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Study of the curvature of surfaces

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We propose to study the curvature of surface more closely than when one limits oneself to considering the radii of osculating circles in sections. We shall first present some considerations in regard to planar sections of surfaces.

I.

1. Pass a vertical plane through a point that is taken arbitrarily on a surface. Upon considering the section, we will have:

$$dx = d\eta \cos \alpha, \qquad dy = d\eta \sin \alpha,$$

and as a result:

$$\frac{dz}{d\eta} = \cos \alpha\,(q \tan \alpha + p),$$

in which η is the abscissa, when measured along the horizontal trace of the secant plane, α is the angle between that plane and the axis of the abscissa, and x, y, z, p, q, … have their usual significance.

By successive differentiations, one will obtain:

$$\frac{d^2 z}{d\eta^2} = \cos^2 \alpha\,(t \tan^2 \alpha + 2\,s \tan \alpha + r),$$

$$\frac{d^3 z}{d\eta^3} = \cos^3 \alpha\,(v \tan^3 \alpha + 3w \tan^2 \alpha + 3\,\omega \tan \alpha + r),$$

and more generally:

$$\frac{d^n z}{d\eta^n} = \cos\alpha \left(\frac{d\,\dfrac{d^{n-1}z}{d\eta^{n-1}}}{dy} \tan\alpha + \frac{d\,\dfrac{d^{n-1}z}{d\eta^{n-1}}}{dx} \right).$$

We represent the right-hand sides of these equations by Z_1, Z_2, Z_3, …, Z_n.

2. If a surface is such that the same value of tan α simultaneously annihilates two consecutive derivatives Z_n and Z_{n+1} at each of its points then that value will annihilate all of their derivatives of order one or higher.

In order to prove that theorem, consider the equations:

(1) $$Z_n = 0, \qquad \frac{dZ_n}{dy}\tan\alpha + \frac{dZ_n}{dx} = 0,$$

which must be true at every point on the surface, by hypothesis. One can differentiate them with respect to each of the variables x and y. When one takes the derivatives with respect to x and eliminates d (tan α) / dx from them, one will have:

$$\frac{dZ_n}{dx}\left(\frac{d^2 Z_n}{dy\,d\tan\alpha}\tan\alpha + \frac{d^2 Z_n}{dx\,d\tan\alpha} + 2\frac{dZ_n}{dy}\right) = \frac{dZ_n}{d\tan\alpha}\left(\frac{d^2 Z_n}{dx\,dy}\tan\alpha + \frac{d^2 Z_n}{dx^2}\right).$$

In the same way, when one differentiates with respect to y and eliminates d (tan α) / dy, one will get:

$$\frac{dZ_n}{dy}\left(\frac{d^2 Z_n}{dy\,d\tan\alpha}\tan\alpha + \frac{d^2 Z_n}{dx\,d\tan\alpha} + 2\frac{dZ_n}{dy}\right) = \frac{dZ_n}{d\tan\alpha}\left(\frac{d^2 Z_n}{dy^2}\tan\alpha + \frac{d^2 Z_n}{dx\,dy}\right).$$

Divide these equations by each other. The ratio of the left-hand sides will be – tan α, by virtue of the second equation (1), and one will have:

$$\frac{d^2 Z_n}{dy^2}\tan^2\alpha + 2\frac{d^2 Z_n}{dx\,dy} + \tan\alpha + \frac{d^2 Z_n}{dx^2} = 0, \qquad Z_{n+2} = 0,$$

while Z_{n+3} and all of its derivatives of order one or higher will obviously be zero, like Z_{n+2}.

If a surface is such that a derivative Z_n has a double root for each of its points then that value of tan α will be a root of dZ_n / dx, and as a result, of Z_{n+1}. It will then annihilate two consecutive derivatives, and from what we just saw, all of the following derivatives.

3. If the first two derivatives Z_1 and Z_2 are zero at the same time then all of them will be, and a horizontal line will pass through each point of the surface, and indeed, if one eliminates tan α from those two derivatives when they are equated to zero then one will find the known equation of surfaces that are generated by a line that is always horizontal.

Now suppose that the second and third derivatives are simultaneously zero:

$$t \tan^2 \alpha + 2s \tan \alpha + r = 0, \tag{2}$$

$$v \tan^3 \alpha + 3w \tan^2 \alpha + 3u \tan \alpha + u = 0. \tag{3}$$

Monge showed that the surface will be a ruled surface in that case. A brief discussion of that here seems necessary.

If just one of the two roots of equation (2) satisfies (3) then one can pass a rectilinear section through each point, and the surface will be skew.

If the two roots of equation (2) are equal then that value of tan α will satisfy equation (3), and the surface will be ruled, and even developable, because it will be easy to recognize that the tangent plane will not change when one displaces the contact point without varying α.

In the case where the two roots of equation (2) both satisfy equation (3), if those roots are real then the surface will be doubly ruled. If they become imaginary then the surface will no longer admit rectilinear sections, but it will always have degree two, because the change can only come from a modification in the relative magnitudes of the numerical coefficients in its equation. If the common roots are equal then the surface, which always has degree two, will become developable.

From that, upon expressing the idea that equation (3) is exactly divisible by equation (2), one will get a partial differential equation that represents second-order surfaces:

$$(u\,t^2 - 3wrt + 2vrs)^2 + (vr^2 - 3u\,t\,r + 2uts)^2 = 0.$$

We can deduce some other consequences from the theorem in article **2**, but the ones that we just pointed out will suffice for this study, in which we shall not consider the derivatives above order three.

II.

4. Let α, β, and γ be the angles that a tangent to the surface at a point under consideration forms with the axes, resp., and let R be the radius of curvature of the normal section that contains that tangent. From a known formula, one will have:

$$R^2 = \frac{p^2 + q^2 + 1}{Z_2^2}, \tag{4}$$

when one sets:

$$Z_2 = t \cos^2 \beta + 2s \cos \beta \cos \alpha + r \cos^2 \alpha .$$

Differentiating this will give:

(5) $$R\,dR = -(p^2+q^2+1)\,\frac{dZ_2}{Z_2^3} + \frac{(ps+qt)\cos\beta + (pr+qs)\cos\alpha}{Z_2^2}\,dS\,,$$

when one sets:

$$Z_3 = v\cos^3\beta + 3w\cos^2\beta\cos\alpha + 3u\cos\beta\cos^2\alpha + u\cos^3\alpha\,,$$

and dS is the differential of arc-length, which is equal to $dx / \cos\alpha$ and $dy / \cos\beta$.

The polynomials that are represented by Z_2 and Z_3 are not the same as the ones in the first part of this study, but when one divides them by $\cos^2\alpha$ and $\cos^3\alpha$, one will find that they are composed from $\cos\beta / \cos\alpha$ in the same way that the former are composed in terms of $\tan\alpha$.

If one differentiates the value of Z_2 then one will find that:

(6) $$dZ_2 = Z_3\,dS + 2\,(s\cos\beta + r\cos\alpha)\,d\cos\alpha + 2\,(t\cos\beta + s\cos\alpha)\,d\cos\beta\,;$$

one must then determine $d\cos\alpha$ and $d\cos\beta$.

The secant plane contains the normal to the surface and the tangent that is determined by the angles α, β, and γ; its equation is:

$$(\cos\beta + q\cos\gamma)\,(x' - x) - (\cos\alpha + p\cos\gamma)\,(y' - y) + (p\cos\beta - q\cos\alpha)\,(z' - z) = 0,$$

in which x', y', z' are variable coordinates.

The differentials $d\cos\alpha$, $d\cos\beta$, $d\cos\gamma$ must be such that the new tangent is in the secant plane. That condition will give:

$$(\cos\beta + q\cos\gamma)\,d\cos\alpha - (\cos\alpha + p\cos\gamma)\,d\cos\beta + (p\cos\beta - q\cos\alpha)\,d\cos\gamma = 0.$$

Since the axes are rectangular and the angles α, β, and γ belong to a tangent, one will have:

(7) $$\begin{cases} \cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1, \\ p\cos\alpha + q\cos\beta - \cos\gamma = 0, \end{cases}$$

from which, one deduces:

$$\cos\alpha\,d\cos\alpha + \cos\beta\,d\cos\beta + \cos\gamma\,d\cos\gamma = 1,$$

$$p\,d\cos\alpha + q\,d\cos\beta - d\cos\gamma + Z_2\,dS = 0.$$

We now have three equations for the determination of $d\cos\alpha$, $d\cos\beta$, $d\cos\gamma$, from which we deduce:

$$d\cos\alpha = -\frac{p}{p^2+q^2+1}\,Z_2\,dS, \qquad d\cos\beta = -\frac{q}{p^2+q^2+1}\,Z_2\,dS\,;$$

when one substitutes those values in equation (6), one will have:

$$dZ_2 = Z_3\, dS - 2\frac{(ps+qt)\cos\beta + (pr+qs)\cos\alpha}{p^2+q^2+1}\, Z_2,$$

and finally, from equations (5) and (4):

$$R\frac{dR}{dS} = \frac{-(p^2+q^2+1)Z_3 + 3Z_2[(ps+qt)\cos\beta + (pr+qs)\cos\alpha]}{Z_2^3}, \tag{8}$$

$$\frac{dR}{dS} = \frac{(p^2+q^2+1)Z_3 - 3Z_2[(ps+qt)\cos\beta + (pr+qs)\cos\alpha]}{Z_2^2\sqrt{p^2+q^2+1}}. \tag{9}$$

It should be remarked that $R\dfrac{dR}{dS}$ is equal to the radius of curvature of the development of the section considered, because if ε is the contingency angle then the radii of curvature of the section and its developments will be dS/ε and dR/ε, resp.

One can construct a parabola that is super-osculated by a curve at a given point when one knows the radius of curvature R and its derivative dR/dS. If $x^2 - 2\,px = 0$ is the equation of the parabola then one will have:

$$p = \frac{R}{\left[1+\left(\frac{1}{3}\frac{dR}{dS}\right)^2\right]^{3/2}}, \qquad x = \frac{R\left(\frac{1}{3}\frac{dR}{dS}\right)}{\left[1+\left(\frac{1}{3}\frac{dR}{dS}\right)^2\right]^{3/2}}.$$

The first equation gives one the parameter of the parabola, while the second one gives the abscissa of the point where super-osculation can be established.

5. – We propose to find the normal sections of a surface that can be super-osculated by a circle. In order to determine them, we must equate the value of dR/dS to zero:

$$\left\{\begin{aligned} &(p^2+q^2+1)\left[\frac{v}{3}\left(\frac{\cos\beta}{\cos\alpha}\right) + w\left(\frac{\cos\beta}{\cos\alpha}\right)^2 + u_{,}\left(\frac{\cos\beta}{\cos\alpha}\right) + \frac{u}{3}\right] \\ &-\left[(ps+qt)\frac{\cos\beta}{\cos\alpha} + (pr+qs)\right]\left[t\left(\frac{\cos\beta}{\cos\alpha}\right)^2 + 2s\frac{\cos\beta}{\cos\alpha} + r\right] = 0. \end{aligned}\right. \tag{10}$$

That equation, which has degree three, will always give at least one real value for $\dfrac{\cos\beta}{\cos\alpha}$, and as a result, a surface that has a normal section that is super-osculated by a circle at any point. It can sometimes have three of them, two of which coincide. Finally, there will be an infinitude of them at some exceptional points. If we replace $\frac{\cos\beta}{\cos\alpha}$ with $\frac{dy}{dx}$ in equation (10) then we will have the differential equation of the curves that are tangent to the normal sections that are super-osculated by a circle at each of their points:

$$
(11)\qquad \left\{
\begin{aligned}
&(p^2+q^2+1)\left[\frac{v}{3}\left(\frac{dy}{dx}\right)+w\left(\frac{dy}{dx}\right)^2+u_1\frac{dy}{dx}+\frac{u}{3}\right]\\
&-\left[(ps+qt)\frac{dy}{dx}+(pr+qs)\right]\left[t\left(\frac{dy}{dx}\right)^2+2s\frac{dy}{dx}+r\right]=0.
\end{aligned}
\right.
$$

Z_2 will be a common factor in equation (10) for the second-degree surface. It represents the rectilinear generators that are, indeed, super-osculated by a circle of infinite radius. The equation will have degree one when one makes that factor vanish. One will then see that there exists just one normal section that is super-osculated by a circle of finite radius at any point of a second-degree surface.

For the general ruled surface, Z_3 and Z_2 will have a common factor of degree one that represented the rectilinear generators. Upon making it vanish, equation (10) will have degree two; sometimes it will have two solutions, but other times it will not.

There exists an infinitude of second-order surfaces that are super-osculated at one of their summits by an arbitrary surface at a given point. Those second-order surfaces traverse the surface in question, and the curves of intersection are obviously tangent to the normal sections that are super-osculated by a circle. If the osculated surface has order two then it will have three lines in common with the osculating surface, namely, two lines and a curve tangent to the normal section that is super-osculated by a circle of finite radius.

6. – We can make the factor $\left[t\left(\frac{dy}{dx}\right)^2+2s\frac{dy}{dx}+r\right]$, which represents the generators, vanish from equation (10) for the second-degree surface, and we will have:

$$
(12)\qquad \left\{
\begin{aligned}
&\left[\frac{v}{3t}(p^2+q^2+1)-(ps+qt)\right]\frac{dy}{dx}\\
&+\left[\frac{u}{3r}(p^2+q^2+1)-(pr+qs)\right]=0.
\end{aligned}
\right.
$$

When we apply that equation to the surface with its center at:

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1,$$

we will find, after various simplifications that present themselves spontaneously, that:

$$\left(1-\frac{c^2}{a^2}\right)\frac{x\,dx}{a^2}+\left(1-\frac{c^2}{a^2}\right)\frac{y\,dy}{b^2} = 0;$$

hence:

$$\left(1-\frac{c^2}{a^2}\right)\frac{x^2}{a^2}+\left(1-\frac{c^2}{a^2}\right)\frac{y^2}{b^2} = C. \tag{13}$$

Hence, the curves that are always tangent to the normal sections that are super-osculated by a circle (as the lines of curvature are to the principal sections) will project onto any of the principal planes of the surface along second-degree curves that are mutually similar. Those projections are hyperbolas in the plane perpendicular to the axis that is parallel to the circular sections, and ellipses on the other two principal planes.

An arbitrary normal section to the surface is super-osculated by a circle at the summits of the surface, but it will not be contacted by one of the lines that is represented by equation (13). Those curves envelop the summits without passing through them. Only two of them will cross at the summits that are on the axis that is parallel to the circular sections. Their projections onto the plane perpendicular to that axis will be composed of two lines that one will obtain when one makes the constant in equation (13) equal to zero. Consequently, they will be planar.

If the surface is one of revolution then the curves will be parallel. If it is a sphere then equation (13) will vanish, and any curve will be a solution to the problem.

Now consider the second-degree surface that has no center:

$$z = \frac{x^2}{a^2}+\frac{y^2}{b^2}.$$

If one substitutes the partial derivatives in equation (12) then one will find, after integration, that:

$$\frac{x^2}{a^2}+\frac{y^2}{b^2} = C$$

and

$$z = \left(1-\frac{b}{a}\right)\frac{x^2}{a}+b\,C.$$

One sees that the projections of the curves are parabolas on the principal plane of the surface and ellipses on the tangent plane to the summit.

7. – As an example, we take the skew surface of the square-threaded screw (*la vis à filets carrés*). It will have the equation:

$$z = h\,\omega,$$

in which h is a constant and ω is the azimuth of the horizontal projection of a point. If one calls the distance from that point to the vertical axis ρ then one will have:

$$\frac{d\omega}{dx} = -\frac{\sin\omega}{\rho}, \qquad \frac{d\omega}{dy} = \frac{\cos\omega}{\rho}, \qquad \frac{d\rho}{dx} = \cos\omega, \qquad \frac{d\rho}{dy} = \sin\omega.$$

When one calculates the partial derivatives of the function z from those formulas and substitutes in equation (9), one will get:

$$\left.\begin{array}{l} \dfrac{h^2+\rho^2}{h^2}\left[\dfrac{1}{3}\cos 3\omega\left(\dfrac{\cos\beta}{\cos\alpha}\right)^3 - \sin 3\omega\left(\dfrac{\cos\beta}{\cos\alpha}\right)^2 - \cos 3\omega\dfrac{\cos\beta}{\cos\alpha} + \dfrac{1}{3}\sin 3\omega\right] \\ -\left(\sin\omega\dfrac{\cos\beta}{\cos\alpha} - \cos\omega\right)\left[\dfrac{1}{2}\sin 2\omega\left(\dfrac{\cos\beta}{\cos\alpha}\right)^2 + \cos 2\omega\dfrac{\cos\beta}{\cos\alpha} - \dfrac{1}{2}\sin 2\omega\right] \end{array}\right\} = 0.$$

For the surface of the square-threaded screw, the curvatures are identical along the different generators. One can then confine oneself to studying what happens for one of them, such as the one whose azimuth is zero. If one sets $\omega = 0$ then one will find that:

$$\text{(14)} \qquad \frac{\cos\beta}{\cos\alpha}\left[\left(\frac{\cos\beta}{\cos\alpha}\right)^2 - \frac{3\rho^2}{h^2+\rho^2}\right] = 0.$$

If one lets μ denote the angle that the axis of the abscissa makes with the projection of the tangent to the normal section that is super-osculated by a circle then one will have:

$$\tan\mu\left(\tan^2\mu - \frac{3\rho^2}{h^2+\rho^2}\right) = 0.$$

We find three values for $\tan\mu$: The value zero indicates the rectilinear generator. We can make it vanish from the beginning of the calculations by exhibiting and suppressing the factor $\left(\cos\omega\dfrac{\cos\beta}{\cos\alpha} - \sin\omega\right)$. The other values of μ are always real, and as a result, two normals sections at any point of the surface will have third-order contact with their osculating circles. If we replace $\tan\mu$ with the analytical expression for the tangent of the angle that a curve makes with its radius vector then we will have the differential equation of the tangent lines to the normal sections that are super-osculated by a circle:

$$\rho\frac{d\omega}{d\rho} = \sqrt{3}\frac{\rho}{\sqrt{h^2+\rho^2}}.$$

Integrating this will give:

$$\rho = \frac{h}{2} e^{(\omega-\omega')/\sqrt{3}} \left(\frac{\rho'}{h} + \sqrt{\frac{\rho'^2}{h^2}+1} \right) - \frac{h}{2} e^{-(\omega-\omega')/\sqrt{3}} \left(\frac{\rho'}{h} + \sqrt{\frac{\rho'^2}{h^2}+1} \right)^{-1} ;$$

ρ' and ω' are the coordinates of the point through which one passes the curve. Upon giving two signs to the radical, in turn, one will find two curves that pass through each point.

The cosines of the angles α, β, and γ are always coupled by equations (7). The second one simplifies because p is zero here. We can calculate cos α with those two equations and equation (14). We will find that its value is $\pm\frac{1}{2}$, so it will result that the lines that we study will meet the rectilinear generators and cut them at angles of sixty degrees.

8. – Equation (11) will simplify even more for surfaces of revolution. Indeed, its two terms will then contain a common factor that represents the parallels. One can exhibit it by taking the partial differential equations for the first three orders of surfaces of revolutions, and eliminates five derivatives with them; for example, p, q, w, $ш$, and v.

We shall go about doing that in a manner that will permit us to discuss the results more simply.

Let:

$$z = f(\rho)$$

be an equation that represents the surface of revolution or its meridian, along which one considers ρ to be a radius vector or an abscissa.

We calculate the partial derivatives using the same device as in article **7**, and after setting ω equal to zero, we substitute their values in equation (11); we will then have:

$$\tan^2 \mu = \rho^2 \frac{\dfrac{dz}{d\rho}\left(\dfrac{d^2z}{d\rho^2}\right)^2 - \dfrac{1}{3}\left[\left(\dfrac{dz}{d\rho}\right)^2+1\right]\dfrac{d^3z}{d\rho^3}}{\rho\dfrac{d^2z}{d\rho^2} - \dfrac{dz}{d\rho}\left[\left(\dfrac{dz}{d\rho}\right)^2+1\right]}, \tag{15}$$

in which μ represents the angle that the axis of the abscissa makes with the projection of the tangent to the normal section that is super-osculated by a circle, as in article **7**,and tan μ replaces $\dfrac{\cos\beta}{\cos\alpha}$.

Equation (15) has degree only two. The coefficient of the third-degree term is zero, which will give a value of ninety degrees for μ, which will indicate the parallels.

If one would like to have the equation of the curves then one would have to replace tan μ in equation (15) with $\rho\dfrac{d\omega}{d\rho}$ and integrate.

If we call the radius of curvature of the meridian g then we will have:

$$g = -\frac{\left[1+\left(\frac{dz}{d\rho}\right)^2\right]^{3/2}}{\frac{d^2z}{d\rho^2}},$$

and upon differentiating:

$$\frac{dg}{d\rho} = -\,3\frac{\left[1+\left(\frac{dz}{d\rho}\right)^2\right]^{1/2}}{\left(\frac{d^2z}{d\rho^2}\right)^2}\left\{\frac{dz}{d\rho}\left(\frac{d^2z}{d\rho^2}\right)^2 - \frac{1}{3}\left[1+\left(\frac{dz}{d\rho}\right)^2\right]\frac{d^2z}{d\rho^2}\right\}.$$

With those values, equation (14) will become:

(16)
$$\tan^2\mu = -\frac{1}{3}\rho^2\frac{\frac{d^2z}{d\rho^2}}{\rho\left[1+\left(\frac{dz}{d\rho}\right)^2\right]^{1/2} + g\,\frac{dz}{d\rho}}.$$

$\tan \mu$ is always zero for the torus and the cone. The meridians of those surfaces will indeed solve the problem. For the other surfaces of revolution, one will find two series of lines that cross while always meeting the meridians at equal angles. Those lines will vanish when the value of $\tan^2 \mu$ is negative. One can derive a geometric method for recognizing when those lines actually exist from equation (16).

If one equates the denominator on the right-hand side of equation (15) to zero then upon integrating one will get a circle with its center on the axis. The surface will then be a sphere. The numerator in equation (15) will become zero at the same time as the denominator, and $\tan \mu$ will be arbitrary.

9. – If one studies the curvature of a surface at a point then one can simplify equations (8) and (9) by supposing that the tangent plane is horizontal. One will then have:

(17)
$$R\frac{dR}{dS} = -\frac{v\sin^3\alpha + 3w\sin^2\alpha\cos\alpha + 3u_{\prime}\sin\alpha\cos^2\alpha + u\cos^3\alpha}{(t\sin^2\alpha + 2s\sin\alpha\cos\alpha + r\cos^2\alpha)^3}$$

and

(18)
$$\frac{dR}{dS} = \frac{v\sin^3\alpha + 3w\sin^2\alpha\cos\alpha + 3u_{\prime}\sin\alpha\cos^2\alpha + u}{(t\sin^2\alpha + 2s\sin\alpha\cos\alpha + r\cos^2\alpha)^2}.$$

If one projects the centers of curvature of the developments of the normal sections onto the tangent plane then one will get a curve that is represented by (15), with an azimuth of α and a radius vector of $R\frac{dR}{dS}$. That curve generally has degree six. It will drop down to fifth degree for skew surfaces, and to fourth degree for the ones that have order two. It will have two asymptotes that are parallel to the asymptotes of the indicatrix.

The curve that is situated in the tangent and is such that its radius vectors are proportional to the derivatives dR / dS has degree only four. Its equation in rectilinear coordinates is:

$$(t\,y^2 + 2s\,xy + rx^2)^2 + v\,y^3 + 3wxy^2 + 2u\,x^2y + ux^3 = 0.$$

The degree of the equation will reduce by one for skew surfaces. For second-order surfaces, the equation will become:

$$t\,y^2 + 2s\,xy + rx^2 + \frac{v}{t}y + \frac{u}{r}x = 0\;;$$

for the umbilics, the curve will be a circle.

One sees that the curvatures of the second-order surfaces and the simply-ruled surfaces define special categories, and that a surface can be super-osculated by a second-order surface or a ruled surface only at exceptional points.