"Sull'equazione differenziale delle superficie applicabili su di una superficie data," Giorn. di mat. de Battaglini **2** (1864), 282-288.

On the differential equation of the surfaces that can be mapped to a given surface

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1. – If one considers a surface to be flexible and inextensible then one will see how a surface can assume different forms and therefore take on the form of another surface. That is, one will see how it can be wrapped around another surface by deformation without breaking or duplicating it. Having said that, the surfaces that can be wrapped around another in the aforementioned way are called *mappable,* and in this brief work, I propose to find the differential equation that will give all of the surfaces that can be mapped to a given one when it is integrated.

That equation was found by BOUR (cf., Journal de l'École Polytechnique, t. XXII) in the case where the coordinates that are adopted on the surface are a system of symmetric imaginary coordinates such that the square of the line element on the surface will have the form:

$$ds^2 = 4\lambda \, du \, dv .$$

Considering the fact that reducing the line element of a surface to one of that form in any case often comes with much difficulty, I therefore propose to find that equation in the case in which the line element has the most general form and then deduce BOUR's equation as a particular case that does not correspond to the other form of ds^2 that one uses more often in science and that coincides better with the surfaces that one considers in the special cases.

2. – In order to solve the problem that I posed, first allow me to recall the conditions that two surfaces must satisfy in order for them to be mappable.

Take a curvilinear coordinate system u, v on a surface S, so the coordinates X, Y, Z of any point on it with respect to three orthogonal axes can be expressed as three functions of u, v, and the square of its line element can be expressed by:

$$ds^2 = E \, du^2 + 2F \, du \, dv + G \, dv^2, \tag{1}$$

in which E, F, G are three functions of u and v such that one will have:

$$(2)\qquad \begin{cases} \left(\dfrac{\partial X}{\partial u}\right)^2+\left(\dfrac{\partial Y}{\partial u}\right)^2+\left(\dfrac{\partial Z}{\partial u}\right)^2=E, \\ \dfrac{\partial X}{\partial u}\dfrac{\partial X}{\partial v}+\dfrac{\partial Y}{\partial u}\dfrac{\partial Y}{\partial v}+\dfrac{\partial Z}{\partial u}\dfrac{\partial Z}{\partial v}=F \\ \left(\dfrac{\partial X}{\partial v}\right)^2+\left(\dfrac{\partial Y}{\partial v}\right)^2+\left(\dfrac{\partial Z}{\partial v}\right)^2=G. \end{cases}$$

In order for a second surface S' to be mappable onto S, it will obviously be necessary and sufficient that the points of S' can be made to correspond to those of S in such a way that the line elements of the two surfaces will prove to be equal to each other in all directions around the corresponding points.

Having said that, take a coordinate system u', v' on the surface S'. It is clear that one can establish the correspondence between the points of S and those of S' in an infinitude of ways, so it will be enough to establish two relations between u, v and u', v':

$$\varphi(u, v, u', v') = 0, \qquad \varphi_1(u, v, u', v') = 0,$$

under which the line element of S' will be expressed by:

$$ds'^2 = E'du^2 + 2F'du\,dv + G'dv^2.$$

However, because the surfaces S and S' are supposed to be mappable, one needs to be able to establish two relations between u, v and u', v' such that it will result that:

$$ds = ds'$$

in any direction around the corresponding points independently; i.e., for increments du and dv. One will then need to have:

$$E' = E, \qquad F' = F, \qquad G' = G.$$

Moreover, if one regards the expressions for E', F', G' in terms of the coordinates X', Y', Z' of a point on S', which are analogous to (2), then one will conclude that if the surfaces S and S' are mappable then when the coordinates X', Y', Z' of the points of S' are expressed in terms of u and v, they will be such that when they are used in place of X, Y, Z in equations (2), those equations will be verified identically, and conversely, when that happens, the surfaces that belong to the coordinates (X, Y, Z), (X', Y', Z') that verify (2) will be mappable.

3. – Now, that is the problem of the search for surfaces that can be mapped to each other that GAUSS posed in the form of equations, and one can say that from the analytical viewpoint, in order to solve it completely, one would have to determine three functions X, Y, Z of the independent variables u and v in a general manner that are coupled with the quantities E, F, G in such a way that the relations (2) will become identities.

The first thing to do then will be to eliminate two of the variables from equations (2) – for example, X, Y – and to then find the differential equation in Z that must be satisfied in order to then go on to the search for the corresponding equations in finite terms by means of integration. The problem in question will be solved completely when one finds those equations, since it would be given in the form of classifying all imaginable surfaces that can be mapped to a given one.

Unfortunately, although the differential equation can be found with the greatest ease, even in the case where the line element has the most general form (1), nonetheless, it seems almost impossible to find the integral equation by any methods that science currently possesses, and one then agrees to just stop after finding that differential equation. That notwithstanding, I believe that is useful to present that differential equation, and I shall then move on to the search for it immediately.

4. – As was just said, one first eliminates the two variables X, Y from equations (2) in order to find the differential equation that Z must satisfy. In order to do that, observe that when one lets p and q denote the partial derivatives of Z with respect to u and v, resp., one can set:

$$(3) \qquad \left\{ \begin{aligned} \frac{\partial X}{\partial u} &= \sqrt{E-p^2}\cos\theta, & \frac{\partial X}{\partial v} &= \sqrt{G-q^2}\cos\varphi, \\ \frac{\partial Y}{\partial u} &= \sqrt{E-p^2}\sin\theta, & \frac{\partial Y}{\partial v} &= \sqrt{E-q^2}\sin\varphi. \end{aligned} \right.$$

One will, in fact, get the first and third of equations (2) upon squaring and summing these, while the other one will become:

$$\cos(\varphi - \theta) = \frac{F - pq}{\sqrt{E-p^2}\sqrt{G-q^2}}$$

with those conventions. Then set:

$$(4) \qquad \sqrt{E-p^2} = \mu, \qquad \sqrt{G-q^2} = \nu, \qquad \frac{F-pq}{\sqrt{E-p^2}\sqrt{G-q^2}} = \lambda,$$

to abbreviate, so the system of equations (3) can be replaced with these equations:

(5)
$$\begin{cases} \dfrac{\partial X}{\partial u} = \mu \cos\theta, \quad \dfrac{\partial X}{\partial v} = \nu \cos\varphi, \\ \dfrac{\partial Y}{\partial u} = \mu \sin\theta, \quad \dfrac{\partial Y}{\partial v} = \nu \sin\varphi, \\ \cos(\varphi - \theta) = \lambda, \end{cases}$$

and the problem will be reduced to eliminating X, Y, θ, φ from those equations.

The elimination of X and Y is accomplished directly by differentiating the first four. We indicate the derivatives with respect to u and v with the subscripts $(_1)$ and $(_2)$, resp., for brevity, and find the equations:

$$\mu_2 \cos\theta - \mu \sin\theta \cdot \theta_2 = \nu_1 \cos\varphi - \nu \sin\varphi \cdot \varphi_1 ,$$

$$\mu_2 \sin\theta + \mu \cos\theta \cdot \theta_2 = \nu_1 \sin\varphi - \nu \sin\varphi \cdot \varphi_1 ,$$

which do not contain X or Y.

If one sets:

(6)
$$\begin{cases} L = \dfrac{\nu_1 \cos(\varphi - \theta) - \mu_2}{\nu} = \dfrac{\nu_1 \lambda - \mu_2}{\nu}, \\ M = \dfrac{\nu_1 - \mu \cos(\varphi - \theta)}{\mu} = \dfrac{\nu_1 - \mu_2 \lambda}{\mu} \end{cases}$$

then one will get:

$$\sin(\varphi - \theta)\, \varphi_1 = L, \qquad \sin(\varphi - \theta)\, \theta_2 = M .$$

If one further sets:

(7)
$$\sin(\varphi - \theta) = h,$$

in which, from the last of (5), h must satisfy the relation:

(8)
$$h^2 = 1 - \lambda^2,$$

then one can further infer that:

(9)
$$\varphi_1 = \frac{L}{h}, \qquad \theta_2 = \frac{M}{h}.$$

However, from the last of (5), one has:

(10)
$$\varphi_1 - \theta_1 = -\frac{\lambda_1}{h}, \qquad \varphi_2 - \theta_2 = -\frac{\lambda_2}{h},$$

so one will have:

(11)
$$\varphi_2 = \frac{M - \lambda_2}{h}, \quad \theta_1 = \frac{L - \lambda_1}{h},$$

and along with (9), that will lead directly to the desired equation.

Indeed, if one differentiates the first of equations (9) with respect to v and the first of (11) with respect to u then one will get the equation:

$$\frac{L_2}{h}-\frac{L\,h_2}{h^2}=\frac{M_1-\lambda_{12}}{h}-\frac{(M-\lambda_2)h_1}{h^2},$$

and with:

$$h\,h_1=-\lambda\,\lambda_1\,,\qquad h\,h_2=-\lambda\,\lambda_2\,,$$

it will be transformed into this one:

$$(L_2-M_1+\lambda_{12})\,(1-\lambda^2)+\lambda\,(L\,\lambda_2-M\,\lambda_1+\lambda_1\,\lambda_2)=0, \tag{12}$$

and since this no longer contains E, F, G, or the partial derivatives of Z and the latter quantities, it will be precisely the equation that we seek.

It should be observed that one will find another equation when one operates on the second of equations (9) and (11) in the same way that one operated on the first ones. However, it cannot be different from (12), since when one assumes that $\varphi_{12}=\varphi_{21}$, one must also have $\theta_{12}=\theta_{21}$.

In order to exhaust the question that I posed, it then remains to substitute equation (12) for L, M, λ and replace their derivatives with the values in E, F, G, Z. The calculations will necessarily be long, but easy. Upon writing equation (12) in the form:

$$(L+\lambda_1)_2-M_1-\lambda^2\,[(L+\lambda_1)_2-M_1]+\lambda\,[L\,\lambda_2-(M+\lambda_2)\,\lambda_1]=0$$

and appealing to (4) and (6), one will finally arrive at the equation:

$$\begin{aligned}
(A)\qquad & 4\,(EG-F^2)(r\,t-s^2)+2\,[(GG_1+FG_2-2GF_2)\,p-(EG_2+FG_1-2FF_2)\,q]\,r \\
&+4\,(E_2\,G\,p+EG_1\,q)\,s+2[(EE_2+E_1F-2EF_1)\,q-(E_1\,G+E_2\,F-2FF_1)\,p]\,t \\
&+(2E_{22}\,G-2G\,G_{11}+4G\,F_{12}+E_2\,G_2+G_1^2-2G_2\,F_1)\,p^2 \\
&+(4FG_{11}+4E_{22}\,F-8FF_{12}+E_1\,G_2-E_2\,G_1+4F_1\,F_2-2E_2\,F_2-2F_1\,G_1)\,p\,q \\
&+(2EG_{11}-2E\,E_{22}+4E\,F_{12}+E_1\,G_1+E_2^2-2E_1\,F_2)\,q^2 \\
&+2\,(E_{22}+G_{11}-2F_{12})(EG-F^2)-E_2^2\,G-EG_1^2-EE_2\,G_2-E_1\,GG_1+2E_1\,F_2\,G+2EF_1\,G_2 \\
&+2E_2\,FF_2+2FF_1\,G_1-E_1\,FG_2+E_2\,FG_1-4FF_1\,F_2=0,
\end{aligned}$$

in which r, s, t are the three second-order partial derivatives:

$$\frac{\partial^2 Z}{\partial u^2}, \qquad \frac{\partial^2 Z}{\partial u \partial v}, \qquad \frac{\partial^2 Z}{\partial v^2}.$$

Observe that this equation is linear in $rt - s^2$ and r, s, t.

5. – That equation will become much simpler in some special coordinate systems.

For $E = G = \lambda$, $F = 0$, it will become:

$$(B)\qquad \begin{aligned} &4\lambda^2 (rt - s^2) + 2\,(\lambda\lambda_1\, p - \lambda\lambda_2\, q)\, r + 4\,(\lambda\lambda_2\, p + \lambda\lambda_1\, q) + 2\,(\lambda\lambda_2\, q - \lambda\lambda_1\, p)\, t \\ &+ (\lambda_1^2 + \lambda_2^2 - 2\lambda\lambda_{11} - 2\lambda\lambda_{22})(p^2 + q^2) - 2\lambda^2(\lambda_{11} + \lambda_{22}) - \lambda(\lambda_1^2 + \lambda_2^2) = 0. \end{aligned}$$

For $E = G = 0$, $F = 2\lambda$, when (A) is divided by $16\lambda^2$, it will become:

$$rt - s^2 - \frac{\lambda_2}{\lambda} qr - \frac{\lambda_1}{\lambda} pt + 2\frac{\lambda_{12}}{\lambda} pq - \frac{\lambda_1\lambda_2}{\lambda^2} pq + 2\frac{\lambda_1\lambda_2}{\lambda^2} - 2\,\lambda_{12} = 0,$$

or, upon adding and subtracting $\frac{\lambda_1\lambda_2}{\lambda^2} pq$:

$$2\,(pq - \lambda)\left(\frac{\lambda_{12}}{\lambda} - \frac{\lambda_1\lambda_2}{\lambda^2}\right) - s^2 + \left(r - p\frac{\lambda_1}{\lambda}\right)\left(t - q\frac{\lambda_2}{\lambda}\right) = 0,$$

and since:

$$\frac{\lambda_{12}}{\lambda} - \frac{\lambda_1\lambda_2}{\lambda^2} = \frac{\partial^2 \log\lambda}{\partial u\,\partial v}, \qquad \frac{\lambda_1}{\lambda} = \frac{\partial \log\lambda}{\partial u}, \qquad \frac{\lambda_2}{\lambda} = \frac{\partial \log\lambda}{\partial v},$$

it can be put into the form:

$$(C)\qquad 2\,(pq - \lambda)\frac{\partial^2 \log\lambda}{\partial u\,\partial v} - s^2 + \left(r - p\frac{\partial \log\lambda}{\partial u}\right)\left(t - q\frac{\partial \log\lambda}{\partial v}\right) = 0.$$

One then recovers the equation to which BOUR arrived upon starting directly from the form of the line element $ds^2 = 4\lambda\, du\, dv$.

When one supposes that $E = 1$, $F = 0$, one will obtain the following equation from (A):

$$(D)\quad 4G\,(r\,t - s^2) + 2\,(GG_1\, p - G_2\, q)\, r + 4G_1\, qs + (G_1^2 - 2GG_1)\,p^2 - 2G_{11}\, q^2 + 2GG_{11} - G_1^2 = 0,$$

and one should observe that, like equation (C), it has the advantage that it does not contain s, whereas the other one did not contain t.

Suppose that the G in (D) is a function of only u, so it will reduce to:

$$(E) \qquad 4G\,(r\,t - s^2) + 2GG_1\,pr + 4G_1\,qs + (G_1^2 - 2GG_1)\,p^2 - 2G_{11}\,q^2 + 2GG_{11} - G_1^2 = 0,$$

and that equation, which characterizes the surfaces that are mappable to a surface of revolution, from WEINGARTEN (cf., Crelle's Journal, t. LIX), also characterizes the surfaces that are the loci of the centers of curvature of the surfaces for which one of the radii of principal curvature is a function of the other one, and in the case where G is a function of degree two with respect to u, they will further represent certain particular ruled surfaces.

If one supposes that E, F, G are constants in (A) then it will reduce to:

$$(F) \qquad r\,t - s^2 = 0,$$

and since it is only for planes that E, F, G will become constants, one will conclude, as one already knows, that this equation characterizes developable surfaces.

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