_1^2}{\beta^2 V_2} = A\,B,
$$

in which A, B denote two constants. From that, one has:

$$V_1 = A\sqrt{\frac{\beta}{V'}}, \qquad U_1 = \frac{AB}{\alpha^2},$$

and finally:

$$\lambda\frac{\partial\varphi}{\partial u} = A\sqrt{\frac{\beta}{V'}},$$

$$\frac{1}{\varphi^2}\frac{\partial\varphi}{\partial u} = B\frac{1}{\alpha^2}\sqrt{\frac{V'}{\beta}}.$$

Upon integrating the last equation, one will get:

(7) $$\varphi = \frac{R}{U+S},$$

in which one has set:

$$R = \frac{1}{B}\sqrt{\frac{\beta}{V'}},$$

$$U = -\int\frac{du}{\alpha^2},$$

and in which S denotes a function of v.

One sees from this procedure that the system (6) can be replaced with the following one:

(8) $$\begin{cases} f = V\varphi, \\ \lambda\dfrac{\partial\varphi}{\partial u} = A\sqrt{\dfrac{\beta}{V'}}, \\ \varphi = \dfrac{R}{U+S}, \\ \dfrac{2\left(\dfrac{\partial\varphi}{\partial v}\right)^2}{\varphi\dfrac{\partial\varphi}{\partial u}} + \dfrac{V''\dfrac{\partial\varphi}{\partial v}}{V'\dfrac{\partial\varphi}{\partial u}} - \dfrac{\dfrac{\partial^2\varphi}{\partial v^2}}{\dfrac{\partial\varphi}{\partial u}} = \alpha\,\alpha'\,\beta^2, \end{cases}$$

in which $\alpha' = d\alpha / du$, and everything reduces to determining the functions V and S by means of the last equation in (8). If one replaces φ with its value then one will have:

$$(U+S)(R\,R''V' - 2R'^2\,V' - R\,R'V'') + 2R\,R'\,S'V' + R^2\,S'V'' - R^2\,S''V' = R^2\,V'\,\beta^2\,U'\,\alpha\,\alpha'.$$

If one differentiates that with respect to u then one will get:

$$\frac{R\,R''V' - 2R'^2\,V' - R\,R'V''}{R^2\,V'\,\beta^2} = \frac{U''\,\alpha\,\alpha' + U'\,\alpha'^2 + U'\alpha\,\alpha'}{U'},$$

or rather:

$$\frac{R R'' V' - 2R'^2 V' - R R' V''}{R^2 V' \beta^2} = \alpha \alpha'' - \alpha'^2 = D ,$$

in which D is a constant.

The equations will become:

$$\frac{D S R V' \beta^2 + 2R' S' V' + R S' V'' - R S'' V'}{R^2 V' \beta^2} = U' \alpha \alpha' - D U = D_1 ,$$

in which D_1 is a new constant.

As a result, if one sets:

$$\text{(9)} \qquad \begin{cases} D = \alpha \alpha'' - \alpha'^2, \\ D_1 = U' \alpha \alpha' - D U \end{cases}$$

then one will have the following equations for determining V and then S:

$$\text{(10)} \qquad R R'' V' - 2R'^2 V' - R R' V'' - D R^2 V' \beta^2 = 0 ,$$

$$\text{(11)} \qquad D S R V' \beta^2 + 2R' S' V' + R S' V'' - R S'' V' - D_1 R V' \beta^2 = 0 .$$

If one differentiates the first equation (9) then one will have:

$$\alpha \alpha''' - \alpha' \alpha'' = 0 ,$$

$$\frac{d}{du} \log \frac{\alpha''}{\alpha} = 0 ,$$

$$\frac{\alpha''}{\alpha} = \text{const.},$$

or rather, as one easily sees:

$$-\frac{1}{C} \frac{\partial^2 C}{\partial u^2} = \text{const.}$$

That constant quantity expresses the total curvature of the surface, from a formula that is due to Gauss ([1]). Thus, the desired transformation can be performed only if the given surface has constant curvature.

We assume that hypothesis is true. The formulas that Darboux ([2]) gave will then allow us to complete the calculations.

If one supposes that the curvature is zero then one will have:

([1]) G. DARBOUX, *Leçons sur la théorie générale des surfaces*, t. II, pp. 416.

([2]) *Ibid.*, pp. 46.

$$C^2 = u^2 .$$

That will give:

$$\alpha = u ,$$
$$\beta = 1 .$$

One will then find that:

$$U = \frac{1}{u} ,$$
$$D = -1 ,$$
$$D_1 = 0 .$$

Equations (10) and (11) will become:

$$3V''^2 - 2V'V'' + 4V'^2 = 0 ,$$
$$S'' + S = 0 .$$

One then deduces that:

$$V = E \tan (v + F) + G ,$$
$$S = H \sin v + K \cos v ,$$

and then:

$$(12) \quad \begin{cases} \varphi = \dfrac{m'(u \sin v) + n'(u \cos v) + p'}{m(u \sin v) + n(u \cos v) + p}, \\ f = \dfrac{m''(u \sin v) + n''(u \cos v) + p''}{m(u \sin v) + n(u \cos v) + p}, \\ \lambda = q[m(u \sin v) + n(u \cos v) + p]^2, \end{cases}$$

in which m, n, p, … are constants.

If one supposes that curvature is positive and equal to $1/a^2$ then one will have:

$$C^2 = a^2 \sin^2 \frac{u}{a} ,$$
$$a = a \sin \frac{u}{a} ,$$
$$\beta = 1 .$$

From that:

$$U = \frac{1}{a \tan \dfrac{u}{a}} ,$$
$$D = -1 ,$$
$$D_1 = 0 .$$

One finds that one has the same equations for determining V, S as in the previous case, and one finally gets:

(13)
$$\begin{cases} \varphi = \dfrac{m'\left(\sin v \tan \dfrac{u}{a}\right) + n'\left(\cos v \tan \dfrac{u}{a}\right) + p'}{m\left(\sin v \tan \dfrac{u}{a}\right) + n\left(\cos v \tan \dfrac{u}{a}\right) + p}, \\ f = \dfrac{m''\left(\sin v \tan \dfrac{u}{a}\right) + n''\left(\cos v \tan \dfrac{u}{a}\right) + p''}{m\left(\sin v \tan \dfrac{u}{a}\right) + n\left(\cos v \tan \dfrac{u}{a}\right) + p}. \end{cases}$$

Finally, if the curvature equals $-1/a^2$ then one will have:

$$U = \frac{2e^{-u/a}}{a\left(e^{u/a} - e^{-u/a}\right)},$$
$$D = -1,$$
$$D_1 = -\frac{1}{a}.$$

The equation for V is the same as in the previous case, and the equation for S is:

$$S'' + S - \frac{1}{a} = 0.$$

It gives:

$$S = H \sin v + K \cos v + \frac{1}{a},$$

and one finds that:

(14)
$$\begin{cases} \varphi = \dfrac{m'\left(\dfrac{e^{u/a} - e^{-u/a}}{e^{u/a} + e^{-u/a}} \sin v\right) + n'\left(\dfrac{e^{u/a} - e^{-u/a}}{e^{u/a} + e^{-u/a}} \cos v\right) + p'}{m\left(\dfrac{e^{u/a} - e^{-u/a}}{e^{u/a} + e^{-u/a}} \sin v\right) + n\left(\dfrac{e^{u/a} - e^{-u/a}}{e^{u/a} + e^{-u/a}} \cos v\right) + p}, \\ f = \dfrac{m''\left(\dfrac{e^{u/a} - e^{-u/a}}{e^{u/a} + e^{-u/a}} \sin v\right) + n''\left(\dfrac{e^{u/a} - e^{-u/a}}{e^{u/a} + e^{-u/a}} \cos v\right) + p''}{m\left(\dfrac{e^{u/a} - e^{-u/a}}{e^{u/a} + e^{-u/a}} \sin v\right) + n\left(\dfrac{e^{u/a} - e^{-u/a}}{e^{u/a} + e^{-u/a}} \cos v\right) + p}. \end{cases}$$

Those are the transformations that we were proposing to obtain.

In the chapter that was cited before, Darboux gave the following equations for geodesic lines:

$$A\,u \cos v + B\,u \sin v + C = 0,$$

$$A \tan \frac{u}{a} \cos v + B \tan \frac{u}{a} \sin v + C = 0 ,$$

$$A \frac{e^{u/a} - e^{-u/a}}{e^{u/a} + e^{-u/a}} \cos v + B \frac{e^{u/a} - e^{-u/a}}{e^{u/a} + e^{-u/a}} \sin v + C = 0 .$$

In the last two, we wrote u / a instead of u in order to make the notations consistent. The eminent geometer added:

> "If one represents the surface on the plane by taking the rectangular coordinates x and y of the point in the plane for the coefficients of A and B in the previous equations then the geodesic lines on the surface will correspond to the lines in the plane… When one has performed *one* representation of the surface in question on the plane, one will get *all of them* by following that representation, no matter what it might be, with the most general homographic transformation in the plane."

Once one has acquired that, it will be sufficient to consider formulas (12), (13), and (14) in order to confirm that the transformations that solve the problem that Appell posed are the ones that transform the lines in the plane into geodesic lines on the surfaces ([1]).

([1]) The results that are contained in this article were the topic of a communication that we had the honor of presenting to the Academy of Sciences (session on 8 December 1890).