"Welche Rolle spielt die Dreidimensionalität des Raumes in den Grundgesetzen der Physik?" Ann. Phys. (Leipzig) 336 (1920), 440-446.

What role does the three-dimensionality of space play in the fundamental laws of physics? ([1])

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Why does our space have exactly three dimensions? ([1]), or in other words: "What *singular* ([2]) phenomena distinguish the physics of R_3 from the ones in the other R_n?" When posed in that way, the questions are perhaps meaningless, but in any event they need to be critiqued. What "is" space? "Is" it three-dimensional? Finally, there is the question of "why," as well as what must one understand the concept of "the" physics of R_4 or R_n to mean?

I will not attempt to give those questions a less-objectionable form. If I can only succeed in discovering more and more singular properties of R_3 then it will ultimately become intrinsically clear what "reasonable" question will lead to the answer that was found already.

§ 1. – Gravity and planetary motion.

As far as the motion of a planet around a central body is concerned, one can establish that a characteristic difference between R_3 and R_2, on the one hand, and all higher R_n, on the other, consists of the stability of the circular orbit. Whereas the motion in R_3 remains finite under a perturbation when the energy is not too big (and that is the case in R_2 even for arbitrary finite amounts of energy), in R_4, R_5, R_6, etc., *it is natural that the circular orbits will also still be possible, but under any ever-so-small perturbation, the planet will follow a spiral that goes into the central body or out to infinity.*

We let the law of attraction in R_n take the form $\kappa \frac{M\,m}{r^{n-1}}$. For $n > 2$, that will belong to a potential energy:

([1]) Abstracted from the treatise of the same name in Versl. d. Akad. van Wetensch. te Amsterdam **26** (1910), pp. 105. (Session on 26. V. 1917) = Proceedings **20**, pp. 200. – The interesting remark that **H. Weyl** made briefly about the four-dimensionality of the spacetime manifold ["Gravitation und Elektrizität," Sitz. preuss. Akad. **26** (1918), pp. 474, top. "Neue Erweitung der Relativitätstheorie," Ann. Phys. **59** (1919), pp. 133, middle.] prompted me to summarize my own remarks here once more, although I an aware that they are very elementary and when taken individually, they are mostly well-known. However, I still hope that perhaps others will be inspired to contribute richer and better material to this fascinating question.

([2]) Cf., on this, "Concluding remark 1."

(1) $$V(r) = -\kappa \frac{M\,m}{(n-2)\,r^{n-2}}\,.$$

Naturally, that corresponds to the assumptions that:

a) The force points towards the center and is a function of only *r*.

b) **Gauss**'s theorem on the flux of the lines of force shall also be true for gravity in R_n .

One will then get the equations of motion:

$$m\frac{d^2 x_h}{dt^2} = -\kappa \frac{M\,m}{r^{n-1}}\frac{x_h}{r} = -\frac{\partial V}{\partial x_h} \qquad (h = 1, \ldots, n).$$

The path is planar. Introducing the polar coordinates r, φ will give the laws of energy and areas:

$$\tfrac{1}{2}m(\dot{r}^2 + r^2\dot{\varphi}^2) + V(r) = E\,, \qquad m\,r^2\,\dot{\varphi} = \Theta\,.$$

Eliminating $\dot{\varphi}$ gives:

(2) $$\dot{r} = \sqrt{\frac{2E}{m} - \frac{2V}{m} - \frac{\Theta^2}{m^2 r^2}} = \frac{1}{r}\sqrt{Ar^2 + B\,r^{4-n} - C^2}\,,$$

in which A, B, C are constants, the first and last of which depend upon the initial data of the motion, to the extent that they include the total energy E and the constant Θ in the law of areas. In order for r to oscillate back and forth between two positive values during the motion, it is necessary that $\dot{r}$ must be real and alternately assume positive and negative values, so the quantity under the radical must be positive between two values of r for which it is zero. The geometric discussion of the case in which that happens will be easy when one draws the curves $y = Ar^2$ and $y = B\,r^{4-n}$, sketches out the total curve, and checks whether it can be cut out from $y = c^2$ ([1]). The case of n = 2 can also be treated analogously, but (1) must be replaced with:

(1′) $$V = \kappa\,M\,m\,\ln r\,,$$

so (2) is replaced with:

(2′) $$\dot{r} = \frac{1}{r}\sqrt{\alpha\,r^2 - \beta\,r^2\ln r - \gamma^2}\,,$$

in which:

$$\alpha = \frac{2\,E}{m}\,, \qquad \beta = 2k\,M\,, \qquad \gamma^2 = \frac{\Theta^2}{m^2}\,.$$

([1]) Cf., the figures in Konikl. Akad. van Wetensch. Amsterdam, *loc. cit.*, "Aanhangsel I."

The result of the discussion is:

n	Circular orbits	Motions between two positive values of r	Motion to infinity
4, 5, …	unstable	impossible!	possible
3	stable	possible [closed orbit]	possible
2	stable	possible [open orbit]	possible!

Remarks:

1. Let us take this opportunity to recall the theorem of **J. Bertrand** (1873) ([1]): The orbits of a material point that are described under the influence of a central force that is a function of only distance will be closed only when the force is either proportional to the distance or inversely proportional to the square of the distance.

2. It is remarkable that the motions of planets correspond to elliptic ones that are also closed in *non-Euclidian* R_3 as long as one simultaneously adapts only the equations of mechanics and the law of attraction to that space ([2]).

3. Naturally, one can also ask what the **Bohr** *atomic model* will become in R_n for $n \neq 3$. If one modifies the law of electrical attraction in the same way that one did for gravity and preserves:

a) The quantization of angular momenta,

b) The Ansatz $\nu = (E' - E) / h$

then one will soon see how sharply *the Bohr model is suited to* R_3 *precisely:* For $n = 4$, one will get a very bad degeneracy ([3]). For $n = 2$, one gets a spectral series with an accumulation point at infinity, since infinite energy is indeed necessary in order to move the electron to infinity in R_2 . Naturally, for $n > 4$, one needs at most the circular orbits, above all. In contrast to R_3, the orbits of increasing quantum numbers will always lie denser and denser around the nucleus, and as a result, the spectral series will again have its accumulation point at infinity ([4]).

([1]) **J. Bertrand**, C. R. Acad. Sci. **77** (1873), pp. 846.
([2]) **H. Liebmann**, *Nichteukl. Geometrie*, (Sammlung Schubert, 2nd ed., 1912, pp. 207)
([3]) *Loc. cit.*, "Aanhangsel II."
([4]) *Loc. cit.*, *ibidem*.

§ 2. – Duality between electric and magnetic fields in R_3 .

In R_n, the electric field is determined by n components, while the magnetic is characterized by $n(n-1)/2$ of them. It is only for $n = 3$ that those two numbers coincide, and that is why the far-reaching duality exists between the two fields.

In fact, let $x_1, \ldots, x_n$, and $x_0 = i\,c\,t$ be the $n + 1$ coordinates of the spacetime manifold, and let $\varphi_0, \varphi_1, \ldots, \varphi_n$ be the components of the retarded potential (which corresponds to the four-potential in the theory of relativity that corresponds to R_3). The $n\,(n - 1) / 2$ components of the rotation:

$$\frac{\partial \varphi_h}{\partial x_k} - \frac{\partial \varphi_k}{\partial x_h} \qquad (h, k = 1, \ldots, n)$$

correspond to the *magnetic* field, and the n components:

$$\frac{\partial \varphi_h}{\partial x_0} - \frac{\partial \varphi_0}{\partial x_h} \qquad (h = 1, \ldots, n)$$

correspond to the *electric* field.

It is known that the force-rotational moment is analogous to the translational-angular velocity.

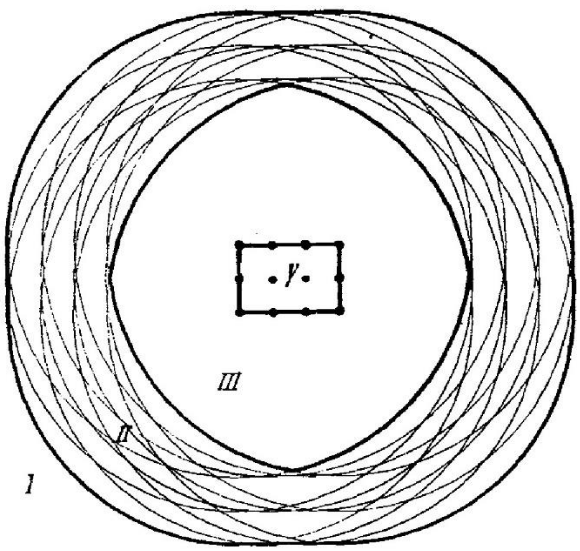

Fig. 1.

§ 2. – The propagation of waves in R_n .

The solutions of the differential equation:

$$\frac{1}{c^2}\frac{\partial^2\varphi}{\partial t^2} - \sum_{h=1}^{n}\frac{\partial^2\varphi}{\partial x_h^2} = 0$$

have the following property in x for $n = 3$: If one has $\varphi = 0$ and $\partial\varphi / \partial t = 0$ everywhere outside of a small region γ at the moment $t = 0$ then one will still have $\varphi = 0$ and $\partial\varphi / \partial t = 0$ everywhere outside of a thin shell between two surfaces (cf., Figure) at every later moment t (but only when it is not taken to be too small), and in the event that γ is sufficiently small, that shell will approach two concentric spheres around γ.

It is known ([1]) that things are different in R_2 : For the propagation of waves on a membrane, one has not only a perturbation in the ring that corresponds to II, but also a (weaker) perturbation in the entire interior region III.

All R_{2n+1} behave like R_3 in that regard, while all R_{2n} behave analogously to R_2. (One therefore cannot apply Huygens's principle in the latter case.) ([2 and 3]) However, among the R_{2n+1}, R_3 is distinguished by a peculiarity that will become obvious when one calculates the retarded potential ([4]), i.e., the integral of the differential equation:

$$\frac{1}{c^2}\frac{\partial^2\varphi}{\partial t^2} - \sum_{h=1}^{n}\frac{\partial^2\varphi}{\partial x_h^2} = \rho\,.$$

For R_3, one gets:

$$\varphi = \frac{1}{c_3}\int\int_{-\infty}^{+\infty}\int d\omega\frac{[\rho]}{r}\,.$$

For R_5 :

$$\varphi = \frac{1}{3c_5}\int\int\int_{-\infty}^{+\infty}\int\int d\omega\left\{\frac{[\rho]}{r^3} + \frac{\frac{1}{c}\left[\frac{\partial\rho}{\partial t}\right]}{r^2}\right\}.$$

For R_7 :

([1]) Cf., e.g., **Rayleigh**, *Theory of Sound*, Chap. XIV, § 275.

([2]) Cf., **Duhem**, *Hydrodynamique*, T. I, Paris, Hermann. **Volterra**, Acta Math. (1894). **Hadamard**, Bull. soc. Franç. de phys. (1906). (Other literature is found there, including the older work of **Hadamard**.) Acta math. **31** (1908), pp. 333. *Leç. sur la prop. des Ondes*, Paris, Hermann, 1903, Chap. VII, § **3**.

([3]) That fact is related to the following computational fact: The formula for the volume of a ball in R_p includes the number π *to the same* power n for $p = 2n$ and $p = 2n + 1$. The analytical relationship between the two facts will become clear when one integrates the wave equation, e.g., with the help of Fourier integrals. One ultimately comes to an integral over the $(p - 1)^{\text{th}}$ power of an expression in a square root. The integrand will then be rational when $p = 2n + 1$ and irrational when $p = 2n$. The same thing happens when one calculates the volume of the sphere.

([4]) Cf., *loc. cit.*, Aanhangsel IV, for a simple derivation of the solutions for R_{2n+1} .

$$\varphi = \frac{1}{5c_7} \int_{-\infty}^{+\infty}\cdots\int d\omega \left\{ \frac{[\rho]}{r^5} + \frac{1}{c}\frac{\left[\frac{\partial \rho}{\partial t}\right]}{r^4} + \frac{1}{c^2}\frac{\left[\frac{\partial^2 \rho}{\partial t^2}\right]}{r^3} \right\} .$$

In that:

$$c_3 = 4\,\pi\,, \qquad c_5 = \frac{8}{3}\pi^2\,, \qquad c_7 = \frac{11}{15}\pi^3$$

are the areas of the unit spheres in R_3, R_5, R_7, resp., and the symbols:

$$[\rho], \quad \left[\frac{\partial \rho}{\partial t}\right], \quad \left[\frac{\partial^2 \rho}{\partial t^2}\right]$$

mean that one must take the values at time $t - v / c$ (the "retarded" value). One sees that: *In contrast to* R_3, *the retarded potentials in* R_5 *and* R_7, *etc., will be determined by not only* ρ, *but also by its differential quotients with respect to time*. Therefore, one must note that for large values of r, so for actual radiation, one will arrive at just the highest differential quotient of ρ, since it is divided by the smallest power of r. An "electron" with a sharply-bounded charge would then radiate fields with higher singularities in R_5, R_7, … as a result of its motion!

Concluding remarks:

1. Here, we have considered only those relationships for which the R_3 will assume a *singular* position in comparison to the other R_n . Above and beyond that, it is naturally quite often instructive to clarify how it is that the triple of dimensions can affect so many things in physics without the transition from R_n to R_3 exhibiting any definitive singular behavior: For example, one has to replace the 4 in the radiation law of **Stefan-Boltzmann** and the 3 in the displacement law of **W. Wien**: $\rho = v^3\ f(v / T)$ with $(n + 1)$ and n, resp., in R_n . The ratio of specific heats of monoatomic gases $c_p / c_v = 1\frac{2}{3}$ must be replaced with $1 + 2/n$, etc.

2. One might be compelled to ask *tentatively* about whether connection to the three-dimensionality of space (or the four-dimensionality of spacetime) exists for any universal constant that appears in physics and is independent of the units of measurement. One might ponder, e.g., the role of the number 8 (and 10?) in the periodic table of the elements [cf., **Born**'s "cubic atomic model" ([1])] and the exponent 2 in the **Balmer** formula.

3. However, yet another whole number intervenes in any triangle in physics: viz., *the exponent "two" in the Pythagorean theorem*. In other words: The homogeneous *quadratic* metric:

([1]) Verh. d. Deutsch. phys. Ges. **20** (1918), pp. 230.

$$ds^2 = \sum_h \sum_k g_{hk}\, dx_h\, dx_k$$

has an overarching significance compared to all other homogeneous forms. Can one not say anything further about the fact that one uses *two*?

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