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Semi-vectors and spinors

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From the great significance that concept of spinors that was introduced by Pauli and Dirac has taken on in molecular physics, it can still not be asserted that the mathematical analysis of this concept up to now satisfies all of the justifiable demands. That is why P. Ehrenfest has urged one of us with great enthusiasm to take on the task of filling this lacuna. Our endeavors have led to a derivation that, in our interpretation, corresponds to all requirements of clarity and naturality, and completely avoid non-intuitive artifices. Therefore, as will be shown in what follows, the introduction of new quantities – viz., "semi-vectors" – will prove to be necessary; these quantities include the spinors, but possess an essentially more intuitive character than the spinors under transformations. In the present paper, we have deliberately restricted ourselves to the representation of the purely formal connection, in order for the mathematical formalism to emerge in full clarity.

The essence of the train of thought that is pursued in this paper may be outlined as follows: Any real Lorentz transformation $\mathfrak{D}$ can be decomposed uniquely into two special Lorentz transformations $\mathfrak{B}$ and $\mathfrak{C}$, whose transformation coefficients $b^i{}_k$ and $c^i{}_k$ are complex conjugate to each other, where the transformations $\mathfrak{B}$ and $\mathfrak{C}$ define groups that are isomorphic to the group ($\mathfrak{D}$) of Lorentz transformations. Semi-vectors are quantities with four complex components that, by assumption, undergo a $\mathfrak{B}$-transformation ($\mathfrak{C}$-transformation, resp.) when a Lorentz transformation is performed. There are special semi-vectors that are characterized by certain symmetry conditions and have only two (instead of four) mutually independent (complex) components. This situation gives rise to the introduction of quantities with only two (complex) components, namely, the Dirac spinors.

§ 1. Rotation and Lorentz transformation.

We think of the space R_4 of special relativity as being referred to Cartesian (not necessarily rectangular) coordinates. The metric tensor (g_{ik}) has well-defined constant components that are numerically invariant under the subsequent transformations that we will have in mind (Lorentz transformations in a broader sense).

In the chosen coordinate system, we consider a vector map:

$$\lambda^{i'} = a^i{}_k\, \lambda^k, \tag{1}$$

which we refer to as a "rotation," when it is length-preserving; i.e., when one always has:

$$g_{ik} \lambda^{i'} \lambda^{k'} = g_{ik}\, \lambda^i\, \lambda^k.$$

For the $a^i{}_k$, this yields the condition:

$$g_{ik}\, a^i{}_p\, a^k{}_q = g_{pq}\ . \tag{2}$$

On the other hand, let:

$$x^{i'} = a^i{}_k\, x^k \tag{3}$$

be a coordinate transformation (with constant $a^i{}_k$); the following transformation laws are valid for the components λ^i (g_{ik}, resp.):

$$\lambda^{i'} = a^i{}_k\, \lambda^k, \tag{4}$$

$$g'_{ik}\, a^i{}_p\, a^k{}_q = g_{pq}\ . \tag{5}$$

We call the transformations (3) that leave the g_{ik} numerically invariant ($g'_{ik} \equiv g_{ik}$) "Lorentz transformations." From (5), the matrix ($a^i{}_k$) of a Lorentz transformation satisfies the equations (2), which we shall derive for the "rotation." This allows us to associate Lorentz transformations that have the same rotation matrix, and study the rotations, instead of the former. The preference for this procedure rests in the fact that the rotation matrix has a tensor character.

Any statement about the "rotation" ($a^i{}_k$) is equivalent to a statement about the "Lorentz transformation" ($a^i{}_k$) with the same matrix.

Let us make a remark about the meaning of the raising and lowering of indices (a tensorial operation) for the transformation matrix ($a^i{}_k$). We would like to represent this in the general example of Riemannian R_n .

Let:

$$x'_i = x'_i\, (x_1, \ldots, x_n) \tag{6}$$

be a point transformation. For the components λ^i of a contravariant vectors at a point one then has:

$$\lambda^{i'} = a^i{}_k\, \lambda^k \qquad \left(a^i{}_k = \frac{\partial x'_i}{\partial x_k} \right). \tag{7}$$

The index (i) in $a^i{}_k$ thus refers to the system of x'_i with the metric components g'_{ik}, and the index (k) to the system of x_i with the metric tensor components g_{ik} . If we observe this then we can raise and lower the indices in (7), and thus write:

$$\lambda'_i = a_{ik}\, \lambda^k \qquad (a_{ik} = g'_{ir}\, a^r{}_k), \tag{7a}$$

$$\lambda'_i = a_i^{\ k}\,\lambda_k \qquad (a_i^{\ k} = g'_{ir}\,g^{ks}\,a^r{}_s), \tag{7b}$$

$$\lambda^{i\prime} = a_i^{\ k}\,\lambda_k \qquad (a^{ik} = g^{ks}\,a^i{}_s); \tag{7c}$$

i.e., all transformations laws for the components of the vector (λ) are included in (7) if one observes the rule for the raising and lowering of indices for $a^i{}_k$.

Naturally, one thus has $g'_{ik} \equiv g_{ik}$ for the Lorentz transformation.

§ 2. The decomposition of second-rank anti-symmetric tensors in R_4 .

Although the reasoning of this paragraph is valid for the general Riemannian R_4, we shall essentially restrict ourselves to the pseudo-Euclidian space of special relativity, which we refer to rectangular coordinates. The metric tensor (g_{ik}) then has the components:

$$\begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{vmatrix}. \tag{8}$$

As is known, there is the fourth-rank covariant tensor in R_4 that is anti-symmetric in all indices [1]:

$$t_{iklm} = \sqrt{g}\ \eta_{iklm}\,, \tag{9}$$

in which $\eta_{1234} = 1$, $g \mid g_{ik} \mid$, or, in contravariant representation:

$$t^{iklm} = \frac{1}{\sqrt{g}}\,\eta^{iklm}\,. \qquad\qquad (\eta^{iklm} = \eta_{iklm}). \tag{10}$$

Thus, for the special choice of coordinates, one will set, in agreement with (8):

---

[1] Proof: From the transformation formula for g_{ik}, it is known that it follows for the determinant that:

$$\sqrt{g'} = \frac{\partial(x_1, x_2, x_3, x_4)}{\partial(x'_1, x'_2, x'_3, x'_4)}\sqrt{g}\ .$$

On the other hand, one has:

$$t'_{iklm} = \frac{\partial(x_p)}{\partial(x'_i)}\frac{\partial(x_q)}{\partial(x'_k)}\frac{\partial(x_r)}{\partial(x'_l)}\frac{\partial(x_s)}{\partial(x'_m)}\sqrt{g}\ \eta_{pqrs} = \sqrt{g}\,\eta_{iklm}\,\frac{\partial(x_1,\ldots,x_4)}{\partial(x'_1,\ldots,x'_4)}\,.$$

The equation to be proved follows from both equations: $t'_{iklm} = \sqrt{g'}\ \eta_{iklm}$. One further has:

$$t^{pqrs} = \sqrt{g}\ g^{pi}\,g^{qk}\,g^{rl}\,g^{sm}\ \eta_{iklm} = \sqrt{g}\cdot\frac{1}{g}\,\eta_{pqrs} = \frac{1}{\sqrt{g}}\,\eta_{pqrs}\,.$$

$$\sqrt{g} = i \tag{11}$$

(i.e., one restricts to pure rotations).

Now, if h_{ik} is a second-rank anti-symmetric tensor (which does not need to be real) then we can associate it with the likewise anti-symmetric tensor ($h_{ik}{}^{x}$):

$$\left.\begin{aligned} h_{ik}{}^{x} &= \tfrac{1}{2}\sqrt{g}\,\eta_{iklm} h^{lm}, \\ h^{ik\,x} &= \frac{1}{2\sqrt{g}}\eta^{iklm} h_{lm}, \text{ resp.} \end{aligned}\right\} \tag{12}$$

Written out more completely, this is:

$$h_{12}{}^{x} = \sqrt{g}\, h^{34}, \qquad h_{34}{}^{x} = \sqrt{g}\, h^{12}, \tag{12a}$$

$$h^{12\,x} = \sqrt{g}\, h_{34}, \qquad h^{34\,x} = \sqrt{g}\, h_{12}\,. \tag{12b}$$

From this, it follows that:

$$(h_{ik}{}^{x})^{x} = h_{ik}\,. \tag{13}$$

Now, there is now a special anti-symmetric tensor h_{ik} such that $h_{ik}{}^{x} = \alpha\, h_{ik}$; according to (13), for such a tensor, $\alpha^2 = 1$.

We call a tensor a *special tensor of the first kind* and denote it by u_{ik} if:

$$u_{ik}{}^{x} = u_{ik}\,. \tag{14}$$

Likewise, should one have:

$$v_{ik}{}^{x} = -\,v_{ik}\,, \tag{15}$$

then this would define a *special anti-symmetric tensor of the second kind.* In the following quantities, we would like to always denote this symmetry character by the symbols u and v. This means, in more complete notation:

$$u_{12} = \sqrt{g}\;\; u^{34}, \qquad u_{34} = \sqrt{g}\;\; u^{12}, \tag{14a}$$

$$v_{12} = \sqrt{g}\;\; v^{34}, \qquad v_{34} = \sqrt{g}\;\; v^{12}, \tag{15a}$$

or, for our special choice of coordinates:

$$u_{12} = -\,i\, u^{34}, \qquad u_{34} = i\, u^{12}, \tag{14b}$$

$$v_{12} = i\, v^{34}, \qquad v_{34} = -\,i\, v^{12}. \tag{15b}$$

From (14), (15), and (12), or also from (14b), (15b), it follows that the complex conjugate of a u_{ik} is a v_{ik} .

The arbitrary anti-symmetric tensor h_{ik} may then be decomposed according to the following schema:

$$h_{ik} = \tfrac{1}{2}(h_{ik} + h_{ik}^{x}) + \tfrac{1}{2}(h_{ik} - h_{ik}^{x}). \tag{16}$$

Since $h_{ik} + h_{ik}^{x}$ is a u_{ik} and $h_{ik} - h_{ik}^{x}$ is a v_{ik}, we have in (16) the unique ^{x}-decomposition of the general second-rank anti-symmetric tensor into a u tensor and a v tensor [1].

If h_{ik} is real then h_{ik}^{x} is pure imaginary and $h_{ik} + h_{ik}^{x}$ is the complex conjugate of $h_{ik} - h_{ik}^{x}$. The special anti-symmetric tensors the first (second, resp.) kind define a linear space: Along with u_{ik}, u'_{ik}, $\alpha u_{ik} + \alpha' u'_{ik}$ also belongs to the totality of all u_{ik}.

Any u_{ik} can be linearly represented by three suitably chosen u_{ik} ; in our special coordinates, one can employ the u_{ik} thus defined for the representation (naturally, relative to the chosen coordinate system):

$$\left.\begin{array}{l}
\underset{1}{u}_{ik}\text{: only } \underset{1}{u}_{12} = -i\underset{1}{u}_{34} = 1 \text{ is different from zero (naturally, } \underset{1}{u}_{21}, \underset{1}{u}_{43} \text{ are also non-zero),} \\
\underset{2}{u}_{ik}\text{: only } \underset{2}{u}_{23} = -i\underset{1}{u}_{14} = 1 \text{ is different from zero,} \\
\underset{3}{u}_{ik}\text{: only } \underset{3}{u}_{31} = -i\underset{3}{u}_{24} = 1 \text{ is different from zero.}
\end{array}\right\} \tag{14c}$$

Since the α in this representation:

$$u_{ik} = \underset{1}{\alpha}\underset{1}{u}_{ik} + \underset{2}{\alpha}\underset{2}{u}_{ik} + \underset{3}{\alpha}\underset{3}{u}_{ik}$$

can be complex, the space is 6-dimensional; naturally, an analogous statement is true for the space of v_{ik}. Relative to our coordinate system, we can define:

$$\underset{\alpha}{v}_{ik} = \underset{\alpha}{\bar{u}}_{ik} \qquad (\alpha = 1, 2, 3) \tag{15c}$$

and thus linearly represent the most general v_{ik}.

§ 3. The decomposition of the Lorentz group.

As in § 1, we again consider the length-preserving map (rotation):

$$\lambda^{i\prime} = a^{i}_{k}\, \lambda^{k}.$$

According to (2), for it one has:

$$\delta^{i}_{k} = a^{i}_{p}\, a_{k}^{p}. \tag{17}$$

From this, it follows that the determinant $|\, a^{i}_{p} \,|$ is different from zero; there is thus an inverse to any rotation. If further follows from the definition that the composition of two rotations is again a rotation. We can therefore speak of the group ($\mathfrak{D}$) of rotations – viz.,

[1] The proof of uniqueness may be based upon the fact that the vanishing of u_{ik} and v_{ik} follows from $u_{ik} + v_{ik} = 0$.

the *Lorentz group*. The individual rotations of the group can also be complex (a^i_k complex).

If we set:

$$a_k{}^p = (a^{-1})^p{}_k \tag{18}$$

then it follows from (17) that:

$$a^i{}_p\,(a^{-1})^p{}_k = \delta^i{}_k\,. \tag{17a}$$

Thus, a_{ki} is the inverse to a_{ik} .

We now consider an infinitesimal rotation:

$$a^i{}_k = \delta^i{}_k + \varepsilon^i{}_k\,,$$

or

$$a_{ik} = g_{ik} + \varepsilon_{ik}\,, \tag{19}$$

which differs from the identity rotation (g_{ik}) only by infinitely small quantities, so the product of two ε is neglected. If one substitutes this into (17) then one obtains the condition:

$$\varepsilon_{ik} = -\,\varepsilon_{ik}\,. \tag{20}$$

(19), (20) characterize the general infinitesimal rotation.

From § 2, we can now decompose the anti-symmetric tensor (ε_{ik}) according to the equation:

$$\varepsilon_{ik} = u_{ik} + v_{ik}\,, \tag{21}$$

where (u_{ik}) and (v_{ik}) are special (infinitesimal) anti-symmetric tensors of the first (second, resp.) kind, in the sense of § 2. If the ε_{ik} are real then u_{ik} and v_{ik} are complex conjugate ($v_{ik} = \bar{u}_{ik}$).

The representation (21) corresponds to the decomposition of the infinitesimal rotation (19):

$$g_{ik} + \varepsilon_{ik} = (g_{ip} + u_{ip})\,(g^p{}_k + u^p{}_k) \tag{22}$$

into two rotations of well-defined type, in which $g^p{}_k$ is written for $\delta^p{}_k$.

Does this decomposition of the infinitesimal elements of the rotation (Lorentz, resp.) group also correspond to a well-defined decomposition of the finite elements of this group?

We would like to temporarily assume this. Thus, if (a_{ik}) is a rotation then there might be two rotations (b_{ik}) and (c_{ik}) that are given by (a_{ik}) such that:

$$a_{ik} = b_{ip}\,c^p{}_k\,. \tag{α}$$

Therefore, b_{ik} (c_{ik} , resp.) might define yet-to-be-determined subgroups of the rotation (Lorentz, resp.) group that are isomorphic, on the basis of the association that is given by (α). The association of b_{ik} with a_{ik} that is given by (α) will be described symbolically by $a_{ik} \rightarrow b_{ik}$.

Analogously, a second Lorentz transformation:

$$a'_{ik} = b'_{ip}\, c'^{\,p}{}_k \tag{β}$$

corresponds to the association:

$$a'_{ik} \to b'_{ik}\,.$$

The condition for isomorphism then requires that the components of the Lorentz transformation are associated with the corresponding components of the b-transformation according to the schema:

$$a_{ir}\, a''^{r}{}_k \to b_{ir}\, b''^{r}{}_k\,.$$

(An analogous statement is true for the subgroup of the c_{ik} .)

Thus, on the basis of (α) and (β), the decomposition:

$$a_{ir}\, a''^{r}{}_k = (b_{ir}\, b'^{p}{}_r)\,(c^{r}{}_q\, c'^{q}{}_k) \tag{γ}$$

shall also exist, and indeed, along with:

$$a_{ir}\, a''^{r}{}_k = (b_{ir}\, c^{p}{}_r)\,(b'^{\,r}{}_q\, c'^{\,q}{}_k), \tag{δ}$$

(γ) and (δ) are naturally fulfilled when and only when:

$$b'^{\,p}{}_r\, c^{r}{}_s = c^{p}{}_r\, b'^{\,r}{}_s\,; \tag{ε}$$

i.e., when any b-rotation commutes with any c-rotation. In the event that a decomposition (α) of the desired kind exists, the condition (ε) must also be satisfied. In addition, it shall correspond to a decomposition of the infinitesimal rotation of the type (22).

Accordingly, we seek to determine (b_{ik}) such that (ε) is fulfilled when we substitute the infinitesimal rotation:

$$c_{ik} = g_{ik} + v_{ik}\,,$$

for c_{ik} , according to (22).

$$b_{ip}(\delta^{p}{}_k + v^{p}{}_k) = (g_{ip} + v_{ip})\, b^{p}{}_k \tag{23}$$

or

$$b_{ip}\, v^{p}{}_k = v_{ip}\, b^{p}{}_k\,, \tag{23a}$$

and indeed for the most general choice of tensor of the second kind v_{ik} , which is established by (15) [(15b), resp.].

The solutions of (23a) for which $|\, b_{ik}\,| \neq 0$ define a group; then, along with b_{ip} and b'_{ip}, $b_{ip}\, b'^{p}{}_k$ also commutes with v_{ik} and has a non-zero determinant. Moreover, the totality of all these solutions includes the identity g_{ik} , and along with b_{ik} , the inverse element $(b^{-1})_{ik}$ exists, which is likewise a solution of (23a) [1].

Our problem is the determination of the structure of the elements of this group, which we would like to denote by ($\mathfrak{B}'$). We again carry out the calculations that this leads to for a coordinate system in which the g_{ik} are given by (8). Due to the tensorial character of (23a), however, the result is independent of the special choice of coordinates.

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[1] One proves this by multiplying (23a) by $(b^{-1})_q{}^i\, (b^{-1})^k{}_r$.

The most general v_{ik} that enters into (23a) can be linearly represented by the $\underset{\alpha}{v}_{ik}$ that were defined in (15c) in the coordinate system that is employed. Therefore, if (23a) is fulfilled for each of these three tensors then this relation is true for any v_{ik} .

We then first substitute $\underset{1}{v}_{ik}$ into (23a) ($\underset{1}{v}_{12} = i\ \underset{1}{v}_{34} = 1$). We arrange the results for i, k = 1, …, 4 into the following tables:

$$
\begin{array}{c|cl}
i & k & \\ \hline
\begin{matrix}1\\2\end{matrix} & \left.\begin{matrix}1\\2\end{matrix}\right\} & b_{12} = -b_{21}, \\
\begin{matrix}3\\4\end{matrix} & \left.\begin{matrix}3\\4\end{matrix}\right\} & b_{34} = -b_{43},
\end{array}
\qquad
\begin{array}{c|cl}
i & k & \\ \hline
\begin{matrix}1\\2\end{matrix} & \left.\begin{matrix}2\\1\end{matrix}\right\} & b_{11} = b_{22}, \\
\begin{matrix}1\\3\end{matrix} & \left.\begin{matrix}3\\1\end{matrix}\right\} & \begin{matrix} b_{14} = ib_{23}, \\ b_{41} = ib_{32}, \end{matrix}
\end{array}
\qquad
\begin{array}{c|cl}
i & k & \\ \hline
\begin{matrix}2\\3\end{matrix} & \left.\begin{matrix}3\\2\end{matrix}\right\} & \begin{matrix} b_{24} = -ib_{23}, \\ b_{42} = -ib_{31}, \end{matrix} \\
\begin{matrix}2\\4\end{matrix} & \left.\begin{matrix}4\\2\end{matrix}\right\} & \begin{matrix} b_{14} = ib_{23}, \\ b_{41} = ib_{32}. \end{matrix}
\end{array}
$$

Since the relations that define $\underset{2}{v}_{ik}$ arise from the ones that define $\underset{1}{v}_{ik}$ by cyclic permutation of 1, 2, 3, when the $\underset{2}{v}_{ik}$ are substituted in (23a) this produces relations that arise from the tables above by cyclic permutation of the indices 1, 2, 3. Then, to any relation (e.g., $b_{1p}\underset{1}{v}^{p1} = \underset{1}{v}_{1p}b^{p1}$) for $\underset{1}{v}_{ik}$ there exists a cyclically associated one for $\underset{2}{v}_{ik}$ (here, $b_{2p}\underset{2}{v}^{p2} = \underset{2}{v}_{2p}b^{p2}$), which yields the relation for b_{ik} that arises from the first one by cyclic permutation of 1, 2, 3.

We thus obtain the relations:

$$
\begin{array}{llll}
b_{23} = -\,b_{32}\,, & b_{22} = b_{33}\,, & b_{42} = ib_{13}\,, & b_{43} = -\,ib_{12}\,, \\
b_{14} = -\,b_{41}\,, & b_{24} = ib_{31}\,, & b_{34} = -\,ib_{21}\,, & b_{11} = -\,b_{44}\,.
\end{array}
$$

$\underset{1}{v}_{ik}$ correspondingly yields:

$$
\begin{array}{llll}
b_{31} = -\,b_{13}\,, & b_{33} = b_{11}\,, & b_{43} = ib_{21}\,, & b_{41} = -\,ib_{23}\,, \\
b_{24} = -\,b_{42}\,, & b_{34} = ib_{31}\,, & b_{14} = -\,ib_{32}\,, & b_{22} = -\,b_{44}\,.
\end{array}
$$

If we set $b_{11} = b_{22} = b_{33} = -\,b_{44} = b$ then we obtain the following results as the summary of all these relations, with hindsight of (14b):

$$b_{ik} = b\,g_{ik} + u_{ik}\,. \tag{24}$$

In (24), we thus have the structure of the most general tensor b_{ik} that commutes with v_{ik} . Since the most general u_{ik} (= $\underset{1}{\alpha}\underset{1}{u}_{ik} + \underset{2}{\alpha}\underset{2}{u}_{ik} + \underset{3}{\alpha}\underset{3}{u}_{ik}$) includes three complex parameters, b_{ik} includes four.

We would like to establish the form of the inverse element of (24) $(b^{-1})_{ik}$. u_{ik} also belongs to the tensors b_{ik} (for $b = 0$); therefore, $u_{ir}\,u^{r}{}_{k}$ is also included in the totality of the

b_{ik} , which indeed defines a group ($\mathfrak{B}'$). However, since $u_{ir}\, u^r{}_k$ is symmetric in i and k, one must have:

$$u_{ir}\, u^r{}_k = \alpha\, g_{ik} \,. \tag{24a}$$

Therefore, as one recognizes by contracting, one has:

$$\alpha = \tfrac{1}{2}\, u_{ir}\, u^r{}_k \,. \tag{24b}$$

We now assume that the inverse element is:

$$(b^{-1})_{ik} = b'\, g_{ik} + c'\, u_{ik} \,.$$

From:

$$g_{il} = b_{ik}\, (b^{-1})^k{}_l = (b\, g_{ik} + u_{ik})\, (b'\, g^k{}_l + c'\, u^k{}_l)$$
$$= (bb' + c'a)\, g_{il} + (bc' + b')\, u_{il} \,,$$

it then follows that:

$$b' = \frac{b}{b^2 + \frac{1}{4} u_{ik} u^{ik}}, \qquad c' = -\,\frac{1}{b^2 + \frac{1}{4} u_{ik} u^{ik}} \;.$$

One thus obtains for the inverse element:

$$(b^{-1})_{ik} = \frac{b}{b^2 + \frac{1}{4} u_{ik} u^{ik}}\, (b\, g_{ik} - u_{ik}) = \frac{1}{b^2 + \frac{1}{4} u_{ik} u^{ik}}\, b_{ki} \,. \tag{25}$$

The map b_{ik} is not a "rotation," in general. Namely, multiplication by b^{ki}, while recalling the definition of the inverse, yields:

$$b_{ki}\, b^{kl} = \delta_i^l (b^2 + \tfrac{1}{4}\, u_{ik}\, u^{ik}).$$

According to (2), b_{ik} then determines rotation (i.e., Lorentz transformation) only when the parameter that appears in (24) corresponds to the condition:

$$b^2 + \tfrac{1}{4}\, u_{ik}\, u^{ik} = 1. \tag{26}$$

The totality of rotations in the group ($\mathfrak{B}'$) will be defined by the "intersection" of the two groups ($\mathfrak{B}'$) and ($\mathfrak{D}$) (viz., the group of rotations). Let this intersection, which is itself a group, be denoted by ($\mathfrak{B}$).

From (24), the infinitesimal element of the group ($\mathfrak{B}'$) reads, in a self-explanatory notation:

$$g_{ik}(1 + \delta b) + \delta u_{ik} \,. \tag{27}$$

From (26), the infinitesimal element of ($\mathfrak{B}$) fulfills the additional condition that $(1 + \delta b)^2 + \frac{1}{4}\, \delta u_{ik}\, \delta u^{ik} = 1$ or $\delta b = 0$. It therefore reads:

$$g_{ik} + \delta u_{ik}\,, \tag{28}$$

in agreement with (22), an equation that indeed defined the starting point for the decomposition of the infinitesimal rotation (except that u_{ik} was written in place of δu_{ik} there).

Precisely as (23a) led to the group ($\mathfrak{B}'$), the relation:

$$c_{ip}\, u^p{}_k = u_{ip}\, c^p{}_k \tag{29}$$

leads to a group ($\mathfrak{C}'$) with the elements c_{ik}. Since every $\bar{v}^p{}_k$ is a $u^p{}_k$, such that both of them can be associated with the other one by means of the relation $\bar{v}^p{}_k = u^p{}_k$, (29) can be regarded as the complex conjugate equation to (23a), whose solutions c_{ik} are thus complex conjugates to those b_{ik} of (23a). From (24), it then emerges that (29) will be solved by:

$$c_{ik} = c\, g_{ik} + v_{ik}\,. \tag{30}$$

In place of (26), here, the additional (necessary and sufficient) condition for the map to be a rotation is the condition:

$$c^2 + \tfrac{1}{4}\, v_{ik}\, v^{ik} = 1. \tag{31}$$

(30), (31) represent the element of the rotation group ($\mathfrak{C}$), which is the intersection of the groups ($\mathfrak{C}'$) and ($\mathfrak{D}$) (viz., the rotation group). The infinitesimal element of ($\mathfrak{C}$) is, according to (28):

$$g_{ik} + \delta v_{ik}\,. \tag{32}$$

We restrict ourselves to any (proper) real Lorentz transformation $a^i{}_k$ that can be decomposed into real infinitesimal transformations.

This obviously defines a true subgroup of the Lorentz group for whose elements (§ 4) the decomposition $a^i{}_k = b^i{}_p \bar{b}^p{}_k$ is true [1].

This subgroup, which we shall be concerned with exclusively in what follows, always contains only one of the two Lorentz transformations ($a^i{}_k$) and ($-\,a^i{}_k$).

When we speak of the Lorentz group ($\mathfrak{D}$) in what follows, we intend this to mean this subgroup of the group of all real Lorentz transformations.

§ 4. Relations between the groups that we defined.

We first show that any element of the group ($\mathfrak{B}'$) commutes with any element of the group ($\mathfrak{C}'$). In fact, one has:

---

[1] A real Lorentz transformation ($a^i{}_k$) has either the decomposition ($b^i{}_p\, \bar{b}^p{}_k$) or ($-\, b^i{}_p\, \bar{b}^p{}_k$). From (§ 4), it follows that $a^i{}_k = b^i{}_p\, c^p{}_k$, namely, $b^i{}_p\, c^p{}_k = \bar{b}^i{}_p \bar{c}^p{}_k = \bar{c}^i{}_p \bar{b}^p{}_k$, hence, one has (§ 4), $b^i{}_p = \pm\, \bar{c}^i{}_p$, moreover.

$$b_{ik}\, c^k{}_l = b_{ik}\,(c\, g^k{}_l + v^k{}_l) = c\, b_{il} + b_{ik}\, v^k{}_l = c\, b_{il} + v_{ik}\, b^k{}_l = (c\, g_{ik} + v_{ik})\, b^k{}_l = c_{ik}\, b^k{}_l\,.$$

Since the "product" of a rotation in $\mathfrak{C}$ with a rotation in $\mathfrak{B}$:

$$a_{il} = b_{ik}\, c^k{}_l\,. \tag{33}$$

includes 3 + 3 = 6 complex parameters, which is just as many as the most general (not real) rotation, one might conjecture that any rotation can be represented as such a product; this was shown for the infinitesimal rotations in § 3. However, any rotation can be represented as the composition of a sequence of infinitesimal rotations, each of which is again a product of an infinitesimal rotation $\mathfrak{B}$ and such a $\mathfrak{C}$.

However, since we know that any $\mathfrak{B}$ commutes with any $\mathfrak{C}$, we can now assume that the permutations in the rotation that is represented by the infinitesimal $\mathfrak{B}$ and $\mathfrak{C}$ are such that first all $\mathfrak{B}$-rotations, and then all $\mathfrak{C}$-rotations follow in sequence. If one unites all $\mathfrak{B}$-rotations in this representation into a single one, and likewise for all $\mathfrak{C}$-rotations then this yields the splitting of the arbitrarily given rotation $\mathfrak{D}$ into:

$$\mathfrak{D} = \mathfrak{B}\mathfrak{C}.$$

If the given (proper) Lorentz transformation is real then each of the infinitesimal rotations that define it can be chosen to be real. Their splitting products $\mathfrak{B}$ and $\mathfrak{C}$ are then complex conjugate, and likewise the finite rotations (Lorentz transformations) $\mathfrak{B}$ and $\mathfrak{C}$ that arise from their composition are, as well.

As a result of the commutability of the b_{ik} and c_{ik}, the association that is induced by (33):

$$a_{il} \to b_{il}\,, \qquad a_{il} \to c_{il}\,,$$

of the elements of the groups ($\mathfrak{B}$) and ($\mathfrak{C}$) to those of the group ($\mathfrak{D}$) is required to be an isomorphism. The proof is obtained from the derivation of the previous paragraphs.

Along with a decomposition:

$$a_{il} = b_{ir}\, c^r{}_l\,,$$

there is always a decomposition:

$$a_{il} = (-\, b_{ir})(-\, c^r{}_l).$$

Are their more decompositions of the type considered for a_{il}? We assert that this is not the case, and indeed we first show this for $a_{il} = g_{il}$.

From:

$$g_{il} = b_{ik}\, c^k{}_l\,, \tag{33a}$$

it follows upon multiplying by $c^i{}_r$ (because $b_{ik}\, b^k{}_l = g_{kr}$) that:

$$b_{lr} = c_{rl}$$

or

$$b\, g_{lr} + u_{lr} = b\, g_{rl} + v_{rl}\ .$$

From this, it follows immediately that $b = c$; $u_{lr} = v_{lr} = 0$. It further follows, by substitution in (33a), that $b^2 = 1$; thus, $b = c = \pm 1$. Therefore, our assertion is proved for $a_{il} = g_{il}$; the single decomposition of the type considered for $a_{il} = g_{il}$ is, in fact:

$$g_{il} = (\pm\, g_{ik})(\pm\, g^k{}_l).$$

Now, let a_{ik} be an arbitrary Lorentz rotation and let:

$$b_{ik}\, c^k{}_l = b'_{ik}\, c'{}^k{}_l$$

be two representations for a_{il} . Multiplication by $b^i{}_p\ c_q{}^l$ yields, if one recalls the fundamental property of rotations:

$$g_{pq} = (b'_{ik}\, b'{}^i{}_p)(c'^k{}_l\, c_q{}^l) = [(b^{-1})'_{ki}\, b^i{}_p][\, c'{}^k{}_l\, (c^{-1})^l{}_q].$$

This is precisely a decomposition of g_{pq} into a $\mathfrak{B}$-rotation and a $\mathfrak{C}$-rotation. From the theorem that we just proved, one then has:

$$(b^{-1})'_{ki}\, b^i{}_p = c'_{kl}\, (c^{-1})^l{}_p = \pm\, g_{kp}\ .$$

From this, it follows, upon multiplication by $b'_r{}^k$ ($c^p{}_r$, resp.), that:

$$\left.\begin{aligned} b_{rp} &= \pm b'_{rp} \\ c'_{kr} &= \pm c_{kr} \end{aligned}\right\} \quad \text{both equations have the same sign.}$$

With that, the assertion is proved.

Aside from this double-valuedness of the sign, the association if $\mathfrak{B}$ ($\mathfrak{C}$, resp.) with the $\mathfrak{D}$ is unique.

Remark: The decomposition of the Lorentz rotation that was performed is true only for pure rotations $|\, a^i{}_k\, | = +\, 1$, so it is not true for reflections; therefore, only pure rotations may be composed from infinitesimal ones. The elements $\mathfrak{B}$ ($\mathfrak{C}$, resp.) are likewise pure rotations.

§ 5. The semi-vector and its invariants.

We refer the space of special relativity to rectangular Cartesian coordinates. The coordinate transformations that take these systems to each other are the Lorentz transformations:

$$x'_i = a^i{}_k\, x_k \qquad (a^i{}_p\, a_{iq} = g_{pq}).$$

The contravariant (covariant, resp.) vector λ^i (λ_i, resp.) is then defined by its transformation law:

$$\lambda'^i = a^i{}_k\, \lambda^k$$

or

$$\lambda'_i = a_i{}^k\, \lambda_k \qquad (a_i{}^k = g_{ip}\, g_{kq}\, a^p{}_q),$$

resp.

Now, however, the Lorentz transformations that take the system of x to the system of x' can be written as the product of a transformation $\mathfrak{C}$ and a transformation $\mathfrak{B}$:

$$a_{ik} = b_i^p\, c_{pk}\,,$$

where b_i^p and c_{pk} are determined completely, up to a (common) change of sign.

The totality of all b_{ik} (c_{ik}) define a subgroup of the Lorentz group that is isomorphic to the Lorentz group relative to the association:

$$a_{ik} \rightarrow b_{ik}\,.$$

This puts us in a position to define new tensorial structures (of rank one and higher) that are defined by the transformations b_{ik} (c_{ik}, resp.) of the group ($\mathfrak{B}$) [($\mathfrak{C}$), resp.]. Indeed, the contravariant semi-vector of the first kind, which we write as $\rho^{\bar{s}}$, might have the components:

$$\rho'^{\bar{r}} = b^r{}_s\, \rho^{\bar{s}} \tag{34}$$

in the x' system. Analogously, for the contravariant semi-vector of the second kind $\sigma^{\bar{s}}$, one has:

$$\sigma'^{\bar{r}} = c^r{}_s\, \sigma^{\bar{s}}\,. \tag{35}$$

Since a_{ik} is a real Lorentz transformation, one has:

$$c^r{}_s = \bar{b}^r{}_s\,.$$

From this, it emerges that the complex conjugate of a contravariant semi-vector of the first kind is a contravariant vector of the second kind, and conversely.

Since ($b^r{}_s$) and ($c^r{}_s$) are themselves Lorentz transformations, the metric tensor g_{ik} is also a semi-tensor of the first kind (and second kind) with transformation-invariant components. We can thus also employ it for the measurement of semi-vectors, as well as the raising and lowering of indices for semi- (and mixed) tensors.

We are then in a position to derive the transformations for covariant semi-vectors $\rho_{\bar{r}}$ ($\sigma_{\bar{r}}$, resp.) from (34) and (35):

$$\rho'_{\bar{r}} = b_r{}^s\, \rho_{\bar{s}} \qquad (b_r{}^s = g_{su}\, g^{rv}\, b^u{}_v), \tag{36}$$

$$\sigma'_{\bar{r}} = c_r{}^s\, \sigma_{\bar{s}} \qquad (b_r{}^s = g_{su}\, g^{rv}\, c^u{}_v). \tag{37}$$

We must indeed observe that a two-valuedness appears in the transformation law for the semi-tensors because of the free choice of sign for the b_{ik} and c_{ik} (for a given a_{ik}). It therefore has no meaning for the covariance of the equations in which semi-tensors appear, as one easily recognizes.

Since b_{ik} corresponds to a special Lorentz transformation, it is to be expected that, in addition to $g_{\bar{s}\bar{t}}$, there are still more (semi-) tensors of the first kind that are numerically invariant under transformations. Which are the simplest?

In order to find them, we need only to go back to the relations (23a), which define the group ($\mathfrak{B}'$):

$$b_i^p \, v_{pk} = v_{ip} \, b^p{}_k \,,$$

where v_{ip} is the most general anti-symmetric tensor of the second kind. Since b_{ik}, as a rotation, satisfies the relation:

$$b^p{}_k \, b_q{}^k = \delta^p{}_q \,,$$

it follows that:

$$v_{iq} = b_i^p \, b_q{}^k \, v_{pk} \,. \tag{38}$$

However, this means that $v_{\bar{i}\bar{q}}$ is a numerically invariant semi-tensor of the first kind.

The numerical invariance of $c g_{\bar{s}\bar{t}} + v_{\bar{s}\bar{t}}$ characterizes the $\mathfrak{B}$-transformations completely, since (23a) and (38) are equivalent for rotations b_i^p.

For two semi-vectors of the first kind $\lambda^{\bar{r}}$, $\mu^{\bar{r}}$, there are, along with the invariants:

$$g_{\bar{s}\bar{t}} \lambda^{\bar{r}} \mu^{\bar{t}} \,, \tag{39}$$

also the characteristic invariants for these quantities:

$$v_{\bar{s}\bar{t}} \lambda^{\bar{r}} \mu^{\bar{t}} \,. \tag{40}$$

If one substitutes (in a rectangular Cartesian coordinate system) $\underset{\alpha}{v}_{\bar{s}\bar{t}}$ ($\alpha = 1, 2, 3$), in sequence, into (40) then one obtains the invariants:

$$\left.\begin{aligned} \underset{1}{v}_{\bar{s}\bar{t}} \lambda^{\bar{s}} \mu^{\bar{t}} &= (\lambda^{\bar{1}} \mu^{\bar{2}} - \lambda^{\bar{2}} \mu^{\bar{1}}) - i(\lambda^{\bar{3}} \mu^{\bar{4}} - \lambda^{\bar{4}} \mu^{\bar{3}}), \\ \underset{2}{v}_{\bar{s}\bar{t}} \lambda^{\bar{s}} \mu^{\bar{t}} &= (\lambda^{\bar{2}} \mu^{\bar{3}} - \lambda^{\bar{3}} \mu^{\bar{2}}) - i(\lambda^{\bar{1}} \mu^{\bar{4}} - \lambda^{\bar{4}} \mu^{\bar{1}}), \\ \underset{3}{v}_{\bar{s}\bar{t}} \lambda^{\bar{s}} \mu^{\bar{t}} &= (\lambda^{\bar{3}} \mu^{\bar{1}} - \lambda^{\bar{1}} \mu^{\bar{3}}) - i(\lambda^{\bar{2}} \mu^{\bar{4}} - \lambda^{\bar{4}} \mu^{\bar{2}}), \end{aligned}\right\} \tag{41}$$

which, along with:

$$g_{\bar{s}\bar{t}} \lambda^{\bar{r}} \mu^{\bar{t}} = \lambda^{\bar{1}} \mu^{\bar{1}} + \lambda^{\bar{2}} \mu^{\bar{2}} + \lambda^{\bar{3}} \mu^{\bar{3}} + \lambda^{\bar{4}} \mu^{\bar{4}} \,, \tag{41a}$$

characterize the semi-vector of the first kind.

It follows in a completely analogous way (already from the fact that semi-tensors of the first and second kind are conjugates) that for the transformations $\mathfrak{C}$ the semi-tensors of the second kind $g_{\bar{s}\bar{t}}$ and the must general tensor $\mu_{\bar{s}\bar{t}}$ remain numerically invariant, a property that characterizes the subgroup ($\mathfrak{C}$) of the rotation group ($\mathfrak{D}$).

Between the semi-vectors of the first kind (μ) and (λ), there exists the relation:

$$\rho\mu_{\bar{s}} = v_{\bar{s}\bar{t}}\lambda^{\bar{t}} \quad (\rho\text{, a scalar}), \tag{42}$$

so, due to the numerical invariance of $v_{\bar{s}\bar{t}}$, it is a numerical relation between the components that is independent of the coordinate system. Therefore, in a rectangular coordinate system, for example (with v set to $\underset{1}{v}$, $\underset{2}{v}$, $\underset{3}{v}$, in sequence), one of the following relations can sensibly (i.e., invariantly) exist:

$$\rho\mu_{\bar{1}} = \lambda^{\bar{2}}, \qquad \rho\mu_{\bar{2}} = -\lambda^{\bar{1}}, \qquad \rho\mu_{\bar{3}} = -i\lambda^{\bar{4}}, \qquad \rho\mu_{\bar{4}} = i\lambda^{\bar{3}}, \tag{42a}$$

$$\rho\mu_{\bar{2}} = \lambda^{\bar{3}}, \qquad \rho\mu_{\bar{3}} = -\lambda^{\bar{2}}, \qquad \rho\mu_{\bar{1}} = -i\lambda^{\bar{4}}, \qquad \rho\mu_{\bar{4}} = i\lambda^{\bar{1}}, \tag{42b}$$

$$\rho\mu_{\bar{3}} = \lambda^{\bar{1}}, \qquad \rho\mu_{\bar{1}} = -\lambda^{\bar{3}}, \qquad \rho\mu_{\bar{2}} = -i\lambda^{\bar{4}}, \qquad \rho\mu_{\bar{4}} = i\lambda^{\bar{2}}. \tag{42c}$$

Analogously, one obtains sensible relations for the semi-vectors of the second kind when one replaces $v_{\bar{s}\bar{t}}$ with $u_{\bar{s}\bar{t}}$ in a relation that is analogous to (42) (when one replaces i with $-i$ in relations that are analogous to (42a), (42b), (42c), resp.).

§ 6. The tensor $E_{\alpha\bar{s}\bar{\bar{t}}}$.

In this paragraph, we seek to find the mixed tensors that exhibit numerical invariance relative to the transformations that correspond to their indices.

There are no numerically-invariant mixed tensors of second rank ($t_{i\bar{s}}$, $t_{i\bar{\bar{s}}}$, $t_{\bar{r}\bar{\bar{s}}}$). The simplest mixed numerically-invariant tensor has the structure $E_{r\bar{s}\bar{\bar{t}}}$; it is of rank three. (Relative to the first index, it is an ordinary tensor, relative to the second one, it is a semi-tensor of the first kind, and relative to the third one, it is a semi-tensor of the second kind.)

For its derivation, we again employ a rectangular coordinate system. As a result of the required numerical invariance, one has for the arbitrary Lorentz transformation:

$$E_{r\bar{s}\bar{\bar{t}}} = a_r{}^l \, b_s{}^m \, c_t{}^n \; E_{l\bar{m}\bar{\bar{n}}}. \tag{43}$$

Since

$$a_r{}^l = b_r{}^p \, c_p{}^l,$$

one also has:

$$E_{r\bar{s}\bar{\bar{t}}} = b_r{}^p \, b_s{}^m \, c_p{}^l \, c_t{}^n \; E_{l\bar{m}\bar{\bar{n}}}. \tag{43a}$$

The numerical invariance of E is also true for the inverse transformation, so one also has:

$$E_{r\bar{s}\bar{\bar{t}}} = a^l{}_r \, b^m{}_s \, c^n{}_t \; E_{l\bar{m}\bar{\bar{n}}}. \tag{43b}$$

The fact that $b_s^{\ m}$, $c_t^{\ n}$ can be replaced with $-b_s^{\ m}$, $-c_t^{\ n}$ has no influence on the validity of (43).

We would now like to determine the form of $E_{r\bar{s}\bar{\bar{t}}}$ from (43). For the sake of calculation, we will omit the bars that characterize the semi-indices, since the type of index in E is recognized from its position.

If $a_r^{\ l}$ is itself chosen to be a $\mathfrak{B}$-transformation then one has $a_r^{\ l} = b_r^{\ l}$, $c_t^{\ n} = \delta_t^n$, such that one obtains from (43):

$$E_{r\bar{s}\bar{\bar{t}}} = b_r^{\ l}\, b_s^{\ m}\, E_{l\bar{m}\bar{\bar{t}}}\,. \tag{44}$$

Likewise, for $a_r^{\ l} = c_r^{\ l}$, $b_s^{\ m} = \delta_s^m$, one gets from (43):

$$E_{r\bar{s}\bar{\bar{t}}} = c_r^{\ l}\, c_s^{\ n}\, E_{l\bar{s}\bar{\bar{n}}}\,. \tag{44a}$$

If one multiples (44) by $b^t_{\ n}$ then it follows that:

$$E_{r\bar{s}\bar{\bar{t}}}\, b^s_{\ u} = b_r^{\ l}\, E_{l\bar{u}\bar{\bar{t}}} = b_{rs}\, E^s_{\ ut}\,. \tag{44b}$$

Conversely, (44), (44a) have (43a) as a consequence, and thus also (43), so they are equivalent to (43).

From (44b), it follows that E_{rst} necessarily has the following form relative to the indices r and s [1]:

$$E_{rst} = g_{rs}\, a_{(t)} + v_{rs(t)}\,. \tag{45}$$

Analogously, it follows from (44a) that:

$$E_{rst} = g_{rs}\, b_{(t)} + u_{rs(t)}\,. \tag{45a}$$

Conversely, these two relations have (44) and (44a) as a consequence, and thus together, also (43), so they are equivalent to (43). If one sets $r = s = t$ in (45) and (45a) (naturally, without summation) then it follows that $a_{(s)} = b_{(s)}$. It further follows that (always without summation):

$$E_{rrr} = g_{rr}\, a_{(r)}\,, \quad E_{rrs} = g_{rr}\, a_{(s)}, \quad E_{rsr} = g_{rr}\, a_{(s)}, \tag{46}$$

and furthermore:

$$E_{rst} + E_{tsr} = 2\, g_{rs}\, a_{(t)}, \qquad\qquad E_{rst} + E_{tsr} = 2\, g_{rt}\, a_{(s)}\,.$$

From each of these equations, it follows for $r \neq s$, $s = t$:

$$E_{rss} = -\, g_{ss}\, a_r \qquad\qquad (r \neq s). \tag{46a}$$

All that is left for us are the E_{rst} with unequal indices. If r, s, t are unequal, and w is a fourth index that is different from all three then it follows from (45) that:

[1] Cf., (38) and the following remarks.

$$E_{rst} = v_{rs(t)} = -\sqrt{g}\;\eta_{rstw}\, v^{tw}{}_{(t)} = \mp\sqrt{g}\; E^{tw}{}_{(t)} = \mp\sqrt{g}\; a^{(w)} = -\sqrt{g}\;\eta_{rstw}\, a^{(w)}, \tag{47}$$

where

$$a^{(w)} = g^{w\alpha}\, a_{(\alpha)}\,.$$

(45a) yielded just this relation. In summary, one obtains:

$$E_{rst} = g_{rs}\, a_{(t)} + g_{rt}\, a_{(s)} - g_{st}\, a_{(r)} - \sqrt{g}\;\eta_{rstw}\, a^{(w)}. \tag{48}$$

This $E_{r\bar{s}\bar{\bar{t}}}$, which is numerically invariant, actually satisfies the equations (45) and (45a), as will be shown forthwith. $a_{(s)}$ are four arbitrarily chosen constants; if they are real then E_{rst} and E_{rts} are complex conjugate.

We show that – as (45) demands:

$$E_{rst} - g_{rs}\, a_{(t)} = (g_{rs}\, g_{tw} - g_{st}\, g_{rw} - \sqrt{g}\;\eta_{rstw}\,)a^{(w)}$$

has the symmetry property of a v_{rs} relative to the indices r, s. Namely, if a_{rs} is an arbitrary anti-symmetric tensor then, from § 2, $a_{rs} - a_{rs}{}^{x}$ is a v_{rs} ; one then has:

$$v_{rs} = a_{rs} - \frac{\sqrt{g}}{2}\,\eta_{rstw}\, a^{(w)},$$

or, when one sets $a_{rs} = (g_{rs}\, g_{tw} - g_{st}\, g_{rw})\, a^{(w)}$:

$$v_{rs} = (g_{rs}\, g_{tw} - g_{st}\, g_{rw} - \sqrt{g}\;\eta_{rstw}\,)a^{(w)}.$$

A comparison yields that $(E_{rst} - g_{rs}\, a_{(t)})$ is a v_{rs} , which was to be proved. The verification that (48) satisfies the condition (45a) is carried our analogously.

§ 7. The simplest system of differential equations for semi-vectors.

The significance of the mixed tensor $E_{r\bar{s}\bar{\bar{t}}}$ resides in the fact that with its help tensors of various types can be related to each other. We thus consider some examples, which we temporarily base upon the R_4 of special relativity, when referred to Cartesian coordinates.

One may construct the ordinary vector:

$$A_r = E_{r\bar{s}\bar{\bar{t}}}\,\chi^{\bar{s}}\psi^{\bar{\bar{t}}} \tag{49}$$

from a semi-vector $\chi^{\bar{s}}$ of the first kind and a semi-vector $\psi^{\bar{\bar{t}}}$ of the second kind. In particular, one can choose the semi-tensor of the second kind to be the conjugate of χ ($\psi^{\bar{\bar{t}}} = \bar{\chi}^{\bar{t}}$):

$$A_r = E_{r\bar{s}\bar{\bar{t}}}\chi^{\bar{s}}\chi^{\bar{\bar{t}}}\,. \tag{49a}$$

The choice of numerical parameter $a_{(w)}$ in E is therefore entirely free (as it also is on the following constructions). One can, moreover, form the following linear system of covariant differential equations for two such semi-vectors (i.e., fields of such vectors):

$$\left.\begin{aligned} E^r{}_{\bar{s}\bar{\bar{t}}}\frac{\partial\chi^{\bar{s}}}{\partial x_r} &= \alpha\psi_{\bar{\bar{t}}}\,, \\ E^r{}_{\bar{s}\bar{\bar{t}}}\frac{\partial\psi^{\bar{\bar{s}}}}{\partial x_r} &= \beta\chi_{\bar{t}}\,, \end{aligned}\right\} \tag{50}$$

where α and β are constants. It can be shown that by eliminating one of the semi-vectors in (50) a system of equations arises that is analogous to the Schrödinger equation [1].

We can further specialize the system (50), again in a natural way, such that we choose ψ to be the complex conjugate semi-vector to χ in (50). We thus obtain the system:

$$E^r{}_{\bar{s}\bar{\bar{t}}}\frac{\partial\chi^{\bar{s}}}{\partial x_r} = \alpha\bar{\chi}_{\bar{\bar{t}}}\,. \tag{51}$$

We can, in a certain sense, speak of (50) as an inexact splitting of a "Schrödinger equation," and (51) as a true one [2].

The raising and lowering of indices in all of these equations comes about by means of the metric tensors g_{st}, $g_{\bar{s}\bar{t}}$, $g_{\bar{\bar{s}}\bar{\bar{t}}}$.

The first peculiarity of this system of equations is the appearance of four arbitrary constants $a_{(w)}$ in E, such that the structure of the system of equations depends upon a choice of them. It will be shown later that this lack of elegance goes away upon the introduction of the Dirac spin quantities.

§ 8. The incorporation of the semi-quantities into the R_4 of general relativity.

From now on, the semi-quantities that are defined at any point of R_4 are referred to an arbitrary, oriented, orthogonal, normed vierbein that is described by the "mixed" tensor:

$$h_{\alpha i}\,. \tag{52}$$

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[1] This rests on the easily proven relation:

$$E^{hr}{}_s\,E^{kps} + E^{kr}{}_s\,E^{kps} = 2\,g_{hk}\,g_{rp}\,a_{(t)}\,a^{(t)}.$$

[2] This equation – which is completed by electromagnetic terms – seems, for that reason, to be inapplicable to the theory of electrons, because it changes under the addition of a gradient of the electrical potential.

If A^i is a contravariant vector then:

$$A_\alpha = h_{\alpha i} A^i \tag{53}$$

is the same vector, when referred to the frame. In the following, the Greek indices will always refer to the vierbein, while Latin indices will refer to the general coordinate system. One then has:

$$g_{ik} = h_{\alpha i} h_{\beta k} g^{\alpha\beta} = h_{1i} h_{1k} + h_{2i} h_{2k} + h_{3i} h_{3k} - h_{4i} h_{4k} \, . \tag{54}$$

For the magnitude of the vector (A), one then has:

$$g_{ik} A^i A^k = g^{\alpha\beta} A_\alpha A_\beta \, ,$$

where

$$g^{\alpha\beta} = g_{\alpha\beta} = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{vmatrix} .$$

A rotation (i.e., a change) of vierbein ($h'^{\alpha}{}_i = a^\alpha{}_\beta \, h^\beta{}_i$) thus corresponds to a local transformation of the local vector according to the equation:

$$A'^{\alpha} = a^\alpha{}_\beta A^\beta . \tag{55}$$

Along with the local vectors, we introduce the semi-vectors $\chi_{\bar\rho}$, $\psi_{\bar{\bar\sigma}}$, referred to the vierbein, which, like the local vectors, only transform by a vierbein rotation, and indeed according to the laws:

$$\chi'_{\bar\sigma} = b_\sigma{}^\rho \, \chi_{\bar\rho} \, , \tag{55a}$$

$$\psi'_{\bar{\bar\tau}} = c_\tau{}^\sigma \, \psi_{\bar{\bar\sigma}} \, , \tag{55b}$$

where

$$a_\alpha{}^\beta = b_\alpha{}^\gamma \, b_\gamma{}^\beta , \tag{56}$$

is the decomposition of the Lorentz transformation according to § 3. Like the local vector, the semi-vector will be measured with the local metric tensor $g_{\alpha\beta}$ ($g_{\bar\alpha\bar\beta}$, $g_{\bar{\bar\alpha}\bar{\bar\beta}}$), which is indeed also numerically invariant under the transformations $b_\alpha{}^\gamma$ and $c_\beta{}^\gamma$.

The introduction of:

$$E^r{}_{\bar\sigma\bar{\bar\tau}} = h_\alpha{}^r \, E^\alpha{}_{\bar\sigma\bar{\bar\tau}} \, , \tag{57}$$

with the help of our E-tensor (§ 6), admits a conversion of the differential equations (50), (51) into the schema of general relativity:

$$\left.\begin{aligned} E^{r}{}_{\bar{\sigma}\bar{\bar{\tau}}}\chi^{\bar{\sigma}}{}_{;r} &= \alpha\psi_{\bar{\tau}}, \\ E^{r}{}_{\bar{\sigma}\bar{\bar{\tau}}}\chi^{\bar{\tau}}{}_{;r} &= \beta\psi_{\bar{\sigma}}, \end{aligned}\right\} \tag{58}$$

$$E^{r}{}_{\bar{\sigma}\bar{\bar{\tau}}}\chi^{\bar{\sigma}}{}_{;i} = \alpha\bar{\chi}_{\bar{\tau}}\,. \tag{59}$$

The invariant derivation of the semi-vectors that is characterized by a semi-colon in these equations [1] shall next be established in such a way that:

$$g_{\bar{\sigma}\bar{\tau},i} = 0, \tag{60}$$

$$g_{\bar{\bar{\sigma}}\bar{\bar{\tau}},i} = 0. \tag{60a}$$

Then and only then can the indices under the differentiation sign be raised and lowered.

Naturally, the introduction of the theory of semi-tensors into the schema of general relativity will be complete when the rules for absolute differentiation of all quantities are established. This shall now come about by means of the following postulates (*A*) to (*D*), where we avail ourselves of the notations:

$$\left.\begin{aligned} \lambda_{\sigma;r} &= \lambda_{\alpha,r} - \lambda_{\beta}P^{\beta}{}_{\alpha r}, \\ \psi_{\bar{\sigma};r} &= \psi_{\bar{\sigma},r} - \psi_{\bar{\beta}}\Gamma^{\beta}{}_{\sigma r}, \\ \chi_{\bar{\bar{\sigma}};r} &= \chi_{\bar{\bar{\sigma}},r} - \chi_{\bar{\bar{\beta}}}\bar{\Gamma}^{\beta}_{\alpha r}. \end{aligned}\right\} \tag{61}$$

The $\bar{\Gamma}$ are chosen to be complex conjugate to the Γ, in order for complex conjugate semi-vectors to remain complex conjugate under differentiation.

(*A*) Naturally, the relation (53) between coordinate vectors and local vectors will not be perturbed by differentiation, from which, it follows that:

$$0 = h_{\alpha i;\,k} \qquad (= h_{\alpha i,\,k} - h_{\alpha r}\left\{ {r \atop ik} \right\} - h_{\beta i}\,P^{\beta}{}_{\alpha k}), \tag{62}$$

or

$$P_{\gamma\alpha k} = h_{\gamma}{}^{i}(h_{\alpha i,k} - h_{\alpha r}\left\{ {r \atop ik} \right\}). \tag{62a}$$

From (62) and $g_{\alpha\beta} = h_{\alpha}{}^{i}\,h_{\beta i}$ (orthonormal vierbein), it follows that $g_{\alpha\beta;\,k} = 0$, and from this, the anti-symmetry of P in the first two indices:

$$P_{\gamma\alpha k} = -\,P_{\alpha\gamma k}\,. \tag{62b}$$

[1] The fact that this system of equations is invariant with respect to the Greek indices (vierbein rotation) and the Latin indices (coordinate transformation)(so it possesses a tensor character) is easy to confirm when one observes the numerical invariance of the E relative to the vierbein rotations, as well as the tensor character of $h_{\sigma\tau}$.

(*B*) This postulate was already established in (60), (60a), and gives, analogous to (62), the condition:

$$\Gamma_{\sigma\tau r} = -\,\Gamma_{\tau\sigma r}\,. \tag{62c}$$

(*C*) The absolute derivative of the numerically invariant semi-tensors of the first (second, resp.) kind $v_{\sigma\tau}$ ($u_{\bar\sigma\bar\tau}$, resp.) shall vanish [1]:

$$0 = v_{\sigma\tau;\,k} = -\,v_{\alpha\tau}\,\Gamma^{\alpha}{}_{\sigma k} - v_{\alpha\sigma}\,\Gamma^{\alpha}{}_{\tau k}\,.$$

This yields:

$$v_{\sigma\alpha}\,\Gamma^{\alpha}{}_{\tau k} = \Gamma_{\sigma}{}^{\alpha}{}_{k}\;v_{\alpha\tau},$$

or

$$v_{\sigma\alpha}\,\Gamma^{\alpha}{}_{\tau k} = \Gamma_{\sigma\alpha k}\;v^{\alpha}{}_{\tau}. \tag{62d}$$

Comparing (62d) with (23a) shows that the Γ have the structure of a (b_{ik}) with respect to the Greek indices, and furthermore, due to their anti-symmetry (62c) and due to (24), they must possess the structure of a (u_{ik}) (cf., (14), (14a), (14b)).

This then yields the fact that the $\bar\Gamma$ must possess the structure of the corresponding (complex conjugate) v_{ik} relative to the first two indices.

In order to make this come out better, in the sequel, we would like to write temporarily:

$$U^{\sigma}{}_{\tau(k)} \quad \text{instead of } \Gamma^{\sigma}{}_{\tau k}\,,$$

$$V^{\sigma}{}_{\tau(k)} \quad \text{instead of } \bar\Gamma^{\sigma}{}_{\tau k}\,.$$

(*D*) The absolute derivative of the numerically invariant local tensor *E* shall vanish:

$$0 = E_{\alpha\sigma\tau;\,k} = -\,(E_{\beta\sigma\tau}\,P^{\beta}{}_{\alpha k} + E_{\alpha\beta\tau}\,\Gamma^{\beta}{}_{\sigma k} + E_{\alpha\sigma\beta}\,\bar\Gamma^{\beta}{}_{\tau k}). \tag{62e}$$

Now, however, due to (45), (45a), the tensor *E* has the structure of a (c_{ik}) with respect to its first two indices and the structure of a (b_{ik}) with respect to the first and third indices. By means of the commutation rules (23a) and (29), one then has the transformation equations:

$$E_{\alpha\beta\tau}\,\Gamma^{\beta}{}_{\sigma k} = E_{\alpha\beta\tau}\,\Gamma^{\beta}{}_{\sigma(k)} = U_{\alpha\beta(k)}\,E^{\beta}{}_{\sigma\tau} = -\,E_{\beta\alpha\tau}\,\Gamma^{\beta}{}_{\sigma k}\,,$$

$$E_{\alpha\sigma\beta}\,\bar\Gamma^{\beta}{}_{\tau k} = E_{\alpha\sigma\beta}\,V^{\beta}{}_{\tau(k)} = V_{\alpha\beta(k)}\,E^{\beta}{}_{\sigma\tau} = -\,E_{\beta\alpha\tau}\,\bar\Gamma^{\beta}{}_{\sigma k}\,.$$

When this is substituted into (62a), it gives:

$$E_{\beta\sigma\tau}\,(P^{\beta}{}_{\alpha k} - \Gamma^{\beta}{}_{\sigma k} - \bar\Gamma^{\beta}{}_{\tau k}) = 0. \tag{62f}$$

However, this then gives [1]:

[1] Wherever the clarity does not suffer, we have omitted the bars on the indices, in order to ease the printing.

$$P_{\beta\alpha k} = \Gamma_{\beta\alpha k} + \bar{\Gamma}_{\beta\alpha k}\,. \tag{62g}$$

With that, we have achieved the assimilation of the semi-vectors into the schema of general relativity.

Remark: In quantum theory, the operator $(;\,\alpha + i\varepsilon\varphi_\alpha)$ plays an important role, where φ_α is interpreted as an electric potential. In order to do justice to this, we temporarily introduce a "slash derivative" (/), along with semi-colon derivative that has been employed up to now, which comes down to the derivative (;) for ordinary (i.e., coordinate and local) vectors. We introduce the notation for semi-vectors:

$$\left.\begin{aligned} \psi_{\bar{\sigma}/k} &= \psi_{\bar{\sigma},k} - \psi_\alpha \Delta^\alpha{}_{\sigma k}\,, \\ \chi_{\bar{\bar{\sigma}}/k} &= \chi_{\bar{\bar{\sigma}},k} - \chi_\alpha \bar{\Delta}^\alpha_{\sigma k}\,, \end{aligned}\right\} \tag{63}$$

where

$$\Delta^\alpha{}_{\sigma k} = \Gamma^\alpha{}_{\sigma k} + i\,\varepsilon\,\delta^\alpha_\sigma\,\varphi_k\,. \tag{63a}$$

One then has:

$$\left.\begin{aligned} \psi_{\bar{\sigma}/k} &= \psi_{\bar{\sigma};k} - i\varepsilon\psi_{\bar{\sigma}}\varphi_k\,, \qquad \psi^{\bar{\sigma}}{}_{/k} = \psi^{\bar{\sigma}}{}_{;k} + i\varepsilon\psi^{\bar{\sigma}}\varphi_k\,, \\ \chi_{\bar{\bar{\sigma}}/k} &= \chi_{\bar{\bar{\sigma}};k} + i\varepsilon\chi_{\bar{\bar{\sigma}}}\varphi_k\,, \qquad \chi^{\bar{\bar{\sigma}}}{}_{/k} = \chi^{\bar{\bar{\sigma}}}{}_{;k} - i\varepsilon\chi^{\bar{\bar{\sigma}}}\varphi_k\,. \end{aligned}\right\} \tag{63b}$$

In place of equations (58), the same equations now appear, in which the only change is that the ";" derivative has been replaced by the "/" derivative.

As one does for the ";" derivative, one also has the following relation for the "/" derivative:

$$E^\alpha{}_{\bar{\sigma}\bar{\bar{\tau}}/k} = 0, \qquad E^{\alpha\bar{\sigma}\bar{\bar{\tau}}}{}_{/k} = 0.$$

If one introduces a real "current vector," in a manner that is analogous to that of Infeld and Van der Waerden [2]:

$$\mathfrak{J}^\alpha = E^\alpha{}_{\sigma\tau}\chi^\sigma\bar{\chi}^\tau + E^{\alpha\sigma\tau}\bar{\psi}_\alpha\psi_\tau\,,$$

and one defines $\mathfrak{J}^\alpha{}_{;\,\alpha}$ $(= \mathfrak{J}^\alpha{}_{/\alpha})$ then this divergence vanishes only if $\alpha + \bar{\beta} = 0$. (The $a_{(\tau)}$ are then assumed to be real.)

---

[1] Namely, by multiplying (62f) by $E_{r\rho}{}^\tau$, while considering the formula that emerge from (48) (Footnote 1, page 18):

$$E^{kr}{}_s\,E^{ips} + E^{kp}{}_s\,E^{irs} = 2\,g^{ik}\,g^{rp}\,a_{(t)}\,a^{(t)}.$$

[2] The authors were friendly enough to send us a copy of their paper "Die Wellengleichung des Elektrons in der allgemeinen Relatitätstheorie," which will be published in a few months. In it, the general relativistic form of the Dirac equations is presented without semi-vectors in a way that is similar to the one that we have pursued.

§ 9. Special semi-vectors (spinors).

We are still not finished with the theory of semi-vectors, since, as we will now show, there are special semi-vectors that have only two independent components. (We thus employ rectangular local coordinates in what follows.)

We first show this for semi-vectors of the first kind, and for the construction of the special semi-vectors, we employ the often-used invariant tensor [1]:

$$\underset{1}{v}_{\bar\sigma\bar\tau} \qquad\qquad (\underset{1}{v}_{\bar 1\bar 2} = i\underset{1}{v}_{\bar 3\bar 4} = 1).$$

With its help, we can associate any semi-vector $\lambda_{\bar\tau}$ with the star vector $\lambda_{\bar\tau}{}^{x}$ according to:

$$\underset{1}{v}_{\bar\sigma\bar\tau}\lambda^{\bar\tau} = \lambda_{\bar\sigma}{}^{x}.$$

In more detail, one has:

$$\lambda_{\bar 1}{}^{x} = \lambda_{\bar 2}, \qquad \lambda_{\bar 2}{}^{x} = -\lambda_{\bar 1}, \qquad \lambda_{\bar 3}{}^{x} = i\lambda_{\bar 4}, \qquad \lambda_{\bar 4}{}^{x} = i\lambda_{\bar 3},$$

which again has the consequence that:

$$(\lambda_{\bar\tau}{}^{x})^{x} = -\lambda_{\bar\tau}. \tag{64}$$

There are now semi-vectors $\lambda_{\bar\sigma}$ whose star vector $\lambda_{\bar\sigma}{}^{x}$ is proportional to the $\lambda_{\bar\sigma}$. In $\lambda_{\bar\sigma}{}^{x} = \rho\,\lambda_{\bar\sigma}$, from (64), one must have $\rho = \pm i$. We call a semi-vector $\lambda_{\bar\sigma}$ an *α-semi-vector*, and write it as $\underset{\alpha}{\lambda}_{\bar\sigma}$ if one has:

$$\underset{\alpha}{\lambda}_{\bar\sigma}{}^{x} = +\underset{\alpha}{\lambda}_{\bar\sigma},$$

and correspondingly, one calls a semi-vector a *β-semi-vector* when one has:

$$\underset{\beta}{\lambda}_{\bar\sigma}{}^{x} = -\underset{\beta}{\lambda}_{\bar\sigma}$$

for it. In more detail, these two relations read:

$$\left.\begin{array}{ll} \underset{\alpha}{\lambda}_{\bar 2} = i\,\underset{\alpha}{\lambda}_{\bar 1}, & \underset{\alpha}{\lambda}_{\bar 2} = -i\,\underset{\alpha}{\lambda}_{\bar 1}, \text{ resp.} \\ \underset{\alpha}{\lambda}_{\bar 4} = \underset{\alpha}{\lambda}_{\bar 3}, & \underset{\alpha}{\lambda}_{\bar 4} = -\underset{\alpha}{\lambda}_{\bar 3}. \end{array}\right\} \tag{64a}$$

One immediately sees that the absolute ";", like the "/" differential, leaves the character of an α-semi-vector (β-semi-vector, resp.) unchanged.

---

[1] If one were to choose, e.g., $\underset{2}{v}$, then that would only correspond to another numbering of the vectors of the vierbein.

Any semi-vector of the first kind can be decomposed additively into an α-semi-vector and a β-semi-vector of the first kind.

In an analogous way, one can, by means of the numerically-invariant tensor $(\underset{1}{u})$ introduce semi-vectors of the second kind that satisfy the conditions:

$$\left.\begin{array}{llll} \alpha-\text{semi-v. of the 2}^{\text{nd}}\text{ kind}: & \underset{\alpha}{\lambda}_{\bar{\bar{2}}} = -i\underset{\alpha}{\lambda}_{\bar{\bar{1}}}; & \beta-\text{semi-v. of the 2}^{\text{nd}}\text{ kind}: & \underset{\beta}{\lambda}_{\bar{\bar{2}}} = i\underset{\beta}{\lambda}_{\bar{\bar{1}}}; \\ & \underset{\alpha}{\lambda}_{\bar{\bar{4}}} = \underset{\alpha}{\lambda}_{\bar{\bar{3}}}; & & \underset{\beta}{\lambda}_{\bar{\bar{4}}} = -\underset{\beta}{\lambda}_{\bar{\bar{3}}}. \end{array}\right\} \tag{65}$$

The notation is chosen in such a way that the complex conjugate quantity to an α-semi-vector (β-semi-vector, resp.) of the one kind is an α-semi-vector (β-semi-vector, resp.) of the other kind.

The α-semi-vectors and β-semi-vectors are indeed two different symmetry types for the semi-vectors (as the symmetric and anti-symmetric tensors are for ordinary tensors of second rank, for example); however, they can (in contrast to the latter) be taken to each other by a simple algebraic operation, so, in a certain sense, they represent a single type (like, e.g., the usual covariant and contravariant vectors) [1].

In fact, if one constructs the semi-vector:

$$\underset{\alpha}{v}_{\bar{\sigma}\bar{\tau}}\underset{\beta}{\psi}^{\bar{\tau}}$$

with the help of the tensor $\underset{\alpha}{v}$ ($\underset{\beta}{v}$, resp.) that is defined in (14c), (15c) then it is an α-semi-vector (which we would like to call $\chi_{\bar{\sigma}}$). Conversely, when $\underset{\alpha}{v}$ acts on an α-semi-vector it gives a β-semi-vector. Due to the fact that (see (24a), (24b)):

$$\underset{\alpha}{v}_{\bar{\sigma}\bar{\tau}}\underset{\beta}{v}^{\bar{\sigma}}{}_{\bar{\rho}} = g_{\bar{\tau}\bar{\rho}}\,,$$

the latter is derivable from the relation:

$$\underset{\alpha}{\chi}_{\bar{\sigma}} = \underset{\alpha}{v}_{\bar{\sigma}\bar{\tau}}\underset{\beta}{\psi}^{\bar{\tau}}\,. \tag{66}$$

The proof of (66) follows from the equation that defines $\underset{\alpha}{v}$:

$$\underset{\alpha}{v}_{23} = iv_{14} = +1$$

when one recalls the symmetry properties of the β-semi-vectors.

Analogously, the α-vectors and β-vectors of the second kind can be associated with each other with the help of the of the $\underset{\alpha}{u}$ that are conjugate to $\underset{\alpha}{v}$.

---

[1] As we will show later, in the Dirac schema, the relation between the α-semi-vectors and the β-semi-vectors will actually represent the spin quantities in this way.

From what we said up to now, it emerges that any semi-vector equation can be decomposed uniquely into an α-equation and a β-equation. It is therefore natural to consider, instead of general semi-quantities (equations, resp.), those of the symmetry type α (β, resp.) that represent both types, although from what we just showed, essentially only one special type.

For the construction of differential equations, we need expressions of the type:

$$E^{r}{}_{\sigma\tau}\,\psi^{\sigma}{}_{;\,r}\,, \qquad h^{ar}\,E_{\alpha\sigma\tau}\,\psi^{\sigma}{}_{;\,r}\,, \text{ resp.}$$

How does one decompose such an expression into an α-tensor and a β-tensor when we construct an α-vector (β-vector, resp.) for it? Obviously, in order to do this one only needs to examine $E_{\varepsilon\sigma\tau}\,\psi^{\sigma}$ (= $E_{\varepsilon\bar{\tau}}$) relative to the index τ [1]. Naturally, we can now decompose the E-tensor (uniquely) relative to the indices σ and τ in the following way:

$$E_{\varepsilon\sigma\tau} = \underset{\alpha}{E_{\varepsilon\sigma\tau}} + \underset{\beta}{E_{\varepsilon\sigma\tau}} \qquad \text{(decomposition relative to the index } \sigma)$$

$$= (\underset{\alpha\alpha}{E_{\varepsilon\sigma\tau}} + \underset{\alpha\beta}{E_{\varepsilon\sigma\tau}}) + (\underset{\beta\alpha}{E_{\varepsilon\sigma\tau}} + \underset{\beta\beta}{E_{\varepsilon\sigma\tau}}) \quad \text{(decomposition of each relative to } \tau).$$

We now perform the decomposition [2]:

$$\left.\begin{aligned} E &= \underset{1}{E} + \underset{2}{E}, \\ E_1 &= \underset{\alpha\beta}{E_{\varepsilon\sigma\tau}} + \underset{\beta\alpha}{E_{\varepsilon\sigma\tau}}, \\ E_2 &= \underset{\alpha\alpha}{E_{\varepsilon\sigma\tau}} + \underset{\beta\beta}{E_{\varepsilon\sigma\tau}}. \end{aligned}\right\} \tag{67}$$

One easily proves that the inner product of two α-semi-vectors (two β-semi-vectors, resp.) vanishes. From this, one infers that relative to τ, $\underset{1}{E}{}_{\varepsilon\sigma\tau}\psi^{\sigma}$ has an α-character (β-character, resp.) when ψ is an α-semi-vector (β-semi-vector, resp.). Conversely, $\underset{2}{E}{}_{\varepsilon\sigma\tau}\psi^{\sigma}$ produces a β-quantity from an α-quantity ψ relative to τ and an α-quantity from a β-quantity ψ. How does one express this decomposition in terms of the constants $a_{(w)}$ that enter into E linearly? In order to do this, we would like to calculate $E_{\varepsilon\sigma\tau}\underset{\beta}{\psi}{}^{\sigma}$ and $E_{\varepsilon\sigma\tau}\underset{\alpha}{\psi}{}^{\sigma}$, since, from our definitions of $\underset{1}{E}$ and $\underset{2}{E}$, the following association exists:

[1] A tensor with a semi-index can naturally be decomposed relative to that index into one of α-type and one of β-type, in any case.

[2] The four special E-tensors that arise from the decomposition of the general E-tensor are numerically invariant, since they are composed of the invariant E and v. They therefore have the general form (48) with (naturally) special $a_{(t)}$.

$$\left.\begin{aligned}
E_{\varepsilon\sigma\tau}\underset{\beta}{\psi}^{\sigma} &= \begin{cases} \underset{\beta}{\chi}_{\varepsilon\tau} \text{ for } E = \underset{1}{E}, \\ \underset{\alpha}{\chi}_{\varepsilon\tau} \text{ for } E = \underset{2}{E}, \end{cases} \\
E_{\varepsilon\sigma\tau}\underset{\alpha}{\psi}^{\sigma} &= \begin{cases} \underset{\alpha}{\chi}_{\varepsilon\tau} \text{ for } E = \underset{1}{E}, \\ \underset{\beta}{\chi}_{\varepsilon\tau} \text{ for } E = \underset{2}{E}. \end{cases}
\end{aligned}\right\} \tag{68}$$

For our purpose, it suffices to examine the aforementioned association for $\varepsilon = 1$. The left-hand side of the first system in (68) yields, by calculating with $\tau = 1, \ldots, 4$:

$$\left.\begin{aligned}
&(a_1 - ia_2)\underset{\beta}{\psi}^{1} + (a_3 + a_4)\underset{\beta}{\psi},^{3} \\
&(a_2 + ia_1)\underset{\beta}{\psi}^{1} + (-ia_4 - ia_3)\underset{\beta}{\psi}^{3}, \\
&(a_3 + a_4)\underset{\beta}{\psi}^{1} + (-a_1 + ia_2)\underset{\beta}{\psi}^{3}, \\
&(a_4 + a_3)\underset{\beta}{\psi}^{1} + (-ia_2 + a_1)\underset{\beta}{\psi}^{3}.
\end{aligned}\right\} \tag{68a}$$

In order for this to be a $\underset{\beta}{\chi}_{\varepsilon\bar{\tau}}$, from (65), one must have $a_3 + a_4 = 0$. This is therefore a necessary condition for $\underset{1}{E}$.

One obtains the corresponding expressions (for $\varepsilon = 1$, $\tau = 1, \ldots, 4$) for the left-hand side of the second system in (68) from (68a) by changing the sign of the second term in all of the parentheses. The argument that is analogous to the above then gives the addition condition $a_3 - a_4 = 0$ for $\underset{1}{E}$.

Therefore, $\underset{1}{E}$ is characterized by $a_3 = a_4 = 0$. Analogously, the corresponding reasoning for $\underset{2}{E}$ from (68), (68a), …, gives $a_1 = a_2 = 0$.

We can now extend the decomposition of $\underset{1}{E}$ ($\underset{2}{E}$, resp.) according to the second and third equations in (67).

Namely, $\underset{\alpha\beta}{E}_{\varepsilon\sigma\tau}$ is an $\underset{1}{E}$ (i.e., $a_3 = a_4 = 0$) of the particular nature:

$$\underset{\alpha\beta}{E}_{\varepsilon\sigma\tau}\underset{\alpha}{\Psi}^{\sigma} = 0.$$

If one denotes the constants in $\underset{\alpha\beta}{E}_{\varepsilon\sigma\tau}$ that correspond to a_1, a_2 by A_1, A_2 then it follows from the (unwritten) system that corresponds to (68a) that $A_1 + iA_2 = 0$.

Analogously, one obtains the four constants B_1, B_2, 0, 0 for the second term in $\underset{1}{E}$ (viz., $\underset{\beta\alpha}{E_{\varepsilon\sigma\tau}}$), where $B_1 - B_2 = 0$. From $a_1 = A_1 + B_1$ and $a_2 = A_2 + B_2$, one obtains $A_1 = \frac{a_1 - ia_2}{2}$, $B_1 = \frac{a_1 + ia_2}{2}$.

An analogous argument gives the splitting of $\underset{2}{E}$. One thus obtains for the four E:

$$\left.\begin{aligned}
\underset{\alpha\beta}{E_{\varepsilon\sigma\tau}} &= \frac{a_1 - ia_2}{2} E_{\varepsilon\sigma\tau}(1, i, 0, 0), \\
\underset{\beta\alpha}{E_{\varepsilon\sigma\tau}} &= \frac{a_1 + ia_2}{2} E_{\varepsilon\sigma\tau}(1, -i, 0, 0), \\
\underset{\alpha\alpha}{E_{\varepsilon\sigma\tau}} &= \frac{a_3 + a_4}{2} E_{\varepsilon\sigma\tau}(0, 0, 1, 1), \\
\underset{\beta\beta}{E_{\varepsilon\sigma\tau}} &= \frac{a_3 - a_4}{2} E_{\varepsilon\sigma\tau}(0, 0, 1, -1),
\end{aligned}\right\} \tag{69}$$

where $E(a_1, a_2, a_3, a_4)$ shall characterize the dependency of E on the constants $a_{(w)}$.

From (69), the most general E may be linearly represented in terms of the four special E (69), which are determined completely by their α, β-character (up to a trivial factor). These four completely determined special E-tensors can, as in (66), take the α, β-semi-vectors to each other [1], so they play a role of a single tensor relative to their applications (like the way that the contravariant tensor is only a different way of writing a covariant tensor in the usual tensor analysis). We can now write down equations (58) for special (α, β) semi-vectors, perhaps with the use of $\underset{\alpha\beta}{E_{\varepsilon\sigma\tau}}$, $\underset{\beta\alpha}{E_{\varepsilon\sigma\tau}}$:

$$\left.\begin{aligned}
\underset{\alpha\beta}{E^{r}{}_{\bar{\sigma}\bar{\bar{\tau}}}}\, \underset{\beta}{\chi}{}^{\bar{\sigma}}{}_{;r} &= \alpha \underset{\beta}{\psi}{}_{\bar{\bar{\tau}}}, \\
\underset{\beta\alpha}{E^{r}{}_{\bar{\sigma}\bar{\bar{\tau}}}}\, \underset{\beta}{\psi}{}^{\bar{\bar{\tau}}}{}_{;r} &= \beta \underset{\beta}{\chi}{}_{\bar{\sigma}}
\end{aligned}\right\} \tag{70}$$

where

$$\underset{\alpha\beta}{E^{r}{}_{\bar{\sigma}\bar{\bar{\tau}}}} = h^{hr} \underset{\alpha\beta}{E_{h\bar{\sigma}\bar{\bar{\tau}}}}. \tag{71}$$

---

[1] One may determine quantities $\underset{\alpha\beta}{E_{\varepsilon\sigma\tau}}$, …, which are equal to the quantities that are given in (69), up to a constant factor, in such a way that they go to each other directly upon performing the corresponding $\underset{2}{\mu}$-operations ($\underset{2}{\nu}$-operations, resp., that are analogous to (66)). It is preferable to employ the E-tensors, thus normed, for the presentation of equations.

§ 10. Connection with spinors.

Since the special semi-vectors (α, β) have only two independent components, it is possible to associate any such semi-vector $\underset{\alpha}{\lambda}_{\bar\sigma}$ with a new sort of quantity with only two components, namely, an *α-spinor of the first kind*, when we set its components equal to:

$$\left.\begin{aligned}\underset{\alpha}{\lambda}_1 (= -i\,\underset{\alpha}{\lambda}_2) &= \underset{\alpha}{p}^1,\\ \underset{\alpha}{\lambda}_3 (= \underset{\alpha}{\lambda}_4) &= \underset{\alpha}{p}^2.\end{aligned}\right\} \tag{72}$$

Correspondingly, we associate the β-semi-vector of the first kind $\underset{\beta}{v}_{\bar\sigma}$ with the β-spinor of the first kind $\underset{\beta}{q}$ according to:

$$\left.\begin{aligned}\underset{\beta}{v}_1 (= i\,\underset{\beta}{v}_2) &= \underset{\beta}{q}_1,\\ \underset{\beta}{v}_3 (= -\underset{\beta}{v}_4) &= \underset{\beta}{q}_2.\end{aligned}\right\} \tag{72a}$$

The placement of the indices in p and q will be justified shortly. We introduce α-spinors and β-spinors of the second kind in a completely analogous way. From the definition, it then follows that the conjugate of an α-spinor (β-spinor, resp.) of the first kind is an α-spinor (β-spinor, resp.) of the second kind.

If one defines:

$$g^{\bar\sigma\bar\tau}\,\underset{\alpha}{\lambda}_{\bar\sigma}\,\underset{\beta}{\lambda}_{\bar\tau} = 2\,(\underset{\alpha}{p}^1\,\underset{\beta}{q}_1 + \underset{\alpha}{p}^2\,\underset{\beta}{q}_2) \tag{73}$$

then the justification for our choice of placement for the indices lies in this equation. With that, it becomes possible to drop the notations α and β.

For two a-quantities $\underset{\alpha}{\lambda}$ and $\underset{\alpha}{\mu}$, we now define the invariant:

$$\underset{2}{v}^{\bar\sigma\bar\tau}\,\underset{\alpha}{\lambda}_{\bar\sigma}\,\underset{\beta}{\lambda}_{\bar\tau} = 2i\,(p^1\,r^2 - p^2\,r^1), \tag{74}$$

where r is the spinor that corresponds to the semi-vector μ. It follows from (74) that:

$$\eta_{\bar\sigma\bar\tau} = \begin{vmatrix} 0 & 1 \\ -1 & 0 \end{vmatrix} \tag{75}$$

is a (covariant) spin tensor that one cares to use as the “metric tensor” in the theory of spinors. From (74), it further follows that the transformation of all spinors is unimodular.

One sees that the theory of spinors comes from the theory of semi-vectors. However, it seems that the semi-vector is preferable to the spinor, as a result of its simpler transformation law.

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