"Sul lavoro di deformazione dei sistemi elastici," Mem. R. Accad. sci. dell'Istituto Bologna (4) **9** (1888), 345-367.

On the work done by deformation in elastic systems

By Prof. **LUIGI DONATI**

(Read at the session on 29 April 1888)

Translated by D. H. Delphenich

The goal of this note is mainly concerned with some propositions in regard to the internal work that is developed under the deformation of elastic bodies, which is then called the *principle of elasticity* or the *theorem of minimum work* that was given by Menabrea, and the theorem of Castigliano that is concerned with the *derivative of the work*, with the aim of shedding some light on the significance and scope of some general principles from the theory of elasticity. In that regard, it does not seem that one finds the requisite clarity and uniformity of ideas, although the propositions are, in themselves, already common and in frequent use in the applications, especially in the science of constructions, and the argument that one finds is treated, or at least touched upon, more or less directly in many publications. Other than the work of MENABREA ([1]) and CASTIGLIANO ([2]), it would be enough for me to cite the work of Prof. CERRUTI ([3]) and Prof. CANEVAZZI ([4]), in which, as in the present writing, one considers the argument from the viewpoint of the theory of elastic potentials. At any rate, I believe that the following considerations can turn out to be not entirely useless, if only because of the form by which the question is treated in its more general aspect.

Let x, y, z denote the orthogonal Cartesian coordinates of the points of a body or system under consideration (in which one always supposes that the *temperature is kept constant*) in its original state, which is denoted by S_0, in which it is not subject to any external force, so it will be found in its natural condition of equilibrium, and let:

$$x + u, \quad y + v, \quad z + w$$

be the coordinates of the same material point in the state S when the body is found to be deformed under the action of external forces, and the components of the displacement of

---

([1]) See L. F. MENABREA, "Nouveau principe sur la distribution des tensions dans les systèmes élastiques," Comptes rendus **46** (1858), and also by the same author: "Étude de Statique physique," Bocca Bros., Turin and Florence, 1868, and *Sulla determinazione delle tensioni e delle pressioni nei sistemi elastici*, Roma, 1875.

([2]) See A. CASTIGLIANO, *Theorie de l'équilibre des systèmes élastiques*, Turin, 1879, and other papers.

([3]) V. CERRUTI, "Sopra un teorema del Sig. Menabrea," Atti della R. Accad. die Lincei (2) **2** (1875).

([4]) S. CANEVAZZI, *Sulla teoria delle travature*, Bologna, 1886.

the point (x, y, z) are denoted, as usual, by u, v, w, which are supposed to continuous functions of the coordinates and small enough in magnitude that they can be treated as differentials in the calculations. Then let a, b, c, f, g, h denote the six components of the deformations in the neighborhood of the point (x, y, z):

$$(1)\qquad \begin{aligned} a &= \frac{\partial u}{\partial x}, & b &= \frac{\partial v}{\partial y}, & c &= \frac{\partial w}{\partial z}, \\ f &= \frac{\partial w}{\partial y}+\frac{\partial v}{\partial z}, & g &= \frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}, & h &= \frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}. \end{aligned}$$

For simplicity, always suppose that the u, v, w represent relative displacements, except for a general motion of the body that one regards as rigid, or that is the same when referred to a system of axes that are fixed in the body. Under those conditions, the u, v, w will be determined completely by means of the values of the a, b, …, h, which are given at all points of the body, and which one supposes to form a *congruent system* (†), by which, we mean a system of admissible values – viz., ones that can effectively represent the components of a *possible deformation* of the body when it is considered in its totality. One knows that in order for that to be true, its values must satisfy six equations at all points of the body that represent the necessary and sufficient conditions for a *given* system of six functions a, b, …, h to be able to satisfy (1).

Let X_x, Y_x, … denote the components of the *stresses*, to use KIRCHHOFF's notation, in which X_n, Y_n, Z_n generally represent the components along the axes of the unit tension that is exerted upon a planar element with normal n. Finally, let X, Y, Z denote the components of the external force that acts upon the masses of the elements of the body per unit mass, let ρ be the density, and let L, M, N denote the components of the unit external force that is applied to the surface of the body.

Let τ denote the (connected) space that is occupied by the body, and let σ denote its bounding surface, whose interior normal will be denoted by n, and let δu, δv, δw denote the infinitesimal variations of the u, v, w that correspond to a *virtual* displacement of the points of the body after it has been deformed. By a well-known transformation [keeping (1) in mind, along with the known relations $X_y = Y_x$, $X_z = Z_x$, $Y_z = Z_y$], one will have:

$$\int_\tau \left\{ \begin{array}{l} \left(\dfrac{\partial X_x}{\partial x}+\dfrac{\partial X_y}{\partial y}+\dfrac{\partial X_z}{\partial z}\right)\delta u \\ +\left(\dfrac{\partial Y_x}{\partial x}+\dfrac{\partial Y_y}{\partial y}+\dfrac{\partial Y_z}{\partial z}\right)\delta v \\ +\left(\dfrac{\partial Z_x}{\partial x}+\dfrac{\partial Z_y}{\partial y}+\dfrac{\partial Z_z}{\partial z}\right)\delta w \end{array} \right\} + \int_\sigma \left\{ \begin{array}{ll} [X_x \cos(nx) + X_y \cos(ny) + X_z \cos(nz)]\,\delta u \\ +\,[Y_x \cos(nx) + \cdots &]\,\delta v \\ +\,[Z_x \cos(nx) + \cdots &]\,\delta w \end{array} \right\} d\sigma$$

$$= -\int_\tau \{X_x\,\delta a + Y_y\,\delta b + \cdots + X_y\,\delta h\}\,d\tau\,.$$

(†) Translator: I. e., a compatible infinitesimal strain.

Set:

(2)
$$F = \frac{\partial X_x}{\partial x} + \frac{\partial X_y}{\partial y} + \frac{\partial X_z}{\partial z},$$

$$G = \frac{\partial Y_x}{\partial x} + \frac{\partial Y_y}{\partial y} + \frac{\partial Y_z}{\partial z},$$

$$H = \frac{\partial Z_x}{\partial x} + \frac{\partial Z_y}{\partial y} + \frac{\partial Z_z}{\partial z}.$$

If one takes into account the relations:

(3)
$$X_n = X_x \cos(nx) + X_y \cos(ny) + X_z \cos(nz),$$

$$Y_n = Y_x \cos(nx) + \ldots,$$

$$Z_n = Z_x \cos(nx) + \ldots$$

then the formula above can be written more briefly as:

(4)
$$\int_\tau \{F\,\delta u + G\,\delta v + H\,\delta w\}\,d\tau + \int_\sigma \{X_n\,\delta u + Y_n\,\delta v + Z_n\,\delta w\}\,d\sigma = -\int_\tau \{X_x\,\delta a + Y_y\,\delta b + \cdots + X_y\,\delta h\}\,d\tau.$$

Along with that, one has the other analogous equation:

(4.*a*)
$$\int_\tau \{u\,\delta F + v\,\delta G + w\,\delta H\}\,d\tau + \int_\sigma \{u\,\delta X_n + v\,\delta Y_n + w\,\delta Z_n\}\,d\sigma = -\int_\tau \{a\,\delta X_x + b\,\delta Y_y + \cdots + h\,\delta X_y\}\,d\tau.$$

In addition, the same process will yield:

(5)
$$\int_\tau \{F\,u + G\,v + H\,w\}\,d\tau + \int_\sigma \{X_n\,u + Y_n\,v + Z_n\,w\}\,d\sigma = -\int_\tau \{X_x\,a + Y_y\,b + \cdots + X_y\,h\}\,d\tau,$$

and more generally:

(6)
$$\int_\tau \{F\,u' + G\,v' + H\,w'\}\,d\tau + \int_\sigma \{X_n\,u' + Y_n\,v' + Z_n\,w'\}\,d\sigma = -\int_\tau \{X_x\,a' + Y_y\,b' + \cdots + X_y\,h'\}\,d\tau,$$

in which the primed quantities refer to any other state S'.

As one knows, the right-hand side of (4) represents the virtual work done by the internal forces, where the left-hand side provides a transformed expression for that work. Suppose that in the state *S*, the body is found to be deformed and in equilibrium under the action of external forces, whose corresponding virtual work is expressed by:

$$\int \{\rho (X\,\delta u + Y\,\delta v + Z\,\delta w)\}\,d\tau + \int \{L\,\delta u + M\,\delta v + N\,\delta w)\}\,d\sigma .$$

If one sums the two works and equates the sum to zero then, from LAGRANGE's principle, one will have the general equation of equilibrium:

$$\int \{(\rho X + F)\,\delta u + (\rho Y + G)\,\delta v + (\rho Z + H)\,\delta w)\}\,d\tau + \int \{(X_n L)\,\delta u + \cdots\}\,d\sigma = 0,$$

which must be verified for an (δu, δv, δw). If one then sets the coefficient equal to zero then one will have:

(7) $$\rho X + F = 0, \quad \rho Y + G = 0, \quad \rho Z + H = 0$$

at the interior points, which are the known indefinite equations of equilibrium, from which, one can pass to the equations of motion by replacing ρX, ρY, ρZ with:

$$\rho\left(X - \frac{d^2u}{dt^2}\right), \qquad \rho\left(Y - \frac{d^2v}{dt^2}\right), \qquad \rho\left(Z - \frac{d^2w}{dt^2}\right),$$

respectively, which include not only the external forces, but also the *forces of inertia* that gives rise to the accelerations, and at the points on the surface, one will have the equations of condition:

(7′) $$X_n + L = 0, \quad Y_n + M = 0, \quad Z_n + N = 0.$$

The quantities *F*, *G*, *H* that are defined by (2) represent the components of the force (per unit volume) that acts upon the individual elements of the body by virtue of the state of stress that exists in it as a result of the resultant action of the contiguous parts, and which are equal and opposite to the external force in a condition of equilibrium, which is expressed by (7) (and in the condition of motion, to that force, when it is modified as was said as a result of the accelerations). The X_n , Y_n , Z_n will then be the components of the stresses that are exerted upon the surface of the body from the inside, which are equal and opposite to the external force that is applied to each element, in the sense of equations (7′). The first one, as well as the second one, represents the elastic reaction that the deformation creates. In what follows, I shall often find it convenient to substitute the aforementioned reactions in the equations in the places where the external forces appear. Their significance will depend solely upon the state of stress in the body, which will often give a more intrinsic and general significance to those relations. For linguistic convenience, they will be denoted by the special name of *elations* (*elaterii*), and when one must distinguish them, the *F*, *G*, *H* will be called *internal elations* and the X_n , Y_n , Z_n will be called *surface elations.*

Starting from the known concepts upon which the notion of energy is based, one assumes that the expression:

$$X_x\,\delta a + Y_y\,\delta b + Z_z\,\delta c + Y_x\,\delta f + X_z\,\delta g + X_y\,\delta h$$

corresponds (at *constant* temperature) to the exact variation of a function that depends uniquely upon the state of deformation in the vicinity of the point (x, y, z), which represents the energy of deformation per unit volume, or as one says, the *elementary elastic potential.*

In the usual way that the theory is presented, one supposes that in the original state S_0 of the body, when it is not subject to external forces and is at rest, all of the elements of that body are found in the natural state, in such a way that any piece of it will keep its own form, even when they are supposed to be isolated and removed from the action of the other ones. The stresses will then be *zero* everywhere, and *the energy of deformation* of any individual element *will be zero.*

For any other state S, if one calls its energy e then one will find that under those conditions, one must have:

$$e = \phi\,(a, b, c, f, g, h), \tag{8}$$

in which ϕ is the symbol of a homogeneous function of degree two that is *always positive* and vanishes only for $a = b = c = \ldots = h = 0$; i.e., in the state S_0 : One can then equate the preceding expression to the variation of that function, and one will have the known relations:

$$X_x = \frac{\partial e}{\partial a}, \qquad Y_y = \frac{\partial e}{\partial b}, \qquad \ldots, \qquad X_y = \frac{\partial e}{\partial h}, \tag{9}$$

which gives the components of the stress as linear functions of the components of the deformations with the reciprocals:

$$a = \frac{\partial e}{\partial X_x}, \qquad b = \frac{\partial e}{\partial Y_y}, \qquad \ldots, \qquad h = \frac{\partial e}{\partial X_y}, \tag{9.a}$$

in which one intends that e should represent the energy that is produced by the stresses or the reciprocal quadratic form to $\phi\,(a, b, \ldots, h)$.

One will then have:

$$X_x\,\delta a + Y_y\,\delta b + \ldots + X_y\,\delta h = \delta\phi\,(a, b, \ldots, h),$$

$$X_x\,a + Y_y\,b + \ldots + X_y\,h = 2\phi\,(a, b, \ldots, h),$$

$$X_x\,a' + Y_y\,b' + \ldots + X_y\,h' = \psi\begin{pmatrix} a, b, \ldots, h \\ a', b', \ldots h' \end{pmatrix} = X'_x\,a + Y'_y\,b + \ldots + X'_y\,h,$$

the primed quantities in the last of these refer to another state S', and $\psi\begin{pmatrix} a,b,\ldots,h \\ a',b',\ldots h' \end{pmatrix}$ denotes the bilinear form that is associated with ϕ (a, b, …, h) and ϕ (a', b', …, h'). When S' is infinitely close to S, one can set $a' = a + \delta a$, …, and reduce the preceding one to:

$$X_x\ \delta a + Y_y\ \delta b + \ldots = a\ \delta X_x + b\ \delta Y_y + \ldots,$$

and from the first one, one will then have:

$$a\ \delta X_x + b\ \delta Y_y + \ldots + h\ \delta X_y = \delta\phi\,(a, b, \ldots, h).$$

If one multiplies by $d\tau$ and integrates then one will deduce the corresponding relation for whole body, which can be transformed by means of formulas (4), (4.*a*), (5), (6) and give:

$$\text{(I)} \qquad \int \Sigma(F\,\delta u)\,d\tau + \int \Sigma(X_n\,\delta u)\,d\sigma = -\delta \int \phi(a,b,\ldots,h)\,d\tau\,,$$

$$\text{(I.}a\text{)} \qquad \int \Sigma(u\,\delta F)\,d\tau + \int \Sigma(u\,\delta X_n)\,d\sigma = -\delta \int \phi(a,b,\ldots,h)\,d\tau\,,$$

$$\text{(II)} \qquad \int \Sigma(Fu)\,d\tau + \int \Sigma(X_n u)\,d\sigma \qquad = -2 \int \phi(a,b,\ldots,h)\,d\tau\,,$$

$$\text{(III)} \qquad \int \Sigma(Fu')\,d\tau + \int \Sigma(X_n u')\,d\sigma = -\int \psi\begin{pmatrix} a,b,\ldots,h \\ a',b',\ldots,h' \end{pmatrix} d\tau = \int \Sigma(F'u)\,d\tau + \int \Sigma(X'_n u)\,d\sigma\,,$$

in which the Σ sign indicates that one must combine the written term with the two analogous terms that relate to the other two components.

The first of these relations once more translates the principle of energy from which we started by means of the equality of the transformed expression for the virtual work done by deformation (δu, δv, δw) for an arbitrary system with the corresponding variation for the energy of deformation of the whole body, whereas the following relation (I.*a*) gives a new expression for the equality for the variation of that energy. (II) implies the value of the total energy of deformation $\int \phi\,(a, b, \ldots, h)\ d\tau$ as a function of the elations and the displacements. Finally, the last one represents a law of reciprocity that refers to the case of equilibrium, and when the elations are replaced with the external forces, it will reduce to a known theorem that was given by BETTI ([1]).

However, one can certainly say that even in the absence of external forces, by virtue of the connections between the various parts of the body, it will be found in a certain state of mutual constriction, and what was supposed in the preceding is therefore not verified – i.e., that in the state S_0, the elements of the body are all in the natural state of zero stress.

([1]) Cf., "Teoria dell'elasticità," in Nuovo Cimento (2), vols. VI, VII, IX, X.

One can seek to once more reduce the question to the preceding case by way of decomposition by replacing the given body or system with the consideration of more neighboring bodies, each of which is taken to satisfy the aforementioned condition and which are coupled to each other by reciprocal stresses. However, here it is better to deal with the more general case, which naturally includes the first.

In order to do that, observe that under the hypothesis that in the state S_0, in which one defines the displacement (u, v, w), the elements of the body are not found in the natural state, so the quantities $a, b, \ldots, h$ that are defined by (1) do not represent the absolute or total deformation around the point (x, y, z), but the *new* deformation that, by the effect of the aforementioned system of displacements, is superimposed with the one that is already present in the state S_0 (which is supposed to be one of stable equilibrium). We confine ourselves to the considering the case in which the latter has the same order of magnitude, so we can replace the $a, b, \ldots, h$ with the differences $\mathrm{a} - \mathrm{a}_0$, $\mathrm{b} - \mathrm{b}_0$, …, $\mathrm{h} - \mathrm{h}_0$, where the new symbols refer to the absolute deformations, whose inverses would take any particle of the body, *when considered separately*, back to the natural state. Since, by hypothesis, the connectivity of the body does not *collectively* submit to that return to the natural state, it will then follow that the $(\mathrm{a}_0, \mathrm{b}_0, \ldots, \mathrm{h}_0)$ or the $(\mathrm{a}, \mathrm{b}, \ldots, \mathrm{h})$ do not constitute a *congruent system* in the sense that was described above. That obviously does not detract from their significance with respect to the individual parts of the body considered in itself.

We then set:

$$a = \mathrm{a} - \mathrm{a}_0, \qquad b = \mathrm{b} - \mathrm{b}_0, \qquad \ldots, \qquad h = \mathrm{h} - \mathrm{h}_0$$

and get:

$$\mathrm{e}_0 = \phi(\mathrm{a}_0, \mathrm{b}_0, \ldots, \mathrm{h}_0), \qquad \mathrm{e} = (\mathrm{a}, \mathrm{b}, \ldots, \mathrm{h})$$

for the values of the unit energy of deformation in the two states S_0 and S, respectively, and:

$$(X_x)_0 = \frac{\partial \mathrm{e}_0}{\partial \mathrm{a}_0}, \qquad (Y_y)_0 = \frac{\partial \mathrm{e}_0}{\partial \mathrm{b}_0}, \qquad \ldots,$$

$$X_x = \frac{\partial \mathrm{e}}{\partial \mathrm{a}}, \qquad Y_y = \frac{\partial \mathrm{e}}{\partial \mathrm{b}}, \qquad \ldots$$

for the respective stresses. If one considers the relation:

$$\begin{aligned}\phi(\mathrm{a}, \mathrm{b}, \ldots) &= \phi(\mathrm{a}_0 + a, \mathrm{b}_0 + b, \ldots, \mathrm{h}_0 + h) \\ &= \phi(\mathrm{a}_0, \mathrm{b}_0, \ldots) + \phi(a, b, \ldots) + \frac{\partial \phi(\mathrm{a}_0, \mathrm{b}_0, \ldots)}{\partial \mathrm{a}_0} a + \ldots\end{aligned}$$

then one will have:

$$\mathrm{e} - \mathrm{e}_0 = \phi(a, b, \ldots) + (X_x)_0\, a + (Y_y)_0\, b + \ldots + (X_y)_0\, h, \tag{8′}$$

which gives the expression for the variation of the energy under the transition from S_0 to S, which can also be negative in some parts of the body, and one will have:

(9) $$X_x - (X_x)_0 = \frac{\partial\phi(a,b,\ldots)}{\partial a}, \qquad Y_x - (Y_x)_0 = \frac{\partial\phi(a,b,\ldots)}{\partial b}, \qquad \ldots$$

for the stresses, in addition.

One will obtain the following system of relations from this:

$$[X_x - (X_x)_0]\ \delta a + [Y_x - (Y_x)_0]\ \delta b + \ldots = \delta\phi\,(a, b, \ldots),$$

$$[X_x - (X_x)_0]\ a + [Y_x - (Y_x)_0]\ b + \ldots = 2\phi\,(a, b, \ldots),$$

$$[X_x - (X_x)_0]\ a' + [Y_x - (Y_x)_0]\ b' + \ldots = [X'_x - (X_x)_0]\,a + [Y'_y - (Y_y)_0]\,b + \ldots$$

$$[X_x - (X_x)_0]\ \delta a + [Y_x - (Y_x)_0]\ \delta b = a\ \delta X_x + b\ \delta Y_y + \ldots,$$

$$a\ \delta X_x + b\ \delta Y_y = \delta\phi\,(a, b, \ldots),$$

which agree with the corresponding relations that are found above and differ from them only by the terms that contain the initial stresses $(X_x)_0$, $(Y_x)_0$, … that are lacking from them.

However, if, as one usually does, one integrates the relations over the entire body, then transforms them with formulas (4) to (6) and notes that the elations *are all zero* in the initial state S_0 , from the definition itself of that state, then all of the terms that relate to S_0 will disappear from the integral equations, and the result will once more be (I), (I.*a*), (II), (III), which will then be valid for the more general case that we now consider, as well.

Along with that, one then has the other relation:

(IV) $$\int \mathrm{e}\,d\tau - \int \mathrm{e}_0\,d\tau = \int \phi(a,b,\ldots)\,d\tau,$$

which is deduced from (8′) by integrating and similarly reducing, and that will show that whereas the variation of the energy under the transition from S_0 to S can be negative for any part of the body, as was said above, the new deformation will necessarily imply an overall *increase* in energy for the body as a whole. It will then have its *minimum* value in the state S_0, which is generally non-zero and depends upon the connectivity of the system, and which can be called the *latent* or *constrained* energy. The expression $\int \phi(a,b,\ldots)\,d\tau$, which represents the excess of energy in the body when it is in any state S over the energy when it is in the state S_0, can also be taken to be a measure of the *total work done by deformation* during the transition from S_0 to S.

Along with the aforementioned equations, we add the following ones, which relate to two different arbitrary states (a, b, …), ($a + \Delta a$, $b + \Delta b$, …), and from the preceding, its deduction will present no difficulties:

(V) $$\int \Sigma(F\,\Delta u)\,d\tau + \int \Sigma(X_n\,\Delta u)\,d\sigma = -\Delta\int \phi(a,b,\ldots)\,d\tau + \int \phi(\Delta a,\Delta b,\ldots)\,d\tau,$$

(V.*a*) $$\int \Sigma (\Delta F\, u)\, d\tau + \int \Sigma (u\, \Delta X_n)\, d\sigma = -\Delta \int \phi(a, b, \ldots)\, d\tau + \int \phi(\Delta a, \Delta b, \ldots)\, d\tau \,,$$

(VI) $$\int \Sigma (\Delta F\, \Delta u)\, d\tau + \int \Sigma (\Delta X_n\, \Delta u)\, d\sigma = -2 \int \phi(\Delta a, \Delta b, \ldots)\, d\tau \,.$$

One returns from (V), (V.*a*) to (I), (I.*a*) by supposing that the states are infinitely close [with the caveat that ϕ (Δa, Δb, …) reduces to a second-order quantity when Δa, Δb, … get smaller], and one returns to (II) from (VI) by supposing that the first state coincides with S_0 . One should note that (V.*a*), along with (I.*a*), preserves its significance and also persists for a *non-congruent system* of variations (Δa, Δb, …) or (δa, δb, …), which is easy to believe if one recalls the manner by which it was deduced. One would like to say that they are also applicable to a variation of the individual parts of the body, when they are regarded as independent, and such that they would not result in a possible deformation of that body as a whole.

Finally, observe that all of those equations (I) and (VI) can also be valid in the case of complex systems that are composed of neighboring parts. Hence, one will not see any sliding between the separation surfaces, or even if one does, if the opposing surface tensions or elations prove to be normal to the surface (which can always happen when one ignores friction) then the parts of the integrals that relate to those surfaces will cancel in the results, and it would be as if no separating surfaces existed and one were dealing with continuous systems.

One infers the proof of the *uniqueness* of the system of displacements (u, v, w) from equation (VI), which will bring about equilibrium with the given external forces, or in our other way of interpreting things, they will develop elations. It is enough to observe that if one supposes that all of the ΔF, ΔX_n are equal to zero and one consequently annuls the left-hand side of that equation then one must also annul $\int \phi(\Delta a, \Delta b, \ldots)\, d\tau$. That will imply that one has ϕ (Δa, Δb, …) = 0 and therefore $\Delta a = \Delta b = \ldots = 0$ at all points of the body. Therefore, there will be just one *congruent* system (a, b, …, h) and thus just one system (u, v, w) that corresponds to the given elations. The converse is also true, and the value of the resulting elations are determined completely when one is given (a, b, …, h) or (u, v, w), *independently of the stress conditions that existed in the state* S_0 *to begin with*, which is a consequence of the fact that the elations are all zero in that state.

It then follows that one has a unique correspondence between elations and displacements, in such a way that if one is given one of them then the other one will prove to be determined completely, and in that sense one can say that the elations are functions of the displacements and *vice versa*. In addition, they are homogeneous linear functions, which will be obvious if one supposes that the displacements vary with a given ratio that is the same for all of them, while the elations all vary in that same ratio.

The work done by deformation of the body under the transition from S_0 to S, when it is given as above or else by $\int \phi(a, b, \ldots)\, d\tau$, will present itself analogously as a homogeneous function of degree two of the system (a, b, …, h) of the components of the deformation. Therefore, it will be completely determined when one is given the a, b, …, h at all points of the body and by taking a, b, … to be equal to ka, kb, …, respectively, everywhere, where k is an arbitrary number, then it will vary by the ratio 1 to k^2.

If one is given that the system (a, b, …, h) is *congruent* – i.e., one treats a deformation of the body that is actually *possible* – then that work done by deformation can also be obviously regarded as a function of the displacement (u, v, w) by means of which the (a, b, …, h) are determined completely, and as such, it will also be homogeneous of degree two. Moreover, it is by virtue of the aforementioned correspondence between displacements and elations that it will follow that the work can also be considered to be a function of the elations that is similarly homogeneous of degree two. Hence, if one starts with (II) and regards the F, …, X_n, … in the left-hand side as functions of the displacements, or *vice versa*, then one will have that work as a function of the displacements or elations, respectively.

The term *function* is used here in a sense that corresponds to an extension of the usual concept, in the sense of a *quantity that depends upon all of the values that one or more functions take in the given region*, and that will be specified by the name of a *function of the region*, for the sake of clarity. We will then say that the u, v, w that relate to a well-defined point are functions of the region of the internal and surface elations, and that are, by and large, functions of the positions of the point or its coordinates and functions of the region of the elations. The same thing will be true conversely for the elations with respect to the displacements.

Such an extension of the concept of a function was taken under consideration in some recent work by Prof. VOLTERRA ([1]), who made it the subject of some interesting studies. It is also convenient for our own purposes to make some brief observations in regard to that type of function, especially as far they relate to the way by which one can hope to extend derivation to them.

Say that a function of a region is *continuous* if giving variations to the function or to functions that it depends upon that are arbitrary, but lower in absolute value than a number k, means that the corresponding variation of that function can be made less than any given quantity by reducing k.

Therefore, suppose that a continuous function Φ that depends upon the function ξ is given in the field ϖ and consider a portion ϖ_p of the region that defines a neighborhood of a point p. Imagine that ξ is given a continuous variation $\Delta\xi$ inside of ϖ_p that is everywhere of equal sign and lower in absolute value to a number k. If $\Delta\Phi$ is the corresponding variation of Φ and one sets:

$$\int_{\varpi_p} \Delta\xi \, d\omega = \varepsilon$$

then as long as there exists a well-defined, finite limit to the ratio:

$$\frac{\Delta\Phi}{\varepsilon}$$

([1]) V. VOLTERRA, "Sopra le funzioni che dipendono da altre funzioni," Rendiconti della R. Accad. dei Lincei, vol. III, fasc. 4, 1887.

for all of the possible variations $\Delta\xi$ as k and ϖ_p tend to zero, we shall say, with Volterra, that the limit in question represents the *derivative of* Φ *at the point p*. That derivative, which I shall also call the *derivative in the region*, will depend upon the position of p and the values of ξ, so it will be a function of the region of ξ and a function of the position of the point.

For a function Φ that depends upon several functions, one can speak in the same way of its partial derivatives of the region with respect to each of them.

That definition of the derivative in the region agrees with the one that one arrives at, as we do here, from much simpler viewpoint that consists of regarding a function in the region as the *limit* of a function of a finite number of variables. Given the continuity of Φ, as above, imagine that the region ϖ is divided into a large number n of parts and assume that as n increases and with a convenient distribution of the parts, one can arrange that in each part ϖ_p where ξ is not constantly equal to zero, it has the same sign everywhere, and its *oscillation* (or the difference between the maximum and minimum) proves to be less than a given number k. If one now supposes that one replaces the variable value ξ everywhere with its mean value $\bar{\xi}_p$, which is given by:

$$\bar{\xi}_p\,\varpi_p = \int_{\varpi_p} \xi\, d\varpi\,,$$

and lets Φ_1 denote the modified values that will result for the function Φ then the difference $\Phi - \Phi_1$ can be made less than any given quantity by decreasing k, as well as increasing n. One will then be led to regard Φ as the limit as n becomes infinite of a function of the n quantities $\xi_1, \xi_2, \ldots, \xi_p, \ldots, \xi_n$, or even better $\lambda_1, \lambda_2, \ldots, \lambda_p, \ldots, \lambda_n$, which result from them by multiplying them by the respective portions of the region:

$$\lambda_p = \bar{\xi}_p\,\varpi_p = \int_{\varpi_p} \xi\, d\varpi\,.$$

If one regards Φ as a function of $\lambda_1, \lambda_2, \ldots$, in that way, and takes its derivative with respect to any λ_p then, as is easy to see, that limit will reduce to the derivative in the region that was defined above. In substance, it represents the limit of the ratio of the variation of the Φ that corresponds to the variation of ξ inside of a small region ϖ_p to the mean of the latter variation multiplied by the volume ϖ_p of the region considered, so it will give the *amount*, so to speak, of the variation of ξ inside ϖ_p.

The complete variation of Φ corresponds to an infinitesimal variation $\delta\xi$ of the function ξ in all of the region, so it will present itself as the limit of the variation of a function of a finite number of variables:

$$\delta\Phi = \lim \sum \frac{\partial\Phi}{\partial\lambda_p}\delta\lambda_p = \lim \sum \frac{\partial\Phi}{\partial\lambda_p}\delta\bar{\xi}_p\,\varpi_p\,,$$

in which (Φ') denotes simply the derivative in the region. One will then have:

$$\delta\Phi = \int (\Phi') \, \delta\xi \, d\varpi \,,$$

which will coincide, in substance, with the expression that VOLTERRA found by using a more rigorous analysis. Conversely, if the expression $\delta\Phi$ has the form:

$$\delta\Phi = \int \theta \, \delta\xi \, d\varpi$$

then one can conclude that:

$$\theta = (\Phi').$$

The same line of reasoning can be extended to the case of a function that depends upon more functions that are given in one or more regions. For example, one of them might be the work done by deformation, when regarded as a function of the displacements or as a function of the interior and surface elations. We would like to refer these brief considerations to the latter precisely, while making no pretense of rigor with respect to the analytical question that one considers in general.

We then commence with our application of the latter to CASTIGLIANO's theorem in regard to the derivative of work. In order to do that, observe that from the aforementioned double dependency of the work done by deformation, the two different expressions for its variations that one infers from (I) and (I.*a*) will match. The former regards the elations as functions of the displacements and gives the variation of the work when it is considered to be a function of the displacements, while the latter regards the displacements as functions of the elations and gives the variation of that work when it is considered to be a function of the elations. From the forms of those expressions, if one remembers what was said above then one will meanwhile conclude directly that:

The derivatives in the region of the work done by deformation with respect to the displacements, when that work is given as a function of the displacements, are equal and opposite to the corresponding elations at any point of the interior and the surface, and the derivatives in the region of that work with respect to the elations, when given as a function of the elations, are equal and opposite to the displacements.

Refer to an elementary portion ϖ_p of the volume or the surface of the body, and note that the produce of the elations (unit force) with ϖ_p will represent the effective forces that are exerted upon the element considered. It will be clear from what was said before that the same proposition can be stated by saying that: The derivatives of the work with respect to the displacements of the element will give the *effective elations* that act upon it, and the derivative with respect to the *effective elations* will give the displacements. The statement is then brought back to ordinary derivatives, and can also be applied to the case of finite forces that act upon its points (limiting case of the forces whose actions are concentrated in a restricted region), such as, e.g., when one treats a deformed body under the action of external forces that are applied to well-defined points of its surface.

Therefore, suppose that the volume and surface of the body are divided into as many elementary portions, and from now on let the common symbol η denote any one of the components of the effective elations that relate to the interior or surface of each portion, and let u denote any one of the components of the displacements. One can write the expression for twice the work done by deformation that was given by (II) more simply in the form:

$$-\sum[\eta\, u],$$

in which the symbol Σ now means that, along with the sum over the three components, it now denotes the sum over all elementary parts of the volume and surface, which generally replaces the integration symbol, and then reduces to a finite number of terms when the η vanish everywhere, except for isolated points where just the preceding statements have a finite value. The two expressions for the variations of that work that are given by (I), (I.*a*) will then take the form:

$$-\sum[\eta\,\delta u], \qquad -\sum[u\,\delta\eta],$$

and twice the preceding proposition [where simply Φ (*u*) and Φ (η) will denote the work done by deformation when it is given as a function of the displacements or the elations, respectively] will generally translate into the formulas:

$$\text{(VII)} \qquad \frac{\partial\Phi(u)}{\partial u} = -\,\eta,$$

$$\text{(VII.}a\text{)} \qquad \frac{\partial\Phi(\eta)}{\partial\eta} = -\,u.$$

Refer to the case of equilibrium and set those elations equal to the external forces with the signs changed. One will then have:

The derivatives of the work done by deformation, when given in terms of the displacements, with respect to the displacement of the various functions, are equal to the external forces that are applied to those points (elements), and the derivatives of that work, when given as a function of the external work, with respect to the forces that are applied to the various points, will give the respective displacements.

That is CASTIGLIANO’s theorem, which he proved for the case of an articulated system, and then generalized, with no further assumptions, by supposing that any body can be regarded as the limit of an articulated system whose vertices correspond to the molecules of the body, and the tensions in the connecting rods correspond to the inter-molecular forces. In addition to its generality, the argument that was given here has the advantage of determining the significance exactly in any case. It appears from this that when one treats the forces that are exerted upon isolated points, in which case, the expression for the work will reduce to a finite number of terms, the statement of the theorem will refer directly to the effective forces that are applied at the isolated points. However, when one treats continuous distributions of forces that act upon either the volumes of the elements of the body or upon their surfaces, the theorem will refer to the forces that are exerted upon elementary portions ϖ_p of the volume or surface (around the point p that one considers) or to the unit forces that result from dividing them by ϖ_p .

The consideration of the elations at the locations of the external forces will then give that theorem a more intrinsic character.

One can note that the first part of (VII) basically represents only the ordinary expression for the characteristic property of the potential (viz., potential energy) of a system, according to which, its derivatives, when taken negatively, will give the forces that are developed in the system. When one regards the function that represents the work done by deformation as the limit of a quadratic form in a finite number of variables, the other part of (VII.*a*) will take the form of a consequence of the first one, by virtue of a known property of reciprocal quadratic forms. That is exactly how one gets the reciprocal relations (9) and (9.*a*) between the elementary deformations and the tensions.

Among the more direct applications of the theorem in question, one has the one that serves to calculate the displacements of the points of the body or system when one knows, in some way, the effective expression for the work as a function of the force. Let us make an observation in regard to that.

If such an express presents itself in a finite form (as would ordinarily happen in effective problems) that treats only the forces that are applied to certain points then one demand to know how one must proceed in order to calculate the displacement of the other points to which no force is applied. In order to answer that, it is enough to observe that the given expression must be considered, in any case, to be a descendent of the general force $-\frac{1}{2}\sum[\eta\, u]$ when the (u) are functions of the (η), in which all of the terms that relate to the parts of the body that do not feel any applied forces disappear, insofar as the corresponding η will be equal to zero. For the calculation of the displacements, one must, as a rule, intend the derivation that one applies to mean the general expression and then set the η equal to zero at the stated points. However, it is clear that (except for the point p at which one seeks the displacement) the result will be the same as if one had supposed that the η were zero at those points from the outset (which would only cancel the terms that would cancel later on), and therefore it is enough to derive the particular expression that relates to the concrete case that one considers. As for the point p, the force that cannot be supposed to be zero from the outset must be differentiated with respect to it, so in the case where no actual force is applied to p, one nonetheless agrees to take the expression for the work that results from supposing that along with the forces that actually exist at the other points, one also applies an arbitrary undetermined force to p and differentiates with respect to it, after which, one gives the value of *zero* to that derivative.

Finally, observe that one can easily give a more general expression to the foregoing by considering an arbitrary system of quantities (of the same order of magnitude) instead of the displacements along the three axes of the points of the body, by means of which one would determine the deformation. If we preserve the same symbol (u) to denote this new quantity then one will also have:

$$2\Phi = -\sum[\eta\, u],$$

in which Φ denotes the work done by deformation and (η) represents the generalized components of the internal forces that tend to vary the (u), and we will then have that those functions are linear and homogeneous in the (u), and conversely. It would be useful to have all of the preceding relations for the new quantities, and therefore, (VII), (VII.*a*), as well. Hence, when one specializes them, one can get the various propositions that one is wont to give by just as many particular theorems.

We now turn to the question of the minimum of work that MENABREA's theorem refers to.

We take into consideration the condition of equilibrium for the deformed body under the action of external forces and briefly let:

$$\sum [X\,\delta u]$$

denote (in the notation that was adopted above) the virtual work that those forces do for an arbitrary system of displacements (δu), in which X denotes any of the components of the effective forces that act upon the elements of the body or its surface. The general equation of equilibrium that is given by LAGRANGE's principle when one takes the work done $\sum [X\,\delta u]$ by the internal forces to be $-\,\delta\Phi$ will take the form:

$$\delta\Phi = \sum [X\,\delta u].$$

If the external forces have a potential, such that:

$$\sum [X\,\delta u] = -\,\delta P \qquad (P = \text{potential energy})$$

then that equation will become:

$$\delta\Phi + \delta P = \delta(\Phi + P) = 0.$$

If we let E denote the total energy of deformation and recall (IV), which will now take the form:

$$\Phi = E - E_0 ,$$

then the same equation can also be written:

$$\delta(E + P) = 0,$$

which expresses the idea that the (stable) equilibrium state corresponds to a minimum of the *total potential energy* that is represented by the sum $E + P$ of the energy of the deformation and the energy of the system of external forces. It is included as a particular case of the known general law of equilibrium that was formulated by DIRICHLET.

However, if one would like to refer to the energy E or the work done by deformation Φ, when considered by themselves, then obviously one cannot speak of a minimum in the absolute sense, except in the special case in which there are no external forces to consider. The total energy will then reduce to just the energy of deformation, which must therefore be a minimum. Indeed, it has already been seen directly that the work done by deformation will be zero in such a case, and the energy E will have its minimum value E_0, which will depend upon only the connectivity of the system.

In the general case, one says that the value of E or Φ must be a minimum that is *compatible with the imposed conditions*. However, such a statement has too much indeterminacy in it, and among other things, it can lead to inexact interpretations. Here is briefly how the question can be posed in general in its proper terminology, in my opinion:

Starting from the state S of equilibrium, consider the variation $\Delta\Phi$ of the work done by deformation for any change (Δa, Δb, …), which will properly be (V), (V.a) in its two forms, with the abbreviated notation:

$$\Delta\Phi = \Phi\,(\Delta a, \Delta b, \ldots) - \sum[\eta\,\Delta u],$$

$$\Delta\Phi = \Phi\,(\Delta a, \Delta b, \ldots) - \sum[u\,\Delta\eta]\ .$$

It is composed of two distinct parts. One part:

$$\Phi\,(\Delta a, \Delta b, \ldots) \qquad \text{or} \qquad \int \phi(\Delta a, \Delta b, \ldots)\,d\tau$$

is common to the two expressions. It is essentially positive and will reduce to a second-order quantity when Δa, Δb, … get smaller. From its form, that part can be called the *relative potential of the deformation*, and its value is given by (VI):

$$2\Phi\,(\Delta a, \Delta b, \ldots) = -\sum[\Delta F\,\Delta u],$$

and one sees that it represents the part of the energy that depends upon the new state of elation that gets superimposed with the preexisting one.

The other part, which can be positive of negative, is expressed by $-\sum[\eta\,\Delta u]$ in the first form and will depend upon the value of the preexisting elations in the state S. In the second form, it is represented by the expression $-\sum[u\,\Delta\eta]$, which is equivalent to the first one by the law of reciprocity when the system (Δa, Δb, …) is *congruent*.

If one takes the difference between $\Delta\Phi$ and the second part, which is represented in its two forms by:

$$\Delta\Phi + \sum[\eta\,\Delta u] \qquad \text{and} \qquad \Delta\Phi + \sum[u\,\Delta\eta],$$

resp., then one will have what one calls the *reduced variation*, which is equal to Φ (Δa, Δb, …), and is therefore an *essentially-positive quantity that will become a second-order quantity when* Δa, Δb, … *get smaller*. If one then equates the first-order part to zero for an infinitesimal variation (δa, δb, …) then one will get back to (I), (I.a), or:

$$\delta\Phi = -\sum[\eta\,\delta u] \qquad \text{and} \qquad \delta\Phi = -\sum[u\,\delta\eta]\ .$$

With the first of these, if one introduces the external forces then under the hypothesis of the existence of a potential P for those forces, one will get the minimum of $\Phi + P$, as above. However, one should note that $\sum[\eta\,\delta u]$ can, in any case, be regarded as the expression for an external work that is produced or a virtual development of the external energy for part of the system, and therefore the expression will contain only quantities that are inherent to that system, so the reduced variation will have an autonomous significance that will also persist when one ignores the external forces and the equilibrium condition. The same thing will be true for the other form.

What was said before about the reduced variation will generally signify a minimum of the work Φ *in the reduced sense*, when one abstracts from the corresponding part of the extrinsic energy. It will then follow, in particular, that one has a proper minimum of that work with respect to all changes *that do not involve the externalization of energy*. That is, Φ will be a minimum in the state S with respect to all of the neighboring states S' such that the transition from S to S' is not connected with a development of external energy. Hence, if one specializes that then one will get a minimum of Φ (or of the energy E) that is subordinate to the conditions:

$$\sum[\eta\,\delta u] = 0 \qquad \text{and} \qquad \sum[u\,\delta\eta] = 0.$$

Leaving aside the first one, which will yield the ordinary potential equations, we focus more especially upon the other, while taking into account the fact that the equation that it refers to is true for congruent variations, as well as for non-congruent ones. As far the congruent variations are concerned, recall that given elations (η) correspond to a unique and well-defined system of displacements and that any system of variation (δu) will therefore necessarily include some variations of the elations. Thus, if one supposes that the elations are given at all points then no congruent variation that is compatible with those values will be possible. That will no longer be true when one treats non-congruent variations, under which, one no longer considers all systems with the same connectivity, but only the individual parts of them, when they are regarded as independent, so it will then be possible to consider the different varied states to correspond to the same elations; i.e., such that all of the $\delta\eta$ are equal to zero. One can therefore no longer speak of the displacements of the points of the body as a whole nor of the work done by deformations being regarded as a function of the displacements or elations, but simply as the sum or integral $\int \phi(a, b, \ldots)\,d\tau$ of the works that relate to the elements of the body that was considered initially.

That caveat is necessary if one is to clearly fix the meaning of MENABREA's theorem, which one can state as:

The work developed under deformation of any body under the action of given external forces is a minimum.

Therefore, if one intends that the external forces (or elations) *should be given at all points* (while keeping the value of *zero* for the points where no force is applied) then no system (δu) would be possible, and there would be no sense in saying that the work is a minimum that is *compatible with the given values*, where one does not consider, more generally, the non-congruent variations either, just the statement that was made just now. More explicitly, the theorem will then say that:

Given elations correspond to just one well-defined system of displacements (u), *i.e., just one congruent system* (a, b, …, h), *which is the one for which the work done by deformation is a minimum with respect to the varied values that result from any non-congruent variation* (δa, δb, …, δh) *that leaves the elations unchanged.*

The consideration of non-congruent variations represents an abstract process in which one regards deformations that are not actually possible. Therefore, when one does not regard *all* of the elations as being given – i.e., one does not take all of the $\delta\eta$ to be zero – while maintaining only the general condition $\sum [u\,\delta\eta] = 0$, it is enough to consider congruent variations, or possible displacements (δu) of the points of the system. From the law of reciprocity, that condition will be equivalent to the other one $\sum [\eta\,\delta u] = 0$, and one will get back to the usual equations of energy. The two forms differ only with respect to the analytical process of application, one of which refers to the (u) and the variables in the function that represents the work, while the other one refers to the (η).

At any rate, in the final analysis, everything is then summarized by the tendency to the minimum of the potential energy, which rules the manner by which the state of tension is transmitted and equilibrated across elastic bodies (like an physical phenomenon, in general) and can be intended to represent the principle of elasticity in the broadest sense. One finds an application of that in the fact that one can determine the modality of the state that corresponds to the special circumstances on the basis of the minimum condition, in any case.

Frequent examples of the practical utility are encountered, above all, in the science of constructions, where one then arrives, often along a simple path, at a knowledge of the elements that relative to a system that is composed of elastic materials, which would remain undetermined if one did not take elasticity into account. As one of the simpler and more interesting examples, we cite the one that is concerned with the determination of the tensions in the supernumerary rods in articulated system, which is also the one in which the discussion that related to MENABREA's theorem was mainly rooted. Without entering into the details or repeating the well-known story of that discussion, recall only that CERRUTI showed, in the cited paper, how the treatment of the problem of articulated systems could be made much simpler by an application of the theorem of the elastic potential, while adding that the theorem of minimum work, when taken in its full generality, would be nothing but that theorem of the potential. To that end, it would be enough for me to recall the preceding observations. On the other hand, as we said before, CASTIGLIANO established the theorem of the derivative of the work for articulated systems, from which, one can deduce the theorem of the minimum as a corollary.

The preceding considerations are general, and also include the case of *constrained systems*, which is a name that is meant to refer to *external* constraints that give rise to external forces, and are distinct from the connections that result from the internal structure of the system, which have already been taken into account expressly. We now conclude precisely by showing how one can treat the question of the determination of the unknown forces that represent the *resistance of the links* in those systems in general with the principles that were presented.

In general, such links consist of a certain number of relations that must be satisfied at the points of the system, and their effect, as is known, is to translate to those points everything that is implied by the introduction of new forces whose values depend upon the undetermined quantities, whose number is equal to the number of those conditions, and one can always make the dependency *linear*.

Hence, consider the totality of *given* forces, which are also denoted by the symbol (X), and the new ones that depend upon the links, which will be denoted by (X'), and the deformed system that is equilibrated under their collective action. One can then regard the displacements (u) of the points of the system as functions of all the forces (X) and (X') or of the corresponding elations (η) and (η'). As one says, the value of (X') or (η') will depend linearly upon the undetermined quantities, which are denoted by (λ) and which are as numerous as the conditions. Therefore, suppose that the *literal* expressions are known as functions of the (η) and (η'). Replace the (u) that are implied in the condition equations with the values of (u) that would result from the equations that would contain the (λ) by means of the (η'), and since the equations are as numerous as the (u), those equations can serve to determine the (u). Afterwards, the (η') or the (X') will also remain to be determined.

That will be true independently of the manner by which one calculates the aforementioned displacements, and therefore also when one supposes that the expression for the work done by deformation Φ is known as a function of the elations (η) and (η'), since one deduces the (u) from that expression by differentiation as in (VII.*a*).

However, when one treats links, properly speaking – or *invariable* links – one can proceed more directly, based upon the proposition that:

The derivatives of the work Φ *with respect to the* (λ) *are equal to zero.*

One proves this immediately by observing that the (X') or (η'), from the very nature of forces of resistance that are developed by invariable links, will satisfy, on the whole, the conditions $\sum[\eta' u] = 0$, $\sum[\eta' \delta u] = 0$, from which, it will also result that:

$$\sum[u \, \delta\eta'] = 0.$$

If one supposes that the (η') are expressed in terms of the (λ) that they depend upon then one will have:

$$\delta\eta' = \sum \frac{\partial \eta'}{\partial \lambda} \delta\lambda .$$

If one substitutes that in the preceding relation then it will take the form:

$$\sum \Lambda \, \delta\lambda = 0,$$

in which $\Sigma = \sum u \dfrac{\partial \eta'}{\partial \lambda}$, and one will then have, in general (VII.*a*), $u = -\partial\Phi / \partial\eta$, so:

$$\Lambda = -\sum \frac{\partial \Phi}{\partial \eta'} \frac{\partial \eta'}{\partial \lambda} = -\frac{\partial \Phi}{\partial \lambda}.$$

Now, the preceding equation must persist for any $\delta\lambda$, which will demand that the Λ are all equal to zero. One will then have:

$$\frac{\partial \Phi}{\partial \lambda} = 0$$

for each of the (λ), which we wished to show.

The (λ) can themselves be considered to be forces, and it is in that sense that the preceding proposition can also be stated by saying that:

The derivatives of the work done by deformation with respect to the forces that are developed by the links are equal to zero.

We then have as many equations as the number of (λ) to be determined, and therefore, from what was said, it is clear that those equations are linear, so one sees that the solution to the problem of the search for the unknown resistances is unique and well-determined.

Now, observe that from the last proposition, one can already conclude with no further analysis that the development of the resistances will come about in such a way that the work done by deformation that result for each part will be a minimum. However, it is easy to see how that follows directly from what was posed above.

Indeed, recall the general condition $\sum [u\,\delta\eta] = 0$ that related to the minimum, which splits off the (η) from the (η') here, by writing:

$$\sum [u\,\delta\eta] + \sum [u\,\delta\eta'] = 0.$$

Note that this will be satisfied if one supposes that all of the (η) are invariant that correspond to the given – or *active* – forces, while one can vary the (η') in any way. All of the $(\delta\eta)$ will then be annulled, so the first sum will vanish, while the second one will also be equal to zero from the characteristic property of resistance that was noted above. It will then result that the work Φ in the state S of equilibrium will be a minimum with respect to any neighboring state S' that corresponds to a variation of the forces that are developed by the links or resistances, while the active forces will remain invariant, and the value of that resistance *will be determined by the condition that renders a minimum for the work* Φ.

With that, one can then suppose that Φ is given as a function of the (η) and the (η'), and that the latter depend upon the (λ). The search for the minimum of Φ will then lead back to the equations:

$$\frac{\partial \Phi}{\partial \lambda} = 0,$$

which will serve to determine the unknown resistances, as above.