"Über Variationsprinzipien in der Elastizitätslehre nebst Anwendungen auf die technische Statik," Zeit. Math. Phys. **63** (1915), 174-192.

On variational principles in the theory of elasticity, with applications to engineering statics

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I.

The principle of virtual displacements, when applied to a deformable body, will yield the known relationship:

$$\int (\sigma_x\,\delta\varepsilon_x + \sigma_y\,\delta\varepsilon_y + \sigma_z\,\delta\varepsilon_z + \tau_x\,\delta\gamma_x + \tau_y\,\delta\gamma_y + \tau_z\,\delta\gamma_z)\,dV - \sum P\cdot\delta w = 0\,, \tag{1}$$

with the usual notations from engineering statics in which σ and τ are the stresses, ε and γ are the associated deformations (distortions), w are the displacements along the line of action of the external forces P, and dV is the differential of spatial volume.

The theorem assumes only equilibrium of the forces and compatibility of the deformations, so it will be valid for *arbitrary deformations.* For *completely elastic* bodies, the first term in the complete variation of a function A, namely, the *deformation work* (energy of distortion, elastic potential). Therefore, from (1):

$$\delta A - \sum P\,\delta w = 0\,,$$

or when one separates the variation symbol and lets the subscript w suggest that the variation acts upon only the deformations:

$$\delta_w[A - \sum P\,w] = 0\,. \tag{2}$$

The bracket expression will then be an extreme value, and indeed a minimum, as is easy to see. The variational principle that is expressed by equation (2), which we would like to refer to as the *first* one, in order to distinguish it from a second one that will be derived later, was found before with a different expression by **Kirchhoff** ([1]).

([1]) **Kirchhoff**, *Vorles. über math. Physik*, *Mechanik*, 1876, Lecture 11, § 5.

If one imagines that A is represented as a function of the displacements w then due to the fact that:

$$\delta_w A = \sum \frac{\partial A}{\partial w_m} \cdot \delta w_m$$

and the theorem that is named after **Green**, one will have:

$$\frac{\partial A}{\partial w_m} = P_m\,, \tag{3}$$

and furthermore, since P are regarded as functions of the w, one will get:

$$\frac{\partial P_m}{\partial w_n} = \frac{\partial P_n}{\partial w_m} \tag{4}$$

by another differentiation with respect to w_n.

That relationship is a counterpart to the known laws of engineering statics that are due to **Maxwell**.

Moreover, **Green**'s theorem will give:

$$A = \sum \int P\,dw \tag{5}$$

upon integrating.

That is the general form of **Clapeyron**'s law of engineering statics. Under the assumption of **Hooke**'s law of deformation, (5) will go to:

$$A = \tfrac{1}{2} \sum P\,w\,. \tag{6}$$

Of all the cited laws, the last one is almost the only one that is applied in engineering statics to any appreciable degree. However, one should take care when using it, since it is by no means correct in all cases. A simpler case in which it does not apply occurs in the buckling of straight rods. If one denotes the **Euler** buckling load by P_{E}, the sag that emerges from the compression of the length of the rod by Δl, and finally, the further sag that emerges from the bending of the rod by w then one will obtain the usual imprecise theory in which an arbitrary deflection is possible when one achieves the **Euler** buckling load, and under the assumption that the domain of validity of **Hooke**'s law has not been exceeded, the work done will be:

$$A = \tfrac{1}{2} P_{\mathrm{E}} \cdot \Delta l + P_{\mathrm{E}} \cdot w\,,$$

which is by no means **Clapeyron**'s law. Even in the more precise theory, which makes a well-defined loading $P_1 > P_{\mathrm{E}}$ belong to each deflection of the rod, there is no agreement with equation

(6) since P_1 is much smaller than P_E, so the additional term in the more precise expression for A will hardly matter.

The basis for the deviation from the general law lies in the fact that two different types of deformations are being combined. In such cases, it would be simplest to start from the first variational principle, which is always appropriate. That is all the more advisable in engineering statics since many problems that are not as simple can be treated quite clearly and concisely with its help.

As an example of the application of the principle, let us cite the determination of the buckling load of a rod when one considered the shearing forces ([1]). With the usual notations (M is the moment, Q is the lateral force), one will have the starting equations ([2]):

$$\delta_w \left[\int_0^1 \left(\frac{M^2}{2EJ} + \frac{\beta Q^2}{2GF} \right) dx - P w \right] = 0 .$$

Since we restrict ourselves to small deflections, the rod axis and the chord can be switched. The work done by the longitudinal force is dropped right from the outset since its influence will be cancelled. If l_0 denotes the original length of the rod then we will have:

$$l = l_0 \left(1 - \frac{P}{EF} \right) .$$

Now, if y represents the ordinate of the bending line then one will have:

$$M = -P y , \qquad Q = \frac{dM}{dx} = -P y' .$$

The sag that is due to deflection alone is:

$$w = \int_0^1 (\sqrt{1 + y'^2} - 1) \, dx \approx \int_0^1 \tfrac{1}{2} y'^2 \, dx .$$

Thus, from the starting equation, one will have:

$$\delta_w \int_0^1 \left[\frac{P y^2}{2EJ} + \frac{\beta P y'^2}{2GF} + \frac{y'^2}{2} \right] dx = 0 .$$

([1]) The problem was solved in a different way (by combining the bending lines) before by **F. Engesser** [Zentralblatt der Bauverwaltung (1891), pp. 483].

([2]) β is the usual coefficient of the work done by shearing forces. It is $\beta = 6/5$ for a rectangular cross-section and $b = 32/27$ for a circular cross-section.

If one varies y then one will get:

$$\frac{P}{EJ}\,y - \frac{\beta P}{GF}\,y'' + y'' = 0\,,$$

from which it will follow, in the known way, that:

$$\frac{P}{F} = \frac{\pi^2 E}{\left(\frac{l}{i}\right)^2 + 25\beta}\,,$$

with $G = 0.4\,E$ and $J = F\,i^2$.

II.

Along with the first variational principle that was just explained, there is yet a *second* one that seems to have remained unknown up to now, even though some of its individual consequences have been much-used in engineering statics. It is the complete counterpart and extension of the first one. It admits some remarkable applications that can be put to use in not only the theory of elasticity, but also in engineering statics.

Before we derive the theorem, recall that the deformations of elastic bodies that occur are so slight that they do not affect the original force distribution noticeably and can then be substituted in the principle of virtual displacements. One then arrives at the well-known theorem of engineering statics ([1]):

$$(7)\qquad \int(\sigma_x\,\bar{\varepsilon}_x + \sigma_y\,\bar{\varepsilon}_y + \sigma_z\,\bar{\varepsilon}_z + \tau_x\,\bar{\gamma}_x + \tau_y\,\bar{\gamma}_y + \tau_z\,\bar{\gamma}_z)\,dV - \sum P\cdot\bar{w} = 0\,,$$

The forces and deformations in that can be assumed to have no mutual dependencies, except that the internal and external forces must be in equilibrium, and the internal and external displacements must be compatible.

In the special case where forces and displacements are paired together, one calls the first term in equation (7) the *virtual* work A_v done by internal forces and arrives at the relationship that the internal virtual work is equal to the external "virtual" work. It is remarkable that one can also substitute the deformations that result from temperature effects in equation (7) with no further discussion.

However, this theorem will also remain correct when one introduces mutually-compatible variations of the forces into it; it will then read:

$$(8)\qquad \int(\varepsilon_x\,\delta\sigma_x + \varepsilon_y\,\delta\sigma_y + \varepsilon_z\,\delta\sigma_z + \gamma_x\,\delta\tau_x + \gamma_y\,\delta\tau_y + \gamma_z\,\delta\tau_z)\,dV - \sum \delta P\cdot w = 0\,.$$

([1]) The theorem defines the foundation for the calculations with elastic systems in the statics of building construction (cf., e.g., **Müller-Breslau**, *Graph. Statik der Baukonstr. II.1* or *Neuere Methoden der Festigkeitslehre*).

That theorem already admits one very general application. In the special case where the first term in the complete variation is a function B, one will get:

$$\delta B - \sum \delta P \cdot w = 0 ,$$

and when one takes the variation symbol out of that expression and lets the subscript P suggest that the variation refers to only the forces:

$$\delta_P [B - \sum P w] = 0 . \tag{9}$$

That theorem shall be referred to as the *second variational principle.* The function B can be written:

$$B = \iint (\varepsilon_x \, d\sigma_x + \varepsilon_y \, d\sigma_y + \varepsilon_z \, d\sigma_z + \gamma_x \, d\tau_x + \gamma_y \, d\tau_y + \gamma_z \, d\tau_z) \, dV .$$

By partial integration, one finds immediately that:

$$B = A_v - A . \tag{10}$$

Therefore, B is a complete integral for completely-elastic bodies. According to **Engesser**, who developed the concept directly from equation (7) ([1]), B is the called the *work done by extension*. One can also call it the "work done by stress," as a counterpart to the work done by deformation. If **Hooke**'s law of deformation is valid and temperature deformations are excluded then due to equations (6) and (7), one will have:

$$B = A .$$

The known laws of engineering statics follow immediately from the second variational principle. If one expresses B as a function of the external forces P then due to the fact that:

$$\delta_P B = \sum \frac{\partial B}{\partial P_m} \delta P_m ,$$

that will imply the theorem of the derivatives of the work done by extension (**Castigliano** and **Engesser**):

$$\frac{\partial B}{\partial P_m} = w_m , \tag{11}$$

or when the bearing C that is necessary to achieve support experiences displacements w_c :

([1]) **Engesser**, "Über statisch unbestimmte Träger bei beliebigen Formänderungsgestze und über der Satz von der kleinsten Ergänzungarbeit," Zeit. d. Arch. u. Ing.-Vereins zu Hannover (1889), pp. 733.

(12)
$$\frac{\partial B}{\partial P_m} = w_m + \sum w_c \cdot \frac{\partial C}{\partial P_m} .$$

When the statically-indeterminate quantities X_a can be considered to be loads and their displacements $w_a = 0$, one will have:

(13)
$$\frac{\partial B}{\partial X_a} = 0 .$$

That is the law of the minimum of the work done by deformation, or really the work done by extension in the ordinary sense of engineering statics.

It follows from equation (11) by a further differentiation with respect to P_n that:

(14)
$$\frac{\partial w_m}{\partial P_n} = \frac{\partial w_n}{\partial P_m} ,$$

which is then **Maxwell**'s law for an arbitrary law of deformation. From **Hooke**'s law, B is a homogeneous function of degree two in P, from which **Clapeyron**'s law will follow immediately, due to equation (11), moreover. Due to that same equation, equation (14) can be expressed simply as the equality of two constants:

$$w_{mn} = w_{nm} ,$$

in the indicial notation that **Müller-Breslau** introduced.

III.

The principle of virtual displacements makes it possible to not only exhibit the relations between forces and given displacements – so the equilibrium conditions – but also to find, conversely, the displacements under known forces. Since the equilibrium conditions can mostly be much simpler to exhibit than the compatibility conditions for deformations, the second variational principle will have a special meaning in the applications as a means for exhibiting such relationships.

The bending lines for straight or curves rods shall be considered initially. Those lines are often necessary for pre-calculating the deflections. However, they are much more important due to the fact that they are the influence lines of static quantities under statically-indeterminate systems, so they will be used in the calculations themselves.

It is known that there are two processes for determining such a bending line. Either one applies the principle of work in a form that follows from the application of equation (11):

$$w_m = \frac{\partial B}{\partial P_m}$$

or the one that follows from that by derivation:

$$w_{m\alpha} = \frac{\partial^2 B}{\partial P_m \, \partial X_\alpha},$$

or one employs a second-order differential equation that can be derived from relations involving the deformations. Both processes have their advantages and disadvantages. With the first one, one will get only isolated ordinates since the expression *B* must ordinarily be calculated in each special case. With the second one, however, one must first integrate and then obtain the entire course of the influence line. Both processes seem completely unconnected, and yet they must go back to the same source. It will be shown in what follows that both paths will emerge from the second variational principle.

The derivation using the first process has been shown before. What is essential in it is that the external forces *P* are chosen to be independent quantities.

Now, one can also introduce other functions of the external forces as independent quantities. What come under consideration for rods are the longitudinal force, the lateral force, and the moment. Since only the independent variables and their differential quotients occur in the integral to be varied (since the lateral force is the differential quotient of the moment and the longitudinal forces come under consideration only for curved rods and can also be expressed in terms of the moment), only the moment remains as something that would be useful as an independent quantity.

1. The differential equation for the bending line of a straight beam when one also considers the shearing forces will initially be exhibited.

From the second variational principle, when *y* denotes the ordinate of the bending line and *p* represents a continuous loading that is all that shall be varied, one will have:

$$\delta_p \int_0^l \left(\frac{M^2}{2EJ} + \frac{\beta Q^2}{2GF} - p\,y \right) dx = 0 .$$

Now, one has:

$$Q = \frac{dM}{dx} = M', \qquad p = -\frac{dQ}{dx} = -M''_p,$$

so

$$\delta_p \int_0^l \left(\frac{M^2}{2EJ} + \frac{\beta M'^2}{2GF} + M''_p\,y \right) dx = 0 .$$

Performing the variation of *M* will yield:

$$\frac{M}{EJ} - \frac{\beta M''}{GF} + y'' = 0$$

or

$$y'' = -\frac{M}{EJ} + \frac{\beta M''}{GF} . \tag{15}$$

One sees immediately that the shearing force alone generates a bending line that is similar to (affinely associated with) the moment line.

In order for the true sense of equation (15) to emerge clearly, some explanatory remarks would be necessary. Namely, the equation is true for any given loading P, so for isolated forces, as well. In order to be able to apply the second variational principle, one imagines applying a continuous loading p, which is the only thing that should be varied, as stated before, and it can be set equal to zero after exhibiting the differential equation. Since the loads P are not varied, the external work that they do can even be dropped from the Ansatz equation. Similarly, only the part M_p of M that depends upon p will be varied. That is explained by the fact that one must set $p = -M''_p$, with the index. The Ansatz equation must properly be written:

$$\int_0^l \left(\frac{M}{EJ}\delta M_p + \frac{\beta M'}{GF}\delta M'_p + \delta M''_p \cdot y \right) dx = 0 .$$

Those considerations will also be true for the following cases.

The given derivation of the differential equation shows that the second differential quotient of the bending line must appear on purely-static grounds. The basis for the fact that it does not give the precise equation with the radius of curvature lies in the fact that the moment, lateral force, and path of integration are referred to the unbent beam axis.

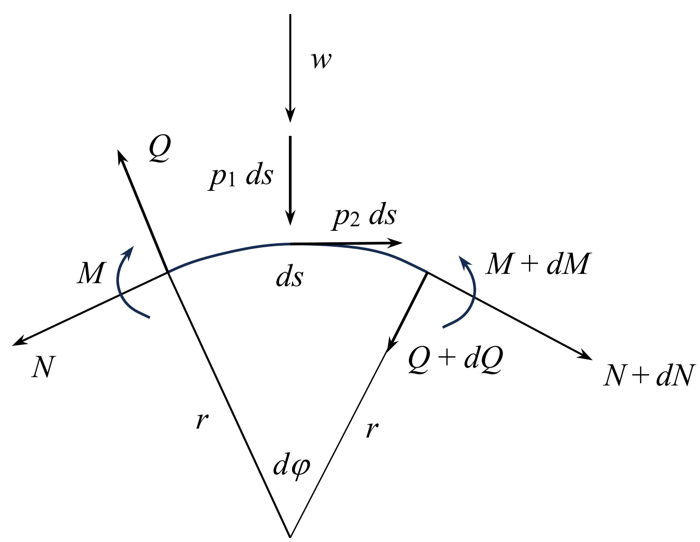

Figure 1.

2. When w is the positive-inward pointing radial displacement, v means the tangential displacement, and p_1 and p_2 are the continuous loadings to be varied, as in Fig. 1, the differential equation for the bending line of a circular ring will follow from:

$$\delta_p \int_0^s \left[\frac{M^2}{2EZ} + \frac{\beta Q^2}{2GF} + \frac{\mathfrak{N}^2}{2EF} - p_1 w - p_2 v \right] ds = 0 .$$

In that, one has:

$$\mathfrak{N} = N - \frac{M}{r} .$$

Z is the value that is introduced for curved rods that deviates only slightly from the moment of inertia ([1]).

The variation of p_1 will yield:

$$\int_0^s \left[\frac{M}{EZ} \cdot \delta M_1 + \frac{\beta Q}{GF} \cdot \delta Q_1 + \frac{\mathfrak{N}}{EF} \cdot \delta \mathfrak{N}_1 - \delta p_1 \cdot w \right] ds = 0 .$$

The subscript 1 is supposed to emphasize the fact that only the terms in the variations that originate in p_1 will need to be considered.

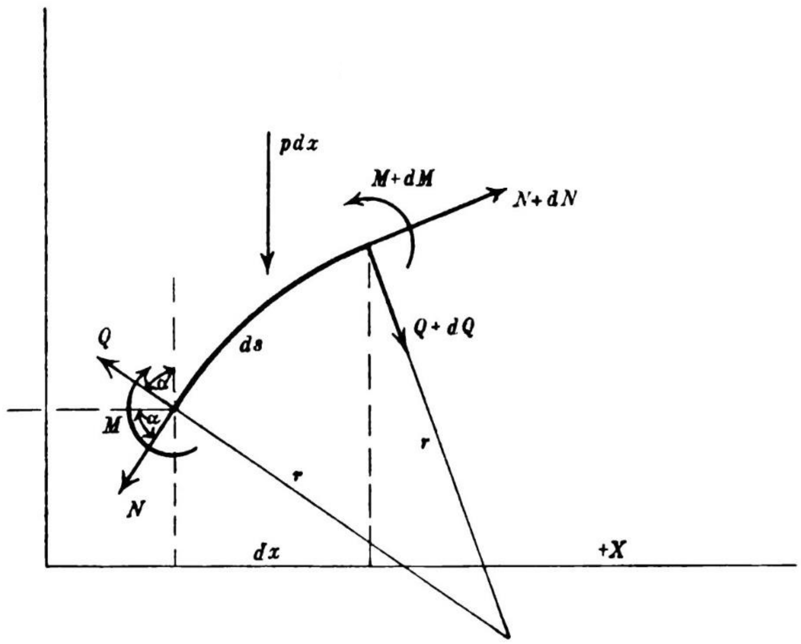

Figure 2.

Now, from Fig. 2, one has:

$$Q = \frac{dM}{ds} = M' ,$$

in general, so:

$$\text{a)} \quad Q_1 = M_1' .$$

Furthermore, from the moment about the center of curvature:

$$dN_1 \cdot r - dM_1 = 0 ,$$

or

([1]) For these formulas, cf., **Müller-Breslau**, *Die neueren Methoden der Festigkeitslehre*, 4th ed., 1913, pp. 236, *et seq.*

$$\frac{d}{ds}\left(N_1 - \frac{M_1}{r}\right) = 0\,,$$

or finally:

$$\mathfrak{N}_1' = 0\,,$$

one will have:

$$\text{b)}\quad \mathfrak{N}_1 = C\,.$$

It will then follow that:

$$N_1 = C + \frac{M_1}{r}\,.$$

Finally, for the forces in the radial direction, one has:

$$p_1\,ds + dQ_1 + N_1\frac{ds}{r} = 0\,,$$

or

$$p_1 = -\,Q_1' - \frac{N_1}{r}\,,$$

so

$$\text{c)}\quad p_1 = -\,M_1'' - \frac{M_1}{r^2} - \frac{C}{r}\,.$$

The basic equation will then be:

$$\int_0^l \left[\frac{M}{EZ}\cdot\delta M_1 + \frac{\beta M'}{GF}\cdot\delta M_1' + \frac{\mathfrak{N}}{EF}\cdot\delta C + w\cdot\delta\left(M_1'' + \frac{M_1}{r^2} + \frac{C}{r}\right)\right] ds = 0\,.$$

Performing the variation will give:

$$\frac{M}{EZ} - \frac{\beta M''}{GF} + w'' + \frac{w}{r^2} = 0\,.$$

The known equation will then follow:

$$w'' + \frac{w}{r^2} = -\,\frac{M}{EZ} - \frac{\beta M''}{GF}\,, \tag{16}$$

which has been extended here.

It is very noteworthy that the elongation of the centerline has no effect.

If one proceeds analogously with p_2 then one will easily find that:

$$\text{a)}\quad Q_2 = M_2'\,.$$

Moreover, from the forces in the radial direction, one has:

$$dQ_2 + N_2 \frac{ds}{r} = 0$$

or

$$N_2 = - M_2'' \cdot r ,$$

so

$$\text{b)} \quad \mathfrak{N}_2 = - M_2'' \cdot r - \frac{M}{r} .$$

From the moment about the center of curvature, one will have:

$$p_2 + N_2' - \frac{M_2'}{r} = 0 ,$$

$$p_2 = - \mathfrak{N}_2' ,$$

$$\text{c)} \quad p_2 = + M''' r + \frac{M'}{r} .$$

Substituting that in the basic equation will give:

$$\int_0^l \left[\frac{M}{E Z} \cdot \delta M_2 + \frac{\beta M'}{G F} \cdot \delta M_2' + \frac{\mathfrak{N}}{E F} \cdot \delta \left(-M_2'' \cdot r - \frac{M_2}{r^2} \right) - v \cdot \delta \left(M_2''' \cdot r + \frac{M_2'}{r^2} \right) \right] ds = 0 ,$$

and from that one will get:

$$\frac{M}{E Z} - \frac{\beta M'}{G F} - \frac{\mathfrak{N}'' r + \mathfrak{N} / r}{E F} + v''' r + \frac{v'}{r} = 0 .$$

With the use of equation (16), that can be easily converted into:

$$- \left(w'' + \frac{w}{r^2} \right) + r \left(- \frac{\mathfrak{N}'' r + \mathfrak{N} / r}{E F} + v''' r + \frac{v'}{r} \right) = 0$$

or

$$\frac{d^2}{ds^2} \left(v' - \frac{w}{r} - \frac{\mathfrak{N}}{E F} \right) + \frac{v' - \frac{w}{r} - \frac{\mathfrak{N}}{E F}}{r^2} = 0 .$$

In addition to the solution:

$$v' - \frac{w}{r} - \frac{\mathfrak{N}}{E F} = C_1 \cos \frac{s}{r} + C_2 \sin \frac{s}{r} = C_1 \cos \varphi + C_2 \sin \varphi ,$$

which corresponds to a parallel displacement, one also has the known solution:

$$v' = \frac{w}{r} + \frac{\mathfrak{N}}{EF} . \tag{17}$$

When all of the forces vanish, equation (16) will also have the solution:

$$w = C_1 \cos \varphi + C_2 \sin \varphi ,$$

so a parallel displacement.

3. For rods of a different form, one would do better to refer the deflections to a rectangular axis-cross. The perpendicular deflection w for an originally curved loaded rod shall be determined.

It easily follows from Fig. 2 that:

$$Q = \frac{dM}{ds} = \frac{dM}{dx} \cdot \frac{dx}{ds} = M' \cos\alpha ,$$

$$N_p = -\, M'_p \sin\alpha + \frac{C}{\cos\alpha} ,$$

$$p = -\, M''_p + C \frac{d\tan\alpha}{d\alpha} .$$

Since $d\alpha$ is negative in Fig. 2, the radius of curvature will be:

$$r = -\frac{d\alpha}{ds} .$$

That will then give:

$$\mathfrak{N} = N - \frac{M}{r} = N + M\frac{d\alpha}{ds} = N + M\frac{d\alpha}{dx}\cos\alpha = N + M\frac{d\sin\alpha}{dx} ,$$

and as a result:

$$\mathfrak{N}_p = -\, M'_p \sin\alpha + M_p \frac{d\sin\alpha}{dx} + \frac{C}{\cos\alpha} .$$

The path of integration is ds, and the independent variable is x so the basic equation will read:

$$\delta_p \int_0^l \left[\frac{M^2}{2EZ\cos\alpha} + \frac{\beta Q^2}{2GF\cos\alpha} + \frac{\mathfrak{N}^2}{2EF\cos\alpha} - p\,w \right] dx = 0 .$$

Substituting the values that were determined will produce:

$$\int_0^l \left[\frac{M}{E\,Z\cos\alpha}\cdot\delta M_p + \frac{\beta\,M'\cos\alpha}{G\,F}\cdot\delta M'_p + \frac{\mathfrak{N}}{E\,F\cos\alpha}\cdot\delta\left(-M'_p\sin\alpha + M_p\frac{d\sin\alpha}{d\alpha} + \frac{C}{\cos\alpha}\right)\right.$$

$$\left. + \ w\cdot\delta\left(M''_p - C\frac{d\tan\alpha}{dx}\right)\right] dx = 0 .$$

That will give:

$$\frac{M}{E\,Z\cos\alpha} - \frac{\beta}{G\,F}\frac{d\,(M'\cos\alpha)}{dx} + \frac{\dfrac{d}{dx}(\mathfrak{N}\tan\alpha) + \mathfrak{N}\cdot\dfrac{d\sin\alpha}{dx\cdot\cos\alpha}}{E\,F\cos\alpha} + w'' = 0 .$$

When one observes the foregoing equations, that is easily converted into ([1]):

$$\text{(18)} \qquad \frac{d^2 w}{dx^2} = -\frac{M}{E\,Z\cos\alpha} - \frac{\beta}{G\,F}\cdot\frac{dQ}{dx} + \frac{1}{E\,F}\left[\frac{d}{dx}(\mathfrak{N}\tan\alpha) - \frac{\mathfrak{N}}{r\cos\alpha}\right].$$

Observe that:

$$Q = \frac{dM}{ds} .$$

The differential equation for horizontal deflections is obtained from this by a reasonable conversion to the other axes.

IV.

The applications of the second variational principle in the previous section assumed Hooke's law. However, it is easy to see that no assumption whatsoever regarding the properties of the deformations will be necessary when one applies the principle the form of equation (8). The results of the variation are then simply relations between the internal deformations, i.e., elongations and shears, and the external ones, namely, the displacements. One will then get the conditions for the compatibility of the internal and external deformations, namely, the so-called "compatibility conditions." If one applies that argument to the first example in the previous section then one will get:

$$\int_0^l [\kappa\cdot\delta M + \gamma\cdot\delta M' + \delta M''\cdot y]\,dx = 0$$

when one denotes the change in curvature that results from the moment by κ and the shear that results from the lateral force by γ, so:

([1]) This formula was already given by **Müller-Breslau** in a different way [*Graph. Statik II.2*, 1908, pp. 513, eqs. 27 and 28. Furthermore: *Neuere Methoden der Festigkeitslehre*, 4th ed., pp. 248, formula (120), which includes a mistake. If one compares formula (60) on pp. 176 then one will see that the factor of *ds* / *dx* should be dropped from the last terms of formulas (120) and (121). Formula (122) is still correct.]

$$+\kappa - \gamma' + y'' = 0 ,$$

i.e.:

(19)
$$\frac{d^2 y}{dx^2} = -\kappa + \frac{d\gamma}{dx} .$$

Since only small deformations can be considered, in principle, one will get only terms of first order in the deformations.

The following example will show how the theorem can be applied to deformable bodies (and even inelastic ones):

1. Exhibit the relations between the elongations and shears and the external displacements *u*, *v*, *w* of a homogeneous body for a rectangular axis-cross.

If *X*, *Y*, *Z* represent body forces (per unit spatial volume) along the three axes then, as is known, the following three equilibrium conditions will exist:

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_z}{\partial y} + \frac{\partial \tau_y}{\partial z} + X = 0 ,$$

$$\frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_x}{\partial z} + \frac{\partial \tau_z}{\partial x} + Y = 0 ,$$

$$\frac{\partial \sigma_z}{\partial z} + \frac{\partial \tau_y}{\partial x} + \frac{\partial \tau_x}{\partial y} + Z = 0 .$$

From the second variational principle, variation of the integral over the entire body will vanish by itself:

$$\int [\varepsilon_x \, \delta\sigma_x + \varepsilon_y \, \delta\sigma_y + \varepsilon_z \, \delta\sigma_z + \gamma_x \, \delta\tau_x + \gamma_y \, \delta\tau_y + \gamma_z \, \delta\tau_z - \delta X \cdot u - \delta Y \cdot v - \delta Z \cdot w] dV = 0 .$$

The work done by the external forces will appear in only the boundary integrals, so they will drop out of the spatial integral.

One introduces the three expressions for *X*, *Y*, *Z* above into the integral:

$$\int \left[\varepsilon_x \, \delta\sigma_x + \cdots + u \cdot \delta\left(\frac{\partial \sigma_x}{\partial x} + \cdots \right) + v \cdot \delta\left(\frac{\partial \sigma_y}{\partial y} + \cdots \right) + w \cdot \delta\left(\frac{\partial \sigma_z}{\partial z} + \cdots \right) \right] dV = 0 .$$

Performing the variations of the six stresses individually will give the equations:

$$\varepsilon_x - \frac{\partial u}{\partial x} = 0 , \qquad \gamma_x - \frac{\partial v}{\partial z} - \frac{\partial w}{\partial y} = 0 ,$$

(20) $$\varepsilon_y - \frac{\partial v}{\partial y} = 0 , \qquad \gamma_y - \frac{\partial w}{\partial x} - \frac{\partial u}{\partial z} = 0 ,$$

$$\varepsilon_z - \frac{\partial w}{\partial z} = 0 , \qquad \gamma_z - \frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} = 0 .$$

Those are the well-known relations for homogeneous infinitesimal deformations. Their derivation then assumes the knowledge of only the initial basis for the calculus of variations, in addition to the equilibrium conditions. It is clear that the body forces can also be assumed to be arbitrarily small.

It is known that one can make the three equilibrium conditions between the stresses inevitable by introducing three stress functions. The substitution of those functions will then give known relations between the deformation quantities, for which a single stress function might be given here since that will suffice for the case of planar problems. Here (when one excludes body forces), one has:

$$\sigma_x = \frac{\partial^2 F}{\partial y^2}, \qquad \sigma_y = \frac{\partial^2 F}{\partial x^2}, \qquad \tau = -\frac{\partial^2 F}{\partial x\,\partial y} .$$

Therefore, since the external loads, in turn, appear in only the boundary integral:

$$\iint \left[\varepsilon_x \cdot \delta \frac{\partial^2 F}{\partial y^2} + \varepsilon_y \cdot \delta \frac{\partial^2 F}{\partial x^2} - \gamma \cdot \delta \frac{\partial^2 F}{\partial x\,\partial y} \right] h\,dx\,dy = 0 ,$$

from which, it will follow that ([1]):

(21) $$\frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial x^2} - \frac{\partial^2 \gamma}{\partial x\,\partial y} = 0 .$$

One will obtain the differential equation for the stress function of the planar problem:

$$\frac{\partial^4 F}{\partial x^4} + 2\frac{\partial^4 F}{\partial x^2\,\partial y^2} + \frac{\partial^4 F}{\partial y^4} = 0$$

from equation (21) when one substitutes the relations between stresses and distortions that follow from Hooke's law. One will arrive at that equation even more simply when one expresses the stresses in terms of the expression for B in terms of F that is defined by Hooke's law and then takes the variation with respect to F.

([1]) Cf., **Love**, *Lehrb. d. Elatizität*, German by **Timpe**, 1907, pp. 59.

2. Derive the compatibility conditions for the planar problem under the assumption that polar coordinates are being used.

With the notations of Fig. 3, when X_r and X_t denote the volume forces per unit spatial volume, as is known ([1]), one will have:

$$\frac{\partial\sigma_r}{\partial r} - \frac{\sigma_r - \sigma_t}{r} + \frac{1}{r}\frac{\partial\tau}{\partial\varphi} + X_r = 0 \,,$$

$$\frac{1}{r}\frac{\partial\sigma_t}{\partial\varphi} + \frac{\partial\tau}{\partial r} + \frac{2\tau}{r} + X_t = 0 \,.$$

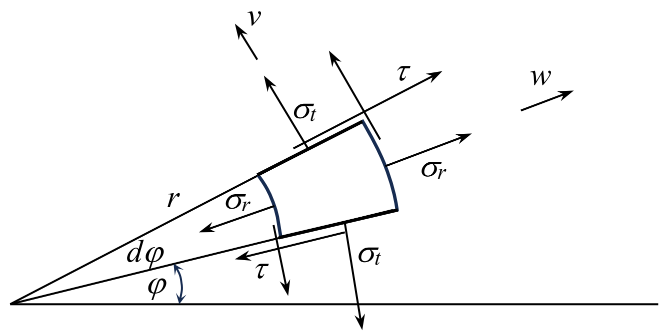

Figure 3.

The second variational principle gives:

$$\iint [(\varepsilon_r \cdot \delta\sigma_r + \varepsilon_t \cdot \delta\sigma_t + \gamma\,\delta\tau - \delta X_r \cdot w - \delta X_t \cdot v)\, r]\, dr\, d\varphi\, h = 0 \,.$$

If one substitutes the values above for X_r and X_t and varies the individual stresses then the equations will follow:

$$\varepsilon_r\, r - \frac{\partial(w r)}{\partial r} + w = 0 \,,$$

$$\varepsilon_t\, r - w - \frac{\partial v}{\partial\varphi} = 0 \,,$$

$$\gamma\, r - \frac{\partial w}{\partial\varphi} - \frac{\partial(v r)}{\partial r} + 2v = 0 \,,$$

or ([2]):

(22)
$$\varepsilon_r = \frac{\partial w}{\partial r} \,,$$

$$\varepsilon_t = \frac{w}{r} + \frac{1}{r}\frac{\partial v}{\partial\varphi} \,,$$

$$\gamma = \frac{1}{r}\frac{\partial w}{\partial\varphi} + \frac{\partial v}{\partial r} - \frac{v}{r} \,.$$

([1]) Cf., **Love**, *Elastizität*, pp. 107.
([2]) Cf., **Love**, *Elastizität*, pp. 66.

V.

The work done by deformation A_0 per unit volume leads to the name of "elastic potential" due to the fact that its partial derivatives with respect to the distortions ε and γ yield the respective stresses, so:

$$\frac{\partial A_0}{\partial \varepsilon_x} = \sigma_x \text{, etc.,} \qquad \frac{\partial A_0}{\partial \gamma_x} = \tau_x \text{, etc.,} \tag{23}$$

That already follows immediately from the explanations for equations (1) and (2). Those relations will also be valid then for an arbitrary law of deformations when the deformations are only completely elastic.

The group of equations:

$$\frac{\partial B_0}{\partial \sigma_x} = \varepsilon_x \text{, etc.,} \qquad \frac{\partial B_0}{\partial \tau_x} = \gamma_x \text{, etc.,} \tag{24}$$

follow from equations (8) and (9) in exactly the same way.

The law of deformation can also be arbitrary here when only complete elasticity is present.

The two variational principles then show a complete parallelism in their consequences, and it is instructive to briefly juxtapose the corresponding theorems once more:

a) If one chooses the stresses and distortions in order to represent the two works A and B then one will arrive at the differential laws (23) and (24) that were just mentioned.

b) The representation of the work done by deformation A in terms of the force displacements w corresponds to the representation of the work done by extension B in terms of the external forces P. It then corresponds to the theorem of **Green** (3) and that of **Castigliano** and **Engesser** (11).

Those are the actual differential relations. It follows from the variational laws that:

c) If one represents the works A and B in terms of the stresses and distortions then one will get the relations between the distortions and displacements [equation (20)] from the second variational principle with the use of the equilibrium conditions. One will get the equilibrium conditions for the stresses in an infinitely-small prism ([1]) from the first variational principle with the use of the relations in equations (20) in an entirely-analogous way.

d) The representation of the work B in terms of functions of the external forces and moments M corresponds to the representation of the work A in terms of functions of the displacements w and the ordinates of the bending line. The buckling problem that was treated in Section I is an example of the latter case.

([1]) Cf., **Love**, *Elastizität*, pp. 198.

One now sees a further parallelism that appears in the same way for each of the two principles: The differential relations in a) correspond to the variational relations in c), and the laws in b) and d) correspond to each other in a similar manner.

The replacement of the stresses in the work done by extension B with stress functions has no actual counterpart in the first variational principle. Such a thing does also not seem to be necessary because the possibility of expressing the six stresses in terms of three stress functions is juxtaposed with the fact that the six distortions can be expressed in terms of the three displacements.

VI.

Whereas the theorems that were cited up to now correspond to each other, but also differ in their content, **Maxwell**'s law [equation (14)]:

$$\frac{\partial w_m}{\partial P_n} = \frac{\partial w_n}{\partial P_m}$$

defines a remarkable exception because it says something that differs from its counterpart [equation (4)] only in form, but not in fact:

$$\frac{\partial P_m}{\partial w_n} = \frac{\partial P_n}{\partial w_m}.$$

That is intrinsically based in the fact that here one is dealing with only differential relations between forces and displacements, so the second series of relations must have the same content as the first. Mathematically, the agreement emerges from the following theorem:

If the series of relations exist between two groups of independent variables P and w:

$$\frac{\partial P_m}{\partial w_n} = \frac{\partial P_n}{\partial w_m}$$

then the series of relations ("its inverse"):

$$\frac{\partial w_m}{\partial P_n} = \frac{\partial w_n}{\partial P_m}$$

must also exist.

The proof of that theorem is simple, and its foundations were given already in the foregoing sections: It follows from the first series that P_m must be the partial differential quotient of a function A with respect to w_m : After a partial integration, $A = \sum\int P\,dw$ will imply that $B = \sum\int w\,dP$ must be a complete integral, so w_m is the partial differential quotient of B with respect to P_m . A repeated differentiation will then give the second series.

Just as Maxwell's theorem also has its extension, **Betti**'s theorem has only a formal counterpart. One finds the generalized form of that theorem to arbitrary laws of deformation in the following way:

One can consider the displacements w to be functions of certain parameters u :

$$w_m = f_m\,(u_1, \ldots, u_i, u_k, \ldots)\,. \tag{25}$$

Each of those parameters u_i characterizes a certain displacement state that can be thought of having been created by a group of forces P_i. Now, it follows from the first variational principle that:

$$\frac{\partial A}{\partial u_i} - \sum \left(P \frac{\partial w}{\partial u_i} \right) = 0\,,$$
$$\frac{\partial A}{\partial u_k} - \sum \left(P \frac{\partial w}{\partial u_k} \right) = 0\,,$$

so after repeated differentiations with respect to u_k and u_i :

$$\frac{\partial}{\partial u_k} \sum \left(P \frac{\partial w}{\partial u_i} \right) = \frac{\partial}{\partial u_i} \sum \left(P \frac{\partial w}{\partial u_k} \right)\,.$$

If one performs the differentiation then after dropping the equal terms on both sides, one will get:

$$\sum \left(\frac{\partial P}{\partial u_k} \cdot \frac{\partial w}{\partial u_i} \right) = \sum \left(\frac{\partial P}{\partial u_i} \cdot \frac{\partial w}{\partial u_k} \right)\,. \tag{26}$$

That is the first form of the generalized **Betti** theorem. If the displacements w themselves were chosen to be the parameters then that would immediately give the counterpart to Maxwell's theorem [equation (4)] since only the forces P_i remain on the left-hand side and only P_k remain on the right, while all of the remaining forces will drop out.

In order to derive the extended counterpart, one considers the forces P to be functions of certain parameters R, so:

$$P_m = \varphi_m\,(R_1, \ldots, R_i, R_k, \ldots)\,. \tag{27}$$

Each of those parameters R_i characterizes a certain state of loading that is associated with a group of displacements w_i. It then follows from the second variational principle that:

$$\frac{\partial B}{\partial R_i} - \sum \left(\frac{\partial P}{\partial R_i} \cdot w \right) = 0 \, ,$$

$$\frac{\partial B}{\partial R_k} - \sum \left(\frac{\partial P}{\partial R_k} \cdot w \right) = 0 \, ,$$

and then:

(26) $$\sum \left(\frac{\partial P}{\partial R_i} \cdot \frac{\partial w}{\partial R_k} \right) = \sum \left(\frac{\partial P}{\partial R_k} \cdot \frac{\partial w}{\partial R_i} \right)$$

will be the second form of the generalized Betti theorem, which corresponds completely to the above. If one chooses the *P* themselves to be parameters then the extended Maxwell theorem [equation (14)] will follow directly since only w_i remains on the left-hand side, while only w_k remains on the right, and all remaining displacements will drop out.

The two forms of Betti's theorem differ only by the meaning that one attributes to the parameters. Mathematically, that difference is irrelevant, and therefore both forms say the same thing, in essence. Naturally, in the applications, one will find it convenient to either make the *w* take the form of *linear* functions of the *u* or make the *P linear* functions of the *R*.

If one uses Hooke's law of deformation as a basis then the agreement between the two forms will become especially clear. That is because due to the law of superposition, the displacements, as well as the forces, will be linear functions of the parameters, which will then give:

$$\frac{\partial P_m}{\partial u_i} = P_{mi} \, , \qquad \frac{\partial P_m}{\partial R_i} = P_{mi} \, ,$$

$$\frac{\partial w_m}{\partial u_i} = w_{mi} \, , \qquad \frac{\partial w_m}{\partial R_i} = w_{mi} \, ,$$

in which the first index gives the point of application of the force or the location of the displacement, while the second one characterizes the group of forces or displacements. It will then follow from (27), as well as (28), that:

(29) $$\sum (P_{mi} \cdot w_{mk}) = \sum (P_{mk} \cdot w_{mi}) \, ,$$

or: The virtual work done by the forces in the loading state *i* as a result of the displacement state *k* is equal to the virtual work done by the forces in the loading state *k* as a result of the displacement state *i*.

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