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On a natural extension of the foundations of the general theory of relativity

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As is known, H. WEYL sought to extend the general theory of relativity by adding a further invariance condition, and in that way he arrived at a theory that has already attracted great interest due to its bold and consistent mathematical structure. That theory is based upon two ideas essentially:

a) In general relativity, the *ratios* of the gravitational potential components $g_{\mu\nu}$ have significantly more fundamental physical meaning that the components $g_{\mu\nu}$ themselves. The totality of all world-directions that point from a world-point and from which light signals start – viz., the light-cone – seems to be given immediately with the space-time continuum. However, that light-cone is determined by the equation:

$$ds^2 = g_{\mu\nu}\, dx_\mu\, dx_\nu = 0,$$

in which only the ratios of the $g_{\mu\nu}$ occur. Above all, only the ratios of the $g_{\mu\nu}$ enter into the electromagnetic equations of the vacuum. By contrast, the quantity ds, which is first determined by the $g_{\mu\nu}$ themselves, does not express merely a property of the space-time continuum, since one requires a material entity (i.e., a clock) in order to measure those quantities. For that reason, one must ask the question: Can the theory of relativity remain unchanged on the basis of the assumption that it is not the quantity ds itself that has an invariant meaning, but only the *equation* $ds^2 = 0$?

b) WEYL's second notion relates to a method of generalizing the RIEMANNian metric, as well as to the physical meaning of the new quantities ϕ_ν that appear in that generalization. The idea can be sketched out in perhaps this way: A metric assumes the translation of line segments (i.e., yardsticks). RIEMANNian geometry further assumes that the behavior (i.e., length) of a yardstick at one location is independent of the manner by which one arrived at that location. That then contains the two assumptions:

I. The existence of translatable yardsticks.
II. The independence of lengths from the path of translation.

WEYL's generalization of the RIEMANNian metric keeps I, but drops II. He allows the measured length of a yardstick to depend upon an integral:

$$\int \phi_\nu \, dx_\nu$$

that is extended along the path of displacement and generally depends upon that path, and in which the ϕ_ν are spatial functions that accordingly determine the metric. In the physical interpretation of the theory, the ϕ_ν will then be identified with the electromagnetic potentials.

With all due admiration for the unity and beauty of WEYL's line of reasoning, it still seems to me that it does not stand up to the test of physical reality. We know of nothing in nature that would be useful for the purpose of measurement when its relative extension depends upon its history. The straightest line that WEYL introduced, as well as the electrical potentials that appear in it, along with the remaining equations of WEYL's theory, do not appear to possess any direct physical interpretation either.

On the other hand, it seems to me that WEYL's idea that was proposed in a) will become more pleasing and natural if one cannot also know *a priori* whether it might lead to a useful physical theory. Given that state of affairs, one can ask whether or not one will arrive at a clear theory when one drops not only assumption II, with WEYL, but also assumption I, from the outset. Now, in what follows, it shall be shown that one will undoubtedly arrive at a theory in which one starts from merely the invariant meaning of the equation:

$$ds^2 = g_{\mu\nu} \, dx_\mu \, dx_\nu = 0$$

without making use of the concept of distance ds or – physically speaking – the concepts of yardstick and chronograph.

In my endeavors to exhibit such a theory, I was effectively supported by my colleague WIRTINGER in Vienna. I asked him whether there was a generalization of the equation of the geodetic line in which only the ratios of the $g_{\mu\nu}$ play a role. He answered me in the following way:

We understand a "RIEMANN tensor" or "RIEMANN invariant" to mean a tensor (invariant, resp.) under arbitrary point transformations whose invariance character is true under the assumption of the invariance of $ds^2 = g_{\mu\nu} \, dx_\mu \, dx_\nu$. We further understand a "WEYL tensor ("WEYL invariant", resp.) of weight n to mean a RIEMANN tensor (invariant, resp.) with the following additional property: The value of the tensor component (invariant, resp.) will be multiplied by λ^n when one replaces the $g_{\mu\nu}$ with $\lambda \, g_{\mu\nu}$, in which λ is an arbitrary function of the coordinates. That condition can be expressed symbolically by the equation:

$$T(\lambda \, g) = \lambda \, T(g).$$

Now, if J is WEYL invariant of weight – 1 that depends upon only the $g_{\mu\nu}$ and their derivatives then:

$$d\sigma^2 = J \, g_{\mu\nu} \, dx_\mu \, dx_\nu \tag{1}$$

will be an invariant of weight 0; i.e., an invariant that depends upon only the ratios of the $g_{\mu\nu}$. The desired generalization of the geodetic line will be then given by the equation:

$$\delta\left\{\int d\sigma\right\} = 0. \tag{2}$$

Naturally, the possibility of solving this equation assumes the existence of a WEYL invariant of the stated kind. WEYL's investigations pointed the way to such a thing. Namely, he showed that the tensor:

$$H_{iklm} = R_{iklm} - \frac{1}{d-2}(g_{il}\,R_{km} + g_{km}\,R_{il} - g_{im}\,R_{kl} - g_{kl}\,R_{im}) + (g_{il}\,g_{km} - g_{im}\,g_{kl})\,R \tag{3}$$

is a WEYL tensor of weight 1. In this, R_{iklm} is the RIEMANN curvature tensor, $R_{km} = g^{il}R_{iklm}$ is the second-rank tensor that emerges from contracting the latter once, R is the scalar that arises from one more contraction, and d is the dimension of the space. That immediately implies that:

$$H = H_{iklm}\,H^{iklm} \tag{4}$$

is a WEYL scalar of weight – 2. One then has that:

$$J = \sqrt{H} \tag{5}$$

is a WEYL invariant of weight – 1. This result, in conjunction with (1) and (2), implies a generalization of the geodetic line according to the method that WIRTINGER gave. Naturally, the question of whether J is the only WEYL invariant of weight – 1 in which no derivatives of the $g_{\mu\nu}$ higher than the second are present has great importance for assessing the meaning of that result.

On the grounds of the developments up to now, it is now easy to assign a WEYL tensor to each RIEMANN tensor, and in that way to exhibit laws of nature in the form of differential equations that no longer depend upon the ratios of the $g_{\mu\nu}$. If we set:

$$g'_{\mu\nu} = J\,g_{\mu\nu}$$

then:

$$d\sigma^2 = g'_{\mu\nu}\,dx_\mu\,dx_\nu$$

will be an invariant that now depends upon only the ratios of the $g_{\mu\nu}$. All RIEMANN tensors that are constructed from $d\sigma$ as fundamental invariants in the usual way will be WEYL tensors of weight 0 as functions of the $g_{\mu\nu}$ and their derivatives. Symbolically, we express that fact as follows: If $T(g)$ is a RIEMANN tensor that can depend upon not only the $g_{\mu\nu}$ and their derivatives, but also upon other quantities (say, the components $\phi_{\mu\nu}$ of the electromagnetic field) then $T(g')$ will be a WEYL tensor of weight 0 when it is considered to be a function of the $g_{\mu\nu}$ and their derivatives. Every law of nature in general relativity that has the form $T(g) = 0$ will then correspond to a law $T(g') = 0$ that involves only the ratios of the $g_{\mu\nu}$. Since a factor remains arbitrary in the $g_{\mu\nu}$, it will be possible to choose it in such a way that one has:

$$J = J_0 , \tag{6}$$

in which J_0 means a constant. $g'_{\mu\nu}$ will then be equal to $g_{\mu\nu}$, up to a constant factor, and the law of nature will once more assume the form:

$$T(g) = 0$$

in the new theory. The whole innovation in comparison to the original form of general relativity then consists of the addition of the differential equation (6) that the $g_{\mu\nu}$ must satisfy.

Here, we have only proposed a logical possibility whose publication might or might not be of use to physics. Whether one or the other case proves to be true must come from further investigations, just like the answer to the question of whether other invariants besides the WEYL invariant $J = \sqrt{K}$ should come under consideration.

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