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GRAVITATIONAL RADIATION

The Riemann tensor in general relativity

BY

Robert Debever
Free University of Brussels

INTRODUCTORY NOTE

The text of this work constituted the object of some lectures that were given at the Institute Henri Poincaré in January and February of 1964. They were given at the invitation of the science faculty of the University of Paris and on the initiative of Mme. M.-A. Tonnelat, who has all of my gratitude.

I also address my strongest thanks to the people and the institutions that have permitted the realization of this cycle of conferences and have encouraged me to publish them.

In this text one finds, aside from a summary of known concepts, a new technique for the calculation of the Riemannian connection and the quantities that are derived from it. The applications that are discussed justify, we feel, the interest that they may present. That part has been realized in collaboration with Mr. M. Cahen and Miss L. Defrise.

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I. Definitions and properties of the Riemann tensor

1. Covariant derivative. Ricci identities. Bianchi identities. – The context will be that of general relativity: a Riemann space V_4 of hyperbolic normal signature +, -, -, -, and subject to the differentiability conditions of A. Lichnerowicz [1] that in local coordinates:

(1.1) $$ds^2 = g_{ab}\, dx^a\, dx^b \qquad (a, b = 0, 1, 2, 3)$$

and V_4 has the structure of a differentiable manifold of class C_2 and the functions $\partial^2_{ab} g_{ab}$ are piecewise C_2.

The covariant derivative – or Riemannian connection – is classically defined by the operator D [2]:

(1.2) $$\mathrm{D}T^a_b = dT^a_b + (T_b{}^f \Gamma^a_{fc} - T^a_f \Gamma^f_{bc})dx^c ,$$

(1.3) $$\Gamma^a_{bc} = \left\{ \begin{matrix} a \\ bc \end{matrix} \right\} = g^{ad}\,\Gamma_{dbc}\,,$$

with:

(1.4) $$\Gamma_{dbc} = [bc, d] = \tfrac{1}{2}(\partial_b g_{cd} + \partial_c g_{bd} - \partial_d g_{bc})\,.$$

The Riemann tensor then appears in the commutation rule for the covariant derivative. One has:

(1.5) $$A_{i,\,jk} - A_{i,\,kj} = R^1{}_{i,\,jk} A_1\,.$$

Hence:

(1.6) $$R_{hijk} = \partial_j \Gamma_{hik} - \partial_k \Gamma_{hij} + \Gamma_{h1j}\ \Gamma^1_{ik} - \Gamma_{h1k}\ \Gamma^1_{ij}\,,$$

or [2]:

(1.7) $$R_{hijk} = \tfrac{1}{2}(\partial^2_{ij} g_{hk} + \partial^2_{hk} g_{ij} - \partial^2_{ik} g_{hj} - \partial^2_{hj} g_{ik}) + \Gamma_{h1j}\Gamma^1_{ik} - \Gamma_{h1k}\Gamma^1_{ij}\,.$$

It results from the definitions that the Riemann tensor possesses the following symmetry properties:

(1.8) $$R_{hijk} = -R_{ihjk} = -R_{hikj} = R_{jkhi}\,,$$

(1.9) $$R_{hijk} + R_{hjki} + R_{hkij} = 0\,.$$

The tensor R_{hijk} thus has 20 components.

Finally, one may verify the Bianchi identities:

(1.10) $$R_{hijk;l} + R_{hikl;j} + R_{hilj;k} = 0\,.$$

We introduce the following notations:

(1.11) $$R_{ij} = R_{aijb}\, g^{ab}\,, \qquad R = R_{ij} g^{ij}\,.$$

R_{ij} is the *Ricci* tensor, and R is the *scalar curvature*.

2. **Geodesic deviation** [3]. – A curve $x^a = x^a(s)$ is a *geodesic curve* if the unit tangent vector to the curve:

(2.1) $$u^a = dx^a/ds$$

has null covariant derivative, namely:

(2.2) $$\frac{\mathrm{D}u^a}{ds} = \frac{du^a}{ds} + u^c \Gamma^a_{cd} u^d = 0\,.$$

Let:

(2.3) $$x^a = x^a(s, v)$$

be a family of curves; the curves v = const. are geodesics, and one supposes that the curves s = const. are orthogonal trajectories. If:

(2.4) $$\eta^a = (\partial x^a/\partial v)\, dv\,, \qquad u^a = \partial x^a/\partial s$$

then one has:

(2.5) $$\eta^a\, u_a = 0\,.$$

The geodesic deviation is defined by:

(2.6) $$\mathrm{D}^2\eta^a/ds^2\,.$$

For two neighboring geodesics it is a measure of the *relative acceleration*. One has:

(2.7) $$\frac{\mathrm{D}}{ds}\left(\frac{\partial x^a}{\partial v}\right) = \frac{\partial^2 x^a}{\partial s\,\partial v} + \frac{\partial x^c}{\partial v}\Gamma^a_{cd}\frac{\partial x^d}{\partial s} = \frac{\mathrm{D}}{dv}\left(\frac{\partial x^a}{\partial s}\right),$$

or furthermore:

(2.8) $$\mathrm{D}\eta^a/ds = (\mathrm{D}u^a/dv)\, dv\,.$$

Recall that:

(2.9) $$A^i{}_{,jk} = A^i_{kj} - A^1 R^i{}_{1,jk}\,.$$

One thus has:

(2.10) $$\frac{\mathrm{D}^2\eta^a}{ds^2} = \frac{\mathrm{D}}{ds}\left(\frac{\mathrm{D}u^a}{dv}\right)dv$$

and:

(2.11) $$\begin{aligned}\frac{\mathrm{D}}{ds}\left(\frac{\mathrm{D}u^a}{dv}\right) &= \frac{\mathrm{D}}{dv}\left(\frac{\mathrm{D}u^a}{ds}\right) - u^1 R^a_{1,cd}\frac{\partial x^c}{\partial v}\frac{\partial x^d}{\partial s}\\ &= \frac{\mathrm{D}}{dv}\left(\frac{\mathrm{D}u^a}{ds}\right) + u^1 R^a_{1,cd}\frac{\partial x^c}{\partial s}\frac{\partial x^d}{\partial v}\\ &= \frac{\mathrm{D}}{dv}\left(\frac{\mathrm{D}u^a}{ds}\right) + R^a_{1,cd}u^1u^c\frac{\partial x^d}{\partial v}\,.\end{aligned}$$

One thus has:

(2.12) $$(\mathrm{D}^2\eta^a/ds^2) + R^a{}_{bcd}\, u^b\, u^d\, \eta^c = 0\,.$$

This formula shows that the gravitational field manifests itself thanks to the Riemann tensor, and it translates into the existence of a relative acceleration between two neighboring observers (i.e., geodesics) [4].

3. **Connection forms** [5]. – The covariant derivative permits us to establish a correspondence (viz., a *parallelism*) between neighboring tangent spaces.

Let T_x and T_{x+dx} be two neighboring tangent spaces, and let **A** and **A** + d**A** be the following elements of T_x and T_{x+dx}:

(3.1) $$\mathbf{A} \in T_x, \qquad \mathbf{A} + d\mathbf{A} \in T_{x+dx} ,$$

respectively. We define the linear map:

(3.2) $$\boldsymbol{\omega} : T_{x+dx} \to T_x , \qquad \boldsymbol{\omega}(\mathbf{A} + d\mathbf{A}) = \mathbf{A} + \mathrm{D}\mathbf{A},$$

in which:

(3.3) $$(\mathrm{D}A)^2 = (dA)^2 + A^b \omega^a{}_b ;$$

in local coordinates:

(3.4) $$\omega^a{}_b = \Gamma^a_{bc}\, dx^c .$$

We denote the matrix (3.4) by **ω**.

The map **ω** is an infinitesimal transformation, and the components $\omega^a{}_b$ are nothing but the components of the images of the basis vectors under the map (3.2).

One knows that the tangent vector spaces are Minkowski spaces, that the map **ω** preserves the metric structure on the tangent spaces, and that the covariant derivative preserves angles.

ω is therefore *an infinitesimal transformation of the homogeneous Lorentz group (or Minkowskian rotations), or furthermore an element of the Lie algebra of this group.*

The connection therefore defined by the given of a 1-form with values in the Lie algebra of the Lorentz group.

Indeed, the definition above is incomplete, and, in particular, in an essential way: Indeed, one must specify *the transformation law of* **ω** under a change of basis in T_x (cf., the techniques of É Cartan that are understood today thanks to the notion of a principal fiber bundle).

For example, let:

(3.5) $$h_b^{(a)}$$

be a coframe in T_x, so it is determined by a basis of four covariant vectors.

The four 1-forms:

(3.6) $$\theta^a = h_b^{(a)}\, dx^b$$

are their components relative to the tangent vector dx^a, and one will have:

$$\omega^a{}_b = \Gamma^a_{bc}\, \theta^c .$$

Let:

(3.7) $$\theta' = T\, \theta$$

be a change of basis:

(3.8) $$\theta^{a'} = T_a^{a'}\, \theta^a .$$

ω satisfies the transformation law:

(3.9) $$\boldsymbol{\omega}' = T\,dT^{-1} + T\,\boldsymbol{\omega}\,T^{-1}\,,$$

or:

(3.10) $$\boldsymbol{\omega}' = -\,dT\,T^{-1} + T\,\boldsymbol{\omega}\,T^{-1}\,,$$

namely:

(3.11) $$\omega^{a'}{}_{b'} = T^{a'}_{c}\,dT^{c}_{b'} + T^{a'}_{c}\,\omega^{c}{}_{d}\,T^{d}_{b'}\,,$$

in which:

(3.12) $$(T^{-1})^{b}_{a'} = T^{b}_{a'}\,.$$

It is this law that assures the tensorial character of covariant derivation.

If **A** is a contravariant vector then:

(3.13) $$\mathbf{A}' = T\,\mathbf{A}\,,$$

(3.14) $$\mathrm{D}\mathbf{A}' = d(T\,\mathbf{A}) + \boldsymbol{\omega}'\,T\,\mathbf{A} = dT\,\mathbf{A} + T\,d\mathbf{A} + T\,\boldsymbol{\omega}\,\mathbf{A}\,.$$

(3.15) $$\boxed{\mathrm{D}(T\,\mathbf{A}) = T\,\mathrm{D}\mathbf{A}}\,.$$

An analogous calculation shows that the 2-form:

(3.17) $$S = \mathrm{D}\theta = d\theta + \boldsymbol{\omega} \wedge \theta\,,$$

or:

(3.18) $$S = d\theta^{a} + \omega^{a}{}_{b} \wedge \theta^{b}\,,$$

which called the *torsion* 2-form, is also a tensorial form with vector values; one has:

(3.19) $$S' = T\,S\,.$$

4. **Riemannian connection in an arbitrary coframe.** – In an arbitrary coframe, let:

(4.1) $$\omega^{a}{}_{b} = \gamma^{a}_{bc}\,\theta^{c}$$

and:

(4.2) $$ds^{2} = g_{ab}\,\theta^{a}\,\theta^{b}\,.$$

The condition that expresses the fact that $\omega^{a}{}_{b}$ defines an infinitesimal Lorentz transformation is written:

(4.3) $$g_{ac}\,\omega^{c}{}_{b} + g_{bc}\,\omega^{c}{}_{a} = dg_{ab}\,,$$

or:

(4.4) $$\gamma_{abc} + \gamma_{bac} = \partial_{c}g_{ab}\,.$$

One will note that $\partial_c\,g_{ab}$ defines a Pfaffian derivative, $dg_{ab} = \partial_c g_{ab}\theta^c$. Let:

(4.5) $$d\theta^{a} = \tfrac{1}{2}\,b^{a}_{bc}\,\theta^{b} \wedge \theta^{c}\,.$$

The Riemannian connection is determined by the supplementary condition that the torsion be annulled:

(4.6) $$d\theta^a + \omega^a{}_b \wedge \theta^c = 0\,.$$

(4.5) and (4.6) allow us to write:

(4.7) $$b^a_{bc} + \gamma^a_{cb} - \gamma^a_{bc} = 0\,,$$

or, furthermore:

(4.8) $$b_{abc} + \gamma_{acb} - \gamma_{abc} = 0\,.$$

(4.8) and (4.4) give:

(4.9) $$\gamma_{acb} + \gamma_{bac} = \partial_c g_{ab} - b_{abc}\,,$$

or, furthermore:

(4.10) $$\gamma_{bac} + \gamma_{cba} = \partial_a g_{bc} - b_{bca}\,,$$

(4.11) $$\gamma_{cba} + \gamma_{acb} = \partial_b g_{ca} - b_{cab}\,.$$

(4.9) – (4.10) + (4.11) gives:

(4.12) $$\gamma_{acb} = (\partial_b g_{ac} + \partial_c g_{ab} - \partial_a g_{bc}) - (b_{abc} + b_{cab} - b_{bca})\,;$$

in a natural frame:

(4.13) $$b_{abc} = 0\,.$$

One thus recovers the formulas of section 1.1.

Orthonormal frames have the properties that:

(4.14) $$ds^2 = (\omega^0)^2 - (\omega^1)^2 - (\omega^2)^2 - (\omega^3)^2$$

and:

(4.15) $$\omega_{ab} + \omega_{ba} = 0\,.$$

We also introduce *normal isotropic frames*, which have the property that:

(4.16) $$ds^2 = 2\,\theta^0\,\theta^3 - 2\,\theta^1\,\theta^2\,.$$

They correspond to a choice of four covariant vectors $h^{(a)}$ such that:

(4.17) $$h^{(0)} \cdot h^{(3)} = -h^{(1)} \cdot h^{(2)} = 1\,,$$

the other scalar products being null.

In Minkowski space, if:

(4.18) $$ds^2 = (dx^0)^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2$$

then one sets:

(4.19) $$\sqrt{2}\,\theta^0 = dx^0 - dx^3, \qquad \sqrt{2}\,\theta^3 = dx^0 + dx^3,$$
$$\sqrt{2}\,\theta^1 = dx^1 + i\,dx^2, \qquad \sqrt{2}\,\theta^2 = dx^1 - i\,dx^2.$$

The antisymmetric tensor η_{abcd} – viz., the volume element with null covariant derivative – is defined by:

(4.20) $$\eta_{abcd} = \sqrt{-g}\,\varepsilon_{abcd}\,, \qquad \eta^{abcd} = \frac{1}{\sqrt{-g}}\,\varepsilon^{abcd}\,,$$

in which ε^{abcd} and ε_{abcd}, the permutation tensors, equal ± 1 whenever *abcd* is an even or odd permutation of the sequence 0, 1, 2, 3, respectively.

In an isotropic frame:

(4.21) $$\eta_{abcd} = i\varepsilon_{abcd}\,, \qquad \eta^{abcd} = i\varepsilon^{abcd}\,.$$

5. **Riemannian curvature.** – The 2-form:

(5.1) $$\Omega = \mathrm{D}\boldsymbol{\omega} = d\boldsymbol{\omega} + \boldsymbol{\omega} \wedge \boldsymbol{\omega}$$

is a tensorial 2-form with values in the Lie algebra of the Lorentz group. Indeed, one has:

(5.2) $$\begin{aligned} \Omega' &= d\boldsymbol{\omega}' + \boldsymbol{\omega}' \wedge \boldsymbol{\omega}' \\ &= (dT \wedge dT^{-1} + dT\,\boldsymbol{\omega}\,T^{-1} + T\,d\boldsymbol{\omega}\,T^{-1} - T\,\boldsymbol{\omega} \wedge dT^{-1}) \\ &\quad + (-\,dT\,T^{-1} + T\,\boldsymbol{\omega}\,T^{-1}) \wedge (T\,dT^{-1} + T\,\boldsymbol{\omega}\,T^{-1})\,, \end{aligned}$$

hence:

(5.3) $$\Omega' = T\,\Omega\,T^{-1},$$

(5.4) $$\Omega^a{}_b = d\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b,$$

(5.5) $$\Omega^a{}_b = \tfrac{1}{2} R^a{}_{bcd}\,\theta^c \wedge \theta^d,$$

so:

(5.6) $$R^a{}_{bcd} = \partial_c \Gamma^a_{bd} - \partial_d \Gamma^a_{bc} + \Gamma^a_{lc}\Gamma^l_{bd} - \Gamma^a_{ld}\Gamma^l_{bc}\,.$$

The $\Omega^a{}_b$ are the components of an infinitesimal Lorentz transformation that is associated with the direction plane $\theta^c \wedge \theta^d$. One has, moreover:

(5.7) $$\Omega_{ab} + \Omega_{ab} = 0\,.$$

The symmetry properties of the index pairs in the Riemann tensor follow immediately.

The symmetry properties of the index triples follow from the identity:

(5.8) $$\Omega^a{}_b \wedge \theta^b = 0\,,$$

which is a consequence of the vanishing of torsion; in order to find (5.8), it suffices to take the covariant differential of (5.4), while taking (5.6) into account.

Finally, the Bianchi identities express that the tensorial form Ω has null covariant derivative:

(5.9) $$D\Omega = d\Omega - \Omega \wedge \boldsymbol{\omega} + \boldsymbol{\omega} \wedge \Omega = 0\ ,$$

or:

(5.10) $$D\Omega^a{}_b = d\Omega^a{}_b - \Omega^a{}_b \wedge \omega^c{}_b + \omega^a{}_c \wedge \Omega^c{}_b = 0\ ;$$

relations (5.9) and (5.10) may be established by exterior differentiation of (5.1).

6. **Curvature trivector. Einstein equations.** – The antisymmetric tensor that is the volume element permits us to define adjoint tensors.

If A is a vector then $\overset{*}{A}$ is a trivector: $\overset{*}{A}_{abc} = \eta_{abcd} A^d$.

If A is a bivector then $\overset{*}{A}$ is also a bivector: $\overset{*}{A}_{ab} = \frac{1}{2}\eta_{abcd} A^{cd}$. Finally, if A is a trivector then $\overset{*}{A}$ is a vector:

$$\overset{*}{A}_a == \frac{1}{3!}\eta_{abcd} A^{bcd}\ .$$

Let $\Theta = -\overset{*}{\Omega} \wedge \theta$. A 3-form, it is a tensorial form, or *curvature trivector* [6], such that:

$$\Theta' = -\overset{*}{\Omega} \wedge \theta' = -T^{-1}\overset{*}{\Omega}\, T^{-1} \wedge T\,\theta = T^{-1}\overset{*}{\Theta}\ .$$

Moreover, it has null covariant derivative because, since the tensor η_{abcd} has null covariant derivative, one also has $D\overset{*}{\Omega} = 0$. One has:

$$\Theta_a = -\overset{*}{\Omega}_{ab} \wedge \theta^b\ ,$$
$$\Theta_a = -\tfrac{1}{2}\eta_{abcd}\,\Omega^{cd} \wedge \theta^b = -\tfrac{1}{4}\eta_{abcd}\,R^{cd}{}_{rs}\,\theta^r \wedge \theta^s \wedge \theta^b\ .$$

Set $\theta^r \wedge \theta^s \wedge \theta^b = \eta^{frsb}\hat{\theta}_f$;

$$\Theta_a = -\tfrac{1}{4}\eta_{abcd}\,\eta^{frsb} R^{cd}{}_{rs}\,\hat{\theta}_f = \tfrac{1}{4}\delta^{frs}_{acd}\,R^{cd}{}_{rs}\,\hat{\theta}_f$$
$$= \tfrac{1}{4}(\delta^f_a\delta^{rs}_{cd} + \delta^r_a\delta^{sf}_{cd} + \delta^s_a\delta^{tr}_{cd})\,R^{cd}{}_{rs}\,\hat{\theta}_f = (R^f_a - \tfrac{1}{2}R\delta^f_a)\hat{\theta}_f = G^f_a\hat{\theta}_f\ .$$

The tensor G^f_a is precisely the Einstein tensor that appears in the left-hand side of the gravitational field equations:

$$G^b_a = R^b_a - \tfrac{1}{2}\delta^b_a R = \kappa T^b_a\ .$$

The fact that the curvature trivector has null covariant derivative is written $\nabla_b G^b_a = 0$, which corresponds to the *conservative* character of this tensor.

7. **A theorem of Élie Cartan** [6]. – The invariant functions that are defined on V_4, or the *scalar invariants* of V_4, are functions of R_{abcd} and their covariant derivatives.

In particular, the *curvature invariants*, which, by definition, depend on at most the second order derivatives of the fundamental tensor g_{ab} are functions of R_{abcd} alone.

8. **Statement of a problem.** – The tangent space at a point of V_4 is a Minkowski space; from this, it results that any type of tensor field on V_4 defines a representation of the group of Minkowskian rotations at each point.

We shall study the structure of the 20-dimensional vector space of Riemann tensors.

II. **The space of bivectors**

9. **Space of bivectors**. – Let:

(9.1) $$F_{ab} = -F_{ba}$$

be a twice-covariant antisymmetric tensor in Minkowski space M_4 ; more generally, F_{ab} will be a tensor field on V_4. Let:

(9.2) $$g_{ab}$$

be the metric tensor on M_4.

The space of bivectors on M_4 is a 6-dimensional vector space M_6. M_6 is a metric space and the norm is defined by:

(9.3) $$F \cdot F = F^2 = \tfrac{1}{4} g_{ab,\,cd}\, F^{ab}\, F^{cd} = \tfrac{1}{4} g^{ab,\,cd}\, F_{ab}\, F_{cd} = \tfrac{1}{2} F_{ab}\, F^{ab} ,$$

in which:

(9.4) $$g_{ab,\,cd} = g_{ab}\, g_{cd} - g_{ad}\, g_{bc} .$$

Moreover, *in an oriented* M_6 the quadratic form:

(9.5) $$F \cdot \overset{*}{F} = \tfrac{1}{4} \eta^{abcd}\, F_{ab}\, F_{cd} = \tfrac{1}{2} \overset{*}{F}{}^{ab} F_{ab}$$

is invariant.

The existence of two quadratic forms in M_6 that are invariant under Minkowskian rotations (det = + 1) is the basis for a representation of the Lorentz group that we specify at the end of the chapter.

Let $\chi^{(a)}$ be a basis of covectors in M_4 and let **A** be a vector of M_4. Then:

(9.6) $$\mathbf{A} = A_a \chi^{(a)} , \qquad \text{that is,} \qquad A_b = A_a \chi_b^{(a)} ,$$

(9.7) $$\mathbf{A} \cdot \mathbf{A} = A_a\, A_b \chi^{(a)} \cdot \chi^{(b)} = A_a\, A_b\, g^{ab}.$$

Let:

(9.8) $$\chi^{(ab)} = \chi^{a} \wedge \chi^{b}$$

be a basis of bivectors; let it be a basis for M_6. A bivector of M_6 is written:

(9.9) $F = \frac{1}{2} F_{ab} \chi^{(ab)}$, that is, $F_{cd} = \frac{1}{2} F_{ab} \chi^{(ab)}_{cd}$.

The metric of M_6 is such that:

(9.10) $$\chi^{(ab)} \cdot \chi^{(cd)} = g^{ab,\, cd} = g^{ab} g^{cd} - g^{ad} g^{bc} .$$

Let:

(9.11) $$\overset{*}{\chi}{}^{(ab)} = \tfrac{1}{4} \eta^{cdab} g_{cdrs} \chi^{(rs)} .$$

One calls bivectors B such that:

(9.12) $\overset{*}{B} = i\, B$, or $\overset{*}{B} = -\, i\, B$,

self-adjoint or *anti-self-adjoint*, respectively.

To any bivector F one can associate the bivectors:

(9.13) $\overset{+}{F} = \frac{1}{2}(F - i\, \overset{*}{F})$, $\overline{F} = \frac{1}{2}(F + i\, \overset{*}{F})$,

which are self-adjoint and anti-self-adjoint, respectively. Indeed:

(9.14) $$\overset{**}{F} = -\, F .$$

We further note that:

(9.15) $$F = \overset{*}{F} + \overline{F} .$$

We note that:

(9.16) $$\overset{*}{F}{}_{ab} = \tfrac{1}{2} \Delta^{cd}_{ab} F_{cd} ,$$

(9.17) $$\Delta^{cd}_{ab} = \tfrac{1}{2} \eta^{rsab} g_{rscd} .$$

One has:

(9.18) $$\tfrac{1}{2} \Delta^{cd}_{ab} \Delta^{fg}_{cd} = -\, \delta^{fg}_{ab} ,$$

$$\tfrac{1}{2} \Delta^{cd}_{ab} \Delta^{fg}_{cd} = \frac{1}{8} \eta_{rsab}\, g^{rscd} \eta_{uvcd}\, g^{uvfg} = \frac{1}{2} \eta_{rsab}\, \eta_{uvcd}\, g^{rc}\, g^{sd}\, g^{uf}\, g^{vg}$$
$$= -\frac{1}{\sqrt{-g}} \eta_{rsab}\, \varepsilon^{rsfg} = -\frac{1}{2} \varepsilon_{rsab}\, \varepsilon^{rsfg} = -\, \delta^{fg}_{ab} .$$

Furthermore, let:

(9.19) $\overset{+}{F}_{ab} = \frac{1}{2} \overset{+}{\Gamma}{}^{cd}_{ab} F_{cd}$,

(9.20) $\overset{+}{F} = \overset{+}{\Gamma} F$,

(9.21) $\overset{+}{\Gamma} = \frac{1}{2}(I - i\, \Delta)$, $\overline{\Gamma} = \frac{1}{2}(I + i\, \Delta)$, One has:

(9.22) $$\overset{+}{\Gamma}\,\overset{+}{\Gamma} = \overset{+}{\Gamma}, \qquad \overset{-}{\Gamma}\,\overset{-}{\Gamma} = \overset{-}{\Gamma}, \qquad \overset{+}{\Gamma}\,\overset{-}{\Gamma} = \overset{-}{\Gamma}\,\overset{+}{\Gamma} = 0,$$

(9.23) $$\overset{+}{\Gamma} + \overset{-}{\Gamma} = I .$$

Theorem. –

(9.24) $$\tfrac{1}{2}(F^2 \pm F \cdot \overset{*}{F})$$

are two quadratic forms of rank 3.

a) *Orthonormal frames.* – Let:

(9.25) $$\chi^a \cdot \chi^b = \eta^{ab}, \qquad \eta^{ab} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix},$$

(9.26) $$A^2 = (A^0)^2 - (A^1)^2 - (A^2)^2 - (A^3)^2,$$
(9.27) $$\eta_{abcd} = \varepsilon_{abcd}, \qquad \eta^{abcd} = -\,\varepsilon^{abcd} \qquad (a, b, c, d = 0, 1, 2, 3).$$

One has:
(9.28) $$F^2 = (F_{23})^2 + (F_{31})^2 + (F_{12})^2 - (F_{01})^2 - (F_{02})^2 - (F_{03})^2 ,$$

(9.29) $$\begin{cases} \overset{*}{F}_{23} = F^{01} = -F_{01}, & \overset{*}{F}_{01} = F^{23} = -F_{23}, \\ \overset{*}{F}_{31} = F^{02} = -F_{02}, & \overset{*}{F}_{02} = F^{31} = -F_{31}, \\ \overset{*}{F}_{12} = F^{03} = -F_{03}, & \overset{*}{F}_{03} = F^{12} = -F_{12}, \end{cases}$$

(9.30) $$F \cdot \overset{*}{F} = -\,2\,F_{23}\,F_{01} - 2\,F_{31}\,F_{02} - 2\,F_{12}\,F_{03},$$

(9.31) $$\begin{cases} \overset{+}{F}_{23} = \tfrac{1}{2}(F_{23} + iF_{01}), & \overset{+}{F}_{01} = \tfrac{1}{2}(F_{01} - iF_{23}) = -i\,\overset{+}{F}_{23}, \\ \overset{+}{F}_{31} = \tfrac{1}{2}(F_{31} + iF_{02}), & \overset{+}{F}_{02} = \tfrac{1}{2}(F_{02} - iF_{31}) = -\,\overset{+}{F}_{31}, \\ \overset{+}{F}_{12} = \tfrac{1}{2}(F_{12} + iF_{03}), & \overset{+}{F}_{03} = \tfrac{1}{2}(F_{03} - iF_{12}) = -\,\overset{+}{F}_{12}. \end{cases}$$

One has:

(9.32) $$\tfrac{1}{2}(F^2 - F \cdot \overset{*}{F}) = [(F_{23})^2 - (F_{01})^2 + 2i\,F_{23}\,F_{01}] + \ldots$$
$$= 2\,[\,(\overset{+}{F}_{23})^2 + (\overset{+}{F}_{31})^2 + (\overset{+}{F}_{12})^2\,] .$$

One may form the following basis of self-adjoint bivectors:

(9.33) $$Z^1 = \tfrac{1}{2}(\chi^{23} - i\chi^{01}), \qquad Z^2 = \tfrac{1}{2}(\chi^{31} - i\chi^{02}), \qquad Z^3 = \tfrac{1}{2}(\chi^{12} - i\chi^{03}),$$

Any bivector may be written:

(9.34) $$F = F_\alpha Z^\alpha + \bar{F}_\alpha \bar{Z}^\alpha , \qquad F_{\lambda\mu} = F_\alpha Z^\alpha_{\lambda\mu} + \bar{F}_\alpha \bar{Z}^\alpha_{\lambda\mu} ,$$

in which:

$$F_\alpha = F_{\beta\lambda} + i\, F_{0\alpha} , \qquad \alpha, \beta, \gamma = (1, 2, 3).$$

The metric of space Z^α is:

(9.35) $$Z^\alpha \cdot Z^\beta = \tfrac{1}{2}\,\delta^{\alpha\beta} .$$

We note that:

(9.36) $$A \cdot A = \tfrac{1}{2}\,[(A_1)^2 + (A_2)^2 + (A_3)^2] .$$

b) *Normal isotropic frames.* – All of the foregoing that relates to the bivector F_{ab} may also be transcribed in terms of differential 1- and 2-forms. We shall do this in a normal isotropic frame.

A normal isotropic frame has the property that:

(9.37) $$\chi^a \cdot \chi^b = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} = i^{ab} .$$

It is defined by four vectors $h^{(0)}$, $h^{(1)}$, $h^{(2)}$, $h^{(3)}$ such that:

(9.38) $$h^{(0)} \cdot h^{(3)} = - h^{(1)} \cdot h^{(2)} = 1,$$

the other scalar products being null. One has:

(9.39) $$g_{ab} = h_a^{(c)} h_b^{(d)}\, i_{cd} .$$

In local coordinates, we denote:

(9.40) $$\theta^a = h_b^{(a)}\, dx^b$$

and:

(9.41) $$h^c_{(a)} h_b^{(d)} = \delta^c_b ,$$

(9.42) $$\eta_{abcd} = i\, \varepsilon_{abcd} , \qquad \eta^{abcd} = i\, \varepsilon^{abcd} , \qquad (a, b, c, d = 0, 1, 2, 3).$$

One has:

(9.43)
$$\begin{cases} \overset{*}{F}_{23} = iF_{23}, & \overset{*}{F}_{01} = iF_{01}, \\ \overset{*}{F}_{31} = -iF_{31}, & \overset{*}{F}_{02} = -iF_{02}, \\ \overset{*}{F}_{12} = -iF_{03}, & \overset{*}{F}_{03} = -iF_{12}. \end{cases}$$

We take the following basis of self-adjoint 2-forms:

(9.44)
$$Z^1 = \theta^2 \wedge \theta^3, \quad Z^2 = \theta^0 \wedge \theta^1, \quad Z^3 = \tfrac{1}{2}(\theta^0 \wedge \theta^3 - \theta^1 \wedge \theta^2) .$$

Now, the space-time metric is:

(9.45)
$$\gamma^{\alpha\beta} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -\frac{1}{2} \end{pmatrix}, \qquad \gamma_{\alpha\beta} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$

One has:

(9.46)
$$Z^\alpha \cdot Z^\beta = \gamma^{\alpha\beta}, \qquad Z^\alpha \cdot \bar{Z}^\beta = 0 , \qquad \bar{Z}^\alpha \cdot \bar{Z}^\beta = \bar{\gamma}^{\alpha\beta} .$$

One has, moreover:

(9.47)
$$Z^\alpha \wedge Z^\beta = \gamma^{\alpha\beta}\, dv, \qquad Z^\alpha \wedge \bar{Z}^\beta = 0 , \qquad \bar{Z}^\alpha \wedge \bar{Z}^\beta = - \bar{\gamma}^{\alpha\beta}\, dv .$$

CONCLUSION. – The space of self-adjoint bivectors Z^α is a complex Euclidian space, which we denote in the sequel by E_3. If:

(9.48)
$$\tfrac{1}{2} F_{ab}\, \theta^a \wedge \theta^b = F_\alpha Z^\alpha + \text{conj.},$$

then:

(9.49)
$$\begin{aligned} &F_1 = F_{23}, && F_2 = F_{01}, && F_3 = F_{03} - F_{12}, \\ &\bar{F}_1 = F_{13}, && \bar{F}_2 = F_{02}, && \bar{F}_3 = F_{03} + F_{12} . \end{aligned}$$

10. **Classification of bivectors. Characteristic isotropic vectors. Invariants**. – The bivectors are classified into two categories, according to whether the invariant:

(10.1)
$$I = F_\alpha F_\alpha \gamma^{\alpha\beta} = \overset{+}{F} \cdot \overset{+}{F}$$

is different from 0 or not. I is an invariant of the Lorentz group, and tensorially it is:

(10.2)
$$I = \tfrac{1}{2}(F_{ab} F^{ab} - i\, F^{ab} \overset{*}{F}{}^{ab}) .$$

If $I \neq 0$ then the bivector is *nonsingular*. If $I = 0$ then the bivector is *singular*, and one has:

$$F_{ab}\, F^{ab} = F^{ab} \overset{*}{F}_{ab} = 0 . \tag{10.3}$$

THEOREM. – With any nonsingular bivector there is associated a pair of characteristic (real) isotropic vector such that:

$$F_{ab}\, \overset{(A)}{k}{}^{b} = I\, \overset{(A)}{k}_{b} \qquad [A = 1, 2; \qquad \text{type } (1, 1)] . \tag{10.4}$$

To any singular bivector there is associated an isotropic vector – or a pair of isotropic vectors that coincide – such that:

$$F_{ab}\, k^b = 0 \qquad [\text{type } (2)]. \tag{10.5}$$

One also has:

$$\overset{*}{F}_{ab}\, k^b_{(A)} = I'\, k_{b(A)} \tag{10.6}$$

in the first case, and:

$$\overset{*}{F}_{ab}\, k^b = 0 \tag{10.7}$$

in the second case.

The proof is particularly simple in an isotropic frame. Any singular bivector:

$$F = F_\alpha\, Z^\alpha + \text{conj.} \tag{10.8}$$

is equivalent to:

$$F = Z^2 + \text{conj.} \tag{10.9}$$

The associated 2-form is therefore written:

$$F = \theta^0 \wedge \theta^1 + \theta^0 \wedge \theta^2 = \theta^0 \wedge (\theta^1 + \theta^2) , \tag{10.10}$$

in which:

$$\theta^0 = k_a\, dx^a, \qquad \theta^1 + \theta^2 = \overset{(1)}{v}_a\, dx^a , \tag{10.11}$$

$$F_{ab} = k_a \overset{(1)}{v}_b - k_b \overset{(1)}{v}_a , \tag{10.12}$$

with:

$$k_a\, k^a = 0 , \qquad k^a \overset{(1)}{v}_a = 0 . \tag{10.13}$$

One has, moreover:

$$\overset{*}{F} = i\, \theta^0 \wedge (\theta^1 - \theta^2) \tag{10.14}$$

and:

$$i\,(\theta^1 - \theta^2) = \overset{(2)}{v}_a\, dx^a , \tag{10.15}$$

with:

$$k^a\, \overset{(2)}{v}_a = \overset{a}{v}_{(2)}\, \overset{(2)}{v}_a = 0 , \tag{10.16}$$

$$k_a = \overset{0}{h}_a , \qquad \overset{(1)}{v}_a = \overset{(1)}{h}_a + \overset{(2)}{h}_a , \qquad \overset{(2)}{v}_a = i\,(\overset{(1)}{h}_a - \overset{(2)}{h}_a) ,$$

hence:

(10.17) $$\overset{*}{F}_{ab} = k_a v_b^{(2)} - k_b\, v_a^{(2)} .$$

One has, at the same time:

(10.18) $$F_{ab}\, k^b = \overset{*}{F}_{ab}\, k^b = 0 .$$

If F is nonsingular then $\overset{+}{F}$ is equivalent to:

(10.19) $$\overset{+}{F} = J \cdot Z^3, \qquad J^2 = -2I .$$

One thus has (†):

(10.20) $$F = \frac{J}{2}(\theta^0 \wedge \theta^3 - \theta^1 \wedge \theta^2) + \text{conj.}$$

$$= \frac{J+\bar{J}}{2}\theta^0 \wedge \theta^3 + \frac{J-\bar{J}}{2}\theta^1 \wedge \theta^2 = \frac{J+\bar{J}}{2}\theta^0 \wedge \theta^3 + i\,\frac{J-\bar{J}}{2} *(\theta^0 \wedge \theta^3).$$

(10.21) $$J = E + i\,H, \qquad F = E\,\theta^0 \wedge \theta^3 + H *(\theta^0 \wedge \theta^3)$$

(10.22) $$E^2 - H^2 = -\tfrac{1}{2} F_{ab}\, F^{ab} , \qquad 2\,E \cdot H = \tfrac{1}{2} F_{ab}\, \overset{*}{F}{}^{ab} .$$

Hence:

(10.23) $$\theta^0 = k_a^{(1)} dx^a , \qquad \theta^3 = k_a^{(2)} dx^a ;$$

(10.24) $$F_{ab} = E\, a_{ab} + H\, \overset{*}{a}_{ab} ,$$

(10.25) $$a_{ab} = (k_a^{(1)}\, k_b^{(2)} - k_a^{(2)}\, k_b^{(1)})/(k^{(1)} \cdot k^{(2)}).$$

If F_{ab} is a non-singular electromagnetic field and u^a is a vector of timelike type that is associated with an observer then:

(10.26) $$E_a = F_{ab}\, u^b, \qquad H_a = -\overset{*}{F}_{ab}\, u^b$$

are the *electric field* and *magnetic field* for the observer u^a, respectively.

If u^a is a vector in the Minkowskian plane of $\overset{(1)}{k}$ and $\overset{(2)}{k}$ then one has:

(10.27) $$u_a = \alpha\, \overset{(1)}{k} + \beta\, \overset{(2)}{k} ,$$
$$E_a = E\,(\alpha\, \overset{(1)}{k}_a - \beta\, \overset{(2)}{k}_a) = E\, e_a, \qquad H_a = -H\, e_a ,$$

† Trans. note: The notation was changed slightly in the last expression of (10.20) for the sake of clarity.

in which e_a is a vector of the same type as u_a that is contained in the plane of $\overset{(1)}{k}$ and $\overset{(2)}{k}$ and orthogonal to u^a . The Minkowskian plane of $\overset{(1)}{k}$ and $\overset{(2)}{k}$ may be characterized by the property that it contains the observers for which the vectors E_a, H_a have the same line of action.

COROLLARY. – *Bivectors associated with a given isotropic vector.* Let k_a be an isotropic vector; it results from the preceding that the set of bivectors that admit k_a for a characteristic vector is given by:

(10.28) $$F_{ab} = A_1 (k_a u_b - k_b u_a) + A_2 a_{ab} + A_3 \overset{*}{a}_{ab} .$$

In a normal isotropic frame, such as:

(10.29) $$k_a = h_a^{(0)} ,$$

$h^{(0)}$ is characteristic if:

(10.30) $$\overset{+}{F}_{ab} k^b = I k_a .$$

Note that:

(10.31) $$k^b = h_{(3)}^b ,$$

(10.32) $$Z_{ab}^{(1)} \cdot h_{(3)}^b = h_a^{(0)} , \qquad Z_{ab}^{(2)} \cdot h_{(3)}^b = 0 , \quad Z_{ab}^{(3)} \cdot h_{(3)}^b = \tfrac{1}{2} h_a^{(0)} .$$

The condition (10.30) may be written:

(10.33) $$A_1 = 0 .$$

To the vector $h^{(0)}$, one can associate the pencil:

(10.34) $$\overset{+}{F} = A_2 Z^2 + A_3 Z^3$$

that is generated by Z^2 and Z^3 .

11. **The energy-momentum tensor.** – To any bivector F_{ab}, one may associate the symmetric tensor:

(11.1) $$T_{ab} = \tfrac{1}{4} g_{ab} F_{cd} F^{cd} - F_{ac} F_b^{\ c} .$$

It enjoys the following properties:

a) Symmetry:

(11.2) $$T_{ab} = T_{ba} .$$

b) Null trace:

(11.3) $$T_{ab} g^{ab} = 0 .$$

c) Involutive character:

(11.4) $$T_a^{\ c}\, T_{cb} = \tfrac{1}{4} g^{ab} I^2 .$$

d) Characteristic isotropic vectors:

(11.5) $$T_{ab}\ k^b_{(A)} = I\, k_{(A)a} .$$

e) Conservative character:

(11.6) $$\nabla_{\mathrm{a}} T^{\ a}_b = 0$$

if:

(11.7) $$\nabla_{\mathrm{b}} \overset{+}{F}{}^{\ a}_b = 0 .$$

T_{ab} is the *energy-momentum tensor* that is associated with the electromagnetic field F_{ab}, and equations (11.7) are nothing but the vacuum Maxwell equations. Note that:

(11.8) $$\overset{*}{F}_{ac}\, \overset{*}{F}{}^{bc} = F_{ab}\, F^{bc} - \tfrac{1}{2} \delta^b_a\, F_{rs}\, F^{rs} .$$

Indeed:

(11.9) $$\overset{*}{F}_{ac}\, \overset{*}{F}{}^{bc} = \tfrac{1}{4}\, \eta_{rsac}\ \eta^{tubc}\, F_{rs}\, F^{tu} = -\tfrac{1}{4} \delta^{tub}_{rsa}\, F_{rs}\, F^{rs} ,$$

in which $\delta^{tub}_{rsa} = 3!\, \delta^{[t}_r \delta^u_s \delta^{b]}_a$. One thus has:

(11.10) $$T_a^{\ b} = \tfrac{1}{2} (F_{ab}\, F^{cb} + \overset{*}{F}_{ac}\, \overset{*}{F}{}^{bc}) .$$

Note, moreover, that:

(11.11) $$T_a^{\ b} = 2\ \overset{+}{F}{}^{\ c}_a \cdot \overline{F}{}^{\ b}_c = 2\, \overline{F}{}^{\ c}_a \cdot \overset{+}{F}{}^{\ b}_c .$$

In the case of a non-singular F, one has, moreover (cf. § 10):

(11.12) $$T_{ab} = -\frac{I}{2}\left(g_{ab} - 2\frac{k_a^{(1)} k_b^{(2)} + k_b^{(1)} k_a^{(2)}}{k^{(1)} \cdot k^{(2)}} \right),$$

in which:

(11.13) $$I^2 = (E^2 + H^2)^2 = (F \cdot F)^2 + (F \cdot \overset{*}{F})^2 ;$$

(10.4) and (10.5) result immediately.

In the singular case:

(11.14) $$T_{ab} = k_a\, k_b .$$

12. **Geometric interpretations.** – 1. *Cayley space* P_3. – In M_4, a hyperplane section is a projective space P_3 that is the Cayley space for the oval quadric Q:

(12.1) $$g_{ab}\,\chi^a\chi^b = 0\,.$$

The covariant vectors have the planes of P_3 for images, and the contravariant vectors, the points of P_3. The points of Q and the planes tangent to Q are the images of isotropic vectors. The interior points are the images of (contravariant) vectors of timelike type, and the exterior points are the vectors of spacelike type; the same is true for the (covariant vector) planes that, when they are not secant, correspond to the hyperplanes of spacelike type, and the secant planes correspond to the hyperplanes of timelike type.

An *orthonormal coframe* is composed of planes that form an auto-polar tetrahedron for Q.

An *isotropic frame* is composed of two real isotropic planes $h_a^{(0)}, h_a^{(3)}$, which are tangent to the points $h_{(3)}^a, h_{(0)}^a$ (isotropic vectors), and the two planes $h_a^{(0)}, h_a^{(3)}$ determine a bivector that has the line of intersection (viz., the bivector $Z^{(2)}$) for its image. The polar line $Z^{(1)}$ is completely determined, as well as the two (complex conjugate) isotropic planes $h_a^{(1)}, h_a^{(2)}$ that have it for their intersection (Fig. 1).

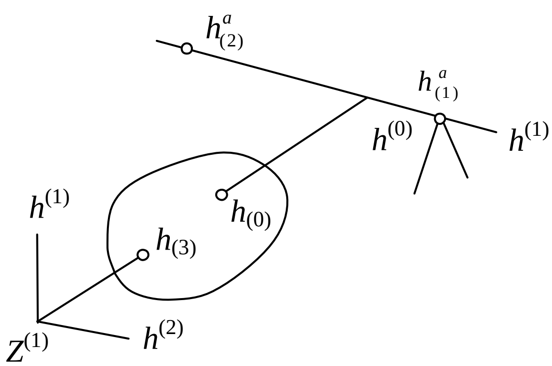


Fig. 1.

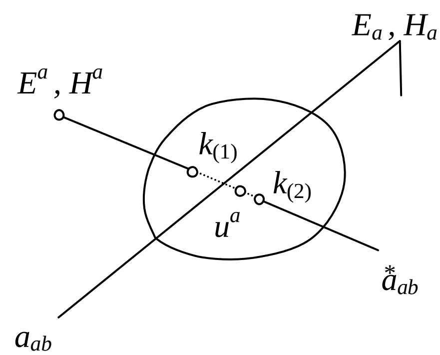


Fig. 2.

One will note that the isotropic frame is determined, up to the normalization conditions (9.38), by the two real isotropic vectors $h^{(0)}$, $h^{(3)}$.

A *non-singular electromagnetic field:*

(12.2) $$F_{ab} = E\,a_{ab} + H\,\overset{*}{a}_{ab}\,,$$

is associated with a definite isotropic frame that is determined by its characteristic vectors $k^{(1)}$, $k^{(2)}$. The observer u^a (a vector of timelike type), which is in the Minkowskian plane $k_{(1)}$, $k_{(2)}$:

(12.3) $$u^a = \lambda\,k_{(1)}^a + \mu\,k_{(2)}^a$$

has the property that the electric and magnetic field vectors that it defines have the same line of action (Fig. 2). One has:

(12.4) $$E_a = F_{ab}\,u^b = E\,(-\lambda\,k_a^{(2)} + \mu\,k_a^{(1)}),$$

(12.5) $$H_a = \overset{*}{F}_{ab}\,u^b = E\,(-\lambda\,k_a^{(2)} + \mu\,k_a^{(1)})\,.$$

Since the planes E_a , H_a coincide they are of spacelike type and are determined by the point u^a and the line a_{ab} . The fields E^a and H^a coincide with the conjugate point on the line $\overset{*}{a}_{ab}$, which joins $k^a_{(1)}, k^a_{(2)}$.

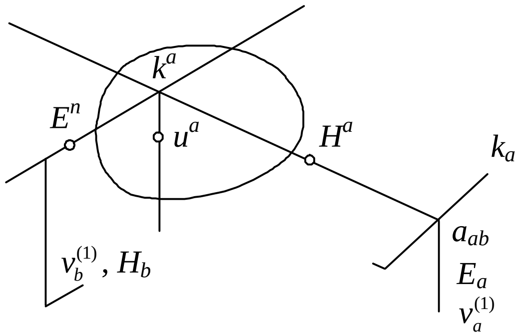


Fig. 3.

In the case of a *singular electromagnetic field* there is degeneracy – viz., $k^a_{(1)} = k^a_{(2)}$ – so a and $\overset{*}{a}$ are two lines tangent to the absolute.

(12.6) $$F_{ab} = k_a v^{(1)}_b - k_b v^{(1)}_a .$$

An observer situated on the intersection of the planes $v^{(1)}$ and $v^{(2)}$, $v^a_{(1)} u^a = v^{(2)}_a u^a = 0$, is again such that E_a, H_a are orthogonal (Fig. 3).

(12.7) $$E_a = F_{ab}\, u^b = -v^{(1)}_a k_b\, u^b ,$$

(12.8) $$H_a = -\overset{*}{F}_{ab}\, u^b = v^{(2)}_a k_b\, u^b ,$$

One has:

(12.9) $$E_a H^a = 0, \qquad E_a E^a = H_a H^a = (k_b u^b)^2.$$

Energy-momentum tensor. – The given of the absolute and the two isotropic vectors $k^{(1)}$, $k^{(2)}$ determine a pencil of quadrics or symmetric tensors:

(12.10) $$\tau_{ab} = \tau_{ba} = \lambda\, g_{ab} + \mu\, (k^{(1)}_a k^{(2)}_b + k^{(2)}_a k^{(1)}_b).$$

In this pencil there exists a quadric that is determined by the condition (called *apolarity*):

(12.11) $$g^{ab} \tau_{ab} = 0$$

so:

(12.12) $$4\lambda + 2\mu\, k^{(1)} \cdot k^{(2)} = 0 .$$

One has:

(12.13) $$\tau_{ab} = \lambda \left(g_{ab} - 2 \frac{k^{(1)}_a k^{(2)}_b + k^{(2)}_b k^{(1)}_a}{k^{(1)} \cdot k^{(2)}} \right).$$

This tensor is involutive and:

(12.14) $$\tau_{ab} \tau^{bc} = \delta^c_a\, \lambda^2 .$$

This energy-momentum tensor corresponds to the case for which $\lambda = -I/2$.

2. *Cayley space P_2*. – The plane sections of E_3 [2] are Cayley projective planes P_2 for the absolute conic γ.

(12.15) $$\gamma_{\alpha\beta} Z^{\alpha} Z^{\beta} = 0 .$$

An *orthonormal coframe* has a triangle that is autopolar for γ as its image in P_2.

An *isotropic frame* is associated with a triangle (Fig. 4) that is formed from two tangents to the absolute ($Z^1 = Z^2 = 0$) and the contact chord ($Z^3 = 0$). The points of the absolute have the rectilinear generators of the absolute Q for their images. Example: to $Z^3 = 0$ is associated the pair of planes whose equations are:

(12.16) $$h_a^{(0)} \chi^a = h_a^{(1)} \chi^a = 0 ,$$

and whose intersection belongs to:

(12.17) $$g_{ab} \chi^a \chi^b = (h_a^{(0)} h_b^{(3)} + h_b^{(0)} h_a^{(3)} - h_a^{(1)} h_b^{(2)} - h_b^{(1)} h_a^{(2)}) \chi^a \chi^b = 0 .$$

The points P_1, P_2 determine two generators $\bar{P}_1, \bar{P}_2$ of the same mode of Q, namely, the conjugate generators (of the contrary modes), hence, two real points $h^{(0)}$, $h^{(3)}$ of the absolute and two complex conjugate points $h^{(1)}$, $h^{(2)}$.

A *non-singular electromagnetic field* has a line for its image:

(12.18) $$A_{\alpha} Z^{\alpha} = 0 ,$$

which is secant to the absolute – for example, $Z^3 = 0$ – and a singular electromagnetic field has a tangent to the absolute for its image – for example, Z^2 (Fig. 5).

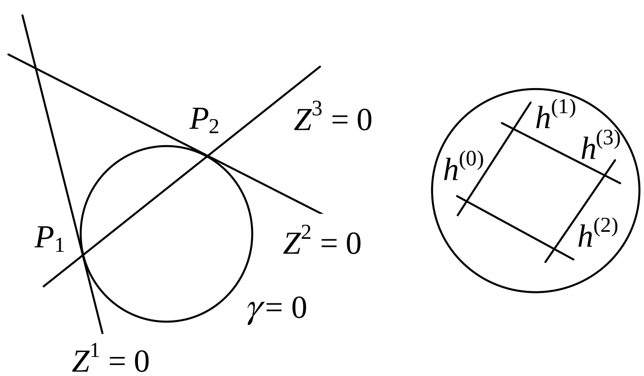


Fig. 4.

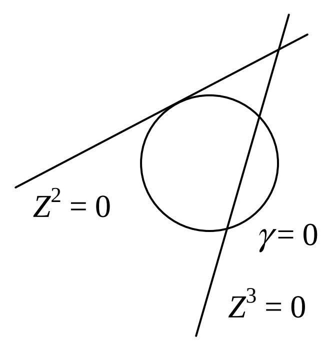


Fig. 5.

Singular electromagnetic field and isotropic vectors. In an isotropic frame, if one lets:

(12.19) $$l = l_a \theta^a$$

be an isotropic vector then one has:

(12.20) $$l_0 \, l_3 = l_1 \, l_2 .$$

The singular bivectors that admit l for a characteristic vector may be written:

(12.21) $$F = (l_a\, \theta^a) \wedge (u_b\, \theta^b)\ ,$$

in which u_a is an arbitrary vector that is not proportional to l. One has:

(12.22) $$\overset{+}{F} = (l_2\, u_3 - l_3\, u_2)\, Z^1 + (l_0\, u_1 - l_1\, u_0)\, Z^2 + (l_0\, u_3 - l_3\, u_0 - l_1\, u_2 + l_2\, u_1)\, Z^3\ .$$

Let:

(12.23) $$l_0\, / l_2 = l_1\, / l_3 = k\ ,$$

One has:

(12.24) $$\overset{+}{F} = (l_2\, u_3 - l_3\, u_2)\, (Z^1 + k\, Z^3) + (l_0\, u_1 - l_1\, u_0)\, (Z^2 + k^{-1}\, Z^3)\ .$$

In P_2, the image of the family $\overset{+}{F}$ is composed of the pencil of lines that pass through P (Fig. 6), where P, a point of γ, is defined by:

(12.25) $$Z^1 + k\, Z^3 = Z^2 + k^{-1}\, Z^3 = 0\ .$$

If:

(12.26) $$Z^1 = \lambda^2, \qquad Z^2 = \mu^2, \qquad Z^3 = \lambda\mu\ ,$$

are the parametric equations of (12.15) then one has:

(12.27) $$\lambda / \mu = k\ .$$

Conversely, if the value of k is known, or furthermore, if one is given a point of g then the associated real isotropic vector l may be written:

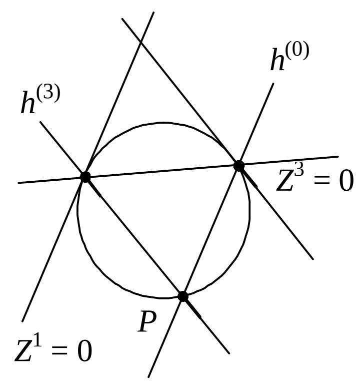


Fig. 6.

(12.28) $$l = (\rho,\ e^{-i\psi},\ e^{i\psi},\ 1/\rho)$$

if:

(12.29) $$k = \rho\, e^{i\psi}\ .$$

If $l = h^{(0)}$, $l_a = (1, 0, 0, 0)$, $k = \infty$ then the point P corresponds to nothing but:

(12.30) $$Z^2 = Z^3 = 0 \qquad (\lambda = 1,\ \mu = 0)\ ;$$

$h^{(3)}$ corresponds to the point:

(12.31) $$Z^1 = Z^3 = 0 \qquad (\lambda = 0,\ \mu = 1)\ .$$

13. **The Lorentz groups and their bivectorial representations.** – We say "Lorentz transformations" or "Minkowski transformations" when we are referring to the linear transformations $\chi^{a'} = L^{a'}_b \chi^b$ that preserve the quadratic form $(\chi^0)^2 - (\chi^1)^2 - (\chi^2)^2 - (\chi^3)^2$.

One calls the transformations with positive determinant *Minkowskian rotations* and denotes them L_+: $\det L_b^{a'} = 1$; one calls the transformations with negative determinants *Minkowskian reversals* and denote them by L_-: $\det L_b^{a'} = -1$.

The rotations and reversals are of two types, $L_+^{\uparrow}, L_+^{\downarrow}$ and $L_-^{\uparrow}, L_-^{\downarrow}$, according to whether they do or do not preserve the time direction.

The four types of Minkowski transformations give rise to five groups:

a) The complete Lorentz group,

b) The restricted Lorentz group $L_+^{\uparrow}$,

c) The group of rotations $L_+^{\uparrow}$ and reversals $L_-^{\uparrow}$, which are transformations that preserve the time direction,

d) The group of rotations $L_+^{\uparrow}$ and reversals $L_-^{\downarrow}$, which are transformations that preserve the spatial orientation,

e) The group of rotations $L_+^{\uparrow}$ and rotations $L_+^{\downarrow}$ that respect or permute both the time direction and the space orientation.

Under the bivectorial representation $M_4 \to M_6$, the Minkowskian rotations have the rotations $Z^{\alpha'} = \theta_\beta^{\alpha'} Z^\beta$, $\bar{Z}^{\alpha'} = \bar{\theta}_\beta^{\alpha'} \bar{Z}^\beta$ for their images; the reversals permute Z and $\bar{Z}$:

$$Z^{\alpha'} = \theta_\beta^{\alpha'} \bar{Z}^\beta, \quad \bar{Z}^{\alpha'} = \bar{\theta}_\beta^{\alpha'} Z^\beta .$$

The group $L_+^{\uparrow}$ is isomorphic to the group of rotations $O_3(\mathbb{C})$.

III. **The Riemann tensor**

14. **Connection and curvature in bivectorial variables** ([1]). – The representation K: $L_+^{\uparrow} \to O_3(\mathbb{C})$ associates $\boldsymbol{\omega}$ with an infinitesimal rotation of $O_3(\mathbb{C})$ that we denote by $\boldsymbol{\sigma}$ ($\sigma^\alpha{}_\beta$).

$\boldsymbol{\sigma}$ is a 1-form with values in the Lie algebra of $O_3(\mathbb{C})$. To formula (4.6):

(14.1) $$d\theta + \boldsymbol{\omega} \wedge \theta = 0 ,$$

which expresses the absence of torsion, one associates the formulas:

(14.2) $$dZ + \boldsymbol{\sigma} \wedge Z = 0 ,$$

and:

(14.3) $$dZ^a + \sigma^\alpha{}_\beta \wedge Z^\beta = 0 \qquad (\alpha, \beta = 1, 2, 3) .$$

[1] The method presented here is unedited. It has been the object of research done in collaboration with Mr. M. Cahen and Miss L. Defrise. The 1-forms $\boldsymbol{\sigma}$ of the infinitesimal rotation correspond to the spinorial coefficients of Newman and Penrose [8].

To the formula:

(14.4) $$\Omega = \mathrm{D}\boldsymbol{\omega} = d\boldsymbol{\omega} + \boldsymbol{\omega} \wedge \boldsymbol{\omega},$$

which defines the curvature 2-form, there corresponds the expression:

(14.5) $$\Sigma = \mathrm{D}\boldsymbol{\sigma} = d\boldsymbol{\sigma} + \boldsymbol{\sigma} \wedge \boldsymbol{\sigma},$$

and:

(14.6) $$\Sigma^{\alpha}{}_{\beta} = d\sigma^{\alpha}{}_{\beta} + \sigma^{\alpha}{}_{\gamma} \wedge \sigma^{\gamma}{}_{\beta},$$

To $\sigma^{\alpha}{}_{\beta}$ ($\Sigma^{\alpha}{}_{\beta}$, resp.), we associate the infinitesimal rotation vector σ^{α} (Σ^{α}, resp.), thanks to:

(14.7) $$\sigma^{\alpha} = \varepsilon^{\alpha\beta\gamma}\sigma_{\beta\gamma}, \qquad \Sigma^{\alpha} = \varepsilon^{\alpha\beta\gamma}\Sigma_{\beta\gamma}.$$

We set:

(14.8) $$\Sigma_{\alpha} = C'_{\alpha\beta} + E_{\alpha\bar{\beta}}\bar{Z}^{\beta}.$$

Here, we give analytical development in an isotropic coframe; the details of this development are given in the Appendix. If:

(14.9) $$d\theta^{\alpha} + \omega^{\alpha}{}_{\beta} \wedge \theta^{\beta} = 0,$$

then one has:

(14.10) $$\begin{cases} dZ^1 = \sigma^3 \wedge Z^1 - \sigma^1 \wedge Z^3, \\ dZ^2 = -\sigma^3 \wedge Z^2 + \sigma^2 \wedge Z^3, \\ dZ^3 = \frac{1}{2}\sigma^2 \wedge Z^1 - \frac{1}{2}\sigma^1 \wedge Z^2, \end{cases}$$

$$\sigma^{\alpha}{}_{\beta} = \begin{pmatrix} -\sigma^3 & 0 & \sigma^1 \\ 0 & \sigma^3 & -\sigma^2 \\ -\frac{1}{2}\sigma^2 & \frac{1}{2}\sigma^1 & 0 \end{pmatrix}.$$

If one refers to the definition of Z^{α}:

(14.11) $$Z^1 = \theta^2 \wedge \theta^3, \quad Z^2 = \theta^2 \wedge \theta^3, \quad Z^3 = \tfrac{1}{2}(\theta^0 \wedge \theta^3 - \theta^1 \wedge \theta^2)$$

then one verifies that one has:

(14.12) $$\omega^{\alpha}{}_{\beta} = \frac{1}{2}\begin{pmatrix} \bar{\sigma}^3 + \sigma^3 & -\bar{\sigma}^2 & -\sigma^2 & 0 \\ \bar{\sigma}^1 & \sigma^3 - \bar{\sigma}^3 & 0 & -\sigma^2 \\ \sigma^1 & 0 & \bar{\sigma}^3 - \sigma^3 & -\bar{\sigma}^2 \\ 0 & \sigma^1 & \bar{\sigma}^1 & -(\sigma^3 + \bar{\sigma}^3) \end{pmatrix},$$

(14.13)
$$\begin{cases} \Sigma^2 = \Sigma_{\ 1} = 2\Omega_{23} = d\sigma^2 - \sigma^2 \wedge \sigma^3, \\ \Sigma^1 = \Sigma_{\ 2} = 2\Omega_{01} = d\sigma^1 + \sigma^1 \wedge \sigma^3, \\ \Sigma^3 = -\tfrac{1}{2}\Sigma_{\ 3} = \Omega_{12} = d\sigma^3 + \tfrac{1}{2}\sigma^1 \wedge \sigma^2, \end{cases}$$

(14.14)
$$\Omega^{\alpha}{}_{\beta} = \tfrac{1}{2}\begin{pmatrix} \bar\Sigma^3 + \Sigma^3 & -\bar\Sigma^2 & -\Sigma^2 & 0 \\ \bar\Sigma^1 & \Sigma^3 - \bar\Sigma^3 & 0 & -\Sigma^2 \\ \Sigma^1 & 0 & \bar\Sigma^3 - \Sigma^3 & -\bar\Sigma^2 \\ 0 & \Sigma^1 & \bar\Sigma^1 & -(\Sigma^3 + \bar\Sigma^3) \end{pmatrix}.$$

If one differentiates formulas (14.3) then one finds, in general, that:

(14.15)
$$\Sigma^{\alpha}{}_{\beta} \wedge Z^3 = 0 ,$$

namely, in an isotropic frame:

(14.16)
$$-\tfrac{1}{2}\Sigma_3 \wedge Z^1 = \Sigma_2 \wedge Z^3, \quad -\tfrac{1}{2}\Sigma_3 \wedge Z^2 = \Sigma_1 \wedge Z^3, \qquad \Sigma_1 \wedge Z^1 = \Sigma_2 \wedge Z^2 .$$

If one refers to formula (9.48) then one finds that:

(14.17)
$$C'_{\alpha\beta} = C'_{\beta\alpha} .$$

From the reality conditions, it results, moreover, that $\bar E_{\alpha\bar\beta}$ is a Hermitian form:

(14.18)
$$\bar E_{\alpha\bar\beta} = E_{\bar\beta\alpha} .$$

One may further set:

(14.19)
$$\Sigma_\alpha = C_{\alpha\beta}\, Z^\beta + \frac{\lambda}{3}\,\gamma_{\alpha\beta}\, Z^\beta + E_{\alpha\bar\beta}\bar Z^\beta ,$$

in which:

(14.20)
$$C_{\alpha\beta}\,\gamma^{\alpha\beta} = 0, \quad \lambda = \gamma^{\alpha\beta}\, C'_{\alpha\beta} .$$

One has:

(14.21)
$$C_{33} = 4\, C_{12} ,$$

(14.22)
$$C'_{33} = C_{33} - \tfrac{2}{3}\lambda , \qquad C'_{12} = C_{12} + \tfrac{1}{3}\lambda ,$$

Example 1 ([9], pp. 379).

$$ds^2 = -\exp(k\,x^1)\{\cos\alpha\left[(dx^0)^2 - (dx^3)^2\right] + 2\sin\alpha\, dx^0\, dx^3\} - (dx^1)^2$$
$$- \exp(-2k\,x^1)\,(dx^2)^2 , \qquad \alpha = \sqrt{3}\,k\,x^1 .$$

One sets:

$$\sqrt{2}\,\theta^0 = \exp\left(\frac{kx^1}{2}\right)\left[\left(\cos\frac{\alpha}{2}+\sin\frac{\alpha}{2}\right)dx^0 - \left(\cos\frac{\alpha}{2}-\sin\frac{\alpha}{2}\right)dx^3\right]$$

$$\sqrt{2}\,\theta^3 = \exp\left(\frac{kx^1}{2}\right)\left[\left(\cos\frac{\alpha}{2}-\sin\frac{\alpha}{2}\right)dx^0 + \left(\cos\frac{\alpha}{2}+\sin\frac{\alpha}{2}\right)dx^3\right]$$

$$\sqrt{2}\,\theta^1 = dx^1 + i\exp(-k\,x^1)\,dx^2\,, \qquad \sqrt{2}\,\theta^2 = dx^1 - i\exp(-k\,x^1)\,dx^2\,;$$

then:

$$ds^2 = 2\,\theta^0\,\theta^3 - 2\,\theta^1\,\theta^2\,,$$

$$\theta^1\wedge\theta^2 = -\,i\exp(-k\,x^1)\,dx^1\wedge dx^2, \qquad dx^1 = \frac{\sqrt{2}}{2}(\theta^1+\theta^3)\,,$$

$$\sqrt{2}\,d\theta^0 = \frac{k}{2}dx^1\wedge\exp\left(\frac{kx^1}{2}\right)[1] + \exp\left(\frac{kx^1}{2}\right)d[1]\,,$$

$$d[1] = \frac{\sqrt{3}}{2}k\,dx^1\wedge[2]\,.$$

One also has:

$$d[2] = -\frac{\sqrt{3}}{2}k\,dx^1\wedge[1]\,,$$

$$\sqrt{2}\,d\theta^0 = \frac{k}{2}(\theta^1+\theta^2)\wedge\theta^0 + \frac{\sqrt{3}k}{2}(\theta^1+\theta^2)\wedge\theta^3$$

$$= \frac{k}{2}(\theta^1+\theta^2)\wedge(\theta^0+\sqrt{3}\,\theta^3)\,,$$

$$\sqrt{2}\,d\theta^3 = \frac{k}{2}(\theta^1+\theta^2)\wedge(\theta^0-\sqrt{3}\,\theta^3),$$

$$\sqrt{2}\,d\theta^1 = k\,\theta^1\wedge\theta^2\,, \qquad \sqrt{2}\,d\theta^2 = -\,k\,\theta^1\wedge\theta^2\,,$$

$$dZ^1 = d\theta^2\wedge\theta^3 - d\theta^3\wedge\theta^2$$

$$= \frac{k}{\sqrt{2}}\left[-\theta^1\wedge\theta^2\wedge\theta^3 + \frac{1}{2}\theta^1\wedge\theta^2\wedge\theta^3 - \frac{\sqrt{3}}{2}\theta^0\wedge\theta^1\wedge\theta^2\right],$$

$$dZ^2 = \frac{k}{\sqrt{2}}\left[-\frac{1}{2}\theta^0\wedge\theta^1\wedge\theta^2 + \frac{\sqrt{3}}{2}\theta^1\wedge\theta^2\wedge\theta^3 - \theta^0\wedge\theta^1\wedge\theta^2\right],$$

$$dZ^3 = \frac{k}{2\sqrt{2}}(-\,\theta^0\wedge\theta^1\wedge\theta^2 - \theta^0\wedge\theta^2\wedge\theta^3)\,,$$

hence:

$$\sigma^1 = -\frac{k}{\sqrt{2}}(\sqrt{3}\,\theta^0 - \theta^3)\,,\quad \sigma^2 = -\frac{k}{\sqrt{2}}(\theta^0 + \sqrt{3}\,\theta^3)\,,\quad \sigma^3 = -\frac{k}{\sqrt{2}}(\theta^1 - \theta^2)\,,$$

$$\Sigma_1 = d\sigma^2 - \sigma^2\wedge\sigma^3 = -\,k^2(\sqrt{3}\,Z^1 + Z^3)\,,$$

$$\Sigma_2 = d\sigma^1 + \sigma^1 \wedge \sigma^3 = -k^2\,(Z^1 - \sqrt{3}\,Z^3)\,,$$
$$\Sigma_3 = -2\,d\sigma^3 + \sigma^1 \wedge \sigma^2 = -4\,k^2\,Z^3\,.$$

Example 2 (type D – Schwarzschild).

$$ds^2 = \left(1-\frac{2m}{r}\right)dt^2 - \frac{1}{1-(2m/r)}\,dr^2 - r^2\,[(d\theta)^2 + \sin^2\theta\, d\Phi^2]\,,$$

$$\left(1-\frac{2m}{r}\right)^{1/2} dt - \left(1-\frac{2m}{r}\right)^{-1/2} dr = \sqrt{2}\,\theta^0\,, \qquad \sqrt{2}\,\theta^1 = r\,(d\theta + i\sin\theta\, d\Phi)$$
$$\left(1-\frac{2m}{r}\right)^{1/2} dt + \left(1-\frac{2m}{r}\right)^{-1/2} dr = \sqrt{2}\,\theta^3\,.$$
$$\sqrt{2}\,\theta^0 = \sqrt{2}\,\theta^3 = \frac{m}{r^2}\left(1-\frac{2m}{r}\right)^{-1/2}\theta^3 \wedge \theta^0,$$
$$\sqrt{2}\,\theta^1 = \frac{m}{r^2}\left(1-\frac{2m}{r}\right)^{-1/2}(\theta^3 - \theta^0)\wedge\theta^1 - \frac{\cot\theta}{r}\,\theta^1\wedge\theta^2\,.$$

$$\sqrt{2}\,dZ^1 = -\left[\frac{1}{r}\left(1-\frac{2m}{r}\right)^{1/2} + \frac{m}{r^2}\left(1-\frac{2m}{r}\right)^{-1/2}\right]\theta^0\wedge\theta^2\wedge\theta^3 + \frac{\cot\theta}{r}\theta^1\wedge\theta^2\wedge\theta^3,$$
$$\sqrt{2}\,dZ^2 = +\left[\frac{1}{r}\left(1-\frac{2m}{r}\right)^{1/2} + \frac{m}{r^2}\left(1-\frac{2m}{r}\right)^{-1/2}\right]\theta^3\wedge\theta^0\wedge\theta^1 + \frac{\cot\theta}{r}\theta^2\wedge\theta^0\wedge\theta^1,$$
$$\sqrt{2}\,dZ^3 = -\frac{1}{r}\left(1-\frac{2m}{r}\right)^{1/2}\theta^1\wedge\theta^2\wedge\theta^3 + \frac{1}{r}\left(1-\frac{2m}{r}\right)^{1/2}\theta^2\wedge\theta^0\wedge\theta^1\,.$$

$$\sigma^1 = -\frac{\sqrt{2}}{r}\left(1-\frac{2m}{r}\right)^{1/2}\theta^2, \qquad \sigma^2 = -\frac{\sqrt{2}}{r}\left(1-\frac{2m}{r}\right)^{1/2}\theta^1,$$
$$\sigma^3 = -\frac{m}{r^2\sqrt{2}}\left(1-\frac{2m}{r}\right)^{-1/2}(\theta^0 + \theta^3) + \frac{1}{\sqrt{2}}\frac{\cot\theta}{r}(\theta^1 - \theta^2)\,.$$

$$\Sigma = \begin{pmatrix} 0 & 2m/r^3 & 0 \\ 2m/r^3 & 0 & 0 \\ 0 & 0 & 8m/r^3 \end{pmatrix}$$

REMARKS. – *a*) The congruences $h_a^{(0)}, h_a^{(3)}$ are integrable. One has:

$$\theta^0 = \left(1-\frac{2m}{r}\right)^{1/2} d\Phi_1\,, \qquad \Phi_1 = t - r - 2m\log(r-2m)\,,$$

$$\theta^3 = \left(1-\frac{2m}{r}\right)^{1/2} d\Phi_2\,, \qquad\qquad \Phi_2 = t - r - 2m \log\,(r - 2m)\,;$$

Φ_1, Φ_2 are the Kruskal variables [10].

b) In an orthonormal frame $r\,t$, $\theta\,t$, $\Phi\,t$:

$$\Sigma = \begin{pmatrix} 2m/r^3 & 0 & 0 \\ 0 & -m/r^3 & 0 \\ 0 & 0 & -m/r^3 \end{pmatrix}.$$

The geodesic deviation (2.12) for an observer $u^a = (1, 0, 0, 0)$ may be written:

$$(\mathrm{D}^2\eta^a / \mathrm{D}s^2) + R^a{}_{00b}\,\eta^b = 0\,,$$

namely:

$$\frac{\mathrm{D}^2\eta^r}{\mathrm{D}s^2} - \frac{2m}{r^3}\eta^r = 0\,, \qquad \frac{\mathrm{D}^2\eta^\theta}{\mathrm{D}s^2} + \frac{2m}{r^3}\eta^\theta = 0\,, \qquad \frac{\mathrm{D}^2\eta^\Phi}{\mathrm{D}s^2} + \frac{2m}{r^3}\eta^\Phi = 0\,.$$

Example 3 (NUT space) [11].

$$ds^2 = f^{\,2}\left(dt + 4l\sin^2\frac{\theta}{2}d\Phi\right)^2 - \frac{1}{f^{\,2}}dr^2 - (r^2 + l^2)\,[d\theta^2 + \sin^2\theta\, d\Phi^2\,]\,,$$

$$f^2 = 1 - 2\frac{r + l^2}{r^2 + l^2}\,.$$

One finds that:

$$\sqrt{2}\,\theta^0 = f(r)\left(dt + 4l\sin^2\frac{\theta}{2}d\Phi\right)^2 - \frac{1}{f}dr$$

$$\sqrt{2}\,\theta^3 = f(r)\left(dt + 4l\sin^2\frac{\theta}{2}d\Phi\right)^2 + \frac{1}{f}dr$$

$$\sqrt{2}\,\theta^2 = \sqrt{r^2 + l^2}\,(d\theta + i\sin\theta\, d\Phi)\,.$$

One also finds that:

$$\sigma^1_2 = \sigma^2_1 = -\sqrt{2}\,f\,(r - i\,l)\,/\,(r^2 + l^2)\,,$$

$$\sigma^3_0 = \sigma^3_3 = \frac{-1}{\sqrt{2}}\left(f' + \frac{i\,l\,f}{r^2 + l^2}\right), \quad \sigma^3_1 = -\sigma^3_2 = \frac{1}{\sqrt{2}}\frac{\cot\theta}{\sqrt{r^2 + l^2}}\,.$$

The other components σ^α_a are null. One further finds that:

$$\Sigma = \begin{pmatrix} 0 & C_{12} & 0 \\ C_{12} & 0 & 0 \\ 0 & 0 & 4C_{12} \end{pmatrix},$$

with:

$$C_{12} = 2\,(m\,r^2 + 2\,l^2\,r - m\,l^2)\,(r - i\,l) / (r^2 + l^2)^3\ .$$

This space contains the preceding one as a particular case if $l = 0$.

Example 4 (spaces with plane waves and parallel rays) [12].

$$ds^2 = -\,2\,dw\,d\bar{w} + 2\,du\,[dv + H(w,\ d\bar{w},\ u)\,du\,]\ .$$

If $\theta^0 = du$, $\theta^3 = dv + H\,du$, $\theta^3 = dw$ then one finds that:

$$\sigma^1 = H_w\,\theta^0\ , \qquad \sigma^2 = \sigma^3 = 0\ .$$

If $H_{w\bar{w}} = 0$ then one finds that:

$$\Sigma = \begin{pmatrix} 0 & 0 & 0 \\ 0 & C_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

with $C_{22} = -\,2\,H_{ww}\,Z^2$.

15. **Introduction to the decomposition of the Riemann tensor. –** To a bivector F_{ab}, one may associate a form that is linear in Z^α, $\bar{Z}^\alpha$:

(15.1) $$F = F_\alpha Z^\alpha + \bar{F}_\alpha \bar{Z}^\alpha\ .$$

To a tensor $R_{ab,cd}$, in view of its symmetry properties, there will be associated a form that is quadratic in Z^α, $\bar{Z}^\alpha$. We set:

(15.2) $$R = C'_{\alpha\beta} Z^\alpha Z^\beta + E_{\alpha\bar{\beta}} Z^\alpha \bar{Z}^\beta + \bar{C}'_{\alpha\beta} \bar{Z}^\alpha \bar{Z}^\beta + E_{\bar{\alpha}\beta} \bar{Z}^\beta Z^\alpha\ ,$$

in which:

(15.3) $$C'_{\alpha\beta} = C'_{\beta\alpha}\ , \qquad E_{\alpha\bar{\beta}} = \bar{E}_{\bar{\beta}\alpha}\ .$$

$C'_{\alpha\beta}$ is a symmetric quadratic form and $E_{\alpha\bar{\beta}}$ is a Hermitian form. We further set:

(15.4) $$C'_{\alpha\beta} = C_{\alpha\beta} + \tfrac{1}{3}\,\lambda\,\gamma_{\alpha\beta}\ ,$$

in which:

(15.5) $$C_{\alpha\beta}\,\gamma^{\alpha\beta} = 0\ , \qquad \lambda = \gamma^{\alpha\beta}\,C'_{\alpha\beta}\ .$$

One immediately sees that the 20-dimensional space of tensors R decomposes into a direct sum of spaces of dimensions 1 for λ, 5 for $C_{\alpha\beta}$ and its conjugate, and 9 for the tensor $E_{\alpha\bar{\beta}}$. One easily verifies that one has:

(15.6) $$C' = C'_{\alpha\beta} Z^a Z^b + \text{conj.} = \tfrac{1}{2}(R - {*R*}) ,$$

(15.7) $$E' = E_{\alpha\bar{\beta}} Z^a Z^b + \text{conj.} = \tfrac{1}{2}(R + {*R*}) ,$$

(15.8) $$\overset{+}{C}' = C'_{\alpha\beta} Z^a Z^b = \tfrac{1}{2}(C' - i\,\overset{*}{C}') ,$$

(15.9) $$*C'* = -C' , \qquad *E'* = E .$$

16. **Tensorial decomposition.** – Let us calculate:

(16.1) $$E_{abcd} = \tfrac{1}{2}(R_{abcd} + R_{\overset{*}{a}b\overset{*}{c}d}) .$$

Note that:

(16.2) $$\eta_{abcd}\ \eta_{rstu} = -\begin{vmatrix} g_{ar} & g_{as} & g_{at} & g_{au} \\ g_{br} & g_{bs} & g_{bt} & g_{bu} \\ g_{cr} & g_{cs} & g_{ct} & g_{cu} \\ g_{dr} & g_{ds} & g_{dt} & g_{du} \end{vmatrix} .$$

We obtain:

(16.3) $$\begin{aligned} R_{\overset{*}{a}b\overset{*}{c}d} &= \tfrac{1}{4}\,\eta_{abcd}\, R^{rstu}\ \eta_{rstu} \\ &= -\,(R_{abcd} + g_{ac}\, R_{bd} + g_{bd}\, R_{ac} - g_{ad}\, R_{bc} - g_{bc}\, R_{ad}\ - \tfrac{1}{2}\, g_{abcd}) \\ &= -\,R_{abcd} - (g_{ac}\, S_{bd} + \ldots), \end{aligned}$$

in which:

(16.4) $$S_{ab} = R_{ab} - \tfrac{1}{4}\, g_{ab}\, R ,$$

(16.5) $$E_{abcd} = -\tfrac{1}{2}(g_{ac}\, S_{bd} + g_{bd}\, S_{ac} - g_{ad}\, S_{bc} - g_{bc}\, S_{ad}) .$$

Note that:

(16.6) $$E_{ab} = g^{bc}\, E_{abcd} = S_{ad} ,$$

(16.7) $$S_{ab}\, g^{ab} = 0 .$$

The spaces $S_{ab} = 0$ are the *Einstein spaces* of the "geometers."
One immediately finds that:

(16.8) $$C'_{abcd} = \tfrac{1}{2}(R_{abcd} - {*R*}_{abcd}) = R_{abcd} + \tfrac{1}{2}(g_{ac}\, S_{bd} + \ldots),$$

and if:

(16.9) $$C'_{abcd} = C_{abcd} + \frac{\lambda}{3}\, g_{abcd}$$

then:

(16.10) $$C_{ab} = 0 ,$$

(16.11) $$C'_{ad} = R_{ad} - S_{ad} = -\,\lambda\, g_{ad} ,$$

(16.12) $$\lambda = -\,R/4 ,$$

and one finally has:

(16.13) $$C_{abcd} = R_{abcd} + \tfrac{1}{2}(\, g_{ac}\, S_{bd} + \ldots) + \frac{R}{12} g_{abcd}\,.$$

The tensor C_{abcd} is the *conformal curvature tensor* of Hermann Weyl. The spaces:

(16.14) $$C_{abcd} = 0$$

are conformally Euclidian, and have a ds^2 of the form:

(16.15) $$ds^2 = e^{2\psi}\,[(dx^0)^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2].$$

One finally has:

(16.16) $$R_{abcd} = C_{abcd} + E_{abcd} + G_{abcd}\,,$$

in which:

(16.17) $$G = - (R/12)\, g_{abcd}\,.$$

$R_{ab,cd}$ may be represented by a 6×6 matrix. Hence, thanks to (16.13), it may be represented in the form of a sum of 6×6 matrices:

(16.18) $$\boldsymbol{R} = \boldsymbol{C} + \boldsymbol{E} + \boldsymbol{G}\,.$$

After passing to the self-adjoint bivectorial variables $Z^\alpha, \bar{Z}^\alpha$ one has [cf. (15.6) and (15.7)]:

(16.19) $$\boldsymbol{R} = \left[\begin{array}{c|c} C_{\alpha\beta} & 0 \\ \hline 0 & \bar{C}_{\alpha\beta} \end{array}\right] + \left[\begin{array}{c|c} 0 & E_{\alpha\beta} \\ \hline \bar{E}_{\alpha\bar{\beta}} & 0 \end{array}\right] - \frac{R}{12}\left[\begin{array}{c|c} \gamma_{\alpha\beta} & 0 \\ \hline 0 & \bar{\gamma}_{\alpha\beta} \end{array}\right],$$

thus:

(16.20) $$\Omega = \boldsymbol{R}\begin{bmatrix} Z \\ \bar{Z} \end{bmatrix} = \tfrac{1}{2}\begin{bmatrix} \Sigma \\ \bar{\Sigma} \end{bmatrix},$$

in which:

(16.21) $$\Sigma_a = C_{ab}\, Z^3 + E_{\alpha\bar{\beta}}\bar{Z}^\beta - \frac{R}{6}\,\gamma_{\alpha\beta}\, Z^\beta\,.$$

It results from paragraph 13 that under the effect of a proper homogeneous Lorentz transformation the components of $C_{\alpha\beta}$ are transformed by the rotations of $O_3(\mathbb{C})$. They are the *irreducible components of the Riemann tensor.*

17. **Petrov classification.** – The Petrov classification finds its origin in the fact that the two forms:

(17.1) $$\gamma_{\alpha\beta}\, Z^\alpha Z^\beta\,, \qquad C_{\alpha\beta}\, Z^\alpha Z^\beta\,,$$

are invariant under $O_3(\mathbb{C})$.

Operating in the 6-dimensional space of pairs of forms:

(17.2) $$R_{abcd}\,,\;\; g_{abcd}\,,$$

Petrov was led ([9], pp. 371) to distinguish three types of Einstein spaces according to the number of elementary divisors of the pair (17.2). It is simpler, as J. Géhéniau [14] has pointed out, to operate in the space E_3 with the pairs (17.1).

If one considers the characteristic matrix:

(17.3) $$D_\alpha^\beta = C_\alpha^\beta - \delta_\alpha^\beta\,\lambda$$

then the classification may be effected by starting with the elementary divisors that exhibit the proper values and certain numerical invariants.

Let λ_1 be a proper value and let l_0 be its multiplicity; $(\lambda - \lambda_0)^{l_0}$ is then a factor of $\det(D_\alpha^\beta)$. Let $(\lambda - \lambda_1)^{l_1}$ be the common factor of the minors of order $n - 1$, $(\lambda - \lambda_2)^{l_2}$, the common factor of the minors of order $n - 2$, etc.; the elementary divisors are $(\lambda - \lambda_1)^{e_0}$, $(\lambda - \lambda_1)^{e_1}$, …, with the exponents $e_0 = l_0 - l_1$, $e_1 = l_1 - l_2$, … Two equivalent pairs have the same elementary divisors, and conversely.

The three cases T_1, T_2, T_3 correspond to the existence of 3, 2, 1 elementary divisors, resp.

Indeed, one distinguishes the different cases by their Segre characteristics, from which one finds the exponents of the elementary divisors and the multiplicities of the roots.

An equivalent classification is based on the behavior of the common roots of $\gamma_{\alpha\beta}\, Z^\alpha Z^\beta = C_{\alpha\beta}\, Z^\alpha Z^\beta = 0$.

Geometrically, the problem under discussion is that of the relative positions of a pair of conics.

Table I summarizes the possible cases and their properties.

18. **Characteristic isotropic vectors**. – If one refers to the corollary in section 10 then to each pair $Z_{ab}^\alpha, \bar{Z}_{ab}^\alpha$ of singular bivectors there is associated a real isotropic vector.

The four points of intersection of:

(18.1) $$\gamma_{\alpha\beta}\, Z^\alpha Z^\beta = C_{\alpha\beta}\, Z^\alpha Z^\beta = 0$$

and thus:

$$\bar{\gamma}_{\alpha\beta} \bar{Z}^\alpha \bar{Z}^\beta = \bar{C}_{\alpha\beta} \bar{Z}^\alpha \bar{Z}^\beta = 0$$

define four *characteristic isotropic vectors* that are associated with the Weyl tensor. They therefore have a conformal significance.

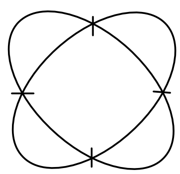
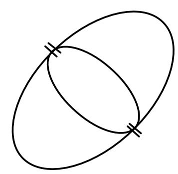
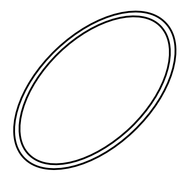
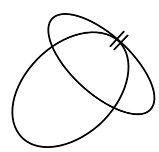
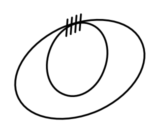
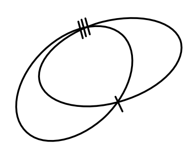
These vectors have already been considered by Ruse [15] and Penrose [17] in the spinorial formalism, and by Debever [15] in tensorial form.

We use a normal isotropic frame to establish the fundamental formulas that relate to characteristic vectors. Therefore, let:

$$(18.2) \qquad \overset{+}{C} = C_{11}\, Z^1 Z^1 + 2\, C_{13}\, Z^1 Z^3 + 2\, C_{12}\, [Z^1 Z^2 + 2\, (Z^3)^2\,] + 2\, C_{23}\, Z^3 Z^3 + C_{22}\, (Z^3)^2 ,$$

$$\overset{+}{\gamma} = 2\, Z^1 Z^2 - 2\, (Z^3)^2 .$$

TABLE 1

[1, 1, 1] [1, 1, 1] I  $(\lambda - \lambda_1), (\lambda - \lambda_2), (\lambda - \lambda_3)$	[(1, 1), 1] (2, 2) D  $(\lambda - \lambda_1), (\lambda - \lambda_1), (\lambda - \lambda_2)$	[(1, 1, 1)] (–) C_0  $(\lambda - \lambda_1), (\lambda - \lambda_1), (\lambda - \lambda_1)$	T_1
	[2, 1] (2, 1, 1) II  $(\lambda - \lambda_1)^2, (\lambda - \lambda_3)$	[2, 1] (4) N  $(\lambda - \lambda_1)^2, (\lambda - \lambda_3)$	T_2
		[3] (3, 1) III  $(\lambda - \lambda_1)^3$	T_3

In order to determine the common solutions of (18.1) and (18.2), set:

$$(18.3) \qquad Z^1 = \lambda^2 , \qquad Z^2 = \mu^2 , \qquad Z^3 = \lambda\, \mu .$$

One will obtain the biquadratic equation:

$$(18.4) \qquad C_{11}\, \lambda^4 + 2\, C_{13}\, \lambda^3 \mu + 6\, C_{12}\, \lambda^2 \mu^2 + 2\, C_{23}\, \lambda \mu^3 + C_{22}\, \mu^4 = 0 .$$

To the isotropic vector $h^{(0)}$, to which we have associated the pencil $A_2 Z^2 + A_3 Z^3$, we associate the point:

(18.5) $$\lambda = 1, \qquad \mu = 0 .$$

$h^{(0)}$ is simply characteristic if:

(18.6) $$C_{11} = 0 , \qquad C_{13} \neq 0 \qquad \text{(type I)};$$

$h^{(0)}$ is doubly characteristic if:

(18.7) $$C_{11} = C_{13} = 0 , \qquad C_{12} \neq 0 \qquad \text{(type II or D)};$$

$h^{(0)}$ is triply characteristic if:

(18.8) $$C_{11} = C_{13} = C_{12} = 0 , \qquad C_{23} \neq 0 \qquad \text{(type III)};$$

$h^{(0)}$ is quadruply characteristic if:

(18.9) $$C_{11} = C_{13} = C_{12} = C_{23} = 0 , \qquad C_{22} \neq 0 \qquad \text{(type N)};$$

If:

(18.10) $$C_{ij} = 0$$

then the space is C_0 or conformally Euclidian. $h^{(0)}$ and $h^{(3)}$ are doubly characteristic if:

(18.11) $$C_{11} = C_{13} = C_{12} = C_{23} = 0 , \qquad C_{12} \neq 0 \qquad \text{(type D)} .$$

Tensorially [cf. (10.32)], one thus has:

(18.12) $$2 \overset{+}{C}_{abcd} \, h^b_{(3)} h^c_{(3)} = -C_{11} h_a^{(2)} h_d^{(2)} - C_{13} (h_a^{(2)} h_d^{(0)} + h_d^{(2)} h_a^{(0)}) + C_{12} h_a^{(0)} h_d^{(0)} .$$

If $h_a^{(0)}$ is simply characteristic then one therefore has, upon setting $h_a^{(0)} = k_a$, $h^a_{(3)} = k^a$:

(18.13) $$C_{abcd} \, k^b \, k^c = k_a \, p_d + k_d \, p_a ,$$

(18.14) $$\overset{*}{C}_{abcd} \, k^b \, k^c = k_a \, p'_d + k_d \, p'_a ;$$

one has:

(18.15) $$p_a \, p^a = - p'_a \, p'^a = - 2 \, C_{13} \bar{C}_{13} , \qquad p_a \, p'^a = 0 .$$

If k^a is doubly characteristic then:

(18.16) $$C_{abcd} \, k^b \, k^c = \tfrac{1}{2} (C_{11} + \bar{C}_{12}) \, k_a \, k_d ,$$

(18.17) $$\overset{*}{C}_{abcd} \, k^b \, k^c = \tfrac{1}{2} (C_{11} - \bar{C}_{12}) \, k_a \, k_d ,$$

If k^a is triply characteristic then:

(18.18) $$\overset{+}{C}_{abcd}\, k^b\, k^c = 0\,,$$

(18.19) $$\overset{+}{C}_{abcd}\, k^d = \tfrac{1}{2} C_{23}\, Z^{(2)}_{ab}\, k_c\,,$$

(18.20) $$C_{abcd}\, k^d = a_{ab}\, k_c\,,$$

(18.21) $$\overset{*}{C}_{abcd}\, k^d = \overset{*}{a}_{ab}\, k_c\,,$$

(18.22) $$\tfrac{1}{2} a_{ab}\, a^{ab} = -\, a_{ab}\, \overset{*}{a}{}^{ab} = -\,(C_{23})^2 + (\bar{C}_{23})^2\,.$$

If k^a is quadruply characteristic then:

(18.23) $$C_{abcd}\, k^d = \overset{*}{C}_{abcd}\, k^d = 0\,.$$

In summary, one has the following tensorial conditions ([2]):

(18.24) $$\mathrm{N}_{abcd}\, k^d = 0 \quad \leftrightarrow \quad \overset{*}{\mathrm{N}}_{abcd}\, k^d = 0$$

(18.25) $$\mathrm{III}_{abc[d}\, k^c\, k_{e]} = 0 \quad \leftrightarrow \quad \overset{*}{\mathrm{III}}_{abc[d}\, k^c\, k_{e]} = 0$$

(18.26) $$\begin{cases} \mathrm{D}_{abc[d}\, k^b k^c k_{e]} = 0 \quad \leftrightarrow \quad \overset{*}{\mathrm{D}}_{abc[d}\, k^b k^c k_{e]} = 0, \\ \mathrm{D}_{abc[d}\, m^b m^c m_{e]} = 0 \quad \leftrightarrow \quad \overset{*}{\mathrm{D}}_{abc[d}\, m^b m^c m_{e]} = 0, \end{cases}$$

(18.27) $$\mathrm{II}_{abc[d}\, k_{e]}\, k^b\, k^e = 0 \quad \leftrightarrow \quad \overset{*}{\mathrm{II}}_{abc[d}\, k_{e]}\, k^b\, k^e = 0\,,$$

(18.28) $$k_{[e}\, \mathrm{I}_{a]\,bc[d}\, k_{f]}\, k^b\, k^c = 0 \quad \leftrightarrow \quad k_{[f}\, \mathrm{I}_{a]\,bc[d}\, k_{e]}\, k^b\, k^c = 0\,.$$

N, III, … indicate the C tensors of the corresponding type.

↔ indicates equivalent conditions; conditions (18.24) to (18.28) each imply the following one. Finally, here is a schema (Fig. 7) that indicates the degeneracies [17].

Fig.7.

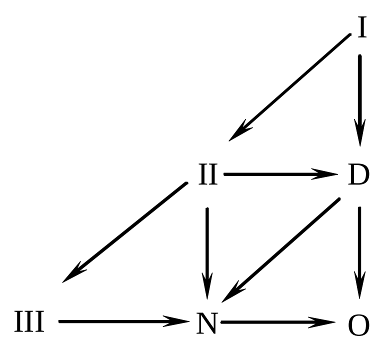


Image of the Petrov classification in P_3. – The characteristic isotropic vectors determine four real points of the absolute and four pairs of conjugate rectilinear generators (Fig. 8).

[2] Here, we have reverted to the very suggestive notations of Sachs [16].

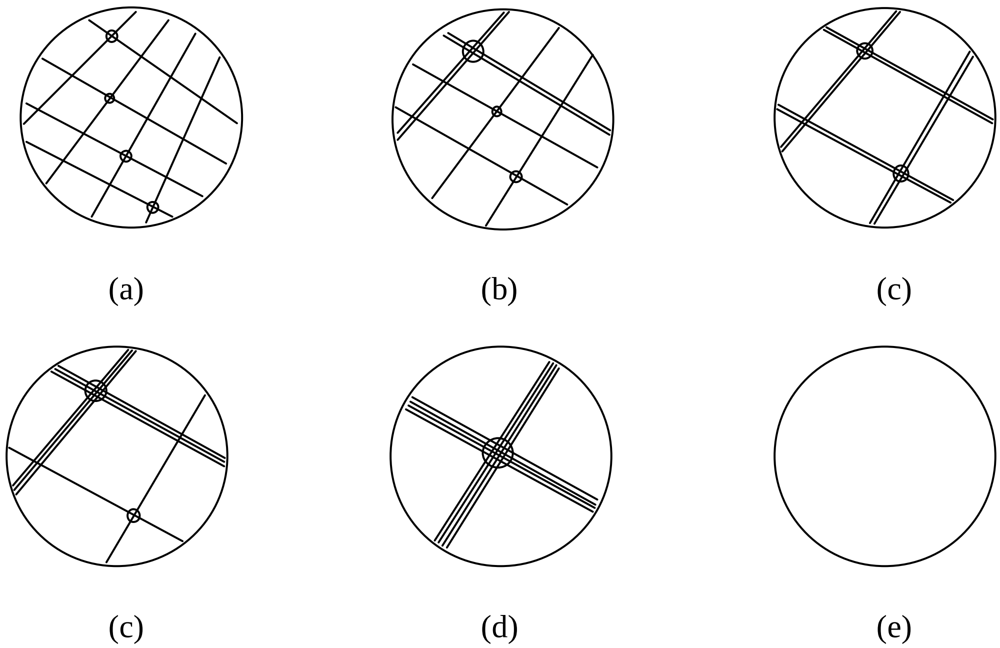


Fig. 8. – a) Type I; b) type II; c) type D; d) type III; e) type N; f) C_0.

If one recalls that a twice-covariant tensor T_{ab} (12.10) is associated with every pair of isotropic vectors then one may predict that one may associate a completely symmetric tensor T_{abcd} with four isotropic vectors; this is the tensor of L. Bel [18]:

$$T_{abcd} = 2(C_{arcs}\, C_b{}^r\, C_d{}^a + \overset{*}{C}_{arcs}\, \overset{*}{C}{}_b{}^r{}_d{}^s) .$$

19. **Particular frames. Reduced forms** [19]. – *Type* I. – We introduce a canonical frame S that is qualified by the following symmetry:

(19.1) $$C_{ab} = \begin{pmatrix} C_{11} & C_{12} & 0 \\ C_{21} & C_{22} & 0 \\ 0 & 0 & 4C_{12} \end{pmatrix}.$$

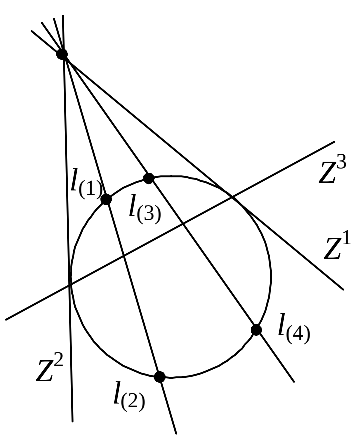


Fig. 9.

Geometrically, in the plane Z^α the frame is uniquely determined if one associates an involution by means of pairs of characteristic vectors. The frame S is such, that the lines that join the pairs $l_{(1)}$, $l_{(2)}$, and $l_{(3)}$, $l_{(4)}$ of the characteristic vectors belong to the pencil $\lambda\, Z^1 + \mu\, Z^2$ (Fig. 9). In the space P_3 the characteristic vectors are four distinct points of the absolute oval quadric, and the line $h^{(0)}$, $h^{(3)}$ of the frame is an axis of (left) involution that permutes $l_{(1)}$, $l_{(2)}$, and $l_{(3)}$, $l_{(4)}$, respectively.

The rest of the frame is indeterminate up to a transformation of the group G_2:

(19.2) $$Z^{1'} = e^{-w} Z^1 , \qquad Z^{2'} = e^{w} Z^2 , \qquad Z^{3'} = Z^3 ,$$
for which one has:
(19.3) $$C'_{12} = C_{12} , \qquad C'_{11}C'_{22} = C_{11}\, C_{22} .$$

One determines the frame by the condition that:

(19.4) $$C_{11} + C_{22} = 0 ,$$
hence:

(19.5) $$C_{\alpha\beta} = \begin{pmatrix} A & B & 0 \\ B & -A & 0 \\ 0 & 0 & 4B \end{pmatrix} .$$

The four characteristic vectors are determined by starting with the biquadratic equation:
(19.6) $$\lambda^4 - \mu^4 + 6\,(B/A)\,\lambda^2\,\mu^2 = 0 .$$

If:
(19.7) $$3\,B/A = -\sinh\gamma$$
then the roots are such, that:
(19.8) $$(\lambda_1/\mu_1)^2 = e^{\gamma} , \qquad (\lambda_2/\mu_2)^2 = e^{-\gamma} ;$$

the four characteristic isotropic vectors may then be expressed as follows:

(19.9) $$l_b^{(a)} = \begin{bmatrix} \rho & e^{-i\Phi} & e^{i\Phi} & 1/\rho \\ \rho & -e^{-i\Phi} & -e^{i\Phi} & 1/\rho \\ 1/\rho & -ie^{i\Phi} & ie^{-i\Phi} & \rho \\ 1/\rho & ie^{i\Phi} & -ie^{-i\Phi} & \rho \end{bmatrix} ,$$

in which $\rho\, e^{i\Phi} = e^{\gamma/2}$. The characteristic vectors have an invariant *bi-ratio*:

(19.10) $$b = (\sinh\gamma - i) / (\sinh\gamma + i) .$$

They form a *harmonic group* if $b^2 = 1, 4$, or ¼, namely:

(19.11) $$\gamma = 0, \quad B = 0 \quad \text{or} \quad B = \pm\, i\, A .$$

They form an *equi-anharmonic group* if $b\bar{b} = 1$, $b + \bar{b} = 1$:

(19.12) $$\sinh^2\gamma = 3, \qquad A^2 = 3\,B^2 .$$

Example I of section 14 corresponds to one such type of vacuum space.

Finally, the four isotropic vectors are coplanar if:

(19.13) $$(B/A + \bar{B}/\bar{A}) = 0 .$$

The *characteristic roots* of $C_{\alpha\beta}$ in the frame S have the property that:

(19.14) $$\begin{vmatrix} A & B-\lambda & 0 \\ B-\lambda & -A & 0 \\ 0 & 0 & 4B+2\lambda \end{vmatrix} = 0 ,$$

hence:

(19.15) $$\lambda_1 = -2B , \quad \lambda_2 = B - iA, \quad \lambda_3 = B + iA .$$

One has:

(19.16) $$\lambda_1 + \lambda_2 + \lambda_3 = 0 .$$

The harmonic spaces therefore correspond to the case in which one of the characteristic roots is null.

The spaces for which the vectors $l_b^{(a)}$ are coplanar correspond to the case in which the proper values have the same modulus.

Singular spaces. – We use the term "singular spaces" to refer to the ones that are not in class I. In the sequel, we specify the formulas in a normal isotropic frame.

If we exclude case C_0 then there exists at least one characteristic isotropic vector at each point of V_4, namely, $h^{(0)}$.

The local frame is determined in the subgroup G_4 of $L_+^{\uparrow}$ that preserves the direction of $h^{(0)}$, namely:

(19.17) $$\begin{cases} \theta^{0'} = e^a \theta^0 \quad (a > 0), \\ \theta^{1'} = e^{ib}(\theta^1 + \bar{\gamma}\theta^0), \\ \theta^{2'} = e^{-ib}(\theta^2 + \gamma\theta^0), \\ \theta^{3'} = e^{-a}(\theta^3 + \gamma\theta^1 + \bar{\gamma}\theta^2 + \gamma\bar{\gamma}\theta^0), \end{cases}$$

or furthermore:

(19.18) $$\begin{aligned} Z^{1'} &= e^{-w}(Z^1 + \gamma^2 Z^2 + 2\gamma Z^3) \qquad (w = a + i\,b), \\ Z^{2'} &= e^{w} Z^2 , \\ Z^{3'} &= \gamma Z^2 + Z^3 . \end{aligned}$$

These transformations induce the following transformations on the connection form:

(19.19) $$\begin{aligned} \sigma^{1'} &= e^{-w}(\sigma^1 + \gamma^2 \sigma^2 + 2\gamma\sigma^3 - 2\,d\gamma) , \\ \sigma^{2'} &= e^{w}\sigma^2 , \\ \sigma^{3'} &= \sigma^3 + \gamma\sigma^2 - dw , \end{aligned}$$

and on the curvature form:

(19.20)
$$\begin{cases} C_{11} = e^{-2w} C'_{11}, \\ C_{13} = 2\gamma e^{-2w} C'_{11} + e^{-w} C'_{13}, \\ C_{12} = \gamma^2 e^{-2w} C'_{11} + \gamma e^{-w} C'_{13} + C'_{12}, \\ C_{23} = 2\gamma^3 e^{-2w} C'_{11} + 3\gamma^2 e^{-w} C'_{13} + 6\gamma C'_{12} + e^{w} C'_{23}, \\ C_{22} = \gamma^4 e^{-2w} C'_{11} + 2\gamma^3 e^{-w} C'_{13} + 6\gamma^2 C'_{12} + 2\gamma e^{w} C'_{23} + e^{2w} C'_{22}. \end{cases}$$

One will note the invariant character of conditions (18.5) and (18.19).

Type D. – If $h^{(0)}$ and $h^{(3)}$ are doubly characteristic vectors then one may construct:

(19.21)
$$C_{\alpha\beta} = \begin{pmatrix} 0 & B & 0 \\ B & 0 & 0 \\ 0 & 0 & 4B \end{pmatrix}.$$

This frame is further determined up to a transformation from the group G_2, since C_{12} is invariant. The complete determination of the frame results indirectly from work of Kerr [20]. Examples 2 and 3 of section 14 correspond to vacuum solution of type D.

Type II. Since $h^{(0)}$ is doubly characteristic and C_{12} is non-null, one may construct:

(19.22)
$$C_{\alpha\beta} = \begin{pmatrix} 0 & C_{12} & 0 \\ C_{12} & 0 & 0 \\ 0 & 0 & 4C_{12} \end{pmatrix}.$$

The frame is determined up to a transformation of the group G_2, and one has:

(19.23)
$$C'_{12} = C_{12}, \qquad C'_{22} = e^{-2w} C_{22}.$$

Since C_{22} is not null, one may determine the frame by the condition:

(19.24)
$$C'_{22} = 1.$$

Other choices may be imposed by the consideration of geodesic congruences.

Type III. Since $h^{(2)}$ is a triply characteristic vector one has:

(19.25)
$$C_{\alpha\beta} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & C_{22} & C_{23} \\ 0 & C_{23} & 0 \end{pmatrix}.$$

The frame is determined up to a transformation of G_4, and one has:

$$C'_{23} = e^{-w} C_{23} , \qquad C'_{22} = 2\gamma e^{-w} C_{23} + e^{-2w} C_{22} .$$

One may determine the frame by conditions such as $C_{22} = 0$, $C_{23} = 1$.

Type N. Since $h^{(0)}$ is quadruply characteristic, one may construct:

$$C_{\alpha\beta} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & C_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and one may restrict oneself to the group G'_2.

$$Z^{1'} = Z^1 + \gamma^2 Z^2 + 2\,\gamma Z^3, \qquad Z^{2'} = Z^2, \qquad Z^{3'} = Z^3 + \gamma Z^2,$$

thanks to the condition $C_{22} = 1$. The complete determination of the frame is related to the work of Kerr [20]. Example 4 of section corresponds to the vacuum spaces of type N.

Remark. Other particular choices of frames may be imposed by the demands of particular problems.

20. **Curvature invariants** [21]. – THEOREM. – In general, there exist 14 curvature invariants.

Indeed, once the frame is chosen, other than R, the coefficients $E_{\alpha\beta}$ – hence, 9 real functions – are invariant. Moreover, $C_{\alpha\beta}$ possesses two complex invariants:

(20.1) $$C_\alpha^\beta C_\beta^\alpha , \qquad C_\alpha^\beta C_\beta^\gamma C_\gamma^\alpha ,$$

respectively.

The *pure invariants* relate to the irreducible components of R_{abcd} :

a) 4 invariants associated with the conformal curvature:

(20.2) $$C_{(2)}^{(1)} = C_{ab}{}^{cd}\, C_{cd}{}^{ab} ,$$

(20.3) $$C_{(2)}^{(2)} = C_{ab}{}^{cd}\, \overset{*}{C}{}_{cd}{}^{ab} ,$$

(20.4) $$C_{(3)}^{(1)} = C_{ab}{}^{cd}\, C_{cd}{}^{rs}\, C_{rs}{}^{ab},$$

(20.5) $$C_{(3)}^{(2)} = C_{ab}{}^{cd}\, \overset{*}{C}{}_{cd}{}^{rs}\, C_{rs}{}^{ab} ,$$

b) 3 invariants associated with the tensor $S_{\alpha\beta}$:

(20.6) $$E_{(1)} = S_{ab}\, S^{ab} , \qquad E_{(2)} = S_{ab}\, S^{bc}\, S_c{}^a , \qquad E_{(3)} = S_{ab}\, S^{bc}\, S_{cd}\, S^{da} ;$$

c) The scalar curvature:

(20.7) $$R\,.$$

The *mixed invariants*, which are 6 in number and fix the ratios of the irreducible components C_{abcd} and E_{abcd} . Let:

(20.8) $$D_{ab} = C_{arsb}\, S^{rs}\,,$$

(20.9) $$\overset{*}{D}_{ab} = \overset{*}{C}_{arsb}\, S^{rs}\,.$$

One may consider the invariants:

(20.10) $$D_{(3)} = D_{ab}\, S^{ab}\,, \qquad \overset{*}{D}_{(3)} = \overset{*}{D}_{ab}\, S^{ab}\,,$$

(20.11) $$D_{(4)} = D_{ab}\, S^{ab}\,, \qquad \overset{*}{D}_{(4)} = \overset{*}{D}_{ab}\, D^{ab}\,,$$

(20.12) $$D_{(5)} = D_{ab}\, D^{bc}\, S_c{}^a\,, \qquad \overset{*}{D}_{(5)} = \overset{*}{D}_{ab}\, \overset{*}{D}^{bc}\, S_c{}^a\,.$$

Summary

(20.13)
$$\begin{array}{c|c|c} \text{Degree} & \text{pure invariant} & \text{mixed invariants} \\ \hline 1 & R & \_ \\ 2 & \mathrm{Tr}\, C^2, \mathrm{Tr}\, C\overset{*}{C}, \mathrm{Tr}\, S^2 & \_ \\ 3 & \mathrm{Tr}\, C^3, \mathrm{Tr}\, C^2\overset{*}{C}, \mathrm{Tr}\, S^3 & \mathrm{Tr}\, DS, \quad \mathrm{Tr}\, \overset{*}{D}S \\ 4 & \mathrm{Tr}\, S^4 & \mathrm{Tr}\, D^2, \quad \mathrm{Tr}\, \overset{*}{D}{}^2 \\ 5 & \_ & \mathrm{Tr}\, DDS, \mathrm{Tr}\, \overset{*}{D}\overset{*}{D}S. \end{array}$$

One may show that the 14 invariants considered are indeed independent. It might nevertheless be interesting to consider the invariants of order higher than 5 or other invariant combinations of the ones above.

Remarks. – a) In a vacuum there are four invariants:

(20.14) $$\mathrm{Tr}\, C^2, \qquad \mathrm{Tr}\, C^3\,, \qquad \mathrm{Tr}\, C\overset{*}{C}\,, \qquad \mathrm{Tr}\, C^2\overset{*}{C}\,.$$

The space is equiharmonic if:

(20.15) $$\mathrm{Tr}\, C^2 = \mathrm{Tr}\, \overset{*}{C}{}^2\,, \qquad \mathrm{Tr}\, C\overset{*}{C} = 0\,,$$

or:

(20.16) $$\mathrm{Tr}\, \overset{+}{C}{}^2 = 0\,.$$

If the space is harmonic then:

(20.17) $$\mathrm{Tr}\,\overset{+}{C}{}^{3} = 0\,.$$

b) In the case for which there exists an electromagnetic field the mixed invariants are determined by the starting with:

(20.18) $$D_{ab} = C_{abcd}\, \underset{(1)}{k}{}^{b}\, \underset{(2)}{k}{}^{c}\,, \qquad \overset{*}{D}_{ab} = \overset{*}{C}_{abcd}\, \underset{(1)}{k}{}^{b}\, \underset{(2)}{k}{}^{c}$$

if the field is non-singular, and the characteristic vectors $\underset{(1)}{k}$ and $\underset{(2)}{k}$, and:

(20.19) $$D_{ab} = C_{abcd}\, k^{b}\, k^{c}\,, \qquad \overset{*}{D}_{ab} = \overset{*}{C}_{abcd}\, k^{b}\, k^{c}$$

if the field is singular.

In an isotropic frame, an electromagnetic field schema must satisfy the following algebraic conditions:

(20.20) $$R = 0\,, \qquad R_{ab}\, R^{ab} = \rho^{2}\, \delta_{a}^{c}\,.$$

One painlessly verifies that it then results that:

(20.21) $$E_{\alpha\beta} = A_{\alpha}\, \bar{A}_{\beta}\,,$$

(20.22) $$\overset{+}{F} = A_{\alpha}\, Z^{\alpha}\,,$$

in which $\overset{+}{F}$ is the electromagnetic field bivector. The MAXWELL equations may be written:

(20.23) $$d(A_1\, Z^1) = 0\,,$$

so one has:

(20.24) $$\begin{cases} A_{1,1} - \frac{1}{2}A_{3,3} + A_1(\sigma_1^3 + \frac{1}{2}\sigma_3^1) - \frac{1}{2}A_2\sigma_3^2 + \frac{1}{2}A_3\sigma_1^2 = 0, \\ A_{1,0} - \frac{1}{2}A_{3,2} + A_1(\sigma_0^3 + \frac{1}{2}\sigma_2^1) - \frac{1}{2}A_2\sigma_2^2 + \frac{1}{2}A_3\sigma_0^2 = 0, \\ A_{2,3} - \frac{1}{2}A_{3,1} + \frac{1}{2}A_1\sigma_1^1 - A_2(\sigma_3^3 + \frac{1}{2}\sigma_1^2) - \frac{1}{2}A_3\sigma_3^1 = 0, \\ A_{2,2} - \frac{1}{2}A_{3,0} + \frac{1}{2}A_1\sigma_0^1 - A_1(\sigma_2^3 + \frac{1}{2}\sigma_0^2) - \frac{1}{2}A_3\sigma_3^1 = 0. \end{cases}$$

21. **Isotropic congruences** [16]. – In an isotropic frame, the real vector fields $h^{(0)}$, $h^{(3)}$ defines a congruence of isotropic curves that are integrals of the field $h^{(0)}$. In tensorial form, one has:

(21.1) $$\nabla_a h_b^{(c)} = -\, h_b^{(r)} h_a^{(s)} \Gamma_{rs}^{c}\,,$$

in which:

(21.2) $$\omega_r^c = \Gamma_{rs}^{c}\, \theta^{s}\,.$$

Recall that $h^a_{(b)}$ denotes the inverse matrix to $h^{(c)}_b$:

(21.3) $$h^a_{(b)}\, h^{(b)}_c = \delta^a_c\,.$$

We fix our attention on the congruence $h^{(0)}$; one has:

(21.4) $$\nabla_a h^{(0)}_b = -\, h^{(r)}_b h^{(s)}_a \Gamma^0_{rs}\,.$$

The coefficients Γ^0_{rs} are directly related to the form of the connection for a given $h^{(0)}$, and are determined up to a transformation of the group G_4 that was considered in section 19. One may define a series of invariants of the congruence.

The congruence is geodesic if:

(21.5) $$h^a_{(3)}\nabla_a h^{(0)}_b = \lambda\, h^{(0)}_b\,;$$

now:

(21.6) $$h^a_{(3)}\nabla_a h^{(0)}_b = -\, h^{(r)}_b \Gamma^0_{rs}$$

and:

(21.7) $$\Gamma^0_{03} = \tfrac{1}{2}(\sigma^3_3 + \bar{\sigma}^3_3)\,, \qquad \Gamma^0_{13} = -\tfrac{1}{2}\bar{\sigma}^2_3\,, \quad \Gamma^0_{23} = -\tfrac{1}{2}\sigma^2_3\,, \quad \Gamma^0_{33} = 0\,.$$

We remark that we have set:

(21.8) $$\sigma^a = \sigma^a_b\,\theta^b\,, \qquad \bar{\sigma}^a = \bar{\sigma}^a_0\theta^0 + \bar{\sigma}^a_3\theta^3 + \bar{\sigma}^a_2\theta^1 + \bar{\sigma}^a_1\theta^2\,.$$

The congruence is geodesic if:

(21.9) $$\sigma^2_3 = 0\,.$$

This condition is equivalent to:

(21.10) $$\theta^0 \wedge dZ^2 = 0\,.$$

This being the case, if:

(21.11) $$\nabla_a h^{(0)}_b = -\, h^{(1)}_b h^{(1)}_a \Gamma^0_{11} - h^{(2)}_b h^{(2)}_a \Gamma^0_{22} - h^{(1)}_b h^{(2)}_a \Gamma^0_{12} - h^{(2)}_b h^{(1)}_a \Gamma^0_{21} + \dots$$

then one has:

(21.12) $$\Gamma^0_{11} = -\tfrac{1}{2}\bar{\sigma}^2_2 = \bar{\sigma}\,, \qquad \Gamma^0_{22} = -\tfrac{1}{2}\sigma^2_2 = \hat{\sigma}\,,$$

(21.13) $$\Gamma^0_{12} = -\tfrac{1}{2}\bar{\sigma}^2_1 = \bar{z}\,, \qquad \Gamma^0_{21} = -\tfrac{1}{2}\sigma^2_1 = z\,.$$

In the group G_4, under the hypothesis that the congruence is geodesic, one has [cf. (19.3)]:

(21.14) $$\sigma'^2_1 = e^a\,\sigma^2_1\,, \qquad \sigma'^2_2 = e^{a+2ib}\,\sigma^2_2\,.$$

One may further choose a in such a fashion that the geodesic congruence $h^{(0)}$ is such that:

(21.15) $$h_a^{(0)} = h_{(3)}^a \nabla_a h_b^{(0)} = 0 \ .$$

The parameter *a* is then *well-defined*, with the reservation that:

(21.16) $$\dot{a} = h_{(3)}^a \partial_3 a = a_{13} = 0 \ .$$

Condition (21.15) amounts to a choice of affine parameter *v* on the isotropic geodesics, a parameter that may be written:

(21.17) $$\frac{dh_{(3)}^a}{dv} + \Gamma_{bc}^a h_{(3)}^b h_{(3)}^c = 0 \ ;$$

v is determined up to a linear transformation with constant coefficients with respect to *v* [16].

By starting with a non-isotropic initial hypersurface, one may then parallel transport the local frames along the geodesics $h_{(3)}$. One then has:

(21.18) $$h_{(3)}^a \nabla_a h_b^{(0)} = h_{(3)}^a \nabla_a h_b^{(1)} = h_{(3)}^a \nabla_a h_b^{(2)} = 0 \ ,$$

hence:

(21.19) $$\sigma_3^1 = \sigma_3^2 = \sigma_3^3 = 0 \ .$$

Under these conditions, one may associate a Lie transport to the congruence $h^{(0)}$; one has:

$$\underset{h_{(3)}}{\mathrm{L}} h_a^{(1)} = h_{(3)}^a \partial_a h_b^{(1)} + h_b^{(1)} \partial_a h_{(3)}^b = h_{(3)}^b (\partial_b h_a^{(1)} - \partial_a h_b^{(1)}) \ .$$

If one refers to the expression for $d\theta^1$ (cf., Appendix) then one finds:

(21.20) $$\underset{h_{(3)}}{\mathrm{L}} h_a^{(1)} = -\tfrac{1}{2}\sigma_1^2 h_b^{(1)} - \tfrac{1}{2}\sigma_2^2 h_a^{(2)} \quad \text{modulo } h_a^{(0)} .$$

One verifies that σ_1^2 and $[\sigma_2^2]$ are then invariants [cf. (21.14)] of the congruence. We set:

(21.21) $$\underset{(3)}{\mathrm{L}} h_a^{(1)} = z h_a^{(1)} + \hat{\sigma} h_a^{(2)} .$$

One has:

(21.22) $$\hat{\sigma} = -\tfrac{1}{2}\sigma_2^2 = - h_{(1)}^b h_{(1)}^a \nabla_a h_b^{(0)} ,$$

(21.23) $$z = -\tfrac{1}{2}\sigma_3^2 = - h_{(1)}^b h_{(2)}^a \nabla_a h_b^{(0)} .$$

If:

(21.24) $$z = \theta + i\ \omega,$$

then it follows that:

(21.25) $$\theta = \tfrac{1}{2} g^{ab} \nabla_a h_b^{(0)} = \tfrac{1}{2} \nabla_a h_{(3)}^a .$$

The formula (21.20) defines a deformation in the Euclidian plane of $h^{(1)}$, $h^{(2)}$, with an instantaneous velocity of:

(21.26) $$\dot{h}_a^{(1)} = \underset{(3)}{\mathrm{L}}\, h_a^{(1)} = z h_a^{(1)} + \hat{\sigma} h_a^{(2)} .$$

One may assume that $\hat{\sigma}$ is real, and if one introduces the Cartesian coordinates x, y on the plane of $h^{(1)}$, $h^{(2)}$ then (21.26) may be written:

(21.27) $$\dot{x} = (\theta + \hat{\sigma})\, x - \omega y , \qquad \dot{y} = \omega x - (\theta - \hat{\sigma})\, y ,$$

This deformation is the result of a *dilatation* (expansion, divergence):

(21.28) $$\dot{x} = \theta\, x , \qquad \dot{y} = \theta\, y ,$$

a *rotation*(curl):

(21.29) $$\dot{x} = -\,\omega y , \qquad \dot{y} = -\hat{\sigma}\, x ,$$

and a *distortion* (shear):

(21.30) $$\dot{x} = \hat{\sigma}\, x , \qquad \dot{y} = -\hat{\sigma}\, y .$$

The distortion is a transformation that preserves areas.

The congruence $h^{(0)}$ is *shear-free geodesic* (s.f.g.) if and only if $\theta^0 \wedge d\theta^0 = 0$:

$$\theta^0 \wedge d\theta^0 = \tfrac{1}{2}[\, -\sigma_3^2\, \theta^0\, \theta^2\, \theta^3 - \bar{\sigma}_3^2\, \theta^0\, \theta^1\, \theta^3 + (\sigma_1^2 - \bar{\sigma}_1^2)\theta^0\, \theta^1\, \theta^2]$$

$$\sigma_3^2 = 0 , \qquad \sigma_1^2 - \bar{\sigma}_1^2 = 0 ;$$

It is therefore geodesic and rotationless.

The vector $h^{(0)}$ is a *Killing vector* if $\nabla_a h_b^{(0)} + \nabla_b h_a^{(0)} = 0$ or $\Gamma_{ab}^0 + \Gamma_{ba}^0 = 0$. It then results that $h^{(0)}$ is *shear-free geodesic* (s.f.g.), and that:

$$\sigma^3 + \bar{\sigma}^3 = (\bar{\sigma}_0^2\, \theta^1 + \sigma_0^2\, \theta^2) .$$

Finally, the vector field $h^{(0)}$ is formed from parallel vectors if and only if $\sigma^2 = 0$.

The field $h^{(3)}$ is geodesic if $\sigma_0^1 = 0$, it is shear-free if $\sigma_1^1 = 0$, and it is s.f.g. if $\sigma^1 \wedge Z^1 = 0$.

Robinson's theorem [23]. – To any singular electromagnetic field that is a solution to the source-free Maxwell equation one may associate an s.f.g. isotropic congruence.

Conversely, to any s.f.g. isotropic congruence one may associate an electromagnetic field that is a solution to the Maxwell equations.

To any field of singular bivectors is associated an isotropic vector field; the associated congruence is s.f.g.

Indeed, let Z^2 be a singular field, and let $h^{(a)}$ be the associated isotropic vector field. Z^2 is a solution to the Maxwell equations if:

(21.31) $$dZ^2 = 0 .$$

If one refers to (14.10) or (20.24) then it results that:

(21.32) $$\sigma_2^2 = \sigma_3^2 = 0 \quad \text{or} \quad \sigma^2 \wedge Z^2 = 0 .$$

Conversely, let $h^{(0)}$ be s.f.g.; one has:

(21.33) $$dZ^2 = \alpha \wedge Z^2 ,$$
in which:
(21.34): $$\alpha = -[(\sigma_2^3 + \tfrac{1}{2}\sigma_0^2)\,\theta^2 + (\sigma_3^3 + \tfrac{1}{2}\sigma_1^2)\,\theta^3] = \alpha_2\,\theta^2 + \alpha_3\,\theta^3 .$$

By differentiating (21.33), it follows that:

(21.34) $$d\alpha \wedge Z^2 = 0 ;$$

hence (cf., Appendix), thanks to the hypotheses, one has:

(21.35) $$\alpha_{2,3} - \alpha_{3,2} = \tfrac{1}{2}\bar{\sigma}_1^2\,\alpha_2 + \tfrac{1}{2}(\sigma_2^3 + \bar{\sigma}_1^3))\,\alpha_3 .$$

This being the case:
(21.36) $$F = e^w\, Z^2$$
satisfies the Maxwell equations if:

(21.37) $$dw \wedge Z^2 + dZ^2 = 0$$
or:
(21.38) $$w_2 = -\,\alpha_2 , \qquad w_3 = -\,\alpha_3 .$$

The integrability conditions for (21.38) result immediately from (21.35).

Remark. – In examples 2 and 3 of section 14, the characteristic vector fields define s.f.g. congruences that are integrable in case 2 and non-integrable in case 3. In example 4, the characteristic vector field is formed from parallel vectors.

IV. **Problems related to the Bianchi equations. Propagation.**

22. **The Bianchi identities.** – If one refers to the definition of the curvature vector (§ 14):
(22.1) $$\Sigma = d\boldsymbol{\sigma} + \boldsymbol{\sigma} \wedge \boldsymbol{\sigma}$$
then one has:
(22.2) $$\mathrm{D}\Sigma = d\Sigma - \Sigma \wedge \boldsymbol{\sigma} + \boldsymbol{\sigma} \wedge \Sigma = 0 .$$

Explicitly, one finds that:

(22.3a) $$d\Sigma_1 + \Sigma_1 \wedge \sigma^3 + \tfrac{1}{2}\Sigma_3 \wedge \sigma^2 = 0\ ,$$

(22.3b) $$d\Sigma_2 - \Sigma_2 \wedge \sigma^3 - \tfrac{1}{2}\Sigma_3 \wedge \sigma^1 = 0\ ,$$

(22.3c) $$d\Sigma_3 - \Sigma_1 \wedge \sigma^1 + \Sigma_2 \wedge \sigma^2 = 0\ .$$

The detailed formulas can be found in the Appendix.

33. **"Singular" vacuum spaces. Goldberg-Sachs theorem** [24]. – One calls spaces *singular* or *algebraically special* when they are not of type I. There always exists a doubly characteristic isotropic congruence, namely $h^{(0)}$; one thus has:

(23.1) $$C_{11} = C_{13} = 0\ .$$

THEOREM. – In the vacuum, the congruence $h^{(0)}$ is s.f.g., and conversely, if $h^{(0)}$ is s.f.g. and $h^{(0)}$ is doubly degenerate then the space is degenerate.

Therefore, let:

(23.2) $$C_{11} = C_{13} = 0\ , \qquad C_{12} \neq 0\ ,$$

so we have to prove that $h^{(0)}$ is s.f.g. One has:

(23.3) $$\Sigma_1 = C_{12}\, Z^2 \quad \text{and} \quad dZ^2 = \alpha \wedge Z^2\ .$$

Thanks to (22.3a):

(23.4) $$\begin{aligned} d\Sigma_1 &= dC_{12} \wedge Z^2 + C_{12}\, \alpha \wedge Z^2 \\ &= -C_{12}\, Z^2 \wedge \sigma^3 - \tfrac{1}{2} C_{23}\, Z^2 \wedge \sigma^2 - 2\, C_{12}\, Z^3 \wedge \sigma^2\ . \end{aligned}$$

One thus has:

(23.5) $$\begin{aligned} \theta^0 \wedge d\Sigma_1 = 0 &= 2\, C_{12}\, \theta^0 \wedge Z^3 \wedge \sigma^2 \\ &= -C_{12}\, \theta^0 \wedge \theta^1 \wedge \theta^2 \wedge \theta^3 = -C_{12}\, \sigma_3^2\, dv\ , \end{aligned}$$

(23.6) $$\begin{aligned} \theta^1 \wedge d\Sigma_1 = 0 &= 2\, C_{12}\, \theta^1 \wedge Z^3 \wedge \theta^2 \\ &= C_{12}\, \theta^0 \wedge \theta^1 \wedge \theta^2 \wedge \theta^3 = C_{12}\, \sigma_2^2\, dv\ , \end{aligned}$$

hence:

(23.7) $$\sigma_2^2 = \sigma_3^2 = 0\ .$$

If $C_{12} = 0$ then one is dealing with type III, and:

(23.8) $$\Sigma_1 = 0\ , \qquad \Sigma_2 = C_{23}\, Z^3, \qquad \Sigma_3 = C_{23}\, Z^3,$$

from which:

(23.9) $$d\Sigma_3 = dC_{23}\, Z^3 = -C_{23}\, Z^3 \wedge \sigma^2\ ,$$

hence:

(23.10) $$\begin{aligned} -C_{23}\, \theta^0 \wedge Z^3 \wedge \theta^2 &= -C_{23}\, \sigma_3^2\, dv = 0\ , \\ -C_{23}\, \theta^1 \wedge Z^3 \wedge \theta^2 &= C_{23}\, \sigma_3^2\, dv = 0\ . \end{aligned}$$

(23.7) then results.

Conversely, assume (23.7). If one refers to the expression for C_{11} (cf., Appendix) then it results from (23.7) that:

(23.11) $$C_{11} = 0 \;;$$

$h^{(0)}$ is thus characteristic.

On the other hand, one finds (cf., Appendix) under the hypothesis (23.7) that:

(23.12) $$C_{13} = -\sigma_{03}^2 + \sigma_{12}^2 + \tfrac{1}{2}\sigma_0^2(\bar{\sigma}_3^3 - \sigma_3^3 + \bar{\sigma}_1^2) + \tfrac{1}{2}\sigma_1^2(\bar{\sigma}_3^1 + \bar{\sigma}_1^3 + \sigma_2^3)\,,$$

(23.13) $$C_{13} = 2(\sigma_{2,3}^3 - \sigma_{3,2}^3) + \sigma_2^3(\bar{\sigma}_3^3 - \sigma_3^3 + \bar{\sigma}_1^2) - \sigma_3^3(\bar{\sigma}_3^1 + \bar{\sigma}_1^3 + \sigma_2^3)\,,$$

Then $4 \times$ (23.12) + (23.13) is:

(23.14) $$5\,C_{13} = 2\,(2\sigma_1^2 - \sigma_3^3)_{,2} - (2\sigma_0^2 - \sigma_2^3)_{,3} + (2\sigma_0^2 - \sigma_2^3)(\bar{\sigma}_3^3 - \sigma_3^3 + \bar{\sigma}_1^2) + (2\sigma_1^2 - \sigma_3^3)(\bar{\sigma}_3^1 + \bar{\sigma}_1^3 + \sigma_2^3)\,.$$

On the other hand, the Bianchi identities (Σ_1, components θ^1, θ^2, θ^3, and θ^0, θ^2, θ^3) give:

(23.15) $$C_{13,\,3} = C_{13}\,(2\sigma_1^2 - \sigma_3^3)\,,$$

(23.16) $$C_{13,\,2} = C_{12}\,(2\sigma_0^2 - \sigma_2^3)\,.$$

One thus has:

(23.17) $$C_{13,\,32} - C_{13,\,23} = C_{13}\,[(2\sigma_1^2 - \sigma_3^3)_{,2} - (2\sigma_0^2 - \sigma_2^3)_{,3}]\,,$$

and, thanks to the commutativity of the derivatives:

(23.18) $$\begin{aligned} C_{13,\,32} - C_{13,\,23} &= -\tfrac{1}{2}C_{13,\,2}\,[(\bar{\sigma}_3^3 - \sigma_3^3 + \bar{\sigma}_1^2) - \tfrac{1}{2}C_{13,\,2}\,(\bar{\sigma}_3^1 + \bar{\sigma}_1^3 + \sigma_2^3)\,,\\ &= -\tfrac{1}{2}C_{13}\,[(2\sigma_0^2 - \sigma_2^3)(\bar{\sigma}_3^3 - \sigma_3^3 + \bar{\sigma}_1^2) + (2\sigma_1^2 - \sigma_3^3)(\bar{\sigma}_3^1 + \bar{\sigma}_1^3 + \sigma_2^3)\,]\,. \end{aligned}$$

Upon substituting (23.18) in (23.17), one finds that:

(23.19) $$C_{13}\,[(2\sigma_1^2 - \sigma_3^3)_{,2} - (2\sigma_1^2 - \sigma_3^3)_{,3} + \tfrac{1}{2}(2\sigma_0^2 - \sigma_2^3)(\bar{\sigma}_3^3 - \sigma_3^3 + \bar{\sigma}_1^2) + \tfrac{1}{2}(2\sigma_1^2 - \sigma_3^3)(\bar{\sigma}_3^1 + \bar{\sigma}_1^3 + \sigma_2^3)\,] = 0$$

or:

(23.20) $$\frac{5}{2}(C_{13})^2 = 0\,.$$

Remark. The hypotheses of s.f.g. isotropic congruences and characteristic isotropic vectors are conformal notions. One may envision weakening the hypothesis of empty space as a way of generalizing the theorem (on this subject, cf. Robinson and Schild [25]).

24. **Singular vacuum spaces. Geodesic propagation. Sachs's theorem** [16]. – We consider a singular vacuum space in which $h^{(0)}$ denotes the doubly characteristic isotropic s.f.g. congruence. One thus has:

(24.1) $$C_{11} = C_{13} = 0\,, \qquad \sigma^2 \wedge Z^2 = 0\,.$$

We choose the parameter v on the geodesics with the property that:

(24.2) $$dh^{(0)}/dv = h^3_{(3)} \nabla_a h^{(0)}_b = 0\,,$$

and we assume that the local frames are parallel displaced along the characteristic geodesics; one has:

(24.3) $$\sigma^1_3 = \sigma^2_3 = \sigma^3_3 = 0\,.$$

By virtue of the equation $E_{1\bar{1}} = 0$ and the hypotheses:

(24.4) $$(dz/dv) + z^2 = 0\,,$$

namely, $(dz/dv) + \theta^2 - \omega^2 = 0$, $d\omega/dv = \theta\,\omega$.

If $\theta = 0$ then one has:

(24.5) $$\omega = 0\,, \quad \theta = 0 \to \omega = 0\,, \qquad z = 0\,.$$

If $\theta \neq 0$, $\omega = 0$ then one may make:

(24.6) $$z = 1/v\,,$$

and if $\theta \neq 0$, $\omega \neq 0$:

(24.7) $$z = 1/(v + i\,a_0)\,,$$

in which a_0 is a constant with respect to v.

It results from the Bianchi identities (in the order Σ_1, $\theta^0 \wedge \theta^1 \wedge \theta^2$, Σ_2, $\theta^0 \wedge \theta^2 \wedge \theta^3$, Σ_2 $\theta^3 \wedge \theta^0 \wedge \theta^1$) that:

(24.8) $$(dC_{12}/dv) + 3\,z\,C_{12} = 0\,,$$

(24.9) $$(dC_{23}/dv) + 2\,z\,C_{23} = 2\,C_{12,1}\,,$$

(24.10) $$(dC_{22}/dv) + z\,C_{22} = \tfrac{1}{2} C_{23,1} - \tfrac{1}{2}\,\sigma^3_1\,C_{23} - \tfrac{3}{2} C_{12}\,\sigma^1_1\,.$$

Hence, if $z \neq 0$:

(24.11) $$\frac{dC_{12}}{dz} - \frac{3}{z} C_{12} = 0\,,$$

(24.12) $$\frac{dC_{23}}{dz} - \frac{2}{z} C_{23} = \frac{-2}{z^2} C_{12,1}\,,$$

(24.13) $$\frac{dC_{22}}{dz} - \frac{1}{z} C_{23} = \frac{1}{2z^2} C_{23,1} + \frac{1}{2z^2}\,\sigma^3_1\,C_{23} + \frac{3}{2z^2} C_{12}\,\sigma^1_1\,.$$

From (24.11), it immediately results that:

(24.14) $$\boxed{C_{12} = z^3 C^0_{12}}\ ,$$

in which C^0_{12} is a constant with respect to *v*.

In order to integrate (24.12), observe that:

(24.15) $$C_{12,13} = C_{12,31} - z\, C_{12,1} + C_{12,3}(\sigma^3_1 + \bar{\sigma}^3_2)$$
$$= -4\, z\, C_{12,1} - 3\, C_{12}\, z_1 - \tfrac{3}{2} z\, C_{12}\, (\sigma^3_1 + \bar{\sigma}^3_2)\ .$$

Hence:

(24.16) $$(C_{12,1})_{,3} + 4\, z\, C_{12,1} = -3\, C_{12}\, A\ ,$$

in which:

(24.17) $$A = z_1 + \tfrac{1}{2}(\sigma^3_1 + \bar{\sigma}^3_2)\, z\ .$$

By virtue of the equations $E_{3\bar{1}} = 0$ and $E_{3\bar{2}} = 0$, one has:

(24.18) $$\sigma^3_{1,3} + z\,\sigma^3_1 = 0\ , \qquad \bar{\sigma}^3_{2,3} + z\,\bar{\sigma}^3_2 = 0\ .$$

Moreover:

(24.19) $$z_{1,3} = -3\, z\, z^1 - \tfrac{1}{2} z^2\, (\sigma^3_1 + \bar{\sigma}^3_2)\ .$$

It results from these relations that:

(24.20) $$\dot{A} = -3\, z\, A\ .$$

One thus has:

(24.21) $$A = A_0\, z^3\ .$$

Furthermore, it results from the relations $\bar{\Sigma}_2 = d\bar{\sigma}^2 - \bar{\sigma}^2 \wedge \bar{\sigma}^3$, which are terms in $\theta^1 \wedge \theta^2$ if $z = \bar{z}$, that:

$$z_1 + \tfrac{1}{2} z(\sigma^3_2 + \bar{\sigma}^3_1) = \tfrac{1}{4}\, \sigma^2_0\, (\bar{\sigma}^2_1 - \sigma^2_1)\ .$$

Hence:

(24.22) $$z = \bar{z} \rightarrow A_0 = 0\ ,$$

and equation (24.16) may be written:

(24.23) $$\frac{d}{dz}(C_{12,1}) - \frac{4}{z} C_{12,1} = 3\, C^0_{12}\, A_0\, z^4\ ,$$

so the general solution is:

(24.24) $$C_{12,1} = B_0\, z^4 + 3\, C^0_{12}\, A_0\, z^5\ .$$

Substituting this solution into equation (24.12) gives:

(24.25) $$\boxed{C_{23} = D_0\, z^2 - B_0\, z^3 - \tfrac{3}{2} A_0\, C^0_{12}\, z^4}$$

INTEGRATION OF EQUATION (24.13). – We calculate:

(24.26) $$D = C_{23,1} - C_{23}\,\sigma_1^3 + 3\,C_{12}\,\sigma_1^1 .$$

a) *Calculation of* A_1. – One has:

(24.27) $$A_3 = -\,3\,z\,A\,, \qquad A_3 = -\,3\,z_1\,A - 3\,z\,A_1\,,$$

$$A_{1,3} + z\,A_1 - \tfrac{1}{2}\,A_3\,(\sigma_1^3 + \bar{\sigma}_2^3) = -\,3\,z_1\,A - 3\,z\,A_1\,,$$

(24.28) $$A_{1,3} + 4\,z\,A_1 = -\,3\,A^2 .$$

hence:

(24.29) $$\frac{dA_1}{dz} - \frac{4}{z}A_1 = 3\,A_0\,z^4 ,$$

(24.30) $$A_1 = A_1^0\,z^4 + 3\,A_0\,z^5 .$$

Remark. If $z = \bar{z}$ then $A_1^0 = A_0 = 0$.

b) *Calculation of* S_1, *where* $S = C_{12,1}$. – Equation (24.16) may be written:

(24.31) $$S_3 + 4\,z\,S = -\,3\,C_{11}\,A\,,$$

where:

(24.32) $$S_{31} + 4\,z_1\,S + 4\,z\,S_1 = -\,3\,S\,A - 3\,C_{12}\,A_1\,,$$

$$S_{1,3} + 5\,z\,S_1 = -\,7\,S\,A - 3\,C_{12}\,[A_1 - \tfrac{1}{2}A\,(\sigma_1^3 + \bar{\sigma}_2^3)]\,.$$

Hence, by virtue of (24.24), (24.21), (24.14), (24.30), and (24.18):

(24.33) $$S_{1,3} + 5\,z\,S_1 = -\,z^7\,M_0 - N_0\,z^8 ,$$

in which $M_0 = N_0 = 0$ is $z = \bar{z}$, from which it follows that:

(24.34) $$S_1 = z^5\,(S_1^0 + M_0\,\mathrm{z} + \tfrac{1}{2}N_0\,z^2)\,.$$

c) *Calculation of* $C_{23,1}$. –

(24.35) $$C_{23,13} = C_{23,31} - z\,C_{23,1} + \tfrac{1}{2}C_{23,3}\,(\sigma_1^3 + \bar{\sigma}_2^3),$$

(24.36) $$C_{23,13} - 3z\,C_{23,1} = 2\,S_1 - 3C_{23}\,A + S\,(\sigma_1^3 + \bar{\sigma}_2^3),$$

(24.37) $$= -\,z^7\,M_0 - N_0\,z^8 ,$$

in which $L_0 = P_0 = 0$ if $z = \bar{z}$, from which:

(24.38) $$C_{23,1} = z^3\,(\alpha_0 + K_0\,z + \tfrac{1}{2}L_0\,z^2 + \tfrac{1}{3}P_0\,z^3)\,.$$

d) It results from the equation $E_{2\bar{1}} = 0$ that:

(24.39) $$\sigma^1_{1,3} + z\,\sigma^1_1 = 0\,,$$

hence:

(24.40) $$\sigma^1_1 = z\,\sigma^{01}_1\,.$$

One thus has:

(24.41) $$M = z^3\,(U_0 + V_0\,z + W_0\,z^2 + Z_0\,z^3\,),$$

in which:

(24.42) $$W_0 = Z_0 = 0 \quad \text{if} \quad z = \bar{z}\,.$$

One may finally integrate equation (24.13), which is written:

(24.43) $$\frac{dC_{22}}{dz} - \frac{1}{z}C_{22} = U_0 + V_0\,z + W_0\,z^2 + Z_0\,z^3\,,$$

namely:

(24.44) $$\boxed{C_{22} = z\,(T_0 + U_0\,z + \tfrac{1}{2}V_0\,z^2 + \tfrac{1}{3}W_0\,z^3 + \tfrac{1}{4}Z_0\,z^4)}$$

Taking into account the results (24.14), (24.25), (24.44), on the integration of the Bianchi equations that regulate the propagation along characteristic isotropic geodesics in a degenerate vacuum space, one finds that:

(24.45) $$(C_{\alpha\beta}) = z\begin{pmatrix} 0 & 0 & 0 \\ 0 & T_0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + z^2\begin{pmatrix} 0 & 0 & 0 \\ 0 & T_0 & D_0 \\ 0 & D_0 & 0 \end{pmatrix} + z^3\begin{pmatrix} 0 & C^0_{12} & 0 \\ C^0_{12} & \frac{1}{2}V_0 & -B_0 \\ 0 & -B_0 & 4C^0_{12} \end{pmatrix}$$
$$+ z^4\begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{3}W_0 & -\frac{3}{2}A_0C^0_{12} \\ 0 & -\frac{3}{2}A_0C^0_{12} & 0 \end{pmatrix} + z^5\begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{4}Z_0 & 0 \\ 0 & 0 & 0 \end{pmatrix};$$

one may also express this result as:

(24.46) $$C_{\alpha\beta}(z) = z\,\mathrm{N}_0 + z^2\,\mathrm{III}_0 + z^3\,\mathrm{II}_0 + z^4\,\mathrm{III}'_0 + z^5\,\mathrm{N}'_0\,,$$

in which N_0, etc., denote tensors $C_{\alpha\beta}$ of the corresponding type that are *constant* along the geodesics. *Moreover*, *if* $z = \bar{z}$ then it results from (24.6), (24.22), (24.42):

(24.47) $$(C_{\alpha\beta}) = \frac{1}{v}\mathrm{N}_0 + \frac{1}{v^2}\mathrm{III}_0 + \frac{1}{v^3}\mathrm{II}_0\,.$$

Finally, if $z = 0$ then one painlessly finds that:

(24.48) $$C_{12} = C^0_{12}\,, \qquad C_{23} = C^0_{23}\,v + C'^0_{23}\,, \qquad C_{22} = C^0_{22}\,v^2 + C'^0_{22}\,v + C''^0_{22}\,,$$

hence:

(24.49) $$(C_{\alpha\beta}) = \mathrm{N}_0\, v^2 + \mathrm{III}_0\, v + \mathrm{II}_0\,.$$

V. **Spinorial representations** [26, 17]

25. **Vector and bivectors in the space of spinors. –** The representation:

(25.1) $$K: L_+^\uparrow \to O_3(\mathbb{C})$$

may be prolonged to:

(25.2) $$O_3(\mathbb{C}) \to SL_2(\mathbb{C})$$

The transformations of $SL_2(\mathbb{C})$ are representable by 2×2 unimodular matrices, namely, the special linear transformations of a complex two-dimensional vector space, or *spinor space* P_1.

The vectors ψ of P_1 will be denoted:

(25.3) $$\psi_{\mathrm{A}} \qquad (\mathrm{A} = 1, 2).$$

Thanks to the anti-involutive matrix:

(25.4) $$\varepsilon_{\mathrm{AB}} = \varepsilon^{\mathrm{AB}} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},$$

(25.5) $$\varepsilon^2 = -1, \qquad {}^{\tau}\varepsilon\,\varepsilon = 1\,,$$

one may raise and lower indices; one will note that:

(25.6) $$\psi_{\mathrm{A}} = \varepsilon_{\mathrm{AB}}\,\psi^{\mathrm{B}} = \varepsilon\,\psi\,,$$

(25.7) $$\psi^{\mathrm{A}} = \varepsilon^{\mathrm{AB}}\,\psi_{\mathrm{B}} = {}^{\tau}\varepsilon\,\psi\,.$$

In view of the complex structure of P_1, one may also introduce:

(25.8) $$\bar{\psi}_{\mathrm{A}} = \psi_{\dot{\mathrm{A}}}\,.$$

Spinors will be tensors over P_1, such as:

(25.9) $$\psi_{\dot{\mathrm{A}}\mathrm{C}\mathrm{D}\dot{\mathrm{E}}\dot{\mathrm{F}}\cdots}\,.$$

The representation $L_+^\uparrow \to SL_2(\mathbb{C})$ appears to originate in the following remark: *To any vector* u_a in Minkowski space, one may associate a *Hermitian matrix* $\psi_{\dot{\mathrm{B}}\mathrm{A}}$ *in a canonical manner.* In an isotropic frame, one sets:

(25.10) $$\psi_{\dot{\mathrm{B}}\mathrm{A}} = \begin{bmatrix} u_0 & u_2 \\ u_1 & u_3 \end{bmatrix}.$$

One has:

(25.11) $$\bar{\psi}_{\dot{\mathrm{B}}\mathrm{A}} = \psi_{\dot{\mathrm{A}}\mathrm{B}} \ .$$

Any unimodular transformation:

(25.12) $$A_{\mathrm{B}}^{\mathrm{A}}$$

in P_1 induces a transformation on $\psi_{\dot{\mathrm{B}}\mathrm{A}}$:

(25.13) $$\bar{\psi}_{\dot{\mathrm{B}}\mathrm{A}} = \bar{A}_{\dot{\mathrm{B}}}^{\dot{\mathrm{C}}} \psi_{\dot{\mathrm{C}}\mathrm{D}} A_{\mathrm{A}}^{\mathrm{D}}$$

which translates into a Lorentz transformation of the u's. One painlessly verifies that one has:

(25.14) $$\begin{cases} u_0' = \bar{A}_1^1 A_1^1 u_0 + \bar{A}_1^2 A_1^1 u_1 + \bar{A}_1^1 A_1^2 u_2 + \bar{A}_1^1 A_1^1 u_3, \\ u_1' = \bar{A}_2^1 A_1^1 u_0 + \bar{A}_2^2 A_1^1 u_1 + \bar{A}_2^1 A_1^2 u_2 + \bar{A}_3^1 A_1^1 u_3, \\ u_2' = \bar{A}_1^1 A_2^1 u_0 + \bar{A}_1^2 A_2^1 u_1 + \bar{A}_1^1 A_2^2 u_2 + \bar{A}_1^2 A_2^2 u_3, \\ u_3' = \bar{A}_2^1 A_2^1 u_0 + \bar{A}_2^2 A_2^1 u_1 + \bar{A}_2^1 A_2^2 u_2 + \bar{A}_2^2 A_2^2 u_3, \end{cases}$$

and:

(25.15) $$u_0' u_3' - u_1' u_2' = (A_1^1 A_2^2 - A_2^1 A_1^2)(\bar{A}_1^1 \bar{A}_2^2 - \bar{A}_2^1 \bar{A}_1^2)\,(u_0 u_3 - u_1 u_2) \ .$$

The unimodular transformations $A_{\mathrm{A}}^{\mathrm{B}}$ therefore induce a Lorentz transformation. The representation (25.2) is $1 \rightarrow 2$, in the sense that the matrices:

(25.16) $$A \ \text{ and } \ -A$$

determine the same Lorentz transformation. To a vector u_a, one further associates the matrices:

(25.17) $$\psi^{\dot{\mathrm{B}}}{}_{\mathrm{A}} = \varepsilon^{\dot{\mathrm{A}}\mathrm{B}} \psi_{\dot{\mathrm{D}}\mathrm{A}} , \qquad \psi^{\dot{\mathrm{B}}\mathrm{A}} = \psi^{\dot{\mathrm{B}}}{}_{\mathrm{C}}\, \varepsilon^{\mathrm{CA}} ,$$

namely, upon starting with (25.10), the matrices:

(25.18) $$\psi_{\dot{\mathrm{B}}\mathrm{A}} = \begin{bmatrix} u_0 & u_2 \\ u_1 & u_3 \end{bmatrix}, \qquad \psi^{\dot{\mathrm{B}}}{}_{\mathrm{A}} = \begin{bmatrix} -u_1 & -u_3 \\ u_0 & u_2 \end{bmatrix}, \qquad \psi^{\dot{\mathrm{B}}\mathrm{A}} = \begin{bmatrix} u_3 & -u_1 \\ -u_2 & u_0 \end{bmatrix} .$$

One will note that, in general:

(25.19) $$u_a = \sigma_a^{\dot{\mathrm{A}}\mathrm{B}} \psi_{\dot{\mathrm{A}}\mathrm{B}} ,$$

(25.20) $$\psi_{\dot{\mathrm{B}}\mathrm{A}} = \sigma_{\dot{\mathrm{A}}\mathrm{B}}^{a} u_a ,$$

in which $\sigma_a^{\dot{A}B}, \sigma^a_{\dot{A}B}$ are tensor-spinors. In an isotropic frame, one has:

(25.21) $$\sigma^0_{\dot{A}B} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \sigma^1_{\dot{A}B} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2_{\dot{A}B} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \sigma^3_{\dot{A}B} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$

(25.22) $$\sigma^{0\dot{A}}{}_B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \sigma^{1\dot{A}}{}_B = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \sigma^{2\dot{A}}{}_B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma^{3\dot{A}}{}_B = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}.$$

Spinors associated with a bivector. – Let u_a and v_a be two vectors, and let $\overset{1}{\psi}$ and $\overset{2}{\psi}$ be the associated matrices; in an isotropic frame one has:

(25.23) $$\overset{1}{\psi}_{\dot{B}A} = \begin{bmatrix} u_0 & u_2 \\ u_1 & u_3 \end{bmatrix}, \qquad \overset{2}{\psi}_{\dot{B}A} = \begin{bmatrix} v_0 & v_2 \\ v_1 & v_3 \end{bmatrix}.$$

To any bivector:

(25.24) $$u_{ab} = u_a\, v_b - u_b\, v_a\,,$$

we associate the matrix:

(25.25) $$\Phi_{DA} = \tfrac{1}{2}(\overset{1}{\psi}{}^{\dot{C}}{}_D\, \overset{2}{\psi}_{\dot{C}A} - \overset{2}{\psi}{}^{\dot{C}}{}_D\, \overset{1}{\psi}_{\dot{C}A})\,,$$

(25.26) $$\Phi_{DA} = \begin{bmatrix} u_0v_1 - v_0v_1 & \tfrac{1}{2}[(u_0v_3 - u_3v_0) - (u_1v_2 - u_2v_1)] \\ \tfrac{1}{2}[(u_0v_3 - u_3v_0) - (u_1v_2 - u_2v_1)] & u_2u_3 - u_3v_2 \end{bmatrix}.$$

In general, if:

(25.27) $$F_{ab} = F_\alpha Z^\alpha + \text{conj.}$$

is an arbitrary bivector then one may associate the matrix:

(25.28) $$\Phi_{DA} = \begin{bmatrix} F_{01} & \tfrac{1}{2}(F_{03} - F_{13}) \\ \tfrac{1}{2}(F_{03} - F_{13}) & F_{23} \end{bmatrix} = \begin{bmatrix} F_2 & \tfrac{1}{2}F_3 \\ \tfrac{1}{2}F_3 & F_1 \end{bmatrix};$$

Φ_{DA} is a symmetric spinor:

(25.29) $$\Phi_{DA} = \Phi_{AD}\,.$$

We set:

(25.30) $$\Phi_{DA} = s_{DA}{}^\alpha F_\alpha\,,$$

(25.31) $$F_\alpha = s^{DA}{}_\alpha\, \Phi_{DA}\,.$$

In an isotropic frame:

(25.32) $$s_{DA}{}^1 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \qquad s_{DA}{}^2 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \qquad s_{DA}{}^3 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Parallel to this, one has:

(25.33) $$\bar{\Phi}_{DA} = \Phi_{\dot{D}\dot{A}} = s_{DA}{}^{\bar{\alpha}}\bar{F}_\alpha\,,$$

and:

(25.34) $$\Phi_{\dot{D}\dot{A}} = \begin{bmatrix} \bar{F}_2 & \tfrac{1}{2}\bar{F}_3 \\ \tfrac{1}{2}\bar{F}_3 & \bar{F}_1 \end{bmatrix} = \begin{bmatrix} F_{02} & \tfrac{1}{2}(F_{03}+F_{12}) \\ \tfrac{1}{2}(F_{03}+F_{12}) & F_{13} \end{bmatrix};$$

one has:

(25.35) $$s_{DA}{}^{\bar{a}} = \bar{s}_{DA}{}^{a} .$$

One may further set:

(25.36) $$\Phi_{DA} = \tfrac{1}{2} S_{DA}{}^{ab} F_{ab} ,$$

(25.37) $$F_{ab} = \tfrac{1}{2} S^{DA}{}_{ab} \Phi_D .$$

In an isotropic frame:

(25.38) $$\begin{cases} S^{23}{}_{AD} = s^1{}_{AD} = S^{\bar{1}\bar{3}}{}_{AD}, & S^{01}{}_{AD} = s^2{}_{AD} = S^{\bar{0}\bar{2}}{}_{AD}, \\ S^{03}{}_{AD} = s^3{}_{AD} = S^{\bar{0}\bar{3}}{}_{AD} = -S^{12}{}_{AD} = S^{\bar{1}\bar{2}}{}_{AD}, & \\ S^{13}{}_{AD} = 0 = S^{\bar{2}\bar{3}}{}_{AD}, & S^{02}{}_{AD} = 0 = S^{\bar{0}\bar{1}}{}_{AD}. \end{cases}$$

In general:

(25.39) $$S_{DA}{}^{ab} = \tfrac{1}{2}(\sigma^{a\dot{C}}{}_D \sigma^b{}_{\dot{C}A} - \sigma^{b\dot{C}}{}_D \sigma^a{}_{\dot{C}A})$$

(25.40) $$S_{DA}{}^{\overset{*}{ab}} = i\, S_{DA}{}^{ab} ,$$

(25.41) $$\tfrac{1}{2} S^{DA}{}_{ab}\, \theta^a \wedge \theta^b = s^{DA}{}_{\alpha} Z^{\alpha} .$$

We further note the identity:

(25.42) $$g^{ab} e_{AD} = \tfrac{1}{2}(\sigma^{a\dot{C}}{}_D \sigma^b{}_{\dot{C}A} + \sigma^{b\dot{C}}{}_D \sigma^a{}_{\dot{C}A}) .$$

Remark. – If u_a is an isotropic vector then the matrix $\psi_{\dot{B}A}$ is singular:

(25.44) $$\det \psi_{\dot{B}A} = 0 .$$

If F_{ab} is a singular bivector then the matrix Φ_{BA} is singular:

(25.44) $$\det \Phi_{BA} = 0 .$$

26. **Geometric interpretation.** – If one regards the space P_1 as a one-dimensional complex projective space then if one considers the one-index spinors ξ^A to be defined up to a factor then P_1 is identified with the conic γ (§12). The use of the spinor ε_{AB} corresponds to the duality on P_1 . The spinors that are associated to vectors and bivectors are then introduced in a natural fashion. First, let:

(26.1) $$F_{ab} = F_a Z^a + \text{conj.}$$

be a real bivector. In the projective plane P_2 the line:

(26.2) $$F_a Z^a = 0$$

defines an involution on the conic γ whose linked points α, β are the points of intersection of (26.2) with γ (Fig. 10). If:

(26.3) $$Z^1 = (\xi^2)^2, \qquad Z^2 = (\xi^1)^2, \qquad Z^1 = \xi^1 \xi^2$$

are the parametric equations of γ then the linked points of the desired involution are given by:

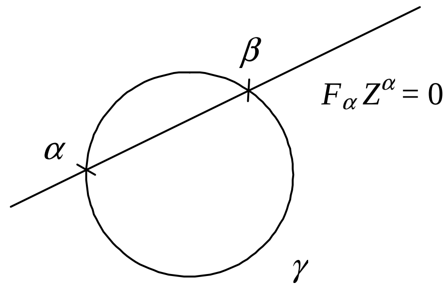


Fig. 10.

(26.4) $$F_2 (\xi^1)^2 + F_1 (\xi^1)^2 + F_3 \xi^1 \xi^2 = 0 ,$$

which one may write as:

(26.5) $$\Phi_{AB} \xi^A \xi^B = 0 ,$$

with:

(26.6) $$\Phi_{AB} = \begin{bmatrix} F_2 & \frac{1}{2} F_3 \\ \frac{1}{2} F_3 & F_1 \end{bmatrix},$$

and we recover the form (25.28). If α_A , β_B are the coordinates of the tangents to γ at α and β for the linked points of the involution Φ_{AB} then:

(26.7) $$\Phi_{AB} = \tfrac{1}{2} (\alpha_A \beta_B + \alpha_B \beta_A) .$$

If the bivector is singular then the line (26.2) is tangent, the points α and β coincide, and one has precisely:

(26.8) $$\Phi_{AB} = \alpha_A \beta_B .$$

Therefore, to any self-adjoint bivector one associates a *pair of points of* γ. To any real bivector one associates the same pair of points of γ and their complex conjugates:

(26.9) $$\alpha_A, \beta_B ; \qquad \alpha_{\dot{A}} , \alpha_{\dot{B}} .$$

Moreover, if one recalls that the points of g are rectilinear generators of the absolute of P_3 then the points of the absolute (the isotropic vectors) will be determined by a pair:

(26.10) $$\alpha_A, \alpha_{\dot{A}}$$

or by a singular Hermitian matrix:

(26.11) $$\psi_{\dot{B}A} = \alpha_{\dot{B}} \alpha_A .$$

In the case of an arbitrary vector, the matrix:

(26.12) $$\psi_{\dot{B}A}$$

geometrically corresponds to a Hermitian bilinear form:

(26.13) $$\psi_{\dot{B}A}\xi^{\dot{B}}\xi^{A} = 0 .$$

Indeed, let:

(26.14) $$2\,\chi^0\chi^3 - 2\,\chi^1\chi^2 = 0$$

be the absolute of P_3, and let:

(26.15) $$u_a\chi^a = 0$$

be the equation of the plane that is associated with the vector u_a. The plane (26.15) intersects the absolute on a conic # whose points each have a pair $\xi^{\dot{B}}$, ξ^{A} as their image. Explicitly, if:

(26.16) $$\chi^0 = \xi^{\dot{1}}\,\xi^1, \qquad \chi^3 = \xi^{\dot{2}}\,\xi^2, \qquad \chi^1 = \xi^{\dot{2}}\,\xi^1, \qquad \chi^2 = \xi^{\dot{1}}\,\xi^2$$

are the parametric equations of (26.14) then on the plane section (26.15), or on #, one has:

(26.17) #: $$u_0\,\xi^{\dot{1}}\,\xi^1 + u_1\,\xi^{\dot{2}}\,\xi^1 + u_2\,\xi^{\dot{1}}\,\xi^2 + u_3\,\xi^3\,\xi^{\dot{3}} = 0$$

or:

(26.18) $$\psi_{\dot{B}A}\xi^{\dot{B}}\xi^{A} = 0 .$$

We thus recover the Hermitian form $\psi_{\dot{B}A}$ that was introduced in (25.10).

27. **Spinorial representation of a tensor.** – In general, a tensor such as $T^a{}_{bc}$ is associated with a spinor with twice the number of indices by the formulas:

(27.1) $$\psi^{\dot{A}B}{}_{\dot{C}D\dot{E}F} = T^a{}_{bc}\,\sigma_a{}^{\dot{A}B}\sigma^b{}_{\dot{C}D}\sigma^c{}_{\dot{E}F} .$$

To the bivector F_{ab} one may associate the spinor:

(27.2) $$\psi_{\dot{A}B\dot{C}D} = F_{ab}\;\sigma^a{}_{\dot{A}B}\sigma^b{}_{\dot{C}D} .$$

If F_{ab} is real and anti-symmetric then one will have:

(27.3) $$\bar{\psi}_{\dot{A}B\dot{C}D} = \psi_{\dot{B}A\dot{D}C} ,$$

(27.4) $$\psi_{\dot{A}B\dot{C}D} = \psi_{\dot{C}D\dot{A}B} ;$$

from this, it results that:

(27.5) $$\psi_{\dot{A}B\dot{C}D} = \Phi_{BD}\,\varepsilon_{\dot{A}\dot{C}} + \Phi_{\dot{B}\dot{D}}\,\varepsilon_{AC} ,$$

with:

(27.6) $$\Phi_{BD} = \Phi_{DB} ;$$

with the help of (27.5) and (27.6), one recovers formulas (25.30) and (25.31).

To the quantities $\sigma^{\alpha}{}_{a}$, which are the components of the connection in bivectorial variables, one may associate the spinor:

$$\sigma^{\mathrm{AB}}{}_{\dot{\mathrm{C}}\mathrm{D}} = s^{\mathrm{AB}}{}_{\alpha}\, \sigma^{a}{}_{\dot{\mathrm{C}}\mathrm{D}}\, \sigma^{\alpha}{}_{a} = \tfrac{1}{2} S^{\mathrm{AB}}{}_{cd}\; \sigma^{a}{}_{\dot{\mathrm{C}}\mathrm{D}}\, \Gamma^{cd}{}_{a} .$$

The spinor $\sigma^{\mathrm{AB}}{}_{\dot{\mathrm{C}}\mathrm{D}}$ has 12 components: $\sigma^{\mathrm{AB}}{}_{\dot{\mathrm{C}}\mathrm{D}} = \sigma^{\mathrm{BA}}{}_{\dot{\mathrm{C}}\mathrm{D}}$; these are the spinorial coefficients of Newman and Penrose [8].

28. **Riemann tensor.** – One associates a spinor with 8 indices with the tensor R_{abcd}. At the basis for the decomposition that we studied in chapter III, one will have:

(28.1) $$C_{abcd} = \sigma_{a}{}^{\dot{\mathrm{A}}\mathrm{B}}\, \sigma_{b}{}^{\dot{\mathrm{C}}\mathrm{D}}\, \sigma_{c}{}^{\dot{\mathrm{E}}\mathrm{F}}\, \sigma_{d}{}^{\dot{\mathrm{H}}\mathrm{G}}\, (\psi_{\dot{\mathrm{A}}\dot{\mathrm{C}}\dot{\mathrm{E}}\dot{\mathrm{H}}}\, \varepsilon_{\mathrm{BD}}\, \varepsilon_{\mathrm{FG}} + \psi_{\mathrm{BDFG}}\, \varepsilon_{\dot{\mathrm{A}}\dot{\mathrm{C}}}\varepsilon_{\dot{\mathrm{E}}\dot{\mathrm{H}}}) ,$$

in which:

(28.2) $$\psi_{\mathrm{ABCD}}$$

is a *completely symmetric* spinor with four indices. It is the image of the *four-point group* defined by the intersection of the conic $C_{\alpha\beta}$ with the conic $\gamma_{\alpha\beta}$. On the other hand:

(28.3) $$E_{abcd} = \sigma_{a}{}^{\dot{\mathrm{A}}\mathrm{B}}\, \sigma_{b}{}^{\dot{\mathrm{C}}\mathrm{D}}\, \sigma_{c}{}^{\dot{\mathrm{E}}\mathrm{F}}\, \sigma_{d}{}^{\dot{\mathrm{G}}\mathrm{H}}\, (\Phi_{\dot{\mathrm{A}}\dot{\mathrm{C}}\mathrm{FH}}\, \varepsilon_{\mathrm{BD}}\, \varepsilon_{\dot{\mathrm{E}}\dot{\mathrm{G}}} + \Phi_{\mathrm{BD}\dot{\mathrm{E}}\dot{\mathrm{G}}}\, \varepsilon_{\dot{\mathrm{A}}\dot{\mathrm{C}}}\varepsilon_{\mathrm{FH}}) ,$$

in which:

(28.4) $$\Phi_{\mathrm{AB}\dot{\mathrm{C}}\dot{\mathrm{D}}} = \Phi_{\mathrm{BA}\dot{\mathrm{C}}\dot{\mathrm{D}}} = \Phi_{\mathrm{AB}\dot{\mathrm{D}}\dot{\mathrm{C}}} = \overline{\Phi}_{\dot{\mathrm{C}}\dot{\mathrm{D}}\mathrm{AB}} .$$

The spinor $\Phi_{\mathrm{AB}\dot{\mathrm{C}}\dot{\mathrm{D}}}$ associated with E_{abcd} is also associated with the quartic that is the intersection of $R_{\alpha\beta}\, \chi^{\alpha}\chi^{\beta}$ and $g_{\alpha\beta}\chi^{\alpha}\chi^{\beta}$. This quartic establishes a *bilinear correspondence between pairs of points* of γ and $\overline{\gamma}$ or between pairs of rectilinear generators of the contrary mode of the absolute $g_{\alpha\beta}\chi^{\alpha}\chi^{\beta} = 0$. This 2-2 correspondence between pairs of points of γ and points of $\overline{\gamma}$ is defined by the spinor $\Phi_{\mathrm{AB}\dot{\mathrm{C}}\dot{\mathrm{D}}}$. Finally, observe that the spinor that is associated with $C'_{\alpha\beta}$, or with:

(28.5) $$C'_{abcd} = C_{abcd} - (R/12)\, g_{abcd}$$

is:

(28.6) $$\Phi_{\mathrm{ABCD}} + (\lambda/3)\, \varepsilon_{\mathrm{AB}}\, \varepsilon_{\mathrm{CD}} ,$$

in which:

(28.7) $$\lambda = - R/2 .$$

APPENDIX

Commutation rules for the Pfaffian derivatives. – They are obtained by starting with Poincaré's theorem: If $A = A_\alpha \theta^\alpha$ is an exact form then it is closed, i.e., $dA = 0$. One has, in turn:

(A.1)
$$\begin{aligned}
d\theta^0 = {} & \tfrac{1}{2}[-\sigma_3^2 Z^1 + (\sigma_1^3 + \bar\sigma_2^3 + \bar\sigma_0^2) Z^2 + (\sigma_3^3 + \bar\sigma_3^3 + \bar\sigma_1^2 - \sigma_1^2) Z^3 \\
& -\bar\sigma_3^2 \bar Z^1 + (\sigma_2^3 + \bar\sigma_1^3 + \sigma_0^2) \bar Z^2 + (\sigma_3^3 + \bar\sigma_3^3 + \sigma_1^2 - \bar\sigma_1^2) \bar Z^3], \\
d\theta^1 = {} & \tfrac{1}{2}[\sigma_2^2 Z^1 + (\bar\sigma_2^1 + \bar\sigma_0^3 - \sigma_0^3) Z^2 + (\bar\sigma_3^1 + \bar\sigma_1^3 - \sigma_2^3 + \sigma_0^2) Z^3 \\
& - (\bar\sigma_3^3 - \sigma_3^3 - \sigma_1^2) \bar Z^1 + \bar\sigma_1^1 \bar Z^2 + (\bar\sigma_3^1 + \sigma_0^2 - \bar\sigma_1^3 - \sigma_2^3) \bar Z^3], \\
d\theta^2 = {} & \tfrac{1}{2}[(\bar\sigma_3^3 - \sigma_3^3 + \bar\sigma_1^2) Z^1 + \sigma_1^1 Z^2 + (\sigma_3^1 + \bar\sigma_0^2 + \bar\sigma_2^3 - \sigma_1^3) Z^3 \\
& -\bar\sigma_2^2 \bar Z^1 + (\sigma_2^1 - \bar\sigma_0^3 + \sigma_0^3) \bar Z^2 + (\sigma_3^1 + \bar\sigma_0^2 + \bar\sigma_2^3 - \sigma_1^3) \bar Z^3], \\
d\theta^3 = {} & \tfrac{1}{2}[(\bar\sigma_3^1 + \sigma_2^3 + \bar\sigma_1^3) Z^1 - \sigma_0^1 Z^2 + (\sigma_0^3 + \bar\sigma_0^3 - \sigma_2^1 + \bar\sigma_2^1) Z^3 \\
& + (\sigma_1^3 + \bar\sigma_2^3 + \sigma_3^1) \bar Z^1 - \bar\sigma_0^1 \bar Z^2 + (\sigma_0^3 + \bar\sigma_0^3 + \sigma_2^1 - \bar\sigma_2^1) \bar Z^3],
\end{aligned}$$

(A.2)
$$\begin{aligned}
dA = {} & [A_{32} - A_{23} - \tfrac{1}{2}\sigma_3^2 A_0 + \tfrac{1}{2}\sigma_2^2 A_1 + \tfrac{1}{2}(\bar\sigma_3^3 - \sigma_3^3 + \bar\sigma_1^2) A_2 \\
& + \tfrac{1}{2}(\bar\sigma_3^1 + \sigma_2^3 + \bar\sigma_1^3) A_3] Z^1 + [A_{10} - A_{01} + \tfrac{1}{2}(\sigma_1^3 + \bar\sigma_2^3 + \bar\sigma_0^2) A_0 \\
& + \tfrac{1}{2}(\bar\sigma_2^1 + \bar\sigma_0^3 - \sigma_0^3) A_1 + \tfrac{1}{2}\sigma_1^1 A_2 - \tfrac{1}{2}\sigma_0^1 A_3] Z^2 \\
& + [(A_{30} - A_{03} - A_{21} + A_{12}) + \tfrac{1}{2}(\sigma_3^3 + \bar\sigma_3^3 + \bar\sigma_1^2 - \sigma_1^2) A_0 \\
& + \tfrac{1}{2}(\bar\sigma_3^1 + \bar\sigma_1^3 - \sigma_2^3 + \sigma_0^2) A_1 + \tfrac{1}{2}(\sigma_3^1 + \bar\sigma_0^2 + \bar\sigma_2^3 - \sigma_1^3) A_2 \\
& + \tfrac{1}{2}(\sigma_0^3 + \bar\sigma_0^3 + \sigma_2^1 - \bar\sigma_2^1) A_3] Z^3 + [A_{31} - A_{13} - \tfrac{1}{2}\bar\sigma_3^2 A_0 \\
& - \tfrac{1}{2}(\bar\sigma_3^3 - \sigma_3^3 - \sigma_1^2) A_1 + \tfrac{1}{2}\bar\sigma_2^2 A_2 + \tfrac{1}{2}(\sigma_1^3 + \bar\sigma_2^3 + \sigma_3^1)] \bar Z^1 \\
& + [A_{30} - A_{03} + \tfrac{1}{2}(\sigma_2^3 + \bar\sigma_1^3 + \sigma_0^2) A_0 + \tfrac{1}{2}\bar\sigma_1^1 A_1 \\
& + \tfrac{1}{2}(\bar\sigma_2^1 - \bar\sigma_0^3 + \sigma_0^3) A_2 - \tfrac{1}{2}\bar\sigma_0^1 A_3] \bar Z^2 + [(A_{30} - A_{03} - A_{21} + A_{12}) \\
& + \tfrac{1}{2}(\sigma_3^3 + \bar\sigma_3^3 + \sigma_1^2 - \bar\sigma_1^2) A_0 + \tfrac{1}{2}(\bar\sigma_3^1 + \sigma_0^2 - \bar\sigma_1^3 + \sigma_2^3) A_1 \\
& + \tfrac{1}{2}(\sigma_3^1 + \bar\sigma_0^2 - \bar\sigma_2^3 + \sigma_1^3) A_2 + (\sigma_0^3 + \bar\sigma_0^3 + \sigma_2^1 - \bar\sigma_2^1) A_3] \bar Z^3 .
\end{aligned}$$

If A is an exact form then it suffices to annul the coefficients of $Z^\alpha, \bar Z^\alpha$ to obtain the commutation relations. One has:
(A.3)

$$\begin{cases}
dZ^1 = (\sigma_0^3 + \tfrac{1}{2}\sigma_2^1)\theta^0 \wedge \theta^2 \wedge \theta^3 + (\sigma_1^3 + \tfrac{1}{2}\sigma_3^1)\theta^1 \wedge \theta^2 \wedge \theta^3 + \tfrac{1}{2}\sigma_0^1\theta^0 \wedge \theta^1 \wedge \theta^2 + \tfrac{1}{2}\sigma_1^1\theta^0 \wedge \theta^1 \wedge \theta^3 \\
dZ^1 = -\tfrac{1}{2}\sigma_2^2\theta^0 \wedge \theta^2 \wedge \theta^3 - \tfrac{1}{2}\sigma_3^2\theta^1 \wedge \theta^2 \wedge \theta^3 - (\sigma_2^3 + \tfrac{1}{2}\sigma_0^2)\theta^0 \wedge \theta^1 \wedge \theta^2 - (\sigma_3^3 + \tfrac{1}{2}\sigma_1^2)\theta^0 \wedge \theta^1 \wedge \theta^3 \\
dZ^1 = \tfrac{1}{2}\sigma_0^2\theta^0 \wedge \theta^2 \wedge \theta^3 + \tfrac{1}{2}\sigma_1^2\theta^1 \wedge \theta^2 \wedge \theta^3 - \tfrac{1}{2}\sigma_2^1\theta^0 \wedge \theta^1 \wedge \theta^2 - \tfrac{1}{2}\sigma_3^1\theta^0 \wedge \theta^1 \wedge \theta^3.
\end{cases}$$

If:
(A.4)
$$B = B_\alpha Z^\alpha , \qquad C = C_\alpha \bar Z^\alpha$$

then:

(A.5) $$dB = \theta^1 \wedge \theta^2 \wedge \theta^3\, [B_{1,1} - \tfrac{1}{2} B_{3,3} + B_1(\sigma_1^3 + \tfrac{1}{2}\, \sigma_3^1) - \tfrac{1}{2} B_2\, \sigma_3^2 + \tfrac{1}{2} B_3\, \sigma_1^2]$$
$$+ \theta^2 \wedge \theta^3 \wedge \theta^0\, [B_{1,0} - \tfrac{1}{2} B_{1,0} + B_1(\sigma_0^3 + \tfrac{1}{2}\, \sigma_2^1) - \tfrac{1}{2} B_2\, \sigma_2^2 + \tfrac{1}{2} B_3\, \sigma_0^2]$$
$$+ \theta^3 \wedge \theta^0 \wedge \theta^1\, [B_{2,3} - \tfrac{1}{2} B_{3,1} + \tfrac{1}{2} B_1\, \sigma_1^1 - B_2(\sigma_3^3 + \tfrac{1}{2}\, \sigma_0^2) - \tfrac{1}{2} B_3\, \sigma_2^1]$$
$$+ \theta^0 \wedge \theta^1 \wedge \theta^2\, [B_{2,2} - \tfrac{1}{2} B_{3,0} + \tfrac{1}{2} B_1\, \sigma_0^1 - B_2(\sigma_2^3 + \tfrac{1}{2}\, \sigma_0^2) - \tfrac{1}{2} B_3\, \sigma_2^1]$$

$$dC = \theta^1 \wedge \theta^2 \wedge \theta^3\, [- C_{1,2} - \tfrac{1}{2} C_{3,3} + C_1(\bar\sigma_2^3 + \tfrac{1}{2}\, \bar\sigma_3^1) - \tfrac{1}{2} C_2\, \bar\sigma_3^2 - \tfrac{1}{2} C_3\, \bar\sigma_2^2]$$
$$+ \theta^2 \wedge \theta^3 \wedge \theta^0\, [C_{2,3} - \tfrac{1}{2} C_{3,2} + \tfrac{1}{2} C_1\, \bar\sigma_2^1 - C_2(\bar\sigma_3^3 + \tfrac{1}{2}\, \bar\sigma_2^2) - \tfrac{1}{2} C_3\, \bar\sigma_3^1]$$
$$+ \theta^3 \wedge \theta^0 \wedge \theta^1\, [C_{1,0} - \tfrac{1}{2} C_{3,1} + C_1(\bar\sigma_0^3 + \tfrac{1}{2}\, \bar\sigma_1^1) - \tfrac{1}{2} C_2\, \bar\sigma_1^2 + \tfrac{1}{2} C_3\, \bar\sigma_0^2]$$
$$+ \theta^0 \wedge \theta^1 \wedge \theta^2\, [- C_{2,1} + \tfrac{1}{2} C_{3,0} - \tfrac{1}{2} C_1\, \bar\sigma_0^1 + C_2(\bar\sigma_1^3 + \tfrac{1}{2}\, \bar\sigma_0^2) + \tfrac{1}{2} C_3\, \bar\sigma_1^1] .$$

Calculation of the curvature.

$$\Sigma_1 = d\sigma^2 - \sigma^2 \wedge \sigma^3 ,$$

(A.6) $\Sigma_1 =$

$$Z^1 [\sigma_{32}^2 - \sigma_{23}^2 - \tfrac{1}{2}\, \sigma_0^2\, \sigma_3^2 + \tfrac{1}{2}\, \sigma_1^2\, \sigma_2^2 + \tfrac{1}{2}\, \sigma_2^2 (\bar\sigma_3^3 - 3 \sigma_3^3 + \sigma_1^2) + \tfrac{1}{2}\, \sigma_3^2 (\bar\sigma_3^1 + 3 \sigma_2^3 + \bar\sigma_1^3)]$$
$$+ Z^2 [\sigma_{10}^2 - \sigma_{01}^2 - \tfrac{1}{2}\, \sigma_0^2 (- \sigma_1^3 + \bar\sigma_2^3 + \bar\sigma_0^2) + \tfrac{1}{2}\, \sigma_1^2 (\bar\sigma_2^1 + \bar\sigma_0^3 + \sigma_0^3) + \tfrac{1}{2}\, \sigma_2^2\, \sigma_1^1 - \tfrac{1}{2}\, \sigma_3^2\, \sigma_0^3)]$$
$$+ Z^3 [\sigma_{30}^2 - \sigma_{03}^2 - \sigma_{21}^2 + \sigma_{12}^2 + \tfrac{1}{2}\, \sigma_0^2 (\bar\sigma_3^3 - \sigma_3^3 + \bar\sigma_1^2) + \tfrac{1}{2}\, \sigma_1^2 (\bar\sigma_3^1 + \bar\sigma_1^3 + \sigma_2^3)$$
$$+ \tfrac{1}{2}\, \sigma_3^2 (3 \sigma_0^3 + \bar\sigma_0^3 - \sigma_2^1 + \bar\sigma_2^1)]$$
$$+ \bar Z^1 [\sigma_{31}^2 - \sigma_{13}^2 - \tfrac{1}{2}\, \sigma_0^2\, \bar\sigma_3^2 - \tfrac{1}{2}\, \sigma_1^2 (\bar\sigma_3^3 + \sigma_3^3 - \bar\sigma_1^2) + \tfrac{1}{2}\, \sigma_2^2\, \bar\sigma_2^2 + \tfrac{1}{2}\, \sigma_3^2 (3 \sigma_1^3 + \bar\sigma_2^3 + \sigma_3^1)]$$
$$+ \bar Z^2 [\sigma_{20}^2 - \sigma_{02}^2 + \tfrac{1}{2}\, \sigma_0^2 (\sigma_1^3 - \bar\sigma_2^3 + \sigma_0^2) + \tfrac{1}{2}\, \sigma_1^2\, \bar\sigma_1^1 + \tfrac{1}{2}\, \sigma_2^2 (\sigma_2^1 - \bar\sigma_0^3 + 3 \sigma_0^3) - \tfrac{1}{2}\, \sigma_3^2\, \bar\sigma_0^1]$$
$$+ \bar Z^3 [- \sigma_{03}^2 + \sigma_{30}^2 - \sigma_{12}^2 + \sigma_{21}^2 + \tfrac{1}{2}\, \sigma_0^2 (\bar\sigma_3^3 - \sigma_3^3 + \bar\sigma_1^2 - \sigma_1^2) + \tfrac{1}{2}\, \sigma_1^2 (\bar\sigma_3^1 + \sigma_0^2 - \bar\sigma_1^3 + \sigma_2^3)$$
$$\tfrac{1}{2}\, \sigma_2^2 (\sigma_3^1 + \bar\sigma_0^2 - \bar\sigma_2^3 + 3 \sigma_1^2) + \tfrac{1}{2}\, \sigma_3^2 (3 \sigma_0^3 + \bar\sigma_0^3 + \sigma_2^1 - \bar\sigma_3^1)],$$

$$\Sigma_2 = d\sigma^1 + \sigma^1 \wedge \sigma^3 ,$$

(A.7) $\Sigma_2 =$

$$Z^1 [\sigma_{32}^1 - \sigma_{23}^1 - \tfrac{1}{2}\, \sigma_0^1\, \sigma_3^2 + \tfrac{1}{2}\, \sigma_1^1\, \sigma_2^2 + \tfrac{1}{2}\, \sigma_2^1 (\bar\sigma_3^3 + \sigma_3^3 + \sigma_1^2) + \tfrac{1}{2}\, \sigma_3^1 (\bar\sigma_3^1 - \sigma_2^3 + \bar\sigma_1^3)]$$
$$+ Z^2 [\sigma_{10}^1 - \sigma_{01}^1 + \tfrac{1}{2}\, \sigma_0^1 (3 \sigma_1^3 + \bar\sigma_2^3 + \bar\sigma_0^2) + \tfrac{1}{2}\, \sigma_1^1 (\bar\sigma_2^1 + \bar\sigma_0^3 - 3 \sigma_0^3) + \tfrac{1}{2}\, \sigma_2^1\, \sigma_1^1 - \tfrac{1}{2}\, \sigma_3^1\, \sigma_0^3)]$$
$$+ Z^3 [- \sigma_{30}^1 + \sigma_{03}^1 + \sigma_{21}^1 - \sigma_{12}^1 + \tfrac{1}{2}\, \sigma_0^1 (3 \sigma_3^3 + \bar\sigma_3^3 + \bar\sigma_1^2 - \sigma_1^2) + \tfrac{1}{2}\, \sigma_1^1 (\bar\sigma_3^1 + \bar\sigma_1^3 - 3 \sigma_2^3 + \sigma_0^2)$$
$$+ \tfrac{1}{2}\, \sigma_2^1 (\sigma_3^1 + \bar\sigma_0^2 + \bar\sigma_2^3 + \sigma_1^3) + \tfrac{1}{2}\, \sigma_3^1 (- \sigma_0^3 + \bar\sigma_0^3 - \sigma_2^1 + \bar\sigma_2^1)]$$
$$+ \bar Z^1 [\sigma_{31}^1 - \sigma_{13}^1 - \tfrac{1}{2}\, \sigma_0^1\, \bar\sigma_3^2 - \tfrac{1}{2}\, \sigma_1^1 (\bar\sigma_3^3 - 3 \sigma_3^3 - \sigma_1^2) + \tfrac{1}{2}\, \sigma_2^1\, \bar\sigma_2^2 + \tfrac{1}{2}\, \sigma_3^1 (- \sigma_1^3 + \bar\sigma_2^3 + \sigma_3^1)]$$
$$+ \bar Z^2 [\sigma_{20}^1 - \sigma_{02}^1 + \tfrac{1}{2}\, \sigma_0^1 (3 \sigma_2^3 + \bar\sigma_1^3 + \sigma_0^2) + \tfrac{1}{2}\, \sigma_1^1\, \bar\sigma_1^1 + \tfrac{1}{2}\, \sigma_2^1 (\sigma_2^1 - \bar\sigma_0^3 + \sigma_0^3) - \tfrac{1}{2}\, \sigma_3^1\, \bar\sigma_0^1]$$
$$+ \bar Z^3 [\sigma_{30}^1 - \sigma_{03}^1 - \sigma_{21}^2 - \sigma_{12}^2 + \tfrac{1}{2}\, \sigma_0^1 (3 \sigma_3^3 + \bar\sigma_3^3 + \sigma_1^2 - \bar\sigma_1^2) + \tfrac{1}{2}\, \sigma_1^1 (\bar\sigma_3^1 + \sigma_0^2 - \bar\sigma_1^3 + 3 \sigma_2^3)$$

$$+\tfrac{1}{2}\,\sigma_2^1\,(\,\sigma_3^1+\bar\sigma_0^2-\bar\sigma_2^3-\sigma_1^2\,)+\tfrac{1}{2}\,\sigma_3^1\,(-\sigma_0^3+\bar\sigma_0^3+\sigma_2^1-\bar\sigma_2^1\,)],$$

$\Sigma_3 = -\,2\,d\sigma^3 - \sigma^1 \wedge \sigma^3$,

(A.8) $\Sigma_3 =$

$$\begin{aligned}
&Z^1\,[2(\sigma_{32}^1-\sigma_{23}^1)+\sigma_0^3\,\sigma_3^2-\sigma_1^3\,\sigma_2^2-\sigma_2^3\,(\bar\sigma_3^3-\sigma_3^3+\sigma_1^2)-\sigma_3^3\,(\bar\sigma_3^1+\sigma_2^3+\bar\sigma_1^3)\\
&\qquad -\sigma_2^1\,\sigma_3^2+\sigma_3^1\,\sigma_2^2\,]\\
&+Z^2\,[2(\sigma_{01}^3-\sigma_{10}^3)+\sigma_0^3\,(\sigma_1^3+\bar\sigma_2^3+\bar\sigma_0^2)-\sigma_1^3\,(\bar\sigma_2^1+\bar\sigma_0^3-\sigma_0^3)-\sigma_2^3\,\sigma_1^1-\sigma_3^3\,\sigma_0^1\\
&\qquad -\sigma_0^1\,\sigma_1^2+\sigma_1^1\,\sigma_0^2\,]\\
&+Z^3\,[2(\sigma_{03}^3+\sigma_{30}^3+\sigma_{21}^3-\sigma_{12}^3)+\sigma_0^3\,(\sigma_3^3+\bar\sigma_3^3+\bar\sigma_1^2-\sigma_1^2)-\sigma_1^3\,(\bar\sigma_3^1+\bar\sigma_1^3-\sigma_2^3+\sigma_0^2)\\
&\qquad +\tfrac{1}{2}\,\sigma_2^3\,(\sigma_3^1+\bar\sigma_0^2+\bar\sigma_2^3-\sigma_1^3)-\sigma_3^3\,(\sigma_0^3+\bar\sigma_0^3-\sigma_2^1+\bar\sigma_2^1)-\sigma_0^1\,\sigma_3^2-\sigma_3^1\,\sigma_0^2\\
&\qquad +\sigma_1^1\,\sigma_2^2-\sigma_2^1\,\sigma_1^2\,]\\
&+\bar Z^1[2(\sigma_{13}^3-\sigma_{31}^3)+\sigma_0^3\,\bar\sigma_3^2+\sigma_1^3\,(\bar\sigma_3^3-\sigma_3^3-\sigma_1^2)-\sigma_2^3\,\bar\sigma_2^2-\sigma_3^3\,(\sigma_1^3+\bar\sigma_2^3+\sigma_3^1)\\
&\qquad -\sigma_1^1\,\sigma_3^2+\sigma_3^1\,\sigma_1^2\,]\\
&+\bar Z^2[2(\sigma_{02}^3-\sigma_{20}^3)-\sigma_0^3\,(\sigma_2^3+\bar\sigma_1^3+\sigma_0^2)-\sigma_1^3\,\bar\sigma_1^1-\sigma_2^3\,(\sigma_2^1-\bar\sigma_0^3+\sigma_0^3)+\sigma_3^3\,\bar\sigma_0^1\\
&\qquad -\sigma_0^1\,\sigma_2^2-\sigma_2^1\,\sigma_0^2\,]\\
&+\bar Z^3[2(\sigma_{03}^3-\sigma_{30}^3+\sigma_{12}^3-\sigma_{21}^3)+\sigma_0^3\,(\sigma_3^3+\bar\sigma_3^3+\sigma_1^2-\bar\sigma_1^2)-\sigma_1^3\,(\bar\sigma_3^1+\sigma_0^2-\bar\sigma_1^3+\sigma_2^3)\\
&\qquad -\sigma_2^3\,(\sigma_3^1+\bar\sigma_0^2-\bar\sigma_2^3+\sigma_1^2)+\sigma_3^3\,(\sigma_0^3+\bar\sigma_0^3+\sigma_2^1-\bar\sigma_2^1)\\
&\qquad -\sigma_0^1\,\sigma_3^2-\sigma_3^1\,\sigma_0^2-\sigma_1^1\,\sigma_2^2+\sigma_2^1\,\sigma_1^2\,]\,.
\end{aligned}$$

One has:

(A.9)
$$R_{ab}-\tfrac{1}{4}g_{ab}\,R=\begin{pmatrix} E_{2\bar2} & \tfrac12 E_{2\bar3} & \tfrac12 E_{3\bar2} & \tfrac14 E_{3\bar3}\\ \tfrac12 E_{2\bar3} & E_{2\bar1} & \tfrac14 E_{3\bar3} & \tfrac12 E_{3\bar1}\\ \tfrac12 E_{3\bar2} & \tfrac14 E_{3\bar3} & E_{1\bar2} & \tfrac12 E_{1\bar3}\\ \tfrac14 E_{3\bar3} & \tfrac12 E_{3\bar1} & \tfrac12 E_{1\bar3} & E_{1\bar1}\end{pmatrix}.$$

Bianchi identities. – If $R = 0$ then the Bianchi identities may be written:

(A.10) $\mathrm{D}\Sigma_1 = \mathrm{D}C_{\alpha\beta} \wedge Z^\beta + \mathrm{D}E_{\alpha\beta}\,\bar Z^\beta = 0$,

in which:

(A.11) $\mathrm{D}C_{\alpha\beta} = dC_{\alpha\beta} - C_{\gamma\beta}\,\sigma^\beta{}_\alpha - C_{\alpha\gamma}\,\sigma^\gamma{}_\beta = C_{\alpha\beta;c}\,\theta^c$,

(A.12) $\mathrm{D}E_{\alpha\beta} = dE_{\alpha\bar\beta} - E_{\gamma\bar\beta}\sigma^\gamma{}_\alpha - E_{\alpha\bar\gamma}\bar\sigma^\gamma_\beta = E_{\alpha\bar\beta;c}\;\theta^c$.

One finds:

(A.13)
$$\begin{cases} C_{\alpha 1;1} - \tfrac{1}{2} C_{\alpha 3;3} = E_{\alpha \bar{1};2} - \tfrac{1}{2} E_{\alpha \bar{3};3}, \\ C_{\alpha 1;0} - \tfrac{1}{2} C_{\alpha 3;2} = -E_{\alpha \bar{2};3} + \tfrac{1}{2} E_{\alpha \bar{3};2}, \\ C_{\alpha 2;3} - \tfrac{1}{2} C_{\alpha 3;1} = E_{\alpha \bar{1};0} + \tfrac{1}{2} E_{\alpha \bar{3};1}, \\ C_{\alpha 2;2} - \tfrac{1}{2} C_{\alpha 3;0} = E_{\alpha \bar{2};1} - \tfrac{1}{2} E_{\alpha \bar{3};0}; \end{cases}$$

(A.14)
$$\begin{cases} \mathrm{D}C_{11} = dC_{11} + 2C_{11}\sigma^3 + C_{13}\sigma^2, \\ \mathrm{D}C_{12} = dC_{12} - \tfrac{1}{2} C_{13}\sigma^1 + \tfrac{1}{2} C_{23}\sigma^2, \\ \mathrm{D}C_{13} = dC_{13} - 2C_{11}\sigma^1 + 3C_{12}\sigma^2 + C_{13}\sigma^3, \\ \mathrm{D}C_{22} = dC_{22} - 2C_{22}\sigma^3 - C_{23}\sigma^1, \\ \mathrm{D}C_{23} = dC_{23} - C_{23}\sigma^3 - 3C_{12}\sigma^1 + C_{22}\sigma^2, \end{cases}$$

(A.15)
$$\begin{cases} \mathrm{D}E_{1\bar{1}} = dE_{1\bar{1}} + E_{1\bar{1}}(\sigma^3 + \bar{\sigma}^3) + \tfrac{1}{2} E_{3\bar{1}}\sigma^2 + \tfrac{1}{2} E_{1\bar{3}}\bar{\sigma}^2, \\ \mathrm{D}E_{1\bar{2}} = dE_{1\bar{2}} + E_{1\bar{2}}(\sigma^3 - \bar{\sigma}^3) + \tfrac{1}{2} E_{3\bar{2}}\sigma^2 - \tfrac{1}{2} E_{1\bar{3}}\bar{\sigma}^1, \\ \mathrm{D}E_{1\bar{3}} = dE_{1\bar{3}} + E_{1\bar{3}}\sigma^3 + \tfrac{1}{2} E_{3\bar{3}}\sigma^2 - E_{1\bar{1}}\sigma^1 + E_{1\bar{2}}\bar{\sigma}^2, \\ \mathrm{D}E_{2\bar{2}} = dE_{2\bar{2}} - E_{2\bar{2}}(\sigma^3 + \bar{\sigma}^3) - \tfrac{1}{2} E_{3\bar{2}}\sigma^1 - \tfrac{1}{2} E_{2\bar{3}}\sigma^1, \\ \mathrm{D}E_{2\bar{3}} = dE_{2\bar{3}} - E_{2\bar{3}}\sigma^3 - \tfrac{1}{2} E_{3\bar{3}}\sigma^1 - E_{2\bar{1}}\bar{\sigma}^1 + E_{2\bar{2}}\bar{\sigma}^2, \\ \mathrm{D}E_{3\bar{3}} = dE_{3\bar{3}} - E_{1\bar{3}}\sigma^1 + E_{2\bar{3}}\sigma^2 - E_{3\bar{1}}\bar{\sigma}^1 + E_{3\bar{2}}\bar{\sigma}^2. \end{cases}$$

$\mathrm{D}\Sigma_1 = 0$ gives $d\Sigma_1 = -\Sigma_1 \wedge \sigma^3 - \Sigma_3 \wedge \sigma^2$. Thus:

$$d(C_{1\alpha} Z^{\alpha}) + (C_{1\alpha} Z^{\alpha}) \wedge \sigma^3 + \tfrac{1}{2}(C_{3\alpha} Z^{\alpha}) \wedge \sigma^2$$
$$= -d(E_{1\bar{\alpha}}\bar{Z}^{\alpha}) - (E_{1\bar{\alpha}}\bar{Z}^{\alpha}) \wedge \sigma^3 - \tfrac{1}{2}(E_{3\bar{\alpha}}\bar{Z}^{\alpha}) \wedge \sigma^2,$$

(A.16)
$$C_{11,1} + C_{11}(2\sigma_1^3 + \tfrac{1}{2}\sigma_3^1) - \tfrac{1}{2} C_{13,3} + C_{13}(\sigma_1^2 - \tfrac{1}{2}\sigma_3^3) - \tfrac{3}{2} C_{12}\sigma_3^2$$
$$= E_{1\bar{1},2} + E_{1\bar{1}}(\bar{\sigma}_1^3 + \sigma_2^3 + \tfrac{1}{2}\sigma_3^1) - \tfrac{1}{2} E_{1\bar{3},3} + \tfrac{1}{2} E_{1\bar{3}}(\bar{\sigma}_1^2 - \sigma_3^2) + \tfrac{1}{2} E_{3\bar{1}}\sigma_2^2$$
$$- \tfrac{1}{4} E_{3\bar{3}}\sigma_3^2 - \tfrac{1}{2} E_{1\bar{2}}\bar{\sigma}_3^2 \qquad (123)$$

(A.17)
$$C_{11,0} + C_{11}(2\sigma_0^3 + \tfrac{1}{2}\sigma_2^1) - \tfrac{1}{2} C_{13,2} + C_{13}(\sigma_0^2 - \tfrac{1}{2}\sigma_2^3) - \tfrac{3}{2} C_{12}\sigma_2^2$$
$$= -\tfrac{1}{2} E_{1\bar{1}}\bar{\sigma}_1^1 + \tfrac{1}{2} E_{1\bar{3},2} - E_{1\bar{2},3} + \tfrac{1}{2} E_{1\bar{3}}(\bar{\sigma}_3^1 + \sigma_2^3) + E_{1\bar{2}}(\bar{\sigma}_3^3 - \sigma_3^3 + \tfrac{1}{2}\bar{\sigma}_1^2)$$
$$+ \tfrac{1}{4} E_{3\bar{3}}\sigma_2^2 - \tfrac{1}{2} E_{3\bar{2}}\sigma_3^2 \qquad (230)$$

(A.18)
$$\tfrac{1}{2} C_{11}\sigma_1^1 - \tfrac{1}{2} C_{13,1} - \tfrac{1}{2} C_{13}(\sigma_3^1 + \sigma_1^3) + C_{12,3} - \tfrac{3}{2} C_{12}\sigma_1^2 + \tfrac{1}{2} C_{23}\sigma_3^2 - (R_{,3}/6)$$
$$= -E_{1\bar{1},0} - E_{1\bar{1}}(\bar{\sigma}_0^3 + \sigma_0^3 + \tfrac{1}{2}\bar{\sigma}_2^1) + \tfrac{1}{2} E_{1\bar{3},1} - \tfrac{1}{2} E_{1\bar{3}}(\bar{\sigma}_0^2 - \sigma_1^3) - \tfrac{1}{2} E_{3\bar{1}}\sigma_0^2$$
$$+ \tfrac{1}{2} E_{1\bar{2}}\bar{\sigma}_2^2 + \tfrac{1}{4} E_{3\bar{3}}\sigma_1^2 \qquad (301)$$

(A.19)
$$\tfrac{1}{2} C_{11}\sigma_0^1 - \tfrac{1}{2} C_{13,0} - \tfrac{1}{2} C_{13}(\sigma_2^1 + \sigma_0^3) + C_{12,2} - \tfrac{3}{2} C_{12}\sigma_0^2 + \tfrac{1}{2} C_{23}\sigma_2^2 - (R_{,2}/6)$$

$$= -\tfrac{1}{2}E_{1\bar{1}}\bar{\sigma}_0^1 - \tfrac{1}{2}E_{1\bar{3},0} - \tfrac{1}{2}E_{1\bar{3}}(\bar{\sigma}_2^1 + \sigma_0^3) + E_{1\bar{2},1} - E_{1\bar{2}}(\bar{\sigma}_2^3 + \tfrac{1}{2}\bar{\sigma}_0^2 - \sigma_1^3)$$
$$-\tfrac{1}{4}E_{3\bar{3}}\bar{\sigma}_0^2 + \tfrac{1}{2}E_{3\bar{2}}\sigma_1^2 . \qquad (012)$$

$D\Sigma_2 = 0$ gives $d\Sigma_2 = \Sigma_2 \wedge \sigma^3 + \tfrac{1}{2}\Sigma_3 \wedge \sigma^1$. Thus:

$$d(C_{2\alpha}Z^\alpha) + (C_{2\alpha}Z^\alpha) \wedge \sigma^3 + \tfrac{1}{2}(C_{3\alpha}Z^\alpha) \wedge \sigma^1$$
$$= -\, d(E_{2\bar{\alpha}}\bar{Z}^\alpha) + (E_{2\bar{\alpha}}\bar{Z}^\alpha) \wedge \sigma^3 + \tfrac{1}{2}(E_{3\bar{\alpha}}\bar{Z}^\alpha) \wedge \sigma^1,$$

(A.20)
$$-\tfrac{1}{2}C_{13}\sigma_1^1 + C_{12,1} + \tfrac{3}{2}C_{12}\sigma_3^1 - \tfrac{1}{2}C_{23,3} + \tfrac{1}{2}C_{23}(\sigma_1^2 + \sigma_3^3) - \tfrac{1}{2}C_{22}\sigma_3^2 - (R_{,1}/6)$$
$$= E_{2\bar{1},2} + E_{2\bar{1}}(\bar{\sigma}_1^3 + \tfrac{1}{2}\sigma_3^1 - \sigma_2^3) - \tfrac{1}{2}E_{2\bar{3},3} + \tfrac{1}{2}E_{2\bar{3}}(\bar{\sigma}_2^2 + \sigma_3^3) - \tfrac{1}{2}E_{2\bar{2}}\sigma_3^2$$
$$-\tfrac{1}{2}E_{3\bar{1}}\sigma_2^1 + \tfrac{1}{4}E_{3\bar{3}}\sigma_3^1 \qquad (123)$$

(A.21)
$$-\tfrac{1}{2}C_{13}\sigma_0^1 + C_{12,0} + \tfrac{3}{2}C_{12}\sigma_2^1 - \tfrac{1}{2}C_{23,3} + \tfrac{1}{2}C_{23}(\sigma_0^2 + \sigma_2^3) - \tfrac{1}{2}C_{22}\sigma_2^2 - (R_{,0}/6)$$
$$= -E_{2\bar{2},3} + E_{2\bar{2}}(\sigma_3^3 + \bar{\sigma}_3^3 + \tfrac{1}{2}\sigma_2^2) + \tfrac{1}{2}E_{2\bar{3},2} + \tfrac{1}{2}E_{2\bar{3}}(\bar{\sigma}_3^1 - \sigma_2^3) - \tfrac{1}{4}E_{3\bar{3}}\sigma_2^1$$
$$-\tfrac{1}{2}E_{2\bar{1}}\bar{\sigma}_1^1 + \tfrac{1}{2}E_{3\bar{2}}\sigma_3^1 \qquad (230)$$

(A.22)
$$\tfrac{3}{2}C_{11}\sigma_1^1 - \tfrac{1}{2}C_{23,1} - C_{23}(\sigma_3^1 - \tfrac{1}{2}\sigma_1^3) + C_{22,3} - C_{22}(2\sigma_3^3 + \tfrac{1}{2}\sigma_1^2)$$
$$= -E_{2\bar{1},0} - E_{2\bar{1}}(\bar{\sigma}_0^3 + \tfrac{1}{2}\bar{\sigma}_2^1 - \sigma_0^3) + \tfrac{1}{2}E_{2\bar{3},1} - \tfrac{1}{2}E_{2\bar{3}}(\bar{\sigma}_0^2 + \sigma_1^3) + \tfrac{1}{2}E_{2\bar{2}}\bar{\sigma}_2^2$$
$$+\tfrac{1}{2}E_{3\bar{1}}\sigma_0^1 - \tfrac{1}{4}E_{3\bar{3}}\sigma_1^1 \qquad (301)$$

(A.23)
$$\tfrac{3}{2}C_{12}\sigma_1^1 - \tfrac{1}{2}C_{23,0} - C_{23}(\sigma_2^1 - \tfrac{1}{2}\sigma_0^3) + C_{22,2} - C_{22}(2\sigma_2^3 + \tfrac{1}{2}\sigma_0^2)$$
$$= E_{2\bar{2},2} - E_{2\bar{2}}(\bar{\sigma}_2^3 + \tfrac{1}{2}\bar{\sigma}_0^2 + \sigma_1^3) - \tfrac{1}{2}E_{2\bar{3},0} - \tfrac{1}{2}E_{2\bar{3}}(\bar{\sigma}_2^1 - \sigma_0^3) + \tfrac{1}{2}E_{2\bar{1}}\bar{\sigma}_0^1$$
$$+\tfrac{1}{4}E_{3\bar{3}}\sigma_0^1 - \tfrac{1}{2}E_{3\bar{2}}\sigma_1^1 \qquad (012)$$

$D\Sigma_3 = 0$ gives $d\Sigma_3 = \Sigma_1 \wedge \sigma^1 - \Sigma_2 \wedge \sigma^2$. Thus:

$$d(C_{3\alpha}Z^\alpha) + (C_{1\alpha}Z^\alpha) \wedge \sigma^1 + \tfrac{1}{2}(C_{2\alpha}Z^\alpha) \wedge \sigma^2$$
$$= -\, d(E_{3\bar{\alpha}}\bar{Z}^\alpha) + (E_{1\bar{\alpha}}\bar{Z}^\alpha) \wedge \sigma^1 - (E_{3\bar{\alpha}}\bar{Z}^\alpha) \wedge \sigma^2,$$

(A.24)
$$-C_{11}\sigma_1^1 + C_{13,1} + C_{13}(\sigma_1^3 + \sigma_3^1) - C_{12,3} + 3C_{12}\sigma_1^2 - C_{23}\sigma_3^2 - (R_{,3}/6)$$
$$= E_{3\bar{1},2} - E_{3\bar{2}}(\tfrac{1}{2}\bar{\sigma}_3^2) - \tfrac{1}{2}E_{3\bar{3},3} + \tfrac{1}{2}E_{3\bar{3}}\bar{\sigma}_1^2 + E_{3\bar{1}}(\bar{\sigma}_1^3 + \tfrac{1}{2}\bar{\sigma}_3^1) - E_{1\bar{1}}\sigma_2^1$$
$$-\tfrac{1}{2}E_{2\bar{3}}\sigma_3^2 + E_{2\bar{1}}\sigma_2^2 + \tfrac{1}{2}E_{1\bar{3}}\sigma_3^1 \qquad (123)$$

(A.25)
$$-C_{11}\sigma_0^1 + C_{13,0} + C_{12}(\sigma_0^3 + \sigma_0^1) - 2C_{12,2} + 3C_{23}\sigma_0^2 - C_{23}\sigma_2^2 - (R_{,2}/6)$$
$$= -E_{3\bar{2},3} + E_{3\bar{2}}(\bar{\sigma}_3^3 + \tfrac{1}{2}\sigma_2^2) + \tfrac{1}{2}E_{3\bar{3},2} + \tfrac{1}{2}E_{3\bar{3}}\bar{\sigma}_3^1 - \tfrac{1}{2}E_{3\bar{1}}\sigma_1^1 - E_{1\bar{2}}\sigma_3^1 - \tfrac{1}{2}E_{1\bar{3}}\sigma_2^1$$
$$-E_{2\bar{2}}\sigma_3^2 + \tfrac{1}{2}E_{2\bar{3}}\sigma_2^2 \qquad (230)$$

(A.26)
$$C_{13}\sigma_1^1 - 2C_{12,1} - 3C_{12}\,\sigma_3^1 + C_{23,3} - C_{23}(\sigma_3^3 + \sigma_1^2) + C_{22}\sigma_3^2 - (R_{,1}/6)$$
$$= -E_{3\bar{1},0} - E_{3\bar{1}}(\bar{\sigma}_0^3 + \tfrac{1}{2}\bar{\sigma}_2^1) + \tfrac{1}{2}E_{3\bar{3},1} - \tfrac{1}{2}E_{3\bar{3}}\bar{\sigma}_0^2 + \tfrac{1}{2}E_{3\bar{2}}\sigma_2^2 + E_{1\bar{1}}\sigma_0^1 + \tfrac{1}{2}E_{2\bar{3}}\sigma_1^2$$
$$-E_{2\bar{1}}\sigma_0^2 - \tfrac{1}{2}E_{1\bar{3}}\sigma_1^1 \qquad (301)$$

(A.27)
$$C_{13}\sigma_1^1 - 2C_{12,0} - 3C_{12}\,\sigma_2^1 + C_{23,2} - C_{23}(\sigma_2^3 + \sigma_0^2) + C_{22}\sigma_2^2 - (R_{,0}/6)$$
$$=$$
$$E_{3\bar{2},1} - E_{3\bar{2}}(\bar{\sigma}_2^3 + \tfrac{1}{2}\bar{\sigma}_0^2) - \tfrac{1}{2}E_{3\bar{3},0} - \tfrac{1}{2}E_{3\bar{3}}\bar{\sigma}_2^1 + \tfrac{1}{2}E_{3\bar{1}}\sigma_0^1 - E_{1\bar{2}}\sigma_1^2 - \tfrac{1}{2}E_{2\bar{3}}\sigma_0^2$$
$$+E_{2\bar{2}}\sigma_1^2 + \tfrac{1}{2}E_{1\bar{3}}\sigma_0^1 \qquad (012)$$

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REFERENCES

[1] A. LICHNEROWICZ, Théories relativistes de la gravitation et de l'électromagnétisme, Masson, Paris, 1955.
[2] L.P. EISENHART, Riemannian geometry, Princeton Univ. Press, 1949.
[3] J.L. SYNGE and A. SCHILD, Tensor calculus, Toronto Univ. Press, 1952.
[4] F.A.E. PIRANI, *Acta phys. Polon.*, v. **13**, 1956, pp. 389.
[5] A. LICHNEROWICZ, Théorie globale des connexions et des groupes d'holonomie, Dunod, Paris, 1955.
[6] E. CARTAN, Leçons sur la géometrie des espaces de Riemann, Gauthier-Villars, Paris, 1955.
[7] E. CARTAN, *J. Math. pur. appl.*, v. **1**, 1922, pp.141.
[8] E. NEWMAN and R. PENROSE, *J. math. Physics*, v. **3**, 1962, pp. 566.
[9] A. Z. PETROV, *in* Recent Developments in Gravitation, Pergamon Press, Oxford, 1962.
[10] M. D. KRUSKAL, *Phys. Review*, v. **119**, 1960, pp. 1743.
[11] E. NEWMAN, L. TAMBURINO, and T. UNTI, *J. math. Physics*, v. **4**, 1963, pp.915.
[12] Cf., for example, J. EHLERS and W. KUNDT, *in* L. WITTEN, Gravitation, J. Wiley, New York, 1962.
[13] A. Z. PETROV, *Dokl. Akad. Nauk SSSR*, v. **105**, 1955, pp. 905.
[14] J. GÉHÉNIAU, *C.R. Ac. Sc.*, v. **224**, 1957, pp.723.
[15] H. S. RUSE, *Proc. Royal Soc. Edinburgh*, v. **62**, 1944, pp. 64; *Proc. London Math. Soc.*, v. **50**, 1948, pp. 75.
[16] R. SACHS, *Proc. Royal Soc.*, [A], v. **264**, 1961, pp. 302.
[17] R. PENROSE, *Ann. Physics*, v. **10**, 1960, pp. 171.
[18] L. BEL, *C.R. Acad. Sc.*, v. **248**, 1959, pp. 1297.
[19] R. DEBEVER, *Bull. Soc. math. Belg.*, v. **10**, 1959, pp. 112; *C.R. Acad. Soc.*, v. **249**, 1959, pp. 1324 and 1744, v. **250**, 1960, pp. 64.
[20] R. P. KERR, *J. math. Mech.*, v. **12**, 1963, pp. 33.
[21] J. GÉHÉNIAU and R. DEBEVER, Les quatorzes invariants de courbure, *in* 50 Jahre Relativitätstheorie, Birkhäuser-Verlag, Bern, 1955, pp. 60.
[22] J. A. WHEELER and C. W. MISNER, *Ann. Physics*, v. **2**, 1957, pp. 525.
[23] I. ROBINSON, *J. math. Physics*, v. **2**, 1961, pp. 290.
[24] J. N. GOLDBERG and R. K. SACHS, *Acta phys. Polon.*, v. **22**, 1962, pp. 13.
[25] I. ROBINSON and A. SCHILD, *J. math. Physics*, v. **4**, 1963, pp. 484.
[26] L. WITTEN, *Phys. Review*, v. **113**, 1959, pp. 357.