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On the transformations of the equations of dynamics

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1. – The work of **Appell** on "homography in mechanics" led **Goursat** to propose the following question:

Suppose that one is given the system of **Lagrange** equations:

$$(A) \qquad \frac{d}{dt}\left(\frac{\partial S}{\partial p_r'}\right) - \frac{\partial S}{\partial p_r} = P_r , \quad \frac{dp_r}{dt} = p_r' \qquad (r = 1, 2, \ldots, n),$$

in which $S = \frac{1}{2}\sum_{r,s} a_{rs}\, p_r'\, p_s'$ ($a_{rs} = a_{sr}$) is a homogeneous quadratic function of the $p_1', p_2', \ldots, p_n'$ whose coefficients are functions of the $p_1, p_2, \ldots, p_n$, and in which the P_r contain only the $p_1, p_2, \ldots, p_n$. *Find* functions $\varphi_1, \varphi_2, \ldots, \varphi_n$, and λ such that when one sets:

$$(1) \qquad \begin{cases} q_r = \varphi_r(p_1, p_2, \ldots, p_n), \\ dt = \lambda(p_1, p_2, \ldots, p_n)\, dt, \end{cases}$$

it will be possible to transform the given system into the other one:

$$(B) \qquad \frac{d}{dt_1}\left(\frac{\partial S_1}{\partial q_r'}\right) - \frac{\partial S_1}{\partial q_r} = Q_r , \quad \frac{dq_r}{dt_1} = q_r' \qquad (r = 1, 2, \ldots, n),$$

in which $S_1 = \frac{1}{2}\sum_{r,s} b_{rs}\, q_r'\, q_s'$ ($b_{rs} = b_{sr}$) is a homogeneous quadratic function of the $q_1', q_2', \ldots, q_n'$ whose coefficients are functions of only the $q_1, q_2, \ldots, q_n$, and in which the Q_r contain only the $q_1, q_2, \ldots, q_n$.

A particularly interesting case of this question was studied recently by **Painlevé** in a long article that was published in the "Journal de Mathématiques pures et appliquées" (1ère fasc., 1894) and related to the transformations that are defined by the formulas:

(2) $$\begin{cases} p_r = q_r \,, \\ dt = \lambda\,(p_1, p_2, \dots, p_n)\, dt_1 \,. \end{cases}$$

That is the case in which the trajectories that are defined by the systems (*A*) and (*B*) have the same equations, but different motions are defined by them.

Painlevé has proved several important theorems on that subject, which do not, however, exhaust the argument, nor do they give any idea of the extent of those transformations and the contribution that they can make to the solution of the problems of mechanics.

Nonetheless, the question can then be said to be solved completely when one has determined the function λ and otherwise determined the conditions that the coefficients a_{rs}, b_{rs} must satisfy in order for the transformation to be feasible.

Starting from those considerations, I propose to solve completely the problem that **Painlevé** treated in the case where the expressions for the semi-*vis vivas* S and S_1 are reducible to *orthogonal form*, and in the case where the forces $P_r\,(p_1, p_2, \dots, p_n)$, $Q_r\,(q_1, q_2, \dots, q_n)$ are, in fact, arbitrary.

2. – With the aim of establishing some preliminary formulas, it is meanwhile necessary to observe, with **Stäckel** (*), that when one develops the left-hand sides of equations (*A*) and (*B*) and takes into account the transformation (1), one can find coefficients $R_r^{(i)}$, which are n^2 in number (r = 1, 2, …, n ; i = 1, 2, …, n) and do depend upon $p_1, p_2, \dots, p_n$ in such a manner that the expressions:

(3) $$\frac{d}{dt_1}\left(\frac{\partial S_1}{\partial q_r'}\right) - \frac{\partial S_1}{\partial q_r} - \sum_{i=1}^{n} R_r^{(i)} \left[\frac{d}{dt}\left(\frac{\partial S}{\partial p_r'}\right) - \frac{\partial S}{\partial p_r}\right]$$

will no longer contain the second derivatives, and consequently reduce to a quadratic form Φ_r in the $p_1', p_2', \dots, p_n'$.

It is enough to annul the coefficients of $\dfrac{d^2 p_k}{dt^2}$ in the expression (3): With that, one will have:

(4) $$\lambda^2 \left[b_{1r}\frac{\partial \varphi_1}{\partial p_k} + b_{2r}\frac{\partial \varphi_2}{\partial p_k} + \dots + b_{nr}\frac{\partial \varphi_n}{\partial p_k} \right] - \sum_{i=1}^{n} R_r^{(i)}\, a_{ki} = 0 \,.$$

If we set k = 1, 2, …, n then we will have n linear equations for defining $R_r^{(1)}, R_r^{(2)}, \dots, R_r^{(n)}$ in which the determinant of the coefficients of the R is the discriminant of S, which is therefore non-zero.

When one calculates the R, the expression (3) will become equal to a quadratic form Φ_r in the $p_1', p_2', \dots, p_n'$, and one will have the following relations:

(*) Crelle's Journal, v. 111.

(5) $$Q_r = \Phi_r + \sum_{i=1}^{n} R_r^{(i)} P_i$$

between the forces P_r and Q_r .

One can deduce from that formula that when the functions λ and φ_r are chosen arbitrarily, forces P_r that are functions of only the coordinates will correspond to forces Q_r that are functions of the coordinates and velocities.

However, if we note that the forces Q_r depend upon only the positions, it will happen that the transformation is specialized by the conditions:

(6) $$\Phi_1 = 0\,, \qquad \Phi_2 = 0\,, \qquad \ldots, \qquad \Phi_n = 0\,.$$

Now, if we desire that those conditions should be satisfied by all possible motions of the system then it is necessary that equations (6) should be true for any $p_1', p_2', \ldots, p_n'$, and that the coefficients of the various powers of p' the should then be equal to zero.

We will then have a much larger number of equations for defining the φ_r and λ. Those equations will generally have a larger number of functions to be determined, in such a way that we cannot realize the transformation without the existence of certain relations between the coefficients a_{rs}, b_{rs} of the form S and S_1 .

When that transformation does exist, a zero force P will correspond to a zero force Q.

The transformations will then preserve geodetic motions.

3. – Having said that, we shall go on to the solution of the problem that we have posed.

We write down the systems (A) and (B), which we shall call (S) and (S_1), from now on:

(S) $$\frac{d}{dt}\left(\frac{\partial S}{\partial p_r'}\right) - \frac{\partial S}{\partial p_r} = P_r\,, \qquad S = \frac{1}{2}\left(\frac{ds}{dt}\right)^2,$$

(S_1) $$\frac{d}{dt_1}\left(\frac{\partial S_1}{\partial q_r'}\right) - \frac{\partial S_1}{\partial q_r} = Q_r\,, \qquad S_1 = \frac{1}{2}\left(\frac{ds_1}{dt_1}\right)^2,$$

and suppose that we have:

$$ds^2 = \sum_{r=1}^{n} a_{rr}\, dp_r^2\,, \qquad ds_1^2 = \sum_{r=1}^{n} b_{rr}\, dq_r^2\,.$$

We say that the systems (S) and (S_1) *correspond*, and also that the linear elements ds, ds_1 .

The conditions that are assumed for (A) and (B) still remain, so I propose to see what the form of the a_{rr}, b_{rr}, and λ must be in order for the system (S) to transform into the system (S_1) by means of the formulas:

$$\begin{cases} q_r = p_r\,, \\ dt = \lambda(p_1, p_2, \ldots, p_n)\, dt_1\,. \end{cases}$$

If we take into account the hypotheses that were made then equations (4) will give:

$$\lambda^2 b_{rr} = R_r^{(r)} a_{rr}$$

in the present case, so:

$$R_r^{(r)} = \lambda^2 \frac{b_{rr}}{a_{rr}},$$

and

$$R_r^{(1)} = R_r^{(2)} = \ldots = R_r^{(r-1)} = R_r^{(r+1)} = \ldots = R_r^{(n)} = 0 .$$

Equations (5) become:

$$Q_r = \Phi_r + \lambda^2 \frac{b_{rr}}{a_{rr}} P_r .$$

If P_r and Q_r take the values that they get from the left-hand sides of the systems (S) and (S_1) then equations (6) will give rise to the system:

$$(7) \quad \begin{cases} \frac{1}{2}\lambda^2 \frac{\partial b_{rr}}{\partial p_r} + \lambda\, b_{rr} \frac{\partial \lambda}{\partial p_r} - \frac{1}{2}\lambda^2 \frac{b_{rr}}{a_{rr}} \frac{\partial a_{rr}}{\partial p_r} = 0 & (r = 1, 2, \ldots, n), \\ \lambda^2 \frac{\partial b_{rr}}{\partial p_r} + \lambda\, b_{rr} \frac{\partial \lambda}{\partial p_r} - \lambda^2 \frac{b_{rr}}{a_{rr}} \frac{\partial a_{rr}}{\partial p_r} = 0 & \begin{pmatrix} r = 1, 2, \ldots, n & \\ & r \neq s \\ s = 1, 2, \ldots, n & \end{pmatrix} \\ \frac{b_{rr}}{a_{rr}} \frac{\partial a_{rr}}{\partial p_r} - \frac{\partial b_{rr}}{\partial p_r} = 0 & \begin{pmatrix} r = 1, 2, \ldots, n & \\ & r \neq s \\ s = 1, 2, \ldots, n & \end{pmatrix}. \end{cases}$$

One sees from this that the solution of the problem comes down to integrating the system (7), which is composed of $n\,(2n - 1)$ differential equations.

Let us integrate it:

The first equation can be written:

$$\frac{\partial}{\partial p_r}(\log b_{rr}) + \frac{\partial}{\partial p_r}(\log \lambda^2) - \frac{\partial}{\partial p_r}(\log a_{rr}) = 0$$

or

$$\frac{\partial}{\partial p_r}\left(\log \lambda^2 \frac{b_{rr}}{a_{rr}}\right) = 0 .$$

That implies:

$$\lambda^2 \frac{b_{rr}}{a_{rr}} = F_r\,(p_1, p_2, \ldots, p_{r-1}, p_{r+1}, \ldots, p_n),$$

in which F_r is the symbol of an arbitrary function that does not contain the p_r .

The second equation can also be written:

$$\frac{\partial}{\partial p_s}(\log b_{rr}) + \frac{\partial}{\partial p_s}(\log \lambda^2) - \frac{\partial}{\partial p_s}(\log a_{rr}) = 0$$

or

$$\frac{\partial}{\partial p_s}\left(\log \lambda^2 \frac{b_{rr}}{a_{rr}}\right) = 0 ,$$

and that equation, which is true for:

$$s = 1, 2, \ldots, r-1, r+1, \ldots, n ,$$

will give:

$$\lambda \frac{b_{rr}}{a_{rr}} = \frac{1}{\Pi_r(p_r)} ,$$

in which the $\Pi_r(p_r)$ is an arbitrary function of only the p_r .

When one takes that result into account, the equation:

$$\lambda^2 \frac{b_{rr}}{a_{rr}} = F(p_1, p_2, \ldots, p_{r-1}, p_{r+1}, \ldots, p_n)$$

will become:

$$\frac{\lambda}{\Pi_r(p_r)} = F_r(p_1, p_2, \ldots, p_{r-1}, p_{r+1}, \ldots, p_n) ,$$

which is valid for $r = 1, 2, \ldots, n$, so it will contain the other one:

(8) $$\lambda = \Pi_1(p_1) F_1 = \Pi_2(p_2) F_2 = \ldots = \Pi_{r-1}(p_{r-1}) F_{r-1} = \Pi_r(p_r) F_r = \ldots = \Pi_n(p_n) F_n .$$

We now direct our attention to the equality:

$$\Pi_{r-1}(p_{r-1}) F_{r-1} = \Pi_r(p_r) F_r ,$$

which can be written:

$$\frac{F_r}{\Pi_{r-1}(p_{r-1})} = \frac{F_{r-1}}{\Pi_r(p_r)},$$

and observe that from the nature of the function F_r, the left-hand side does not contain p_r, and from the nature of the function F_{r-1}, the right-hand side does not contain p_{r-1} . Therefore, in order for the equality to be true, it is necessary that the two sides should be equal to a certain function:

$$\varphi_{r,r-1}(p_1, p_2, \ldots, p_{r-2}, p_{r+1}, \ldots, p_n)$$

that contains neither p_{r-1} nor p_r .

If we appeal to those considerations then we can deduce the following relations from the system (8):

$$\frac{F_r}{\Pi_1} = \frac{F_1}{\Pi_r} = \varphi_{r1}\,(p_1, p_2, \ldots, p_{r-1}, p_{r+1}, \ldots, p_n)\,,$$

$$\frac{F_r}{\Pi_2} = \frac{F_2}{\Pi_r} = \varphi_{r2}\,(p_1, p_2, \ldots, p_{r-1}, p_{r+1}, \ldots, p_n)\,,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\frac{F_r}{\Pi_s} = \frac{F_s}{\Pi_r} = \varphi_{rs}\,(p_1, p_2, \ldots, p_{s-1}, p_{s+1}, \ldots, p_{r-1}, p_{r+1}, \ldots, p_n)\,,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\frac{F_r}{\Pi_n} = \frac{F_n}{\Pi_r} = \varphi_{rn}\,(p_1, p_2, \ldots, p_{r-1}, p_{r+1}, \ldots, p_n)\,,$$

in which the φ are symbols of arbitrary functions, and the indices r, s in the functions φ_{ri} indicate that the variables p_r and p_s are missing from them.

Now, one can infer from those relations that:

$$F_r$$

(which is independent of the p_r) is that function whose quotient, when divided by $\Pi_2\,(p_1)$, is independent of p_1, when divided by $\Pi_2\,(p_1)$, is independent of $p_2, \ldots, \Pi_3\,(p_2)$, and when divided by $\Pi_n\,(p_n)$ is independent of p_n .

Therefore, it will necessarily have the form:

$$F_r = C \cdot \Pi_1\,(p_1) \cdot \Pi_2\,(p_2) \ldots \Pi_{r-1}\,(p_{r-1}) \cdot \Pi_{r+1}\,(p_{r+1}) \ldots \cdot \Pi_n\,(p_n)\,,$$

in which C is an arbitrary constant.

That expression will exhibit the forms of the $F_1, F_2, \ldots, F_n$ when one sets:

$$r = 1, 2, \ldots, n$$

in succession.

If one takes those results into account, along with the system (8), then it will follow that λ has the value:

$$\lambda = C \cdot \Pi_1\,(p_1) \cdot \Pi_2\,(p_2) \ldots \Pi_n\,(p_n)\,.$$

Therefore, one infers from:

$$\lambda \frac{b_{rr}}{a_{rr}} = \frac{1}{\Pi_r(p_r)}$$

that:

$$b_{rr} = \frac{a_{rr}}{C \cdot \Pi_1 \Pi_2 \cdots \Pi_{r-1} \Pi_r^2 \Pi_{r+1} \cdots \Pi_n} .$$

Having done that, we solve the system that is contained in the third equation of the system (7), which is:

$$\frac{b_{rr}}{a_{rr}} \frac{\partial a_{ss}}{\partial p_r} - \frac{\partial b_{ss}}{\partial p_r} = 0 \qquad \begin{pmatrix} r = 1, 2, \ldots, n & \\ & r \neq s \\ s = 1, 2, \ldots, n & \end{pmatrix}.$$

Replace the b_{rr} and b_{ss} in that with their values that were found above, and one will have:

$$\frac{1}{\Pi_r^2} \frac{\partial a_{ss}}{\partial p_r} = \frac{1}{\Pi_s} \frac{\Pi_r \frac{\partial a_{ss}}{\partial p_r} - a_{ss} \frac{\partial \Pi_r}{\partial p_r}}{\Pi_r^2}$$

or

$$(\Pi_r - \Pi_s) \frac{\partial a_{ss}}{\partial p_r} = a_{ss} \frac{\partial \Pi_r}{\partial p_r},$$

i.e.:

$$\frac{\partial}{\partial p_r} (\log a_{ss}) = \frac{\partial}{\partial p_r} [\log (\Pi_r - \Pi_s)] .$$

Integrating the latter will give:

$$a_{ss} = (\Pi_r - \Pi_s)\ \psi_r ,$$

in which ψ_r denotes a function that is independent of p_r .

If one successively sets:

$$r = 1, 2, \ldots, n$$

in the formula that was just deduced then one will get the relations:

$a_{ss} = (\Pi_1 - \Pi_s)\ \psi_1$, in which ψ_1 is independent of p_1 ,
$a_{ss} = (\Pi_2 - \Pi_s)\ \psi_2$, " ψ_2 " p_2 ,
……………………………………………………………..
$a_{ss} = (\Pi_r - \Pi_s)\ \psi_r$, " ψ_r " p_r ,
……………………………………………………………..
$a_{ss} = (\Pi_n - \Pi_s)\ \psi_n$, " ψ_n " p_n .

One deduces from those relations that a_{ss} contains the p_r only in the factor $(\Pi_r - \Pi_s)$, and it can therefore be put into the form:

$$a_{ss} = (\Pi_1 - \Pi_s)\ (\Pi_2 - \Pi_s) \ldots (\Pi_r - \Pi_s) \ldots (\Pi_n - \Pi_s)\ P_s^{(s)} ,$$

in which $P_s^{(s)}$ is an arbitrary function of only p_s . Now, with no loss of generality in the results, one can suppose that $P_s^{(s)}$ = 1, which is then equivalent to replacing the p_r with other variables $p^{(r)}$ that are coupled with the latter by relations:

$$p^{(r)} = \int \sqrt{P_r^{(r)}}\, dp_r \ .$$

Therefore, we will assume that:

$$a_{ss} = (\Pi_1 - \Pi_s)\,(\Pi_2 - \Pi_s)\,\ldots\,(\Pi_r - \Pi_s)\,\ldots\,(\Pi_n - \Pi_s) \qquad (r \neq s).$$

If one substitutes that value for a_{ss} in the expression that was found for b_{ss} then one will get:

$$b_{ss} = \frac{(\Pi_1 - \Pi_s)(\Pi_2 - \Pi_s)\cdots(\Pi_r - \Pi_s)\cdots(\Pi_n - \Pi_s)}{\Pi_1\,\Pi_2\cdots\Pi_{s-1}\,\Pi_s^2\,\Pi_{s+1}\cdots\Pi_n} \qquad (r \neq s).$$

Meanwhile, one should observe that the value of a_{ss} will not change when one replaces Π_r with $\Pi_r + h$, in which h is a constant.

On the contrary, b_{ss} will become a function of h, and will vary when h varies.

Therefore, any value of a_{ss} that satisfies the integrated system of equations will correspond to an infinitude of values of b_{ss} that all satisfy that same system.

What was just said will allow us to state the following proposition:

If the expressions for the semi-vis vivas S and S_1 are reducible to orthogonal form then the system (S) *of* ***Lagrange*** *equations can be transformed into the system* (S_1) *by using the formulas:*

$$q_r = p_r \,,$$
$$dt = \lambda\,(p_1, p_2, \ldots, p_n)\, dt_1$$

when one sets:

$$S = \frac{1}{2}\left(\frac{ds}{dt}\right)^2 = \sum_{r=1}^{n} a_{rr}\left(\frac{dp_r}{dt}\right)^2,$$

$$S_1 = \frac{1}{2}\left(\frac{ds_1}{dt_1}\right)^2 = \sum_{r=1}^{n} b_{rr}\left(\frac{dq_r}{dt}\right)^2 = \sum_{r=1}^{n} b_{rr}\left(\frac{dp_r}{dt}\right)^2$$

then one will have:

$$a_{rr} = (\Pi_1 - \Pi_s)\,(\Pi_2 - \Pi_s)\,\ldots\,(\Pi_n - \Pi_s)\,,$$

$$b_{ss} = \frac{(\Pi_1 - \Pi_r)(\Pi_2 - \Pi_r)\cdots(\Pi_n - \Pi_s)}{C\,(\Pi_1 + h)(\Pi_2 + h)\cdots(\Pi_{r-1} + h)(\Pi_r + h)^2\,(\Pi_{r+1} + h)\cdots(\Pi_n + h)}\,,$$

respectively, as well as:

$$\lambda = C\,(\Pi_1 + h)\,(\Pi_2 + h)\,\ldots\,(\Pi_n + h)\,.$$

In that case, and only in that case, the proposed transformation will make the forces P_r in the system (S), which are functions of only the coordinates, correspond to the forces Q_r in the system (S_1), which are likewise functions of only the coordinates.

The dependency of the Q_r *on the* P_r *is defined by the formula:*

$$Q_r = \lambda^2 \frac{b_{rr}}{a_{rr}} P_r \ ,$$

or by:

$$(9) \qquad Q_r = C\,(\Pi_1 + h)\,(\Pi_2 + h)\,\ldots\,(\Pi_{r-1} + h)\,(\Pi_{r+1} + h)\,\ldots\,(\Pi_n + h)\,P_r \qquad (r = 1, 2, \ldots, n)\ .$$

4. – We shall now study a particularly important case of those transformations.

Suppose that the P_r are derived from a potential function U, such that one will have:

$$P_r = \frac{\partial U}{\partial p_r}\ .$$

It does not necessarily follow from that hypothesis that the Q_r must also admit a potential: In general, that is not true.

However, one can ask:

Does there exist a form for the function U *for which if one has* $P_r = \dfrac{\partial U}{\partial p_r}$ *then one will also have* $Q_r = \dfrac{\partial U_1}{\partial p_r}$?

For that to be true, it is necessary that the expression:

$$Q_1\,dp_1 + Q_2\,dp_2 + \ldots + Q_n\,dp_n$$

must be an exact differential and that one must therefore have:

$$\frac{\partial Q_r}{\partial p_s} = \frac{\partial Q_s}{\partial p_r} \qquad \left(\begin{matrix} r = 1, 2, \ldots, n \\ s = 1, 2, \ldots, n \end{matrix} \quad r \neq s\right).$$

Now:

$$Q_r = C\,(\Pi_1 + h)\,(\Pi_2 + h)\,\ldots\,(\Pi_{r-1} + h)\,(\Pi_{r+1} + h)\,\ldots\,(\Pi_n + h)\,\frac{\partial U}{\partial p_r}\ ,$$

$$Q_s = C\,(\Pi_1 + h)\,(\Pi_2 + h)\,\ldots\,(\Pi_{r-1} + h)\,(\Pi_{r+1} + h)\,\ldots\,(\Pi_n + h)\,\frac{\partial U}{\partial p_s}\,.$$

Therefore:

$$\frac{\partial Q_r}{\partial p_s} = \frac{\partial Q_s}{\partial p_r}$$

will give:

$$\frac{\partial}{\partial p_s}\left[(\Pi_s + h)\frac{\partial U}{\partial p_r}\right] = \frac{\partial}{\partial p_r}\left[(\Pi_r + h)\frac{\partial U}{\partial p_s}\right]$$

or

(10)
$$\frac{\partial^2\,(\Pi_r - \Pi_s)U}{\partial p_r\,\partial p_s} = 0 \qquad \begin{pmatrix} r = 1, 2, \ldots, n & \\ & r \neq s \\ s = 1, 2, \ldots, n & \end{pmatrix}.$$

The determination of the function U is then reduced to the integration of the system (10). In order to perform such an integration, consider the following n equations:

$$\frac{\partial^2\,(\Pi_r - \Pi_1)U}{\partial p_r\,\partial p_1} = 0\,,$$
$$\frac{\partial^2\,(\Pi_r - \Pi_2)U}{\partial p_r\,\partial p_2} = 0\,,$$
$$\frac{\partial^2\,(\Pi_r - \Pi_3)U}{\partial p_r\,\partial p_s} = 0\,,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\frac{\partial^2\,(\Pi_r - \Pi_s)U}{\partial p_r\,\partial p_s} = 0\,,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\frac{\partial^2\,(\Pi_r - \Pi_n)U}{\partial p_r\,\partial p_n} = 0\,.$$

When those equations are integrated, that will give:

$$(\Pi_r - \Pi_1)\,U = \varphi_r^{(1)}(p_1, p_2, \ldots, p_{r-1}, p_{r+1}, \ldots, p_n) + \varphi_1^{(r)}(p_2, p_3, \ldots, p_n)\,,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$(\Pi_r - \Pi_s)\,U = \varphi_r^{(s)}(p_1, p_2, \ldots, p_{r-1}, p_{r+1}, \ldots, p_n) + \varphi_s^{(r)}(p_1, p_2, \ldots, p_{s-1}, p_{s+1}, \ldots, p_n)\,,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$(\Pi_r - \Pi_n)\,U = \varphi_r^{(n)}(p_1, p_2, \ldots, p_{r-1}, p_{r+1}, \ldots, p_n) + \varphi_n^{(r)}(p_2, p_3, \ldots, p_{n-1})\,,$$

in which $\varphi_r^{(s)}$ is the symbol for an arbitrary function of the variables, where the index r indicates that the variable p_r is missing from it.

Now, an examination of the preceding relations will show that U must be expressible as the sum of two functions.

The first one contains the variables in an arbitrary way, except for p_r, which must enter into only the factors:

$$\frac{1}{\Pi_r - \Pi_1}, \qquad \frac{1}{\Pi_r - \Pi_2}, \qquad \ldots, \qquad \frac{1}{\Pi_r - \Pi_s}, \qquad \ldots, \qquad \frac{1}{\Pi_r - \Pi_n}.$$

The second one contains p_r arbitrarily, and will be a function of $p_1, p_2, \ldots, p_{r-1}, p_{r+1}, \ldots, p_n$, such that it will not contain p_1 when it is multiplied by $(\Pi_r - \Pi_1)$, it will not contain p_2 when it is multiplied by $(\Pi_r - \Pi_2)$, …, and it will not contain p_n when it is multiplied by $(\Pi_r - \Pi_n)$.

That second function will then have the form:

$$\frac{f_r(p_r)}{(\Pi_r - \Pi_1)(\Pi_r - \Pi_s)\cdots(\Pi_r - \Pi_n)},$$

in which $f_r(p_r)$ is an arbitrary function of only p_r.

One can then write:

$$U = F_r(p_1, p_2, \ldots, p_{r-1}, p_{r+1}, \ldots, p_n) + \frac{f_r(p_r)}{(\Pi_r - \Pi_1)(\Pi_r - \Pi_s)\cdots(\Pi_r - \Pi_n)},$$

in which F_r is an arbitrary function of the $p_1, p_2, \ldots, p_{r-1}, p_{r+1}, \ldots, p_n$ and contains the p_r only in the factors:

$$\frac{1}{\Pi_r - \Pi_1}, \qquad \frac{1}{\Pi_r - \Pi_2}, \qquad \ldots, \qquad \frac{1}{\Pi_r - \Pi_n}.$$

However, the argument is valid for all values that r can assume, i.e.:

$$1, 2, \ldots, n.$$

It will then follow that p_1 can enter arbitrarily into only the term:

$$\frac{f_1(p_1)}{(\Pi_1 - \Pi_2)(\Pi_1 - \Pi_3)\cdots(\Pi_1 - \Pi_n)},$$

p_2 can enter arbitrarily into only the term:

$$\frac{f_2(p_2)}{(\Pi_2-\Pi_1)(\Pi_2-\Pi_3)\cdots(\Pi_2-\Pi_n)},$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$

and that U must necessarily have the form:

$$U=\sum_{r=1}^{n}\frac{f_r(p_r)}{(\Pi_r-\Pi_1)(\Pi_r-\Pi_s)\cdots(\Pi_r-\Pi_n)}$$

then, in which f_1 (p_1), f_2 (p_2), …, f_n (p_n) are completely-arbitrary functions of only the indicated arguments.

We then deduce the theorem:

If the forces P_r admit a potential function U of the form:

$$U=\sum_{r=1}^{n}\frac{f_r(p_r)}{(\Pi_r-\Pi_1)(\Pi_r-\Pi_s)\cdots(\Pi_r-\Pi_n)}$$

then the forces Q_r will also admit a potential function in that case, and only in that case.

The problems that are defined by the function U are characterized by an important property.

In order to recall it, it is necessary to invoke a theorem by **Stäckel** that was generalized by my friend Dr. **Burgatti** and relates to the problems that are endowed with potentials that admit n quadratic integrals.

Let Φ denote the following determinant:

$$\Phi=\begin{vmatrix}\varphi_{1,1} & \varphi_{1,2} & \cdots & \varphi_{1,n}\\ \varphi_{2,1} & \varphi_{2,2} & \cdots & \varphi_{2,n}\\ \vdots & \vdots & \ddots & \vdots\\ \varphi_{n,1} & \varphi_{n,2} & \cdots & \varphi_{n,n}\end{vmatrix},$$

in which $\varphi_{r,s}$ is an arbitrary function of only p_r, in general, and let $\Phi_s^{(r)}$ denote the algebraic complement of the element $\varphi_{r,s}$ in the determinant Φ.

The theorem to which we alluded is the following one:

If $ds^2=\sum_{i,\lambda}a_{i\lambda}\,d\rho_i\,d\rho_\lambda$ *is reducible to the form* $ds^2=\sum_{i=1}^{n}\frac{\Phi}{\Phi_i^{(n)}}dp_i^2$ *, and if there exists a force function U_n that is reducible to the form:*

$$U_n = \frac{1}{\Phi} \sum_{i=1}^{n} \Phi_i^{(n)} f_i(p_i),$$

in which f_i *is an arbitrary function of only* p_i*, in general, then in addition to the vis viva:*

$$\sum_{i=1}^{n} \frac{\Phi}{\Phi_i^{(n)}} \left(\frac{dp_i}{dt} \right)^2 = 2\,(U_n + \alpha_n) \qquad (\alpha_n \text{ is a constant}),$$

there will also exist $n - 1$ *integrals that are quadratic in* p'*, i.e.:*

$$\sum_{i=1}^{n} \frac{\Phi\,\Phi_i^{(\lambda)}}{(\Phi_i^{(n)})^2} \left(\frac{dp_i}{dt} \right)^2 = 2\,(U_\lambda + \alpha_\lambda) \qquad (\lambda = 1, 2, \ldots, n - 1),$$

in which:

$$U_\lambda = \sum_{i=1}^{n} \frac{\Phi_i^{(\lambda)}}{\Phi} f_i(p_i) .$$

Now, a simple inspection of the form of the coefficients a_{rr} that appear in the element under consideration:

$$ds^2 = \sum_{r=1}^{n} a_{rr}\, dp_r^2$$

will show that this linear element coincides with the one considered in the preceding theorem, except that the determinant Φ will take the following form:

$$\Phi = \begin{vmatrix} 1 & 1 & \cdots & 1 \\ \Pi_1 & \Pi_2 & \cdots & \Pi_n \\ \Pi_1^2 & \Pi_2^2 & \cdots & \Pi_n^2 \\ \vdots & \vdots & \ddots & \vdots \\ \Pi_1^{n-1} & \Pi_2^{n-1} & \cdots & \Pi_n^{n-1} \end{vmatrix},$$

in which Π_r^s is the same Π_r that appears in the coefficients a_{rr}, raised to the power s.

One then concludes that the problems that are defined by the force function:

$$U = \sum_{r=1}^{n} \frac{f_r(p_r)}{(\Pi_r - \Pi_1)(\Pi_r - \Pi_s)\cdots(\Pi_r - \Pi_n)}$$

admit n integrals that are quadratic in the dp / dt.

The *transformed* problems also admit an equal number of integrals that are quadratic in the $\frac{dq_r}{dt_1} = \frac{dp_r}{dt_1}$.

As one can see, the results that were presented above contain all of the ones that were obtained by **Appell**, **Dautheville**, Dr. **Picciati** in regard to the problems with two degrees of freedom.

I have attempted to derive some consequences and applications of those results.

Rome, 26 February 1895.

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