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Unified Theory of Gravitation and Electricity

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Up till now, the general theory of relativity was first and foremost a rational theory of gravitation and the metric properties of space. However, for the treatment of electromagnetic phenomena, one must be satisfied with a largely superficial incorporation of MAXWELL's theory into the relativistic schema. In addition to the quadratic metric form of gravitation one must introduce a linear form that is logically independent of it, one whose coefficients represent the potentials of the electromagnetic field. In the tensor equations of the gravitational field one finds, in addition to the curvature tensor, the covariantly written MAXWELL tensor of the electromagnetic field, which is linked to the latter only superficially and in a logically arbitrary way by a plus sign. This is quite painful, since MAXWELL's theory, as a field theory in the first approximation, is founded on empirical facts that are generally quite rich. It is not presently known whether the linearity of MAXWELL's actually does not correspond with reality, but only that the true equations of electromagnetism diverge from MAXWELL's for strong fields.

For that reason, the theoretician must take the trouble to formulate a presentation of general relativity in a logically unified theory of both fields. However, one cannot maintain that the appreciable efforts that have been applied to the problem heretofore have produced a satisfactory result. Since the onset of quantum mechanics, physics has generally turned away from these problems, as if it is assumed that the problems are, in a certain sense of the word, completely unsolvable within the framework of a field theory. In contrast to this opinion, we shall give a theory here that we believe represents a completely satisfactory and definitive solution, aside from quantum considerations. One arrives at the old formulas of gravitation and electricity in a new and thoroughly unified way. It shows that MAXWELL's equations, as they were introduced at the onset of general relativity, can be regarded as rigorous equations in the same sense as the gravitational equations in empty space.

The theory to be presented here connects psychologically with the well-known theory of KALUZA, but it avoids extending the physical continuum to one of five dimensions. KALUZA described the total field in a five-dimensional space by means of a five-dimensional metric tensor $g_{\mu\nu}$, in which $g_{11}, \ldots, g_{44}$ played the physical role of gravitational potentials, whereas $g_{15}, \ldots, g_{45}$ were interpreted as the electromagnetic potential, and the meaning of g_{55} was left open. In order to properly extend the four-dimensionality of the spacetime continuum the continuum was assumed to be "cylindrical" with respect to the coordinate x^5 $\left(\frac{\partial g_{ik}}{\partial x^5} = 0\right)$. KALUZA then succeeded in obtaining equations in an uncontrived way that agreed with the known gravitational

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[†] Translated by D.H. Delphenich.

equations (MAXWELL equations for the electromagnetic field, resp.) in the first approximation by proposing equations for the five-dimensional space that were completely analogous to the pure gravitational field equations of general relativity. The equations of geodetic lines in a five-dimensional space then represent the equations of motion for an electrically charged mass point.

The unsatisfactory aspect of KALUZA's theory resides in the assumption of a five-dimensional continuum, since the world of our experience seems to be four-dimensional. Furthermore, a cylindricality conditions seems hardly natural from the standpoint of a relativistic five-dimensional theory. Also, this theory does not correctly arrive at the constancy of the ratio of the electrical to the ponderable mass of a moving mass point. Finally, as we have already pointed out, it does not suggest a physical meaning for the component g_{55} of the metric tensor.

All of these difficulties will be avoided in the theory to be presented here, since one still remains in the four-dimensional continuum, but one introduces vectors that have five components and correspondingly, tensors whose indices range from 1 to 5. How this is possible will be explained in what follows. If this formal difficulty is overcome then one obtains the complete theory in a way that is entirely analogous to the path that was originally embarked upon in general relativity for the statement of the laws of a pure gravitational field or the statement of the general field laws in KALUZA's theory.

§1. Four-vectors and five-vectors.

At each point of a four-dimensional RIEMANNIAN space one is not only given a four-dimensional vector space V_4 that is composed of covariant and contravariant vectors, but also a five-dimensional linear vector space V_5. Let the components of a contravariant vector in the latter space be denoted by, e.g.:

$$a^i \qquad (i = 1, \ldots, 5).$$

Greek indices shall denote the components of a five-vector, and Latin ones, a four-vector. We denote the coordinates of a RIEMANNIAN space by x_i ($i = 1, \ldots, 4$).

By a five-dimensional (linear) vector space, we mean that each of its (contravariant) vectors is determined by five numbers a^ι, and that in a neighborhood of a^i, first, addition and, second, multiplication by a pure number (scalar) is defined in the usual way, but not outside that domain. Therefore, $\alpha a^\iota + \beta b^\iota$ (α and β are pure numbers) are the components of a vector in the vector space V_5 when this is true for a^ι and b^ι. The vector whose components are all zero is called the null vector.

A coordinate transformation in V_5 corresponds to equations of the form:

$$a^\iota = M^\iota{}_\tau \bar{a}^\tau, \tag{1}$$

in which:

$$|M^\iota{}_\tau| \neq 0. \tag{2}$$

The $M^{\iota}{}_{\tau}$ are generally functions of the x_i. Due to the homogeneous form of this transformation, the operations of forming the sum of five-vectors or multiplication by a scalar are independent of any special choice of coordinates.

We shall call quantities b_ι that are changed by a transformation (1) in such a way that for each a^ι:

$$b_\iota a^\iota \tag{3}$$

is invariant the components of a covariant five-vector. The operations of forming the sum and multiplication by a scalar are also meaningful for covariant vectors.

Just as the vectors of V_4 were measured by means of the metric tensor of R_4:

$$g_{ik}, \qquad \text{resp., } g^{ik} \qquad (g_i{}^k = \delta_i{}^k),$$

similarly, there is also a metric tensor (a non-degenerate symmetric tensor) for the five-vectors a^ι, b_ι:

$$g_{\iota\kappa}, \qquad \text{resp., } g^{\iota\kappa} \qquad (g_\iota{}^\kappa = \delta_\iota{}^k),$$

and we would now like to say, imprecisely, that a five-vector a has a contravariant (a^ι) and a covariant (a_ι) notation for which one has:

$$\left.\begin{aligned} a_\iota &= g_{\iota\kappa} a^k \\ a^\iota &= g^{\iota\kappa} a_\kappa \end{aligned}\right\}. \tag{4}$$

Up to this point, no relation exists between the five-vectors (a^ι) in V_5 and the four-vectors a^i in V_4. For that reason, we now introduce a "mixed" tensor:

$$\gamma_\iota{}^k, \tag{5}$$

that takes a vector in V_5 to a vector in V_4, and conversely:

$$a^k = \gamma_\iota{}^k a^\iota \tag{6}$$

$$b_\iota = \gamma_\iota{}^k b_k . \tag{7}$$

By means of the metric tensor for V_4 and V_5, we can write the projection tensor in the following equivalent ways:

$$\gamma_\iota{}^k,\ \gamma_{\iota k},\ \gamma^{\iota k},\ \gamma^\iota{}_k, \tag{8}$$

in which, e.g.:

$$\gamma^\iota{}_k = g^{\iota\sigma} g_{kl}\, \gamma_\sigma{}^l .$$

We will now consider the association that was established in (6) and (7) more closely. We make the assumption that the rank of the matrix:

$$\| \gamma_\iota{}^k \|$$

is equal to four. A four-dimensional vector space in V_5 is then given by $\gamma_\iota{}^k v_k$ (v_k arbitrary), which we call the "distinguished plane A."

The relation:

$$\gamma_\iota^k v_k = 0 \tag{9}$$

has only the solution $v_k = 0$, whereas:

$$\gamma_\iota^k A^\iota = 0 \tag{10}$$

has only the solution (up to a given factor) of the normal vector A^ι of the distinguished plane A ((10) then implies that $\gamma_\iota{}^k\, v_k A^\iota$ vanishes for any v_k). We will call A^ι the "distinguished direction in V_5," and we normalize it by setting ([1]):

$$g_{\iota\kappa} A^\iota A^\kappa = 1. \tag{11}$$

From (10) and (6), the distinguished direction A^ι is associated with the null vector in V_4, whereas (7) associated each vector (b_k) of V_4 with a vector of the distinguished plane in V_5.

The five-tensor ($g_{\iota\kappa}$), along with the mixed tensor ($\gamma^\iota{}_k$), determines the four-tensor ($g_{\iota\kappa}\, \gamma^\iota{}_i\, \gamma^\kappa{}_k$); from this fact, we shall assume that it is identical with the metric tensor g_{ik} on V_4. It thus satisfies the relation ([2]):

$$g_{\iota\kappa}\, \gamma^\iota{}_i\, \gamma^\kappa{}_k = g_{ik}\,. \tag{12}$$

The following relations are equivalent to (12):

$$\gamma_{\kappa p}\; \gamma^\kappa{}_q = g_{pq} \tag{12a}$$
$$\gamma_\kappa^p\; \gamma^\kappa{}_q = \delta^p{}_q \tag{12b}$$
$$\square^{\iota\kappa} \gamma_\iota^p\; \gamma_\kappa{}^q = g^{pq}\,, \tag{12c}$$

etc. We next compute the quantities:

$$\gamma_\kappa{}^p\; \gamma^\iota{}_p = \Sigma_\kappa{}^\iota. \tag{13}$$

Upon multiplying by $\gamma^\kappa{}_q$ and taking (12b) into account, we obtain:

([1]) However, this association also implies an assumption about the character of the metric on V_5.

([2]) If $g_{\iota\kappa}$ is non-degenerate and $\| \gamma^\iota{}_p \|$ has rank four then g_{pq} is also non-degenerate.

Proof: We show that from $g_{pq}x^p = 0$, …, (α), it follows that $x^p = 0$. From (a), it follows as a result of (12) that:

$$0 = g_{\iota\kappa}\, \gamma^\kappa{}_q \sigma^\iota = \sigma_\kappa \gamma^\kappa{}_q,$$

in which we have set:

$$\sigma^\iota = \gamma^\iota{}_p \sigma^\square\,.$$

From this, it follows that $\sigma_\kappa = \rho A_\kappa$, thus, also that $\sigma^\iota = \rho A^\iota$.

By setting both expressions for σ^ι equal to each other, it follows that:

$$\rho A^\iota = \gamma^\iota{}_p \sigma^\square,$$

from which, upon multiplying by A_ι (10) gives us that $\rho = 0$, and furthermore, the vanishing of $\sigma^\square$ follows from (9).

$$\gamma^{\iota}{}_q = \gamma^{\kappa}{}_q \Sigma_{\kappa}{}^{\iota},$$

or:

$$\gamma^{\kappa}{}_q (\delta^{\iota}{}_{\kappa} - \Sigma_{\kappa}{}^{\iota}) = 0.$$

From this, it follows that:

$$\delta^{\iota}{}_{\kappa} - \Sigma_{\kappa}{}^{\iota} = \rho^{\iota} A_{\kappa}. \tag{14}$$

Moreover, it follows from (13) upon multiplying by A^{κ} that:

$$\Sigma_{\kappa}{}^{\iota} A^{\kappa} = 0.$$

Thus, if we multiply (14) by A^{κ}, we obtain:

$$A^{\iota} = \rho^{\iota},$$

from which, (14) becomes:

$$\Sigma_{\kappa}{}^{\iota} = \delta^{\iota}{}_{\kappa} - A^{\iota} A_{\kappa}.$$

As a result (13) takes the form:

$$\gamma_{\kappa}{}^{p}\, \gamma^{\iota}{}_{p} = \delta^{\iota}{}_{\kappa} - A^{\iota} A_{\kappa}\,, \tag{13a}$$

or, if we lower the index ι:

$$g_{pq}\gamma_{\kappa}{}^{p}\, \gamma_{\iota}{}^{q} + A_{\kappa} A_{\iota} = g_{\kappa\iota}. \tag{13b}$$

In equations (12) and (13b), we have arrived at the relations that link the metric in V_4 with the one in V_5.

The coupling between vectors (b_k) in V_4 and vectors b_{ι} in V_5 that is expressed by (7) is one-to-one. If we multiply (7) by $\gamma^{\iota}{}_p$ and take (12b) into account then we obtain:

$$b_r = \gamma^{\iota}{}_r\, b_{\iota}, \tag{14}$$

which therefore completely determines the vector b_r in V_4.

On the other hand, if we multiply (6) by $\gamma^{\sigma}{}_k$ then (13a) gives:

$$\gamma^{\sigma}{}_k\, a^k = (\delta_{\iota}{}^{\sigma} - A_{\iota} A^{\sigma})\, a^{\iota} = a^{\sigma} - \rho A^{\sigma}, \quad (\rho = A_{\iota}\, a^{\iota}),$$

thus:

$$a^{\sigma} = \gamma^{\sigma}{}_k\, a^k + \rho A^{\sigma}.$$

The five-vector (a^{σ}) is therefore determined from the four-vector (a^k) only up to the addition of a vector that is proportional to (A^{σ}).

Tensors of arbitrary rank may be defined relative to V_5 in a manner that is completely analogous to the way they are defined relative to V_4, except that they are covariantly related by transformations of type (1). Likewise, one may construct mixed tensors, which are characterized by Latin and Greek indices; the projection tensor $\gamma^{\iota}{}_k$ is an example of this. An explicit statement for the construction of tensors from other tensors by summation, multiplication, and "reduction" (i.e., contraction) is not needed; the latter operation can only be naturally performed on two indices of the same type (i.e., two Latin indices or two Greek ones).

§ 2. Absolute Differential Calculus.

Absolute differential and absolute derivative of a five-vector: The transformation (1):

$$A^\iota = M^\iota{}_\tau \, \bar{a}^\iota,$$

is no longer true at a point da^ι $(= a^\iota(x^i + dx^i) - a^\iota(x^i))$ away from a^ι, since the $M^\iota{}_\tau$ are generally functions of x^i. We then introduce the absolute differential by applying the three-index quantities $\Gamma^\iota_{\pi q}$ (ι and π range from 1 to 5, and q ranges from 1 to 4):

$$\mathfrak{d}a^\iota = da^\iota + \Gamma^\iota_{\pi q} a^\pi dx^q, \tag{15}$$

in which we have:

$$\mathfrak{d}a^\iota = M^\iota{}_\tau \, \mathfrak{d}a^\tau,$$
$$da^\iota = d\bar{a}^\tau + \Gamma^\tau_{\pi q} a^\pi dx^q.$$

From this, we obtain the following transformation law for the Γ's:

$$M^\pi{}_\sigma \Gamma^\iota_{\pi q} = M^\iota{}_\tau \Gamma^\tau_{\pi q} - M^\iota{}_{\sigma,q} \quad \left(M^\iota{}_{\sigma,q} = \frac{\partial}{\partial x^q} M^\iota{}_{\sigma,}\right). \tag{16}$$

Likewise, we define the absolute differential of a covariant vector by:

$$\mathfrak{d}b_\iota = db_\iota - \Gamma^\pi_{\iota q} b_\pi \, dx^q, \tag{17}$$

in which this definition is chosen such that:

$$d(b_\iota \, a^\iota) = \mathfrak{d}b_\iota \, a^\iota + b_\iota \, \mathfrak{d}a^\iota. \tag{18}$$

The absolute differential of a five-tensor of arbitrary rank is then defined in the usual way. The absolute differential of a five-tensor is a five tensor of the same type.

We now denote the coefficients of dx^q in $\mathfrak{d}a^\iota$ *by* $a^\iota{}_{;q}$, such that we have:

$$a^\iota{}_{;\,q} = a^\iota{}_{,\,q} + \Gamma^\iota_{\pi q} a^\pi; \tag{15a)}$$

$a^\iota{}_{;q}$ behaves like a covariant four-vector under coordinate transformations (of x^i). $a^\iota{}_{;q}$ is therefore a mixed tensor, like $\gamma^\iota{}_q$, for instance. Therefore, mixed tensors arise from five-tensors under differentiation.

Absolute differentiation of four-vectors: If (τ^i) is a vector in V_4 then we define the absolute derivative as in RIEMANNIAN geometry through the equation:

$$\tau^i{}_{;\,q} = \tau^i{}_{,\,q} + \begin{Bmatrix} i \\ pq \end{Bmatrix} \tau^p, \tag{19}$$

in which the $\begin{Bmatrix} i \\ pq \end{Bmatrix}$ are the CHRISTOFFEL symbols that one constructs from the metric tensor g_{ik} on V_4. Therefore, let $T^{\cdot\cdot}_{\cdot;q}$ denote the corresponding RICCI derivative whenever $T^{\cdot\cdot}_{\cdot}$ has only Latin indices.

On the other hand, if $S^{\cdot\cdot}_{\cdot}$ is a mixed tensor then we define:

$$S^{\cdot\cdot}_{\cdot;q} = S^{\cdot\cdot}_{\cdot,q} + \Sigma\,(\bullet), \tag{20}$$

in which the sum has exactly as many summands as tensor indices, and, in fact:

$$\begin{array}{llll}
\text{a Greek index} & S^{\cdot\cdot\tau}_{\cdot\cdot} & \text{corresponds to a summand} & +\Gamma^{\tau}_{\sigma q} S^{\cdot\cdot\sigma}_{\cdot\cdot} \\
\text{“} & S^{\cdot\cdot}_{\cdot\cdot\pi} & \text{“} \qquad \text{“} & -\Gamma^{\sigma}_{\pi q} S^{\cdot\cdot}_{\cdot\cdot\sigma} \\
\text{a Latin index} & S^{\cdot\cdot i}_{\cdot\cdot} & \text{“} \qquad \text{“} & +\begin{Bmatrix} i \\ pq \end{Bmatrix} S^{\cdot\cdot p}_{\cdot\cdot} \\
\text{“} & S^{\cdot\cdot\cdot}_{\cdot\cdot p} & \text{“} \qquad \text{“} & -\begin{Bmatrix} r \\ pq \end{Bmatrix} S^{\cdot\cdot\cdot}_{\cdot\cdot r}\,.
\end{array}$$

This is an extension of the absolute differential calculus that was introduced by WAERDEN and BARTOLOTTI in a formally analogous case, and is chosen such that the following rules of calculation are valid:

$$\left.\begin{aligned} (A^{\cdot\cdot}_{\cdot} B^{\cdot})_{;q} &= A^{\cdot\cdot}_{\cdot;q} B^{\cdot} + B^{\cdot}_{;q} A^{\cdot}_{\cdot\cdot} \\ (A^{\cdot\cdot}_{\cdot} + B^{\cdot\cdot}_{\cdot})_{;q} &= A^{\cdot\cdot}_{\cdot;q} + B^{\cdot\cdot}_{\cdot\,;q} \\ \rho_{;q} &= \rho_{,q} \quad (\rho = \text{scalar}) \end{aligned}\right\}. \tag{21}$$

Naturally, just as in the ordinary RICCI calculus, we have here that:

$$0 = g_{ik;\,q} = g^{ik}{}_{;\,q} = \delta^k_{i\,;\,q}. \tag{22}$$

If a five-vector a^ι satisfies the relations:

$$\eth a^\iota = 0 = da^\iota + \Gamma^{\iota}_{\pi q} a^\pi dx^q,$$

along a curve segment in the space of x^i then we say that the vector is parallel displaced along the curve segment. The five-vector changes along the curve according to the equation:

$$da^\iota = -\,\Gamma^{\iota}_{\pi q} a^\pi dx^q. \tag{23}$$

Furthermore, the equation:

$$da^\iota = -\begin{Bmatrix} i \\ pq \end{Bmatrix} a^\pi dx^q \tag{24}$$

defines the parallel displacement of a four-vector (a^i) along a curve.

§ 3. Determination of the three-index symbol $\Gamma^{\iota}_{\pi q}$.

A manifold of the type that we have in mind here is now given when the tensors ($g_{\iota\kappa}$) and (γ^k_ι) are given, which then determine the four-dimensional metric tensor g_{pq} according to (12c). However, for a given GAUSSIAN coordinate system the tensors ($g_{\iota\kappa}$) and (γ^k_ι) are not entirely arbitrary, since the choice of coordinates in V_5 can be made arbitrarily. For that reason, (1) implies that of the 15 + 20 components of these tensors, only 25 can be chosen arbitrarily, such that for a given GAUSSIAN coordinate system, as in the case of the old gravitation theory, only 10 of them actually remain to characterize the manifold. The latter is therefore regarded as completely characterized when the quantities $\Gamma^{\iota}_{\pi q}$ are defined, which we shall now do. For that purpose, we introduce three axioms.

If (a^ι), resp. (a_ι), is a five-vector then one has the relation:

$$a_\iota = g_{\iota\kappa}\, a^\iota.$$

If we construct the absolute differential then we obtain:

$$\mathfrak{d} a_\iota = g_{\iota\kappa}\, \mathfrak{d} a^\kappa + a^\kappa\, \mathfrak{d} g_{\iota\kappa}.$$

From $\mathfrak{d} a_\iota = 0$, it then follows that $\mathfrak{d} a^\kappa = 0$ (and conversely) only when one has assumed that $\mathfrak{d} g_{\iota\kappa} = 0$, or that:

$$\mathfrak{d} g_{\iota\kappa;\, j} = 0. \tag{I}$$

Axiom I implies that the statement that the absolute differential of a five-vector (in a particular direction) vanishes has a well-defined meaning. This axiom can also be stated in the equivalent form that under a parallel displacement of two five-vectors (a^ι) and (b^κ), the form:

$$g_{\iota\kappa}\, a^\iota\, b^\kappa$$

remains unchanged. Axiom (I) gives 60 equations for the 100 quantities $\Gamma^{\iota}_{\pi q}$.

There is a one-to-one correspondence between a vector (a^ι) in the distinguished plane and a vector (a^k) in V_4:

$$a^\iota = \gamma^\iota{}_k\, a^j.$$

When (a^k) is displaced (not parallel) arbitrarily along a curve segement C in the four-dimensional space these equations imply that the vector a^ι in the distinguished plane A will be displaced along with it in a completely determined way, which implies the equation:

$$\mathfrak{d} a^\iota = \gamma^\iota{}_j\, \mathfrak{d} a^j + a^j\, \mathfrak{d}\gamma^\iota{}_j. \tag{25}$$

Our second axiom is the following one: under parallel displacement of (a^k) (i.e., $\eth a^j = 0$) the absolute differential of the vectors that are displaced by it lies in the distinguished direction A^ι. This means that in (25) when $\eth a^j = 0$ the $\eth a^\iota$ must be proportional to A^ι, or that (for arbitrary a^k, dx^q):

$$a^k \, dx^q \, \gamma^\iota_{j\,;\,q}$$

must be proportional to A^ι. It must then be true that:

$$\gamma^\iota_{j\,;\,q} = A^\iota F_{kq}, \tag{II}$$

in which F_{kq} is a four-tensor of rank two.

Our third axiom is a specialization of the following one: If the parallel displacement of the vector (a^k) proceeds in its proper direction ($dx^q = \rho a^q$) then so will the vector (a^ι) in the distinguished plane that is associated with it ($\eth a^\iota = 0$). This implies the condition:

$$0 = \gamma^\iota_{j\,;\,q} \, a^k \, a^q = A^\iota F_{kq} \, a^k \, a^q,$$

or, since a^k is an arbitrary vector:

$$F_{kq} = - F_{qk} \,. \tag{III}$$

The consistency of axioms I, II, and III will be proved later.

If one multiplies (II) by A_ι then from (11) and (10) one obtains:

$$F_{kq} = A_\iota \, \gamma^\iota_{j\,;\,q} = -\gamma^\iota{}_q \, A_{\iota\,;\,j} \,. \tag{26}$$

Upon multiplying by $\gamma_\sigma{}^q$, (13a) further yields:

$$\gamma_\sigma{}^q \, F_{kq} = - \left(\delta_\sigma^\iota - A_\sigma A^\iota\right) A_{\iota\,;\,q} = - A_{\sigma;\,q} + A_\sigma A^\iota A_{\iota\,;\,q}$$

or, due to the vanishing of $A^\iota A_{\iota\,;\,q}$ ($= \frac{1}{2} \, (A^\iota A_\iota)_{;\,q}$):

$$A_{\sigma\,;\,q} = \gamma_\sigma{}^k F_{qk} \,. \tag{27}$$

The ultimate conclusion of this section is the following one: If one subjects the Γ’s to the simple and reasonable conditions I, II, III then they are not determined completely for given g_{ik} and $\gamma_\iota{}^k$, but only up to an anti-symmetric tensor F_{kq} that one is free to choose. It will be shown that this, together with the RIEMANNIAN metric tensor g_{ik} , completely determines the properties of manifold in question.

It is instructive to recast the problem that is treated here in the context of a RIEMANN space R_m that is embedded in a RIEMANN space R_n of higher dimension. Then there would be two metrics defined in this problem: one of them belongs to R_m and the other one belongs to R_n. R_m corresponds to the four-dimensional space, whereas, in our case, instead of R_n, we are given only a vector space of dimension n (= 5) at every point of R_m.

$x^\iota = x^\iota (y^1, \ldots, y^m)$ ($\iota = 1, \ldots, n$) is the analytical representation of the subspace. A point of R_m is associated with the metric $g_{pq}\, dy^p\, dy^q$ in R_m and the metric $g_{\iota\kappa}\, dx^\iota\, dx^\kappa$ in R_n. $\frac{\partial x^\iota}{\partial y^p} = \gamma^\iota{}_p$ is the projection tensor here. A^ι, which is defined by $\frac{\partial x^\iota}{\partial y^p} A_\iota = 0$, is the normal to R_m at the point in question. Relations I and II are also valid in this case. However, F_{pq} is symmetric for this problem, and in the case $n = m + 1$ it is known as the "second fundamental form." It is then ultimately axiom III that distinguishes the spatial structure that we consider here from the one in the problem of the embedded manifold. Thus, we also find the difference between the momentum that follows from our theory and the one that follows from KALUZA's theory.

§ 4. Concerning the straightest lines in V_5.

If we parallel displace a vector (a^ι) in V_5 in the direction $a^k = \gamma_\iota{}^k\, a^\iota$ that is associated with it in V_4 then a curve will be determined in coordinate space, whose equation we will now derive.

For an appropriate choice of parameter t we can set $a^k = \gamma_\iota{}^k\, a^\iota = \frac{dx^k}{dt}$. From:

$$a^k = \gamma_\iota{}^k\, a^\iota,$$

it then follows by differentiation that since $\mathfrak{d} a^\iota = 0$:

$$\mathfrak{d} a^k = \gamma_\iota{}^k{}_{;r}\, a^\iota\, a^r\, dt$$

or, from II:

$$\frac{d^2 x^k}{dt^2} + \begin{Bmatrix} k \\ pq \end{Bmatrix} \frac{dx^p}{dt} \frac{dx^q}{dt} = A_\iota\, a^\iota\, F^k{}_r \frac{dx^r}{dt}\ . \tag{28}$$

However, one now has:

$$\frac{\mathrm{d}(A_\iota a^\iota)}{\mathrm{d}t} = A_{\iota\,;\,p}\, a^p\, a^\iota + A_\iota \frac{\mathrm{d}a^\iota}{\mathrm{d}t},$$

in which the second term on the right-hand side vanishes due to the fact that $\mathfrak{d} a^\iota = 0$. However, the first term on the right-hand side also vanishes. (27) then implies that:

$$A_{\iota\,;\,p}\, a^p\, a^\iota = \gamma_\iota{}^k\, F_{pk}\, a^p\, a^\iota = F_{pk}\, a^p\, a^k = 0.$$

Thus, $A_\iota\ a^\iota = z$ is constant along the curve, such that (28) can be written in the form:

$$\frac{d^2x^k}{dt^2} + \begin{Bmatrix} k \\ pq \end{Bmatrix} \frac{dx^p}{dt}\frac{dx^q}{dt} = z\, F^k_{\ r} \frac{dx^r}{dt} \qquad (z = \text{const.}). \tag{28a}$$

What the parameter t entails is that along the curve $\mathfrak{d}(g_{\iota\kappa}\, a^\iota\, a^\kappa) = 0$, hence, $g_{\iota\kappa}\, a^\iota\, a^\kappa =$ const., or, from (13b):

$$g_{pq}\frac{dx^p}{dt}\frac{dx^q}{dt} + z^2 = \text{const.}$$

or:

$$g_{pq}\frac{dx^p}{dt}\frac{dx^q}{dt} = \text{const.}$$

By restriction, we can therefore introduce the arclength, which is defined by:

$$ds^2 = -\, g_{pq}\, dx^p\, dx^q ,$$

as a parameter for the timelike curves in (28a), from which the constant ρ that appears in (28a) is also fixed.

Equation (28a) corresponds precisely to the relativistic equation of motion for an electrically charged mass point, not merely approximately, as in KALUZA's theory. In particular, it is noteworthy that in our theory the ratio ρ of the electrical to the ponderable mass must be strictly constant.

§ 5. Curvature and V_5.

The integrability conditions for parallel displacement:

$$\mathfrak{d}a^q = 0,$$

or:

$$da^\sigma = -\Gamma^\sigma_{\iota p}\, a^\iota\, dx^p$$

resp., take the form:

$$P^\sigma_{\ \iota qp}\, a^\iota = 0, \tag{29}$$

in which:

$$P^\sigma_{\ \iota qp}\, a^\iota = -\Gamma^\sigma_{\iota q,p} + \Gamma^\sigma_{\iota p,q} + \Gamma^\sigma_{\tau q}\Gamma^\tau_{\iota p} - \Gamma^\sigma_{\tau p}\Gamma^\tau_{\iota q}. \tag{30}$$

From (29), it follows that the vanishing of (30) has an invariant character. The proof that $P^\sigma_{\ \iota qp}$ is a mixed tensor of the type that is expressed by the indices is achieved in the following way: We consider the two-dimensional manifold $x^q = x^q(u, v)$, which defines the two directions $\frac{\partial x^i}{\partial u}, \frac{\partial x^i}{\partial v}$ at each of its points. If the point in question is given, along with the five-vector a^ι in a neighborhood of it, then we construct:

$$\frac{\mathrm{d}}{\mathrm{d}v}\left(\frac{\mathrm{d}a^\iota}{\mathrm{d}u}\right)-\frac{\mathrm{d}}{\mathrm{d}u}\left(\frac{\mathrm{d}a^\iota}{\mathrm{d}v}\right).$$

We have:

$$\frac{\mathrm{d}a^\iota}{\mathrm{d}u}=\frac{\partial a^\iota}{\partial u}+\Gamma^\iota_{\pi q}a^\pi\frac{\partial x^q}{\partial u},$$

and furthermore:

$$\frac{\mathrm{d}}{\mathrm{d}v}\left(\frac{\mathrm{d}a^\iota}{\mathrm{d}u}\right)=\left(\frac{\partial a^\iota}{\partial u}+\Gamma^\iota_{\pi q}a^\pi\frac{\partial x^q}{\partial u}\right)_{,v}+\Gamma^\iota_{\sigma p}\left(\frac{\partial a^\sigma}{\partial u}+\Gamma^\sigma_{\pi q}a^\pi\frac{\partial x^q}{\partial u}\right)\frac{\partial x^p}{\partial v}.$$

From this, it follows that:

$$\frac{\mathrm{d}}{\mathrm{d}v}\left(\frac{\mathrm{d}a^\iota}{\mathrm{d}u}\right)-\frac{\mathrm{d}}{\mathrm{d}u}\left(\frac{\mathrm{d}a^\iota}{\mathrm{d}v}\right)=P^\sigma{}_{\iota pq}\,a^\pi\,\frac{\partial x^p}{\partial v}\,\frac{\partial x^q}{\partial u}, \tag{31}$$

from which the tensor character of $P^\sigma{}_{\iota pq}$ emerges.

Naturally, in a space with the sort of structure that we are examining there also exists the usual RIEMANN curvature that one constructs from the g_{ik}, just as one also has geodetic lines that are constructed from the g_{ik}. However, the newly acquired knowledge resides precisely in the fact that the definitive structures for the physical laws are the same as the ones that are obtained from parallel displacement of five-vectors by means of the Γ's.

On the other hand, it is clear that the mathematical model thus defined can only relate to physical laws in the context of the four-dimensional space (the tangent space V_4, resp.) Thus, the fact that the latter definitive expressions are ultimately expressible in terms of only the four-tensors g_{ik} and F_{ik}, is connected with the fact that $\gamma^\iota{}_\kappa$ and $g_{\iota\kappa}$ no longer appear explicitly in these expressions (cf. eq. (28a)); these facts will be more clearly distinguished from each other in the next section.

To that end, it is appropriate to look for the relation that exists between the five-curvature (30) and the (RIEMANNIAN) four-curvature. We start with equations (II) and (27) ([1]):

$$\gamma_{\iota k\,;\,q}=A_\iota\,F_{kq} \tag{II}$$
$$A_{\iota;\,p}=\gamma_{\iota k}\,F_p{}^k. \tag{27}$$

By repeated absolute differentiation, one obtains:

$$\gamma_{\iota k\,;\,q\,;\,p}=F_{kq\,;\,p}A_\iota+F_{kq}\,F_p{}^r\,\gamma_{\iota r} \tag{IIa}$$
$$A_{\iota\,;\,p\,;\,q}=F_{kq}\,F_p{}^r\,\gamma_{\iota r}+\gamma_{\iota k}\,F_p{}^k{}_{;\,q} \tag{27a}$$

and thus:

$$\gamma_{\iota k\,;\,q\,;\,p}-\gamma_{\iota k\,;\,p\,;\,q}=A_\iota\,(F_{kq\,;\,p}-F_{kp\,;\,q})+\gamma_{\iota r}\,(F_{kq}\,F_p{}^r-F_{kp}\,F_q{}^r) \tag{IIb}$$
$$A_{\iota\,;\,p\,;\,q}-A_{\iota\,;\,q\,;\,p}=\gamma_{\iota r}\,(F_p{}^k{}_{;\,q}-F_{kp}\,F_q{}^k{}_{;\,p}). \tag{27b}$$

([1]) The Γ's are uniquely computable from equations (II) and (27). If $F_{pq}+F_{qp}=0$ then, as one easily sees, axioms I and III are satisfied, which shows their consistency. Now, only the validity of I remains to be proved, but this follows easily from an application of (13b).

By explicit evaluation of the left-hand side of these equations one obtains from a single computation:

$$\gamma_{\iota k\,;\,q\,;\,p} - \gamma_{\iota k\,;\,p\,;\,q} = P^{\sigma}{}_{\iota qp}\,\gamma_{\sigma k} - R^{\,r}{}_{\iota pq}\,\gamma_{\iota r} \tag{32}$$

$$A_{\iota\,;\,p\,;\,q} - A_{\iota\,;\,q\,;\,p} = P^{\sigma}{}_{\iota qp}\,A_{\sigma}\,, \tag{33}$$

in which R refers to the RIEMANNIAN four-curvature:

$$R^{\,r}{}_{\iota pq} = -\begin{Bmatrix} r \\ kq \end{Bmatrix}_{,p} + \begin{Bmatrix} r \\ kp \end{Bmatrix}_{,q} + \begin{Bmatrix} r \\ tq \end{Bmatrix}\begin{Bmatrix} t \\ kp \end{Bmatrix} - \begin{Bmatrix} r \\ tp \end{Bmatrix}\begin{Bmatrix} t \\ kq \end{Bmatrix}. \tag{34}$$

From (IIb), (27b), (32), and (33) one obtains the desired relations:

$$P^{\sigma}{}_{\iota qp}\,\gamma_{\sigma k} = A_{\iota}\,(F_{kq\,;\,p} - F_{kp\,;\,q}) + \gamma_{\iota r}\,\{R^{\,r}{}_{\iota pq} + F_{kq}\,F_{p}{}^{r} - F_{kp}\,F_{q}{}^{r}\,\} \tag{35}$$

$$P^{\sigma}{}_{\iota qp}\,A_{\sigma} = \gamma_{\iota r}\,(F_{q}{}^{r}{}_{;\,p} - F_{p}{}^{r}{}_{;\,q}). \tag{36}$$

Multiplying (35) by $\gamma^{\tau k}$, and taking into account the fact that $\gamma_{\sigma k}\,\gamma^{\tau k} = \gamma_{\sigma}{}^{k}\,\gamma^{\tau}{}_{k} = \delta_{\sigma}{}^{\tau} - A_{\sigma}A^{\tau}$, along with (36), yields:

$$\begin{aligned} P^{\tau}{}_{\iota qp} = {} & -\gamma_{\iota r}\,A^{\tau}(F_{p}{}^{r}{}_{;\,q} - F_{q}{}^{r}{}_{;\,p}) + \gamma^{\tau k}\,A_{\iota}\,(F_{kq\,;\,p} - F_{kp\,;\,q}) \\ & + \gamma_{\iota r}\,\gamma^{\tau k}\,(R^{\,r}{}_{\iota pq} + F_{kq}\,F_{p}{}^{r} - F_{kp}\,F_{q}{}^{r}\,)\,. \end{aligned} \tag{37}$$

It must be remarked that the (BIANCHI) identity is also valid for the five-curvature tensor:

$$P^{\tau}{}_{\iota pq;\,r} + P^{\tau}{}_{\iota qr;\,p} + P^{\tau}{}_{\iota rp;\,q}\ ;\ 0. \tag{38}$$

It is simplest to prove this by considering a transformation of the coordinates (the x^i) that makes the $\begin{Bmatrix} r \\ pq \end{Bmatrix}$ vanish, and a the transformation (1) of the five-coordinates the $\Gamma^{\tau}_{\sigma q}$ vanish, as well, which is possible, according to (16).

Furthermore, we use (37) to construct the contractions:

$$P_{\iota p} = \gamma_{\tau}{}^{q}\,P^{\tau}{}_{\iota qp} = A_{\iota}\,F_{p}{}^{k}{}_{;\,k} + \gamma_{\iota r}\,(R^{\,r}{}_{r} - F_{kp}\,F^{kr}\,) \tag{39}$$

$$P = \gamma^{\iota p}\,P_{\iota p} = R - F_{kp}\,F^{kp}, \tag{40}$$

and [1]:

$$U_{\iota p} = P_{\iota p} - \tfrac{1}{4}\,\gamma_{\iota p}\,(P + R) = A_{\iota}\,F_{p}{}^{k}{}_{;\,k} + \gamma_{\iota}{}^{r}\,\{(R_{rr} - \tfrac{1}{2}g_{rp}R) - (F_{kr}\,F^{k}{}_{p} - \tfrac{1}{4}\,F_{kp}\,F^{kr}\,)\}. \tag{41}$$

---

[1] The RIEMANNIAN R is introduced into definition (41), even though it is not derived from algebraic operations on the five-curvature tensor. The justification for this is given by the identity (45), in which the derivation of R is (implicitly) expressed in terms of five-tensors.

Finally, if we multiply (37) by $-A_\tau \gamma^\iota_i$, cyclically permute the indices $i\ p\ q$, add, and divide by two, then this yields the anti-symmetric tensor:

$$N_{ipq} = F_{pi\,;\,q} + F_{iq\,;\,p} + F_{qp\,;\,i} = F_{pi\,,\,q} + F_{iq\,,\,p} + F_{qp\,,\,i}\,. \tag{42}$$

§ 6. The field equations.

In the following, we shall address both of the well-known identities:

$$(R_r^p - \tfrac{1}{2}\delta_r^p\, R)_{;\,p}\ ;\ 0 \tag{43}$$

$$(F_{kr}\, F^{kp} - \tfrac{1}{4}\delta_r^p\, F_{kl}\, F^{kl})_{;\,p}\ ;\ F_{kr}\, F^{kp}{}_{;\,p} + \tfrac{1}{2}(F_{pk\,;\,r} + F_{kr\,;\,p} + F_{rp\,;\,k})\, F^{kp}\,, \tag{44}$$

the first of which is most comfortably derived by twice contracting the BIANCHI identity for the usual curvature tensor, whereas the second one is easily verified directly. With the help of (43), (44), (II) and (27), one obtains, by taking the divergence in the index p in (41):

$$U_\iota^p{}_{;\,p} - \tfrac{1}{4}\gamma_\iota{}^r\, N_{rkp}\, F^{kp}\ ;\ 0\,. \tag{45}$$

If one thus proposes the following field equations:

$$U_{ip} = 0\,, \tag{46}$$

$$N_{rkp} = 0 = F_{pk\,;\,r} + F_{kp\,;\,r} + F_{pr\,;\,k}\,, \tag{47}$$

then the identity (45) exists between them. If one multiplies (41) by $\gamma^\iota{}_q$ on one side and by A^ι on the other then one recognizes that (46) splits into the equations:

$$(R_{qp} - \tfrac{1}{2}g_{qp}\, R) - (F_{kq}\, F^k{}_p - \tfrac{1}{4}g_{qp}\, F_{kl}\, F^{kl}) = 0\,, \tag{46a}$$

$$F^{pk}{}_{;\,k} = \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^k}(\sqrt{g}F^{pk}) = 0\,, \tag{46b}$$

which no longer includes the g_{ik} and F_{ik}, just as one could also say about (47). Thus, we have arrived at the same equations that have already been regarded as the field equations of gravitation and electricity in general relativity up till now, except that equation (46) subsumes the gravitational equations and the first MAXWELL equation in a single system of equations and all three systems of equations need to be connected with the "curvature." There are no corpuscles in this theory, or – what amounts to the same thing – they are included as singularities.

§ 7. Introduction of special coordinates in V_5.

Among all of the possible choices for a coordinate system in V_5, one of them seems most natural: the one that makes $\gamma^\iota{}_p$ equal to $\delta^\iota{}_p$ and A^ι equal $\delta^\iota{}_5$ ($\delta^\iota{}_p$ equals 1, resp., 0, whenever $\iota = p$ or $\iota \neq p$, resp. $\delta^\iota{}_5$ equals 1, resp., 0, whenever $\iota = 5$ or $\iota \neq 5$, resp.).

If:

$$a^\iota = M^\iota{}_\tau \, \bar{a}^\tau$$

is a transformation to new five-coordinates then we have:

$$\gamma^\iota{}_p = M^\iota{}_\tau \, \bar{\gamma}^\tau{}_p$$
$$A^\iota = M^\iota{}_\tau \, \bar{A}^\tau .$$

Thus, if we would like to have $\bar{\gamma}^\tau{}_p = \delta^\iota{}_p$, $\bar{A}^\tau = \delta^\tau{}_5$ then we only need to choose:

$$M^\iota{}_p = \gamma^\iota{}_p$$
$$M^\iota{}_5 = A^\iota .$$

From the equation:

$$\bar{b}_\kappa = M^\iota{}_k \, b_\iota ,$$

it then also follows that:

$$\bar{\gamma}_\kappa^{\,p} = \delta_\kappa^p \qquad (= 1 \text{ for } \kappa = p; = 0 \text{ for } \kappa \neq p)$$
$$\bar{A}_\kappa = \delta_\kappa^5 \qquad (= 1 \text{ for } \kappa = 5; = 0 \text{ for } \kappa \neq 5) .$$

When we omit the bar by an appropriate transformation we then have:

$$\gamma^\iota{}_p = \delta^\iota{}_p , \qquad \gamma_\iota{}^p = \delta_\iota{}^p , \qquad A^\iota = \delta^\iota{}_5 , \qquad A_\iota = \delta_\iota^5 . \tag{48}$$

If we now perform a transformation on the space coordinates x^ι then since (48) continues to be true we must transform the five-coordinates along with them, in order that the a^ι, resp., a_ι ($\iota = 1, \ldots, 4$) behave like the components of a contravariant, resp., covariant four-vector, whereas a^5, resp., a_5, transform as an invariant.

However, it would be false to conclude that a five-vector can be regarded as a sort of "sum" of a four-vector and a scalar. Two quantities are equal only when there is no well-defined operation that produces differing results from them. For instance, the absolute differential of a five-vector is different from a vector that "equals" one of the form: four-vector + scalar.

In the new coordinate system one encounters the notational difficulty of distinguishing the first four components of the five-vector a^ι from the corresponding components of the four-vector $a^k = \gamma_\iota{}^k \, a^\iota$, which are numerically equal to them. We shall write $\underline{a}^k$ for the first four components of the five-vector a^κ. We then have the numerical relation:

$$\underline{a}^k = a^k, \tag{49}$$

in place of the relation:

$$\gamma_\iota^k \, a^\iota = a^k \, .$$

Analogously, we shall write $T_{\cdot\cdot\underline{k}}^{\cdot}$ when we mean one of the first four index values of $T_{\cdot\cdot\kappa}^{\cdot}$.

In our coordinate system the distinguished plane A has the equation $a^5 = a_5 = 0$. From the fact that $A_\iota \, a^\iota = g_{\iota\kappa} \, a^\iota \, a^\kappa = 0$ for vectors in the plane A, we conclude that:

$$g_{5\kappa} = 0 \, , \tag{50}$$

and, from the fact that $g_{\iota\kappa} A^\iota A^\kappa = 1$:

$$g_{55} = 1. \tag{51}$$

Here, equation (12) takes the form:

$$g_{\underline{i}\underline{k}} = g_{ik} \, . \tag{52}$$

We regard the parallel displacement of five-vectors as being characterized by the Γ's.

Axiom I ($g_{\iota\kappa\,;\,q} = 0$), with the help of (52), (50), (51), gives:

$$\left.\begin{aligned} g_{jk,\,q} - \Gamma^s_{\underline{j}q} g_{sk} - \Gamma^s_{\underline{k}q} g_{js} &= 0 \\ -\Gamma^s_{\underline{j}q} g_{sk} - \Gamma^5_{\underline{k}q} &= 0 \\ -\Gamma^5_{5q} &= 0 \end{aligned}\right\} . \tag{53}$$

Axiom II said that under parallel displacement of the four-vector a^i the invariant increment $\eth a^i$ of the vector $a^\iota = \gamma^\iota_k \, a^k$ that is associated with it in the plane A points in the direction A^ι in order that $\eth a^i$ should vanish for our choice of coordinate system.

From the fact that:

$$da^i + \begin{Bmatrix} i \\ pq \end{Bmatrix} a^p \, dx^q = 0$$

it thus follows that:

$$da^{\underline{i}} + \Gamma^{\underline{i}}_{\underline{p}q} a^{\underline{p}} dx^q = 0 \qquad\qquad (a^5 = 0) \, ,$$

or, since $a^{\underline{i}} = a^i$:

$$\Gamma^{\underline{i}}_{\underline{p}q} = \begin{Bmatrix} i \\ pq \end{Bmatrix} \, . \tag{54}$$

Axiom III says: If we parallel displace the vector a^k in its proper direction then the invariant increase $\eth a^i$ of the vector $a^\iota = \gamma^\iota_k \, a^k$ that is associated with it in the plane A also vanishes. Along with $\eth a^i$, $\eth a^5$ must also vanish:

$$\eth a^5 = da^5 + \Gamma^{\underline{i}}_{\underline{p}q} a^{\underline{p}} dx^q = 0 \, ,$$

which, since $a^5 = 0$ (and $da^5 = 0$) and $a^{\underline{p}} = a^p$ has the consequence that:

$$\Gamma^5_{\underline{p}q} = -\Gamma^5_{q\underline{p}} \, , \tag{55}$$

and indeed Γ^5_{pq} is the expression for the quantity that we denoted by F_{pq} (the electromagnetic field strength). (53), (54), (55) show that the Γ's are completely determined by the g_{ik} and F_{ik} (for a fixed coordinate system in V_5).

The use of special coordinates has the advantage that the equations that result from the elimination of the superfluous field variables can be written simply. However, one must distinguish the index 5 in order to do this, from which the number of equations increases, which complicates one's understanding of the naturally formal connection. For this reason, we have employed general coordinates in V_5 from the outset in our presentation; however, it should be remarked that we were first led to this theory by considerations that were thoroughly similar to the ones in this section. We therefore dispense with them in order for the usual previously-given considerations and results to carry over into the special coordinate system.

§ 8. Field equations and the law of motion.

It will now be shown that the field equations that we postulated in § 6 have a natural relationship with the law of motion that was presented independently of them in § 4. Thus, one must observe that the nature of material particles is not defined in this theory, such that they must be treated as singular points. Instead of this, it is simpler to add a fictitious term to the equations that expresses the density of matter; thus, one can concern oneself with continuous functions, which is computationally simpler.

We assume that equation (47) is exactly valid everywhere, even where "matter" is present (excluding magnetic matter). It then follows from (45) that the equation:

$$U_\iota{}^p{}_{;\,p} = 0 \tag{56}$$

is satisfied exactly everywhere. By contrast, we must insert the mixed tensor of matter density into the right-hand side of (46). By analogy with the primitive Ansatz for this, which represents irrotational (dust-like) matter in the old gravitation theory, in place of (46), we set:

$$U^{\iota p} = \mu\, \xi^\iota\, \xi^p\,. \tag{57}$$

(ξ^ι) is a five-vector that is associated with the four-vector $(\gamma_\iota{}^p\, \xi^\iota)$, or (ξ^p). From (56), one obtains:

$$(\mu\, \xi^p)_{;\,p}\, \xi^\iota + \mu\, \xi^\iota{}_{;\,p}\;\; \xi^p = 0\,. \tag{58}$$

From (57), it follows that μ is then determined once one normalizes the "magnitude" of ξ^ι, i.e., when one sets:

$$\xi^\iota\, \xi_\iota = \text{const.} \tag{59}$$

Then, since $\xi_\iota\, \xi^\iota{}_{;\,p} = \frac{1}{2}(\xi_\iota\, \xi^\iota\,)_{;\,p} = 0$ multiplying (58) by ξ_ι gives:

$$(\mu\, \xi^p)_{;\,p} = 0\,, \tag{60}$$

from which (58) becomes:

$$\xi^{\iota}{}_{;p}\;\xi^{p} = 0. \tag{61}$$

We now turn our attention to curves in the coordinate space that are "tangent" to the ξ^{p}-field; thus, for a suitable choice of parameter:

$$\frac{dx^{p}}{dt} = \xi^{p}\,(x^1, \ldots, x^4) \tag{62}$$

is the system of differential equations that defines these curves.

Equation (61) then states that along such a curve, one has $\delta\xi^{\iota} = 0$, i.e., that ξ^{ι} is parallel displaced in the direction that is associated with it ($\xi^{p} = \gamma^{\iota}{}_{p}\,\xi^{\iota}$).

The curves (62) that result from this displacement were treated in § 4, and satisfy the system of equations:

$$\frac{d^2x^k}{dt^2} + \left\{ {k \atop pq} \right\} \frac{dx^p}{dt}\frac{dx^q}{dt} = \rho\, F^{k}{}_{r}\,\frac{dx^r}{dt}\,, \qquad \rho = \text{const.} \tag{63}$$

From equation (60) one can further show that these curves represent the "trajectories of matter." If we consider a filament formed of one such curve then (60) (the equation of continuity) states that the vanishing, resp., non-vanishing, of the density function μ is propagated along the filament, or, more precisely, that the "mass" is constant along such a filament, which is, however, equivalent in content to the assertion.

The theory that was presented here gives the equations of the gravitational field and the electromagnetic field informally in a unified way; by contrast, they presently give us no understanding of the structure of particles that would be comparable to that which is summarized in the facts of quantum theory.

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