"Généralisation du théoreme de Poisson," C. R. Acad. Sci. Paris **148** (1909), 610-612.

# Generalization of Poisson's theorem

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Presented by H. Poincaré

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Consider a *generalized canonical system:*

$$\frac{dx_i}{\sum_k \frac{\nu_{ki}}{\nu}\frac{\partial H}{\partial x_k}} = dt \qquad (i, k = 1, \ldots, 2m), \tag{1}$$

in which $\nu_{ki}$ is the minor of $N_{ki} \equiv \frac{\partial N_k}{\partial x_i} - \frac{\partial N_i}{\partial x_k}$ in the skew-symmetric determinant $\nu$ that is formed from the $N_{ik}$. That determinant will be supposed to be non-zero. The $N_k$ and $H$ are functions of $2m$ variables $x_1, \ldots, x_{2m}$ .

The system (1) enjoys the following remarkable properties:

I. It admits the *relative integral invariant:*

$$J_1 \equiv \int \sum_{i=1}^{2m} N_i\, \delta x_i \ .$$

By virtue of (1), one will have:

$$\frac{d}{dt}\sum_{i=1}^{2m} N_i\, \delta x_i = \delta\left( H + \sum_{i=1}^{2m}\sum_{k=1}^{2m} \frac{\nu_{ki}}{\nu}\frac{\partial H}{\partial x_k} N_i \right) .$$

The expression $\sum_{i=1}^{2m} N_i\, \delta x_i$ is a Pfaff form of *class* $2^m$.

II. The square root of $\nu$ is a *Jacobi multiplier* of (1).

III. Knowing that a relative integral invariant of class two belongs to a *system of* $2m$ *equations* will permit one to write it in the generalized canonical form.

IV. *The characteristic function H* is an invariant of the system (1).

V. An *arbitrary* change of variables $x_1, \ldots, x_{2m}$ *preserves* the generalized canonical form of equations (1). If $x'_1, \ldots, x'_{2m}$ are new variables then one will have:

$$\frac{dx'_i}{\sum_k \frac{\nu'_{ki}}{\nu'} \frac{\partial H'}{\partial x'_k}} = dt \qquad (i, k = 1, \ldots, 2m),$$

in which:

$$\frac{\nu'_{ki}}{\nu'} = \sum_{\alpha=1}^{2m} \sum_{\beta=1}^{2m} \frac{\partial x'_k}{\partial x_\alpha} \frac{\partial x'_i}{\partial x_\beta} \frac{\nu_{\alpha\beta}}{\nu},$$
$$H' = H.$$

VI. Poincaré showed ([1]) that any *relative* integral invariant of order $p$ can be considered to be the sum of an *absolute* invariant of order $p$ and an *exact* differential form of order $p$. Here, $p = 1$. One will then find the *generalized Jacobi function:*

$$V = V_0 + \int_{t_0}^{t} \left( H + \sum_{i=1}^{2m} \sum_{k=1}^{2m} \frac{\nu_{ki}}{\nu} \frac{\partial H}{\partial x_k} N_i \right) dt.$$

One cannot deduce *the general form* of the Jacobi equation from it.

VII. By generalizing the process that Poincaré indicated, one can deduce an *absolute* invariant of order two $I_2$, a *relative* invariant of order three $J_3$, etc., from $J_1$. They are particularly interesting in context of the Pfaff problem when $J_1$ is an *absolute* invariant.

VIII. *Generalization of Poisson's theorem. – If* $\rho_1$ *and* $\rho_2$ *are two invariants of the system* (1) *then:*

$$\{\rho_1, \rho_2\} = \sum_{i=1}^{2m} \sum_{k=1}^{2m} \frac{\nu_{ki}}{\nu} \frac{\partial \rho_1}{\partial x_i} \frac{\partial \rho_2}{\partial x_k} \tag{2}$$

*will also be an invariant of* (1).

If one replaces the parentheses $(\rho_1, \rho_2)$ in the left-hand side of the Poisson *identity* with the expressions $\{\rho_1, \rho_2\}$ then one will no longer get a result that is identically zero. On the contrary, Poincaré's new proof ([2]) extends perfectly to the generalized Poisson theorem.

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([1]) **Poincaré**, *Les méthodes Nouvelles de la Mécanique céleste*, t. III, pp. 14.
([2]) *Loc. cit.*, t. I, pp. 169.

IX. The expression (2) is identical to the *first mixed differential parameter of the bilinear form:*

$$\sum_i \sum_k N_{ik} \delta_1 x_i \, \delta_2 x_k \quad .$$

The *second differential parameter* $\Delta_2$ ($f$) of that same bilinear form and an arbitrary function $f$ is zero.

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