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The sufficient conditions for the extremum in the theory of the Mayer problem

By

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In Volume 58 of these Annalen (*), **Mayer** extended the so-called **Hilbert** independence theorem to the case of the extremum of a simple integral with arbitrarily-many functions and condition equations, and in that way produced the means to derive the sufficient conditions for an extremum quite simply. In what follows, analogous considerations will be developed for the most general problem in the calculus of variations with one independent variable, which will be referred to as the *Mayer problem* and reads as follows:

> "Among all continuous functions $y_0, y_1, \ldots, y_n$ of the independent variable x that fulfill the $r + 1$ given first-order differential equations:

(1) $$\varphi_k(x, y_0, y_1, \ldots, y_n, y_0', y_1', \ldots, y_n') = 0 \qquad (k = 0, 1, \ldots, r < n)$$

> identically, and the last n of which possess given values for two given values x_0 and x_1 of x, but the first one y_0, only for $x = x_0$, find the ones that are associated with a greatest or smallest value of the function y_0 at the location $x = x_1$ (**)."

§ 1.

If the values of the functions y_i for $x = x_0$ and $x = x_1$ are denoted by y_{i0} and y_{i1}, resp., then one deals with determining the extremum of y_{01} for prescribed values of:

$$y_{00}, y_{10}, \ldots, y_{n0} \qquad \text{and} \qquad y_{01}, y_{11}, \ldots, y_{n1}.$$

(*) Cf., Leipziger Berichte, 1903.

(**) **A. Mayer**, "Die Lagrangesche Multiplicatorenmethode," Leipziger Berichte (1895).

If one sets:

(2) $$\Omega = \sum_{k=0}^{r} \lambda_k \varphi_k$$

then the $n + 1$ functions y_i and the $r + 1$ multipliers λ_k will be determined by equations (1) and (*):

(3) $$\frac{d}{dx}\frac{\partial \Omega}{\partial y_i'} = \frac{\partial \Omega}{\partial y_i} \qquad (i = 0, 1, \ldots, n).$$

We call the manifolds:

$$y_1 = y_1(x), \; y_2 = y_2(x), \; \ldots, \; y_n = y_n(x)$$

that are obtained in that way *extremals*. The points $(x_0, y_{10}, y_{20}, \ldots, y_{n0})$ and $(x_1, y_{11}, y_{21}, \ldots, y_{n1})$, which might be referred to as 0 and 1, resp., together with the prescribed initial value y_{00} of the function y_0, determine an extremal C that yields a well-defined final value y_{01} for that function.

We now consider the *comparison curves* C':

(4) $$y_1 = \bar{y}_1(x), \quad y_2 = \bar{y}_2(x), \quad \ldots, \quad y_n = \bar{y}_n(x),$$

which all go through the points 0 and 1 and are subject to the following restriction, in addition: The substitution of the values (4) in equations (1) will not imply a contradiction and will yield a well-defined function $y_0 = \bar{y}_0(x)$ for prescribed initial values y_{00}. For $x = x_1$, we will then get a well-defined final value $\bar{y}_{10}$ on every comparison curve C', and we must then deal with ascertaining the conditions under which the difference:

$$\Delta y_{01} = \bar{y}_{01} - y_{01} = (\bar{y}_{01} - y_{00}) - (y_{01} - y_{00})$$

will possess a constant sign for all comparison curves under consideration. It is easy to see that we further have:

(5) $$\Delta y_{01} = \int_{C'} y_0' \, dx - \int_C y_0' \, dx,$$

in which the integrals extend over the arc 01 of the comparison curve C' and the extremal C, resp., and the initial value of y_0 is assumed to be equal to y_{00}.

We now consider any q-parameter family ($q \leq n$) of extremals that all correspond to the initial value y_{00} and include the extremal C. By eliminating the q parameters upon which the functions $y_i(x)$ their derivatives $y_i'(x)$ depend, we will get (among other things) the equations:

(*) **A. Mayer**, *loc. cit.*

$$(6)\qquad \begin{aligned} y_0 &= Y(x, y_1, y_2, \ldots, y_q), \\ y_i' &= p_i\,(x, y_1, y_2, \ldots, y_q) \end{aligned} \qquad (i = 0, 1, \ldots, n),$$

which are fulfilled identically for all extremals of the family. When we differentiate the first of those equations, we will get:

$$(7)\qquad y_0' = \frac{dY}{dx} = \frac{\partial F}{\partial x} + \sum_{h=1}^{q} \frac{\partial Y}{\partial y_h} y_h'$$

for every extremal of the family, and as a result:

$$\int_C y_0'\, dx = \int_C \frac{dY}{dx}\, dx = Y\,(x_1, y_{11}, \ldots, y_{q1}) - Y\,(x_0, y_{10}, \ldots, y_{q0})\,,$$

or since the last integral is obviously independent of the path of integration:

$$(8)\qquad \int_C y_0'\, dx = \int_C \left(\frac{\partial Y}{\partial x} + \sum_{h=1}^{q} \frac{\partial Y}{\partial x} y_h' \right) dx\,.$$

If the values of the derivatives y_i' in equations (6) are substituted in equation (7) then, as one will easily see, one will get an identity:

$$(9)\qquad p_0 = \frac{\partial Y}{\partial x} + \sum_{h=1}^{q} \frac{\partial Y}{\partial x} y_h'\,.$$

If one substitutes the value $\partial Y / \partial x$ from that into equation (8) and substitutes the value of the integral thus-obtained into equation (5) then one will ultimately get:

$$(10)\qquad \Delta y_{01} = \int_{C'} E\, dx\,,$$

in which the expression E is defined by the equation:

$$(11)\qquad E = y_0' - p_0 - \sum_{h=1}^{q} \frac{\partial Y}{\partial y_h} (y_h' - p_h)\,.$$

With an appropriate choice of the family of extremals, and therefore the function Y, formulas (10) and (11) will allow one to immediately extend the theories of Weierstrass and Hilbert to the problem in question (*).

(*) Cf., the presentation of **Kneser**, *Lehrbuch*, §§ 59-61.

§ 2.

If we now return to the differential equations (1) and (3) of the Mayer problem then we should point out that the number of arbitrary constants that the complete integration of those equations entails is equal to 2 (n + 1), but one of those constants is inessential for the problem, such that the functions y_i and the ratios $\lambda_k : \lambda_0$ will include only $2n + 1$ constants (*).

If we now demand that the initial conditions:

$$y_i = y_{i0} \qquad \text{for} \qquad x = x_0 \qquad (i = 0, 1, \dots, n)$$

are satisfied then only n arbitrary constants will remain, and we will get an n-parameter family of extremals that includes the extremal C. If we then set:

$$(12) \qquad \frac{\lambda_s}{\lambda_0} = -\mu_s \qquad (s = 1, 2, \dots, r)$$

then the following equations will exist for the aforementioned family of extremals:

$$(13) \qquad \begin{aligned} &y_1 = y_1(x, a_1, a_2, \dots, a_n), \quad y_2 = y_2(x, a_1, a_2, \dots, a_n), \quad \cdots \quad y_n = y_n(x, a_1, a_2, \dots, a_n), \\ &y_0 = y_0(x, a_1, a_2, \dots, a_n), \quad y'_i = y'_i(x, a_1, a_2, \dots, a_n), \qquad \mu_s = \mu_s(x, a_1, a_2, \dots, a_n) \end{aligned}$$

$$(i = 0, 1, \dots, n\,;\ s = 1, 2, \dots, r),$$

and the parameters $a_1, a_2, \dots, a_n$ can be determined as functions of $x, y_1, y_2, \dots, y_n$ from the first n of those equations for all values of the variables in question for which the Jacobian determinant:

$$(14) \qquad \frac{\partial(y_1, y_2, \dots, y_n)}{\partial(a_1, a_2, \dots, a_n)}$$

does not vanish. When one substitutes the values of the parameters $a_1, a_2, \dots, a_n$ thus-obtained into the remaining equations (13), one will arrive at the relations:

$$(15) \qquad y_0 = Y(x, y_1, y_2, \dots, y_n), \qquad y'_i = p_i(x, y_1, y_2, \dots, y_n), \qquad \mu_s = \pi_s(x, y_1, y_2, \dots, y_n),$$

$$(i = 0, 1, \dots, n\,;\ s = 1, 2, \dots, r),$$

which are fulfilled identically for all extremals of the family.

As is known, (**) the function Y satisfies a first-order partial differential equation whose characteristics coincide with the extremals, and the following identities will then exist:

(*) **A. Mayer**, *loc. cit.*
(**) **A. Mayer**, *loc. cit.* § **2**.

(16) $$\frac{\partial Y}{\partial y_h} = -\frac{\dfrac{\partial \bar{\Omega}}{\partial p_h}}{\dfrac{\partial \bar{\Omega}}{\partial p_0}} \qquad (h = 1, 2, \ldots, n) ,$$

in which the y_0, y_i', μ_s on the right-hand side are replaced with their values Y, p_i, π_s, resp., in equations (15); that replacement is suggested by an overbar on Ω. If we apply the general developments of § **1** to the extremal family that was just considered, while employing the identities (16) in so doing, then the expression E that is defined by equation (11) will assume the following form:

(17) $$E = \frac{1}{\dfrac{\partial \bar{\Omega}}{\partial p_0}} \cdot \sum_{i=0}^{n} \frac{\partial \bar{\Omega}}{\partial p_h} (y_i' - p_i) .$$

If we now assume that one of equations (1) has been solved for y_0' and the value thus-obtained is substituted in the remaining equations, by which the system (1) will assume the form:

(1*) $$\begin{aligned} &\varphi_0 = y_0' - \psi(x, y_0, y_1, \ldots, y_n, y_1', \ldots, y_n') = 0, \\ &\varphi_s (x, y_0, y_1, \ldots, y_n, y_1', \ldots, y_n') = 0 \qquad (s = 1, 2, \ldots, r), \end{aligned}$$

then, as is easy to see, we will get:

$$\frac{\partial \Omega}{\partial y_0'} = \lambda_0 ,$$

and as a result, when we set:

(18) $$\omega(x, y_0, y_1, \ldots, y_n, y_1', \ldots, y_n') = \psi + \sum_{s=1}^{r} \pi_s \varphi_s ,$$

we will get:

$$\frac{\dfrac{\partial \bar{\Omega}}{\partial p_h}}{\dfrac{\partial \bar{\Omega}}{\partial p_0}} = -\frac{\partial \omega(x, Y, y_1, \ldots, y_n, p_1, \ldots, p_n)}{\partial p_n} \qquad (h = 1, 2, \ldots, n) .$$

If we further consider the fact that the equation $\varphi_0 = 0$ is, on the one hand, true for every comparison curve C', but on the other hand, it is also true for every extremal of our family, from which the identity will follow that:

$$p_0 = \psi(x, Y, y_1, \ldots, y_n, p_1, \ldots, p_n) ,$$

then we will ultimately arrive at the formula:

(19)
$$E = \psi(x, y_0, y_1, \ldots, y_n, y_1', \ldots, y_n') - \psi(x, Y, y_1, \ldots, y_n, p_1, \ldots, p_n) - \sum_{h=1}^{n} \frac{\partial \psi(x, Y, y_1, \ldots, y_n, p_1, \ldots, p_n)}{\partial p_h}(y_h' - p_h).$$

Since the equations $\varphi_s = 0$ are true for all comparison curves C', as well as for all extremals of the family, the expression E can also be put into the equivalent form:

(19*)
$$E = \omega(x, y_0, y_1, \ldots, y_n, y_1', \ldots, y_n') - \omega(x, Y, y_1, \ldots, y_n, p_1, \ldots, p_n) - \sum_{h=1}^{n} \frac{\partial \omega(x, Y, y_1, \ldots, y_n, p_1, \ldots, p_n)}{\partial p_h}(y_h' - p_h).$$

Now, it is clear from the developments in §§ **1** and **2**, with no further discussion, that the arc 01 of the extremal C certainly yields an extremum when, on the one hand, the Jacobian determinant (14) is continually non-zero, and on the other hand, the expression E (19) possess a constant sign for all comparison curves that come under consideration. Moreover, if equations (1) are assumed to be analytic and infinite values of the y_i' are excluded, such that the only comparison curves that come under consideration are the ones for which the $|y_i'|$ all remain below a certain limit then continuity considerations will imply, in a known way, that the foregoing sufficient conditions for the extremum can also be replaced with the following one: The point on the extremal C that is "conjugate" to 0 shall lie beyond 1, and at every point of the arc 01 of the extremal, the expression E shall be continually positive or negative without vanishing for arbitrary values of the y_i', except for the case of "orderly" vanishing (i.e., $y_i' = p_i$). For the weak extremum, it is sufficient that the latter condition should be fulfilled for only those values of the y_i' that deviate from the p_i sufficiently little.

§ 3.

We now ask about the extent to which the sufficient conditions that were just given are also necessary for the presence of an extremum. Since we shall restrict ourselves to the case of $n = 1$ in what follows, we would like to show that, just as is true for the usual problem of the calculus of variations, the following conditions are necessary for the presence of a minimum (maximum):

1. The point that is conjugate to 0 shall not lie in the interior of the arc 01 of the extremal.

2. One must have $E \geq 0$ ($E \leq 0$) at all points of the arc.

Since we have taken $n = 1$, only two functions y_0, y_1 are present here, which we will denote by u and y, resp., and they satisfy the equation:

$$u' = \psi(x, u, y, y') \quad . \tag{1**}$$

The set of all extremals that go through the point 0 and correspond to the initial value u_0 define a one-parameter family, and we will have:

$$u = Y(x, y) , \qquad y' = p(x, y) \tag{15*}$$

for every extremal of that family.

Formulas (10) and (19) of §§ **1** and **2** assume the form:

$$\Delta u_1 = \int_{C'} E \, dx , \tag{10*}$$

$$E = \psi(x, u, y, y') - \psi(x, Y, y, p) - \frac{\partial \psi(x, Y, y, p)}{\partial p}(y' - p) \quad , \tag{19**}$$

in which u_1 denotes the final value of the function u, and the Jacobian determinant (14) will be equal to the derivative of y with respect to the parameter a of the extremal family, which immediately explains the fact that the point that is conjugate to 0 is the contact point of the extremal C with the envelope of the family of extremals.

The proof of the necessity of the first of the conditions that were given above will be achieved when one shows that one will arrive at the same final value u_1 at the point 1 when one employs a path that consists of the arc 02 (Fig. 1) of any extremal of the aforementioned family and the path 21 of the envelope, instead of the extremal C. The validity of that assertion will be clear with no further analysis when one imagines that one can set:

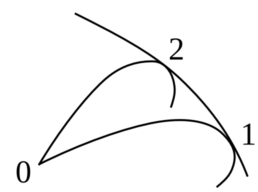

Figure 1.

$$u = Y(x, y)$$

on the arc 21, because under that assumption, on the one hand, the function u will take the same value at the point 2 that is does on the extremal 02, but on the other hand, equation (1^{**}) will be satisfied. Namely, the replacement of:

$$u = Y(x, y)$$

in that equation will yield (*):

$$\frac{\partial Y}{\partial x} + \frac{\partial Y}{\partial y} p = \psi(x, Y, y, p) , \tag{20}$$

and one will see with no further analysis that (20) will be an identity when one writes down equation (1^{**}) for the extremal family that was considered above.

We now turn to a consideration of the conditions that were given at the beginning of this section, and we would like to restrict ourselves to the case of the minimum. (The case of the

(*) Obviously, the second of equations (15^*) will be true on the envelope.

maximum is resolved analogously.) The usual proof of the necessity of that condition is not immediately applicable here, since the extremal family that the foregoing developments are based upon depends essentially upon the initial value of the function *u*. However, if we assume that not only the arc 01 of the extremal, but also every segment 03 of that arc, will yield a minimum for the value of the function *u* at its endpoint 3 then it will obviously suffice to carry out the proof for the endpoint of the arc 01, and that can proceed as usual. Namely, let $E < 0$ at the point 1 for any value of y', that is, for any line element at the point 1. We then consider a curve that goes through the point 1 and whose tangent at that point coincides with the line of the line element. Let 2 be a point on that curve (Fig. 2) that is at a sufficiently-small distance from the point 1 and whose abscissa is less than the abscissa x_1 of 1. If we connect that point with the point 0 by an extremal of the family that was considered above and employ the line-path 021 as a comparison curve then $E = 0$ on 02 and $E <$ on 21, as long as the point 2 is chosen to be sufficiently close to the point 1. We accordingly get $\Delta u_1 < 0$ from formula (10^*), which is contrary to the assumption that the extremal 01 should give a minimum.

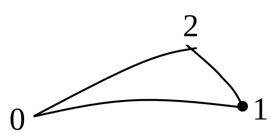

Figure 2.

All that remains now is to justify the assumption that we just made about all internal points of the extremal arc 01. That will be accomplished as soon as the proof is provided that every internal point 3 of the arc 01 will be a "minimal point" when that is the case for the endpoint 1. We call the point 3 a *minimal point* when the extremal arc 03 yields a minimum for the value u_0 of the function *u* at the point 3. The validity of the assertion that was made is clear with no further discussion for the ordinary problem of the calculus of variations. The proof can be carried out for the case of the Mayer problem when one restricts oneself to a weak minimum in perhaps the following way:

Let 3 (Fig. 3) not be a minimal point. One can then draw a "comparison curve" *C* in an arbitrarily-small neighborhood of the extremal curve 03 such that Δu_3 proves to be negative. The absolute value of Δu_3 can then be chosen to be arbitrarily small, because *u* is determined from equation (1^{**}), and according to our assumption, the values of *y* and y' along the curve *C* deviate arbitrarily little from the values of those quantities along the extremal arc 03. We now consider any point to the right of 3 along the extremal arc 01. In order to introduce no new notations, we assume that this point coincides with the point 1 in Fig. 3. If we introduce a comparison curve that takes the form of the line-path that consists of the curve *C* and the extremal arc 31 then we can calculate the value of the function *u* that is associated with the point 1 in the following way: Set *y* and y' equal to their values along the extremal 01 in equation (1^{**}) and integrate the differential equation that is obtained in that way for a prescribed value $\bar{u}_3$ of the function *u* at the point 3. We will ultimately get:

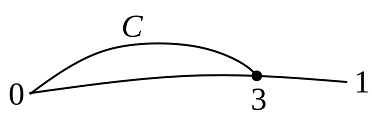

Figure 3.

$$u = \varphi(x, \bar{u}_3) , \tag{21}$$

and the value of *u* at the point 1 is equal to $\varphi(x_1, \bar{u}_3)$. If the curve *C* coincides with the extremal arc 03 then $\bar{u}_3 = u_3$ and $\varphi(x_1, u_3) = u_1$. From our assumption, $\varphi(x_3, u_3)$ equals u_3, so the derivative:

$$\frac{\partial \varphi(x, u_3)}{\partial u_3}$$

will equal 1 at the point 3. When the point 1 is chosen to be sufficiently close to the point 3, we can see with no further analysis that from continuity considerations, that same derivative will be positive at that point in any case, and as a result that $\bar{u}_1 = \varphi(x_1, \bar{u}_3)$ will increase and decrease along with $\bar{u}_3$ in the neighborhood of $\bar{u}_3 = u_3$. Moreover, since $\Delta u_3 < 0$ for the comparison curve *C*, we will also have $\Delta u_1 < 0$, and as a result, the point 1 cannot be a minimal point. We have thus proved that all points of a certain region to the right of a point that is not a minimal point will likewise not be minimal points. When we start from the point 3, such that any point of the aforementioned region is employed and the same construction is applied, etc., we will always get closer and closer to the endpoint of the extremal arc 01. Now, two cases are conceivable *a priori*: Either we ultimately attain the endpoint of the arc, or we do not go beyond a certain limiting point in the interior of the arc. In the first case, the endpoint 1 would not be a minimal point, which would be contrary to our assumption, and that would prove the impossibility of the existence of a point in the interior of the arc 01 that is not a minimal point. In the second case, the aforementioned limiting point would certainly be a minimal point, while none of the points of a certain neighborhood to the left of that point would be minimal points. However, such a distribution of points is impossible, as will be shown by the following: In fact, let 1 (Fig. 3) be a minimal point and let 3 be a point that is not a minimal point, but is chosen to be arbitrarily close to 1. Now consider an arbitrary curve *C* that runs sufficiently close to the extremal and employ a comparison curve that takes the form of the line-path that consists of *C* and the extremal arc 31. From our assumption, Δu_1 will certainly prove to be positive for that comparison curve. In order to calculate $\Delta u_3 = \bar{u}_3 - u_3$, we can proceed as we did above in order to calculate $\bar{u}_1$ from $\bar{u}_3$, and in that way, we will arrive at the result that Δu_3 is positive for *every* comparison *C* at the same time as Δu_1, which contradicts our assumption that 3 is not a minimal point.

The desired proof is complete with that, and the question that was raised at the beginning of the section has been resolved, at least for the case of a weak extremum.

Moscow, 22 May 1905.