

The speed of light and the statics of the gravitational field

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In a paper that appeared last year ⁽¹⁾, starting from the hypothesis that the gravitational field and the state of acceleration of a coordinate system are physically equivalent, I inferred some consequences that are in very good agreement with the results of the theory of relativity (viz., the theory of relativity of uniform motion). However, it was shown that the validity of one of the fundamental laws of that theory, namely, the law of the constancy of the speed of light, can claim to be valid only for space-time domains of constant gravitational potential. Despite the fact that this result excluded the general applicability of the Lorentz transformation, it should not deter us from pursuing consequences of choosing that path. At the very least, my opinion in regard to the hypothesis that the “acceleration field” is a special case of the gravitational field seems so likely to be true, especially when one recalls the consequences in regard to the gravitational mass of the energy content that were inferred before in the latter paper, that a more precise analysis of the consequences of that equivalence hypothesis would seem to be in order.

Since then, **Abraham** has presented a theory of gravitation ⁽²⁾ that includes the consequences of my first paper as special cases. However, in what follows we will see that **Abraham**’s system of equations cannot be brought into agreement with the equivalence hypothesis and that its concept of space and time cannot be maintained, even from a purely-mathematical standpoint.

1. – Space and time in the acceleration field.

The reference system K (coordinates x, y, z) is found to be in a state of uniform acceleration in the direction of its x -coordinate. Let that acceleration be a uniform in the **Born** sense, i.e., let the acceleration of its origin relative to a system that is unaccelerated, relative to which the points of K possess no velocity at all (an infinitely-small velocity, resp.), be a constant quantity. According to the equivalence hypothesis, such a system K is equivalent to a system at rest in which one finds a mass-free static gravitational field ⁽³⁾ of a certain kind. The spatial measurement of K happens by means of yardsticks that possess equal lengths (when compared to each other in the rest state at the aforementioned location for K). The laws of geometry shall be valid for lengths that are measured in that way, so they will also be valid for the relationships between the coordinates $x, y,$

⁽¹⁾ A. Einstein, Ann. Phys. (Leipzig) **4** (1911), pp. 35.

⁽²⁾ M. Abraham, Phys. Zeit. **13**, no. 1 (1912).

⁽³⁾ One must imagine that the masses that produce that field are at infinity.

z and other lengths. It is not obvious that this convention is legitimate since it includes physical assumptions that might possibly prove to be incorrect. For example, it does not seem likely that it is true in a uniformly-rotating system, in which the Lorentz contraction would imply that the ratio of the circumference to the diameter would have to be different from π when one applies our definition of length. The yardstick, as well as the coordinate axes, are imagined to be rigid bodies. That is permissible, despite the fact that according to the theory of relativity, rigid bodies cannot exist in reality. One can then think of the rigid measuring devices as being replaced with a large number of small non-rigid bodies that are arranged with respect to each other in such a way that they exert no forces of repulsion on each other, which will keep each of them in place. We imagine that the time t in the system K is measured by clocks that have such a nature and such a fixed arrangement at spatial points of the system K that the time interval (as measured by those clocks) that a light ray needs in order to arrive at a point B in the system K from a point A does not depend on the time-point of the emission of the light ray at A . It will be shown further that one can make a consistent definition of simultaneity such that all light rays that pass through a point A in K will possess the same speed of propagation independently of the direction at A relative to the *readings* on the clocks that one gets by continuation.

We now imagine that the reference system $K(x, y, z, t)$ is considered from an unaccelerated reference system (of constant gravitational potential) $\Sigma(\xi, \eta, \zeta, \tau)$. We assume that the x -axis continually calls along the ξ -axis and the y -axis continually lies parallel to the η -axis, while the z -axis continually lies parallel to the ζ -axis. That assumption is possible under the assumption that the state of *acceleration* does not affect the form of K relative to Σ . We shall use that physical assumption as a basis. It implies that for arbitrary τ , we must have:

$$(1) \quad \begin{cases} \eta = y, \\ \zeta = z, \end{cases}$$

such that all we still need to look for is the relationship that exists between ξ and τ , on the one hand, and between x and t , on the other. Both reference systems might coincide at time $\tau = 0$. In any event, the desired equations of the substitution must have the form:

$$(2) \quad \begin{cases} \xi = \lambda + \alpha t^2 + \dots \\ \tau = \beta + \gamma t^2 + \delta t^2 + \dots \end{cases}$$

The coefficients of these series, which are valid for sufficiently-small positive and negative values of τ , are regarded as unknown functions of x , for the time being. When we restrict ourselves to the terms that were written down, we will get:

$$(3) \quad \begin{cases} d\xi = (\lambda' + \alpha' t^2) dx + 2\alpha t dt, \\ d\tau = (\beta' + \gamma' t^2 + \delta' t^2) dx + (\gamma + 2\delta t) dt \end{cases}$$

upon differentiation.

In the system Σ , we think of time as being measured in such a way that the speed of light will be equal to 1. We can then write the equation of a shell that propagates with the speed of light from an arbitrary space-time point, when we restrict ourselves to an infinitely-small neighborhood of that space-time point, in the form:

$$d\xi^2 + d\eta^2 + d\zeta^2 - d\tau^2 = 0 .$$

The same shell must have the equation:

$$dx^2 + dy^2 + dz^2 - c^2 dt^2 = 0$$

in the system K . The equations of the substitution (2) must be such that those two equations are equivalent. Due to (1), that requires the identity:

$$(4) \quad d\xi^2 - d\tau^2 = dx^2 - c^2 dt^2 .$$

If one sets the expressions for dx and dt in the left-hand side of this equation equal to unity and sets the coefficients of dx^2 , dt^2 , and $dx dt$ equal to each other on the left-hand and right-hand sides then one will get the equations:

$$\begin{aligned} 1 &= (\lambda' + \alpha' t^2)^2 - (\beta' + \gamma' t^2 + \delta' t^2)^2, \\ -c^2 &= 4\alpha^2 t^2 - (\gamma + 2\delta t)^2, \\ 0 &= (\lambda' + \alpha' t^2) \cdot 2\alpha t - (\beta' + \gamma' t^2 + \delta' t^2)(\gamma + 2\delta t) . \end{aligned}$$

Those equations are valid at t identically, up to higher powers of t , in such a way that the terms that were omitted from (2) can have no influence, so the first equation is valid up to the second power of t , and the second and third ones are valid up to the first power of t . That will imply the equations:

$$\begin{aligned} 1 &= \lambda'^2 \beta'^2, & 0 &= \beta' \gamma', & 2\lambda \alpha' - \gamma'^2 - 2\beta' \delta' &= 0, \\ -c^2 &= -\gamma^2, & 0 &= \gamma \delta, \\ 0 &= \beta' \gamma, & 0 &= 2\alpha \lambda' - 2\beta' \delta - \gamma \gamma' . \end{aligned}$$

Since γ cannot vanish, it will follow from the first equation in the third row that $\beta' = 0$. β is a constant then that we can set equal to zero by a suitable choice of time origin. Furthermore, the coefficient γ must be positive, so from the first equation in the second row:

$$\gamma = c .$$

From the second equation in the second row:

$$\delta = 0 .$$

Since β' vanishes and one can assume that x increases with ξ , it will follow from the first equation in the first row that:

$$\lambda' = 1 ,$$

so if one is to have $x = 0$, $\xi = 0$ for $t = 0$ then:

$$\lambda = x .$$

Finally, when one employs the relations that were found above, the third equation in the first row and the second equation in the third row will imply the differential equations:

$$\begin{aligned} 2\alpha' - c'^2 &= 0 , \\ 2\alpha - c c' &= 0 . \end{aligned}$$

When we denote integration constants by c_0 and a , it will follow from the latter equations that:

$$\begin{aligned} c &= c_0 + a x , \\ 2\alpha &= a (c_0 + a x) = a c . \end{aligned}$$

The desired substitution is ascertained by that for sufficiently-small values of t . When one neglects third and higher powers of t , one will have the equations:

$$(4) \quad \left\{ \begin{array}{l} \xi = x + \frac{ac}{2} t^2 , \\ \eta = y , \\ \zeta = z , \\ \tau = ct , \end{array} \right.$$

by which, the speed of light c in the system K , which can depend upon only x , but not t , will be given by the relation that was just derived as:

$$(5) \quad c = c_0 + a x .$$

The constant c_0 depends upon the rate at which the clock that one measures time with ticks at the origin of K . One gets the meaning of the constant a in the following way: When one recalls (5), the first and fourth of equations (4) yield the equation of motion:

$$\xi = \frac{a}{2c_0} \tau^2$$

for the origin ($x = 0$) of K . a / c_0 is then the acceleration of the origin of K relative to Σ when measured in time units in which the speed of light is equal to 1.

**§ 2. – Differential equation of the static gravitational field.
Equation of motion of a material point in a static gravitational field.**

It already emerges from the previous paper that a relationship exists between c and the gravitational potential of a static gravitational field, or in other words, that the field is determined by c . In those gravitational fields that correspond to the acceleration field that was considered in § 1, from (5) and the equivalence principle, the equation:

$$(5.a) \quad \Delta c = \frac{\partial^2 c}{\partial x^2} + \frac{\partial^2 c}{\partial y^2} + \frac{\partial^2 c}{\partial z^2} = 0$$

is fulfilled, and that suggests that we have assumed that this equation is valid in every mass-free static gravitational field ⁽¹⁾. In any event, that equation is the simplest one that is compatible with (5).

It is easy to exhibit the presumably-valid equation that would correspond to **Poisson's**. Namely, it follows immediately from the meaning of c that c is determined only up to a constant factor that depends upon how one measures t at the origin of K with a suitable clock. The equation that corresponds to **Poisson's** must then be homogeneous in c . The simplest equation of that kind is the linear equation:

$$(5.b) \quad \Delta c = k c \rho ,$$

when k is understood to mean the (universal) constant of gravitation and ρ is the density of matter. The latter must be defined such that it is already given by the mass distribution, i.e., it is independent of c for given matter in the spatial element. We can achieve that when we set the mass of a cubic centimeter of water equal to 1, which might also be found to be in a gravitational potential. ρ will then be the ratio of the mass that is found in a cubic centimeter to that unit.

We now seek to ascertain the law of motion for a material point in a static gravitational field. To that end, we shall seek the law of motion of a force-free material point that moves in the acceleration field that was considered in § 1. That law of motion in the system Σ is:

$$\begin{aligned} \xi &= A_1 \tau + B_1 , \\ \eta &= A_2 \tau + B_2 , \\ \zeta &= A_3 \tau + B_3 , \end{aligned}$$

in which the A and B are constant. By means of (4), those equations will go to the equations:

$$x = A_1 c t + B_1 - \frac{ac}{2} t^2 ,$$

⁽¹⁾ In a paper that will follow shortly, it will be shown that equations (5.a) and (5.b) cannot be correct exactly. However, in this article, they will be employed provisionally.

$$\begin{aligned}y &= A_2 c t + B_2 , \\z &= A_3 c t + B_3 ,\end{aligned}$$

which are true for sufficiently-small t . Upon repeatedly differentiating the first equation, when one sets $t = 0$ in it, one will get the two equations ⁽¹⁾:

$$\begin{aligned}\dot{x} &= A_1 c , \\ \ddot{x} &= 2A_1 \dot{c} - a c .\end{aligned}$$

When one eliminates A_1 from those two equations, it will follow that:

$$c \ddot{x} - 2 \dot{c} \dot{x} = - a c^2 ,$$

or the equation:

$$\frac{d}{dt} \left(\frac{\dot{x}}{c^2} \right) = - \frac{a}{c^2} .$$

In an analogous way, it results that the other two components satisfy the equations:

$$\begin{aligned}\frac{d}{dt} \left(\frac{\dot{y}}{c^2} \right) &= 0 , \\ \frac{d}{dt} \left(\frac{\dot{z}}{c^2} \right) &= 0 .\end{aligned}$$

Initially, those three equations are true at the instant $t = 0$. However, they are true in general, since that time-point is not distinguished from any other one by anything except for the fact that we have made it the starting point for our series development. The equations that are found in that way are the desired equations of motion of the force-free moving point in a constant acceleration field. If we consider that $a = \partial c / \partial x$ and that $(\partial c / \partial y) = (\partial c / \partial z) = 0$ then we can also write those equations in the form:

$$(6) \quad \left\{ \begin{aligned} \frac{d}{dt} \left(\frac{\dot{x}}{c^2} \right) &= - \frac{1}{c} \frac{\partial c}{\partial x} , \\ \frac{d}{dt} \left(\frac{\dot{y}}{c^2} \right) &= - \frac{1}{c} \frac{\partial c}{\partial x} , \\ \frac{d}{dt} \left(\frac{\dot{z}}{c^2} \right) &= - \frac{1}{c} \frac{\partial c}{\partial x} . \end{aligned} \right.$$

The x -axis is no longer distinguished in this form for the equations; both sides have a vector character. For that reason, we must probably also regard those equations as the equations of motion

⁽¹⁾ The terms in (2) that were dropped have no effect on the result of that double differentiation and subsequent setting of t to zero.

of a material point in a static gravitational field when the point is subject to only the influence of gravity.

The relationship of the constant k that appears in (5.b) to the gravitational constant K in the usual sense next follows from (6). Namely, in the case of a speed that is less than c , one has from (6) that:

$$\ddot{x} = -c \frac{\partial c}{\partial x} = -\frac{\partial \Phi}{\partial x},$$

such that (5.b) will go to:

$$\Delta \Phi = k c^2 \rho$$

when one neglects certain terms. One then has:

$$K = k c^2.$$

The gravitational constant is not a constant then, but only the quotient K / c^2 is constant.

If we multiply equations (6) by $\dot{x} / c^2$, $\dot{y} / c^2$, $\dot{z} / c^2$, in succession and add them then when we set:

$$q^2 = \dot{x}^2 + \dot{y}^2 + \dot{z}^2,$$

we will get:

$$\frac{d}{dt} \left(\frac{1}{2} \frac{q^2}{c^4} \right) = -\frac{\dot{c}}{c^3} = \frac{d}{dt} \left(\frac{1}{2c^2} \right)$$

or

$$\frac{d}{dt} \left[\frac{1}{c^2} \left(1 - \frac{q^2}{c^2} \right) \right] = 0,$$

or

$$(7) \quad \frac{c}{\sqrt{1 - \frac{q^2}{c^2}}} = \text{const.}$$

That equation includes the law of energy for the material point that moves in a stationary gravitational field. The left-hand side of that equation depends upon q in precisely the same way that the energy of the material point depends upon q in the usual theory of relativity. We must then regard the left-hand side of the equation as the energy E of the point, up to a factor (that depends upon only the mass-point itself). Obviously, that factor is equal to the mass m , in the sense that was established above, because that definition of mass was established independently of the gravitational potential. One then has:

$$(8) \quad E = \frac{mc}{\sqrt{1 - \frac{q^2}{c^2}}},$$

or approximately:

$$(8.a) \quad E = m c + \frac{m}{2c} q^2.$$

It next emerges from the second terms of that development that the quantity that we have deferred to as energy possesses a dimension that deviates from the more familiar one. Correspondingly, the unit of the individual energy quantities will also be different, namely, it will be c times smaller than it is in the system that is familiar to us. Furthermore, the “kinetic energy,” which generally cannot be separated from the gravitational energy using (8), taken rigorously, depends upon not only m and q , but also on c , i.e., on the gravitational potential. (8) further implies the important result that the energy of the point at rest in the gravitational field is $m c$. If we would like to preserve the relation:

$$\text{force} \cdot \text{path length} = \text{energy supplied}$$

then the force $\mathfrak{K}$ that is exerted on the material point at rest will be:

$$\mathfrak{K} = - m \text{ grad } c .$$

We would now like to derive the equations of motion for a material point in an arbitrary gravitational field for the case in which other forces act on the point besides gravity. We remark that equations (6) are not similar to the equations of motion that are true in relativistic mechanics. However, if we multiply them by the left-hand side of (7) then we will get the equations:

$$(6.a) \quad \frac{d}{dt} \left\{ \frac{\frac{\dot{x}}{c}}{\sqrt{1 - \frac{q^2}{c^2}}} \right\} = - \frac{\frac{\partial c}{\partial x}}{\sqrt{1 - \frac{q^2}{c^2}}}, \text{ etc.,}$$

which are equivalent to equations (6). Except for the factor $1 / c$ in the numerator, which is irrelevant in the ordinary theory of relativity, the left-hand side has precisely the same form that it has in the ordinary theory of relativity. For that reason, we will have to refer to the quantity in brackets as the x -component of the quantity of motion (for a point of mass 1). Furthermore, we have just shown that $-\partial c / \partial x$ must be regarded as the x -component of the force that is exerted by the gravitational field on an arbitrary moving mass-point of mass 1. The force that is exerted by the gravitational field on an arbitrary moving mass-point differ from it by only a factor that vanishes with q . The equation that was just presented then leads one to set that force $\mathfrak{K}_g$ equal to

$-\frac{\partial c / \partial x}{\sqrt{1 - q^2 / c^2}}$. The right-hand side of the equation presented will then be $\mathfrak{K}_g$. The time derivative

of the impulse will then be equal to the applied force. If another force $\mathfrak{K}$ acts on the point then one will have to add a term $\mathfrak{K} / m$ to the right-hand side of the equation, such that the equation of motion of a point of mass m will assume the form:

$$(6.b) \quad \frac{d}{dt} \left\{ \frac{m \frac{\dot{x}}{c}}{\sqrt{1 - \frac{q^2}{c^2}}} \right\} = - \frac{m \frac{\partial c}{\partial x}}{\sqrt{1 - \frac{q^2}{c^2}}} + \mathfrak{K}_x, \quad \text{etc.}$$

However, that equation is permissible only when the law of energy is fulfilled in the form:

$$\mathfrak{K} q = \dot{E}.$$

That can be accomplished in the following way:

If one writes (6.b) in the form:

$$\frac{d}{dt} \left\{ \frac{\dot{x}}{c} E \right\} + \frac{1}{c} \frac{\partial c}{\partial x} E = \mathfrak{K}_x, \quad \text{etc.}$$

and multiplies those equations by $\dot{x}/c^2, \dots$, in succession, then one will find that:

$$\frac{1}{2} \frac{q^2}{c^4} \dot{E} + \frac{1}{2} E \frac{d}{dt} \left(\frac{q^2}{c^4} \right) + E \frac{\dot{c}}{c^3} = \frac{\mathfrak{K} q}{c^2}.$$

That will imply the desired relation when one considers the fact that, from (8), one will have:

$$\frac{q^2}{c^4} = \frac{1}{c^2} - \frac{m^2}{E^2}$$

and

$$\frac{d}{dt} \left(\frac{q^2}{c^4} \right) = - \frac{\dot{c}}{c^3} + \frac{m^2 E}{E^3}.$$

The relationship between force and the law of energy-impulse then remains preserved.

§ 3. – Remarks on the physical meaning of the static gravitational potential.

If we measure the speed of light in a space with an almost constant gravitational potential when we measure time by means of a certain clock that makes light traverse a closed path with a well-defined length then we will always obtain the same number for the speed of light, independently of how large the gravitational potential is in the space where we perform that measurement ⁽¹⁾.

⁽¹⁾ The clock that is employed in order to measure time is therefore always the same one. It is always brought to the position where c is to be ascertained.

That follows immediately from the equivalence principle. When we say that the speed of light at a point is c / c_0 greater than it is at a point P_0 then that will mean that we must appeal to a clock that runs c / c_0 slower at P , where we measure time ⁽¹⁾, than the clock that is employed to measure time at P_0 in the event that the ways that both clocks would work at the same location are comparable to each other. In other words: A clock will run faster when we bring it to a location where c is greater. That dependency of the rate of passage of time on the gravitational potential (c) is true for the rate at which arbitrary processes proceed. That was explained already in the previous article.

Similarly, the tension in a spring that is stretched in a certain way, and above all, the force (energy, resp.) in an arbitrary system, always depends upon how large c is found to be at a location in the system. That emerges easily from the following elementary argument: When we successively experiment in several small spatial regions of varying c and continually appeal to the same clock, the same yardstick, etc., we will find the same regularities with the same constants everywhere, except for possible differences in the intensities of the gravitational field. That follows from the equivalence principle. As a clock, we can appeal to perhaps two mirrors at a distance of 1 cm apart, when we count the number of times a light signal goes back and forth between them. We would then operate with a type of local time that **Abraham** denoted by l . It is then related to the universal time by:

$$dl = c dt .$$

If we measure the time by l then we will assign a certain velocity dx / dl to a spring of mass m that has been stretched in a certain way by means of the energy of deformation, and independently of how large c is at a location where that process takes place. One has:

$$\frac{dx}{dl} = \frac{dx}{c dt} = a ,$$

in which a is independent of c . However, from (8), the kinetic energy that corresponds to that motion can be set equal:

$$\frac{m}{2c} q^2 = \frac{m}{2c} \left(\frac{dx}{dt} \right)^2 = \frac{m}{2c} a^2 c^2 = \frac{ma^2}{2c} \cdot c .$$

The energy of the spring is then proportional to c , and there is equality between energy and force for any system.

That dependency has a direct physical meaning. I imagine, e.g., a massless wire that stretched between two points P_1 and P_2 with different gravitational potentials. One of two equally-composed springs is stretched to a point P_1 on the wire, while the second one is stretched to P_2 , in such a way that equilibrium exists. However, the elongations l_1 and l_2 that the two springs experience in that way will not be equal, since the equilibrium condition will read ⁽²⁾:

⁽¹⁾ Namely, we measure the time that was denoted by t in the equations.

⁽²⁾ It is generally assumed in that that no forces act on the stretched massless spring in the gravitational field. That will be founded in an article that will follow shortly.

$$l_1 c_1 = l_2 c_2 .$$

Finally, let it be mentioned that equation (5.b) also agrees with this general result. It will, in fact, follow from that equation and the fact that the gravitational force that acts on a mass m equals $-m \text{ grad } c$ that the force $\mathcal{R}$ of attraction between two masses that are found at a distance r from each other in a potential c is given by:

$$\mathcal{R} = c k \frac{m m'}{4\pi r^2},$$

in the first approximation. That force is also proportional to c then. If we further imagine a “gravitational clock” that consists of a mass m that orbits around a fixed mass m' at a constant distance R under the action of only the gravitational field then, according to (6.b), that will happen in accord with the equations:

$$m \ddot{x} = c \mathcal{R}_x, \quad \text{etc.},$$

in the first approximation. It will then follow that:

$$m \omega^2 R = c^2 k \frac{m m'}{4\pi R^2}.$$

The rate ω at which the gravitational clock takes is then proportional to c , which should be true for clocks of any type.

§ 4. – General remarks in regard to space and time.

How does the foregoing theory relate to the older theory of relativity (i.e., to the theory of a universal c)? In **Abraham**’s opinion, the equations of the **Lorentz** transformation must be true, as before, in the infinitely small, i.e., they should give an xt -transformation such that:

$$dx' = \frac{dx - v dt}{\sqrt{1 - \frac{v^2}{c^2}}},$$

$$dt' = \frac{-\frac{v}{c^2} dx + dt}{\sqrt{1 - \frac{v^2}{c^2}}}.$$

dx' and dt' must be complete differentials. The following equations must then be true:

$$\frac{\partial}{\partial t} \left\{ \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \right\} = \frac{\partial}{\partial x} \left\{ \frac{-v}{\sqrt{1 - \frac{v^2}{c^2}}} \right\},$$

$$\frac{\partial}{\partial t} \left\{ \frac{-\frac{v}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}} \right\} = \frac{\partial}{\partial x} \left\{ \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \right\}.$$

Now let the gravitational field in the unprimed system be a static one. c is then an arbitrarily-given function of x , but it is independent of t . Should the primed system be a “uniformly-moving” one, then v would have to be independent of t for a fixed x in any case. The left-hand sides of the equations would then have to vanish, and therefore, the right-hand sides, as well. However, the latter is impossible, since for arbitrarily-given functions c of x , both right-hand sides cannot be made to vanish when one suitably chooses v as a function of x . In that way, it is then proved that one cannot establish the Lorentz transformation for infinitely-small space-time regions either as soon as one abandons the universal constancy of c .

It seems to me that the space-time problem consists of the following: If one restricts oneself to a region of constant gravitational potential then the laws of nature will take on a distinctly simpler and invariant form when one refers them to a space-time system of those manifolds that are coupled to each other by the Lorentz transformations with constant c . If one does not restrict oneself to the regions of constant c then the manifold of equivalent systems, as well as the manifold of transformations that leave the laws of nature unchanged, will become larger, but the laws themselves will become more complicated.

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