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On the invariant theory of systems of Pfaff equations

(Second communication)

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In a prior publication (these Berichte, 1889, pp. 157-176), I had already developed some theorems on systems of **Pfaff** equations. I started from the fact that every such system admits two different conceptions (*loc. cit.*, pp. 158). Namely, one can first consider it to be a system of differential equations, but secondly it can also be interpreted by saying that it determines a family of infinitely many infinitesimal transformations. The basic idea of my paper consisted of the fact that I employed that double nature of systems of **Pfaff** equations in order to define systems of equations that are invariantly coupled with a given system of **Pfaff** equations.

The fecundity of the method that was used has become quite clear to me in the meantime. In what follows, I would like to communicate some new theorems to which I have arrived by applying that method. However, before I move on to the derivation of the theorems in question, I will summarize a series of general remarks in § **1** on the concept of "systems of **Pfaff** equations." Thus, § **1** contains nothing new, but only serves to orient one for what comes later.

§ 1.

Let:

$$(1)\qquad dx_{m+k} - \sum_{\mu=1}^{m} \mathfrak{a}_{\mu,m+k}(x_1,\ldots,x_n)\,dx_\mu = 0 \qquad (k = 1, \ldots, n-m)$$

be a system of $n - m$ independent **Pfaff** equations in the n variables $x_1, \ldots, x_n$. If we interpret the $dx_1, \ldots, dx_n$ in it as the infinitely-small increments that the variables $x_1, \ldots, x_n$ experience under an infinitesimal transformation then our system (1) will determine a family of infinitesimal transformations. When one employs the abbreviation:

$$A_\mu f = \frac{\partial f}{\partial x_\mu} + \sum_{j=1}^{n-m} \mathfrak{a}_{\mu,m+j}\,\frac{\partial f}{\partial x_{m+j}} \qquad (\mu = 1, \ldots, m),$$

the most general infinitesimal transformation of that family will take the form:

(2) $$\sum_{\mu=1}^{m} \chi_\mu (x_1, \dots, x_n)\, \mathfrak{A}_\mu f \;,$$

in which the χ_μ denote arbitrary functions of $x_1, \dots, x_n$. The symbol (2) is then basically just a new analytical representation of the system of **Pfaff** equations (1).

One can also link the system (1) with an intuitive picture. Namely, every point in the space of $x_1, \dots, x_n$ will be assigned a plane bundle of ∞^{m-1} directions of advance $dx_1, \dots, dx_n$ by equations (1). If one imagines an associated bundle of directions of advance at each point in space then one will obtain a figure that is the precise geometric image of the system (1). One will arrive at the same figure naturally when one imagines constructing all of the directions of advance at each point in space that assign the totality of all infinitesimal transformations (2) to the point.

In what follows, we will often apply that geometric interpretation of the system (1), which goes back to **Lie**.

The system of linear partial differential equations:

(3) $$A_1 f = 0, \dots, A_m f = 0$$

is invariantly coupled with the family of infinitesimal transformations (2), and therefore with the system (1), as well, and indeed that coupling is one-to-one and invertible. For that reason, in many cases, one can operate with the system (3), and the result can then be immediately adapted to the family of infinitesimal transformations (2) and the system (1).

It is obvious that the geometric interpretation of the system of **Pfaff** equations (1) is also applicable to the system of linear partial differential equations (3).

In what follows, we would like to assume that the system of Pfaff equations (1) *is not integrable without restriction.* That demand stems from the fact that the m equations (3) cannot define a complete m-parameter system. In other words, the expressions:

$$(\mathfrak{A}_\mu \, \mathfrak{A}_\nu) \qquad (\mu, \nu = 1, \dots, m)$$

cannot all vanish identically.

Finally, we would like to introduce a terminology that is quite convenient in many cases. Namely, when two infinitesimal transformations $Y f$ and $Z f$ are related to each other in such a way that $Z f$ can be represented in the form:

(4) $$Z f = Y f + \sum_{\mu=1}^{m} \varphi_\mu (x_1, \dots, x_n)\, \mathfrak{A}_\mu f,$$

then we would like to say that those two infinitesimal transformations are congruent as moduli relative to $\mathfrak{A}_1 f, \dots, \mathfrak{A}_\mu f$, so we write the congruence:

$$Z f \equiv Y f \qquad (\text{mod } \mathfrak{A}_1 f, \dots, \mathfrak{A}_\mu f)$$

in place of equation (4).

§ 2.

We choose any of the infinitesimal transformations of the family (2), say, this one:

$$U f = \sum_{\mu=1}^{m} u_\mu(x_1, \dots, x_n)\,\mathfrak{A}_\mu f,$$

and apply it to the system of equations (3). In that way, we will obtain a system:

$$\mathfrak{A}_\mu f + \delta t\,(\mathfrak{A}_\mu U) = 0 \qquad (\mu = 1, \dots, m) \tag{5}$$

that is infinitely close to the system (3) and which is invariantly coupled with the system (3) and the infinitesimal transformation $U f$; it is, so to speak, a simultaneous covariant of the two.

If the systems (3) and (4) are coupled with each other then that will imply a new system of equations:

$$\mathfrak{A}_\mu f = 0 \qquad (\mathfrak{A}_\mu U) = 0 \qquad (\mu = 1, \dots, m),$$

and due to the congruence:

$$(\mathfrak{A}_\mu U) \equiv \sum_{\nu=1}^{m} u_\nu\,(\mathfrak{A}_\mu \mathfrak{A}_\nu) \qquad (\text{mod } \mathfrak{A}_1 f, \dots, \mathfrak{A}_m f),$$

it can take the form:

$$\mathfrak{A}_1 f = 0, \qquad u_1\,(\mathfrak{A}_\mu \mathfrak{A}_1) + \dots + u_m\,(\mathfrak{A}_\mu \mathfrak{A}_m) = 0 \qquad (\mu = 1, \dots, m), \tag{6}$$

or when we set $(\mathfrak{A}_\mu \mathfrak{A}_\nu) = \mathfrak{B}_{\mu\nu} f$, the form:

$$\mathfrak{A}_\mu f = 0, \qquad u_1\,\mathfrak{B}_{\mu 1} f + \dots + u_m\,\mathfrak{B}_{\mu m} f = 0 \qquad (\mu = 1, \dots, m). \tag{6′}$$

It is self-explanatory that this system is also a simultaneous covariant of (3) and $U f$, or what amounts to the same thing, a simultaneous covariant of (1) and $U f$.

We would like to express the results obtained somewhat differently.

Namely, if $dx_1, \dots, dx_n$ are the infinitely-small increases that $x_1, \dots, x_n$ experience under the infinitesimal transformation $U f$ then equations of the form:

$$dx_1 = u_1 \cdot \delta t, \dots, dx_n = u_m \cdot \delta t$$

will exist, and in addition:

$$dx_{m+k} = \sum_{\nu=1}^{m} \mathfrak{a}_{\nu,m+k}\, dx_\nu \quad (k = 1, \dots, n - m).$$

If we now imagine that equations (1) determine the most general infinitesimal transformation $U f$ that belongs to the family (2) then we will see the following: Instead of saying "the system of equations (6′) is coupled invariantly with the system (1) and with the infinitesimal transformation $U f$," we can also say "by combining the equations:

$$\mathfrak{A}_\mu f = 0, \qquad u_1 \mathfrak{B}_{\mu 1} f + \ldots + u_m \mathfrak{B}_{\mu m} f = 0 \qquad (\mu = 1, \ldots, m) \tag{7}$$

with the equations (1), we will obtain a system of equations that is invariantly coupled with the system (1)." Hence, we have the:

Theorem 1:

The system of Pfaff equations:

$$dx_{m+k} - \sum_{\mu=1}^{m} \mathfrak{a}_{\mu,m+k}(x_1, \ldots, x_n)\, dx_\mu = 0 \qquad (k = 1, \ldots, n - m) \tag{1}$$

is invariantly coupled with the system of equations:

$$\begin{cases} dx_{m+k} - \sum\limits_{\mu=1}^{m} \mathfrak{a}_{\mu,m+k}(x_1, \ldots, x_n)\, dx_\mu = 0, \\ \mathfrak{A}_\mu f = 0, \quad dx_1 \mathfrak{B}_{\mu 1} f + \cdots + dx_m \mathfrak{B}_{\mu m} f = 0 \quad (k = 1, \ldots, n - m; \mu = 1, \ldots, m). \end{cases} \tag{8}$$

In this, $\mathfrak{A}_\mu f$ and $\mathfrak{B}_{\mu\nu} f$ mean expressions of the following form:

$$\mathfrak{A}_\mu f = \frac{\partial f}{\partial x_\mu} + \sum_{k=1}^{n-m} \mathfrak{a}_{\mu,m+k} \frac{\partial f}{\partial x_{m+k}},$$

$$\mathfrak{B}_{\mu\nu} f = (\mathfrak{A}_\mu\ \mathfrak{A}_\nu) = \sum_{k=1}^{n-m} (\mathfrak{A}_\mu\, \mathfrak{a}_{\nu,m+k} - \mathfrak{A}_\nu\, \mathfrak{a}_{\mu,m+k}) \frac{\partial f}{\partial x_{m+k}}.$$

The system (1) should not be integrable without restriction, so the expressions $\mathfrak{B}_{\mu\nu} f$ should not all vanish identically. It follows from this that as long as one does not assign entirely special values to the $dx_1, \ldots, dx_m$, there will not be more than m of equations (7) that are mutually independent. We can then assume that the system of equations (7) contains precisely $m + h > m$ independent equations for general values of $dx_1, \ldots, dx_m$, or what amounts to the same thing, that in fact all $(h + 1)$-rowed determinants in the matrix:

$$\begin{vmatrix} \sum\limits_{\nu=1}^{m} dx_\nu \cdot \mathfrak{B}_{1\nu}\, x_{m+1} & \sum\limits_{\nu=1}^{m} dx_\nu \cdot \mathfrak{B}_{1\nu}\, x_n \\ \sum\limits_{\nu=1}^{m} dx_\nu \cdot \mathfrak{B}_{m\nu}\, x_{m+1} & \sum\limits_{\nu=1}^{m} dx_\nu \cdot \mathfrak{B}_{m\nu}\, x_n \end{vmatrix} \tag{9}$$

vanish identically, but not all h-rowed ones.

We shall try to interpret the system of equations (8) intuitively.

Let $x_1, \ldots, x_n$ be a point in general position, and let $dx_1, \ldots, dx_n$ be a system of values that satisfies equations (1), although we would next like to assume that not all h-rowed determinants in the matrix (9) vanish for the values of $dx_1, \ldots, dx_n$ in question. We then consider any direction $dx_1 : \ldots : dx_n$ in general position in the bundle of ∞^{m-1} directions that is associated with the point $x_1, \ldots, x_n$ by the system (1). The system of equation (8) reduces to (7) for the chosen system of values $dx_1, \ldots, dx_n$. With the assumptions that were made, the latter includes precisely $m + h$ independent equations, so the point $x_1, \ldots, x_n$ will be assigned a plane bundle of ∞^{m+h-1} directions that obviously go through the aforementioned bundle of ∞^{m-1} directions. That then implies the following:

If B is the plane bundle of ∞^{m-1} directions that is assigned to the point $x_1, \ldots, x_n$ by the system of Pfaff equations (1) *then any direction in general position that is contained in B will correspond to a plane bundle of ∞^{m+h-1} directions that go through B.*

On the other hand, we consider a direction $dx_1, \ldots, dx_n$ for which all h-rowed determinants in the matrix (9) vanish. Such a direction obviously corresponds to a plane bundle of at most ∞^{m+h-2} directions that goes through B. At the same time, it is clear that the totality of all directions $dx_1 : \ldots : dx_n$ for which the given condition is fulfilled has an invariant relationship to the point $x_1, \ldots, x_n$ and the bundle B. We will then get:

Theorem 2:

If all $(h + 1)$*-rowed determinants vanish identically in the matrix* (9), *but not all h-rowed ones, and if $D_1, \ldots, D_q$ are the h-rowed matrices that do not vanish identically then the system of equations:*

$$(10) \qquad dx_{m+k} - \sum_{\mu=1}^{m} \mathfrak{a}_{\mu,m+k}\, dx_\mu = 0, \qquad D_j = 0 \qquad (k = 1, \ldots, n-m,\ j = 1, \ldots, q)$$

is invariantly coupled with the system of Pfaff equations:

$$(1) \qquad dx_{m+k} - \sum_{\mu=1}^{m} \mathfrak{a}_{\mu,m+k}\, dx_\mu = 0, \qquad (k = 1, \ldots, n-m)\,.$$

Naturally, one can also set the h'-rowed ($h' < h$) determinants in (9) equal to zero, instead of the h-rowed ones.

§ 3.

In order to apply the results of the previous paragraphs to a special case, we consider a system of three **Pfaff** equations in six variables:

$$dx_{3+k} - \sum_{\mu=1}^{3} \mathfrak{a}_{\mu,3+k}\, dx_\mu\, (x_1, \ldots, x_6)\, dx_\mu = 0, \qquad (k = 1, 2, 3)\,. \tag{11}$$

The system of three independent linear partial differential equations that is invariantly coupled with (11) reads:

$$\mathfrak{A}_\mu f = \frac{\partial f}{\partial x_\mu} + \sum_{k=1}^{3} \mathfrak{a}_{\mu,3+k} \frac{\partial f}{\partial x_{3+k}} = 0 \qquad (\mu = 1, \ldots, 3)\,.$$

We would further like to set:

$$(\mathfrak{A}_1\ \mathfrak{A}_2) = \mathfrak{B}_3 f, \qquad (\mathfrak{A}_2\ \mathfrak{A}_3) = \mathfrak{B}_1 f, \qquad (\mathfrak{A}_3\ \mathfrak{A}_1) = \mathfrak{B}_2 f\,.$$

Theorem **1**, pp. 4 then implies that:

The system of equations:

$$\left\{\begin{array}{c} dx_{3+k} - \sum_{\mu=1}^{3} \mathfrak{a}_{\mu,3+k} dx_\mu = 0 \quad (k = 1, \ldots, 3), \\ \mathfrak{A}_1 = 0, \quad \mathfrak{A}_2 = 0, \quad \mathfrak{A}_3 = 0, \\ dx_1 : dx_2 : dx_3 = \mathfrak{B}_1 f : \mathfrak{B}_2 f : \mathfrak{B}_3 f \end{array}\right. \tag{12}$$

is invariantly coupled with the system (11).

Now, one must distinguish three cases here, since the number of mutually-independent equations in:

$$\mathfrak{B}_1 f = 0, \qquad \mathfrak{B}_2 f = 0, \qquad \mathfrak{B}_3 f = 0 \tag{13}$$

can be either three or two or one. We would like to discuss each of those cases in succession.

I. *The three equations* (13) *are mutually independent.*

The system (11) assigns a plane bundle B of ∞^2 directions to each point $x_1, \ldots, x_6$. Now, every direction of B is associated with a plane bundle of ∞^4 directions that go through B by means of (12). In that way, one gets a one-to-one, invertible projective relationship between the ∞^2 directions of B and the ∞^2 plane bundles of ∞^4 directions that go through B. Theorem **2** on pp. 5 yields nothing here.

II. *An identity of the form:*

$$\mathfrak{B}_3 f = \alpha_1 (x)\, \mathfrak{B}_1 f + \alpha_2 (x)\, \mathfrak{B}_2 f \tag{14}$$

exists between $\mathfrak{B}_1 f$, $\mathfrak{B}_2 f$, *and* $\mathfrak{B}_3 f$, *while* $\mathfrak{B}_1 f = 0$ *and* $\mathfrak{B}_2 f = 0$ *are independent of each other.*

In that case, Theorem **2** will say that the system (11) is invariantly coupled with the system of four **Pfaff** equations:

$$\left\{\begin{aligned} dx_{3+k} - \sum_{\mu=1}^{3} \mathfrak{a}_{\mu,3+k} dx_\mu &= 0 \quad (k = 1, 2, 3), \\ dx_3 - \alpha_1\, dx_1 - \alpha_2\, dx_2 &= 0. \end{aligned}\right. \tag{15}$$

Any point $x_1, \ldots, x_6$ will be assigned a plane pencil of ∞^1 directions by (15) that lies in the bundle B that is determined by (11).

III. *There exist two identities of the form:*

$$\mathfrak{B}_2 f = \beta_2 (x)\, \mathfrak{B}_1 f, \qquad \mathfrak{B}_3 f = \beta_3 (x)\, \mathfrak{B}_1 f\,. \tag{16}$$

In that case, Theorem **2** will yield a simultaneous system:

$$\left\{\begin{aligned} dx_{3+k} - \sum_{\mu=1}^{3} \mathfrak{a}_{\mu,3+k} dx_\mu &= 0 \quad (k = 1, 2, 3), \\ dx_2 - \beta_2\, dx_1 = 0, \quad dx_3 - \beta_3\, dx_1 &= 0 \end{aligned}\right. \tag{17}$$

that is invariantly coupled with (11) and the fact that every point is assigned a direction that lies in B.

A family of infinitesimal transformations is defined by the simultaneous system (15), whose symbol reads:

$$\chi (x_1, \ldots, x_6)\, (\mathfrak{A}_1 f + \beta_2\, \mathfrak{A}_2 f + \beta_3\, \mathfrak{A}_3 f\,), \tag{18}$$

in which χ is an arbitrary function. If one denotes the infinitesimal transformation (18) briefly by $X f$ and one defines the expressions $\mathfrak{A}_1$, $\mathfrak{A}_2$, $\mathfrak{A}_3$ then upon considering the identities (16), one will immediately find that:

$$(X\, \mathfrak{A}_\mu) \equiv 0 \qquad (\text{mod } \mathfrak{A}_1 f, \mathfrak{A}_2 f, \mathfrak{A}_3 f) \qquad\qquad (\mu = 1, 2, 3)\,;$$

in other words: The system $\mathfrak{A}_1 f = 0$, $\mathfrak{A}_2 f = 0$, $\mathfrak{A}_3 f = 0$ remains invariant under all infinitesimal transformations of the form (18). Naturally, that is likewise true for the system of **Pfaff** equations (11).

§ 4.

Now for another application of the developments in **§ 2**.

On pp. 33, *et seq*. of my dissertation (Math. Ann. XXIII, pp. 1, *et seq*.), I determined all transformations of an $(m + n)$-fold extended space R_{m+n} under which the n-fold extended manifolds of that space will be exchanged with each other in such a way that first-order contact is preserved. At the time, I found (by some very laborious calculations) that in the case of $m > 1$, there were no other actual transformations with that property than the point transformations of R_{m+n}. I will now show that this result can be obtained easily from the theorems in **§ 2**.

If $z_1, \ldots, z_n, x_1, \ldots, x_n$ are point coordinates in R_{m+n}, and one sets:

$$\frac{\partial z_\mu}{\partial x_i} = p_{\mu i} \qquad (\mu = 1, \ldots, m \,;\, i = 1, \ldots, n)$$

then one will be dealing with the problem of finding all transformations of the $m + n + mn$ variables:

$$z_i \,,\, x_i \,,\, p_{\mu i} \qquad (\mu = 1, \ldots, m \,;\, i = 1, \ldots, n)$$

that leave the system of m **Pfaff** equations:

$$dz_\mu - \sum_{i=1}^{n} p_{\mu i}\, dx_i = 0 \quad (\mu = 1, \ldots, m) \tag{19}$$

invariant.

The system (19) corresponds to a system of $n + mn$ linear partial differential equations that read as follows:

$$\left\{\begin{aligned} \mathfrak{A}_i f &= \frac{\partial f}{\partial x_i} + \sum_{\nu=1}^{m} p_{\nu i} \frac{\partial f}{\partial z_i} = 0, \\ \mathfrak{A}_{\mu i} f &= \frac{\partial f}{\partial p_{\mu i}} = 0 \qquad (\mu = 1, \ldots, m; i = 1, \ldots, n). \end{aligned}\right. \tag{20}$$

It then follows from Theorem **1**, pp. 4 that the system of equations:

$$\left\{\begin{aligned} & dz_\mu - \sum_{j=1}^{m} p_{\nu j}\, dx_j = 0, \\ & \mathfrak{A}_i f = 0, \quad \mathfrak{A}_{\mu i} f = 0, \\ & \sum_{j=1}^{m} p_{\nu j} \frac{\partial f}{\partial z_\nu} = 0, \quad dx_i \frac{\partial f}{\partial z_\mu} = 0 \quad (\mu = 1, \ldots, m; i = 1, \ldots, n) \end{aligned}\right. \tag{21}$$

is invariantly coupled with the system (19).

In order to be able to apply Theorem **2**, pp. 5, we must define the matrix (9), which takes the form:

$$(22)\qquad \begin{vmatrix} dp_{11} & dp_{12} & \cdots & dp_{1n} & dx_1 & 0 & \cdots & 0 & dx_2 & 0 & \cdots & dx_n & 0 & \cdots & 0 \\ dp_{21} & dp_{22} & \cdots & dp_{2n} & 0 & dx_1 & \cdots & 0 & 0 & dx_2 & \cdots & 0 & dx_n & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ dp_{m1} & dp_{m2} & \cdots & dp_{mn} & 0 & 0 & \cdots & dx_1 & 0 & 0 & \cdots & 0 & 0 & \cdots & dx_n \end{vmatrix}$$

in the present case, and then be must set all m-rowed determinants in that matrix equal to zero and append the equations that are obtained to the system (19). In that way, we will get the following system of equations:

$$(23)\qquad \begin{cases} dz_1 = 0, \quad \cdots \quad dz_m = 0, \quad dx_1 = 0, \quad \cdots \quad dz_n = 0, \\ \qquad\qquad \Delta_1 = 0, \quad \cdots \quad \Delta_0 = 0, \end{cases}$$

which is invariantly coupled with (19). In that system, $\Delta_1, \ldots, \Delta_q$ denote the m-rowed determinants of the matrix:

$$(24)\qquad \begin{vmatrix} dp_{11} & \cdots & dp_{1n} \\ \cdots & \cdots & \cdots \\ dp_{m1} & \cdots & dp_{mn} \end{vmatrix},$$

but naturally, the equations $\Delta_1 = 0, \ldots, \Delta_q = 0$ will occur only when $m \leq n$.

Any transformation of the variables z_μ, x_i, $p_{\mu i}$ that leaves (19) invariant must also leave the system (23) invariant. Now, if $m > n$ then (23) will have the simple form:

$$(25)\qquad dz_1 = 0, \ldots, dz_m = 0, \quad dx_1 = 0, \ldots, dx_n = 0 .$$

On the other hand, if $m \leq n$, but > 1 then the Δ will be entire functions of order two or higher of the $dp_{\mu i}$, and it will then be clear that a transformation that leaves the system (23) invariant must also leave invariant the system of equations that arises when one drops the equations $\Delta_1 = 0, \ldots, \Delta_q = 0$. We then get:

Theorem 3:

If $m > 1$ then any transformation of the variables z_μ, x_i, $p_{\mu i}$ that leaves the system:

$$(19)\qquad dz_\mu - \sum_{i=1}^{n} p_{\mu i}\, dx_i = 0 \quad (\mu = 1, \ldots, m)$$

invariant will also leave the system:

$$(25)\qquad dz_1 = 0, \ldots, dz_m = 0, \quad dx_1 = 0, \ldots, dx_n = 0$$

invariant, and it will then be an extended point transformation of the space $z_1, \dots, z_m, x_1, \dots, x_n$.

The considerations above also imply a remarkable result for $m = 1$. Namely, they show that the **Pfaff** system:

$$dz - \sum_{i=1}^{n} p_i\, dx_i = 0 \tag{26}$$

is invariantly coupled with the system:

$$dz = 0, \qquad dx_i = 0, \qquad dp_i = 0 \qquad (i = 1, \dots, n), \tag{27}$$

and indeed that relation is obviously valid under not just transformations of the variables z, x, p alone, but also under all transformations in $z, x_1, \dots, x_i, p_1, \dots, p_n$, and arbitrarily many other variables $u_1, \dots, u_q$. One then has:

Theorem 4:

Any transformation in the variables $z, x_1, \dots, x_i, p_1, \dots, p_n, u_1, \dots, u_q$ *that leaves the Pfaff equation:*

$$dz - \sum_{i=1}^{n} p_i\, dx_i = 0 \tag{26}$$

invariant will also leave the system:

$$dz = 0, \qquad dx_i = 0, \qquad dp_i = 0 \qquad (i = 1, \dots, n) \tag{27}$$

invariant, and will then transform the $2n + 1$ *variables* $z, x_1, \dots, x_n, p_1, \dots, p_n$ *amongst themselves.*

Some time ago, **A. V. Bäcklund** proved that **Lie**'s contact transformations of the space $z, x_1, \dots, x_n$ are the only transformations of that space under which n-fold extended manifolds that have m^{th}-order contact will go to other such things (see Math. IX, pp. 297, *et seq.*). Now, it is very easy to prove **Bäcklund**'s theorem on the basis of Theorem **4** and Theorem **4** on pp. 165 of previous paper (these Berichte, 1889). I would then like to content myself by showing that for the case of transformations that preserve second-order contact, so for the case of osculation transformations. One proceeds similarly in the general case.

An osculation transformation of the space $z, x_1, \dots, x_n$ is a transformation of the variables:

$$z, x_i, p_i, p_{ik} = p_{ki} \qquad (i, k = 1, \dots, n)$$

that leaves the system of **Pfaff** equations:

$$(28) \qquad \left\{ \begin{aligned} & dz - \sum_{k=1}^{n} p_k \, dx_k = 0, \\ & dp_i - \sum_{k=1}^{n} p_{ik} \, dx_k = 0 \quad (i = 1, \ldots, n) \end{aligned} \right.$$

invariant. Now the system of linear partial differential equations:

$$\frac{\partial f}{\partial x_i} + p_i \frac{\partial f}{\partial z} + \sum_{k=1}^{n} p_{ki} \frac{\partial f}{\partial p_k} = 0, \qquad \frac{\partial f}{\partial p_{ji}} = 0 \qquad (i, j = 1, \ldots, n)$$

is invariantly coupled with the system (28), and so from Theorem **4** of the previous article, also with the system:

$$\frac{\partial f}{\partial x_i} + p_i \frac{\partial f}{\partial z} = 0, \qquad \frac{\partial f}{\partial p_i} = 0, \qquad \frac{\partial f}{\partial p_{ji}} = 0 \qquad (i, k = 1, \ldots, n),$$

or what amounts to the same thing: the **Pfaff** equation:

$$(26) \qquad dz - \sum_{i=1}^{n} p_i \, dx_i = 0 .$$

Any transformation of the variables z, x_i, p_i, p_{ik} that leaves (28) invariant will also leave the **Pfaff** equation (26) invariant then. As a result, from Theorem **4**, pp. 10, it will transform the variables $z, x_1, \ldots, x_n, p_1, \ldots, p_n$ among themselves. In other words, it will consist of an extension of a **Lie** contact transformation of the space $z, x_1, \ldots, x_n$.

One can show in an entirely similar way that the point transformations of the R_{m+n} ($m > 1$) are the only transformations of that space that take n-fold extended manifolds that have p^{th} order contact to other ones.

§ 5.

We shall once more take up the general investigation.

Let $U f$ and $V f$ be any two infinitesimal transformations of the family (2), and indeed:

$$U f = \sum_{\mu=1}^{m} u_\mu(x) \mathfrak{A}_\mu f , \qquad V f = \sum_{\mu=1}^{m} v_\mu(x) \mathfrak{A}_\mu f .$$

That will imply that:

$$(U\, V) = \left(\sum_{\mu=1}^{m} u_\mu \, \mathfrak{A}_\mu f, \sum_{\nu=1}^{m} v_\nu \, \mathfrak{A}_\nu f \right)$$

$$\equiv \sum_{\mu,\nu=1}^{m} u_\mu\, v_\nu\, (\mathfrak{A}_\mu\, \mathfrak{A}_\nu) \qquad (\text{mod } \mathfrak{A}_1 f, \ldots, \mathfrak{A}_m f,)$$

or

$$(U\,V) = \tfrac{1}{2} \sum_{\mu,\nu=1}^{m} (u_\mu\, v_\nu - u_\nu\, v_\mu)(\mathfrak{A}_\mu\, \mathfrak{A}_\nu) \qquad (\text{mod } \mathfrak{A}_1 f, \ldots, \mathfrak{A}_m f)\ .$$

Now, if the equation $(U\,V) = 0$ is invariantly coupled with $U f$ and $V f$ then the system of equations:

$$\mathfrak{A}_1 f = 0, \ldots, \mathfrak{A}_m f = 0, \qquad (U\,V) = 0$$

will be a simultaneous covariant of $U f$, $V f$, and of the system $\mathfrak{A}_1 f = 0, \ldots, \mathfrak{A}_m f = 0$, or what amounts to the same thing, the system of equations:

$$\mathfrak{A}_1 f = 0, \ldots, \mathfrak{A}_m f = 0, \qquad \sum_{\mu,\nu=1}^{m} (u_\mu\, v_\nu - u_\nu\, v_\mu)(\mathfrak{A}_\mu\, \mathfrak{A}_\nu) = 0$$

will be a simultaneous covariant of $U f$, $V f$, and the system of **Pfaff** equations (1).

We can express the result somewhat differently when we introduce the infinitely-small increases that $U f$ and $V f$ experience. The increases that $x_1, \ldots, x_n$ experience under $U f$ might be called $dx_1, \ldots, dx_n$, and the corresponding increases under $V f$: $\delta x_1, \ldots, \delta x_n$. From considerations that are similar to then ones on pp. 3, *et seq.*, we find that:

Theorem 5:

The system of Pfaff equations:

$$dx_{m+k} - \sum_{\mu=1}^{m} \mathfrak{a}_{\mu,m+k}\, dx_\mu = 0 \qquad (k = 1, \ldots, n - m)$$

is invariantly coupled with the system of equations:

$$(29) \qquad \left\{ \begin{array}{ll} dx_{m+k} - \sum_{\mu=1}^{m} \mathfrak{a}_{\mu,m+k}\, dx_\mu = 0, & (k = 1, \ldots, n - m) \\ \delta x_{m+k} - \sum_{\mu=1}^{m} \mathfrak{a}_{\mu,m+k}\, \delta x_\mu = 0, & \\ \mathfrak{A}_1 f = 0, \quad \cdots, \quad \mathfrak{A}_m f = 0 & \\ \sum_{\mu,\nu=1}^{m} (dx_\mu\, \delta x_\nu - dx_\nu\, \delta x_\mu)(\mathfrak{A}_\mu\, \mathfrak{A}_\nu) = 0. & \end{array} \right.$$

That theorem has a simple intuitive sense. Namely, if $x_1, \ldots, x_n$ is any point, and $dx_1, \ldots, dx_n$ and $\delta x_1, \ldots, \delta x_n$ are any two directions that belong to the bundle of ∞^{m-1} directions that is determined by (1) then the system of equations (29) will reduce to:

$$\text{(30)} \qquad \begin{cases} \mathfrak{A}_1 f = 0, \quad \cdots, \quad \mathfrak{A}_m f = 0 \\ \sum_{\mu,\nu=1}^{m} (dx_\mu\, \delta x_\nu - dx_\nu\, \delta x_\mu)(\mathfrak{A}_\mu\, \mathfrak{A}_\nu) = 0, \end{cases}$$

and the latter will obviously associate the point $x_1, \ldots, x_n$ with a bundle of ∞^m directions that go through B, in general. If one further imagines that $dx_1, \ldots, dx_m$ and $\delta x_1, \ldots, \delta x_m$ can be regarded as the homogeneous coordinates of two selected directions at B, and that the $dx_\mu\ \delta x_\nu - dx_\nu\ \delta x_\mu$ are the homogeneous coordinates of the plane pencil that is determined by the two directions $dx_1, \ldots, dx_m$ and $\delta x_1, \ldots, \delta x_m$ then one will see the following: Every plane pencil of ∞^1 directions that is contained in B and does not have a special position is associated with a plane bundle of ∞^m directions that goes through B.

If one wishes to find all pencils that lie in B that do not correspond to a plane bundle of ∞^m directions then one needs only to determine the $dx_\mu\ \delta x_\nu - dx_\nu\ \delta x_\mu$ in such a way that the infinitesimal transformation:

$$\sum_{\mu,\nu=1}^{m} (dx_\mu\, \delta x_\nu - dx_\nu\, \delta x_\mu)(\mathfrak{A}_\mu\, \mathfrak{A}_\nu)$$

vanishes identically. In that way, one finds that the defining equations of the pencil in question:

$$\sum_{\mu,\nu=1}^{m} (dx_\mu\, \delta x_\nu - dx_\nu\, \delta x_\mu)(\mathfrak{A}_\mu\, \mathfrak{a}_{\nu,m+k} - \mathfrak{A}_\nu\, \mathfrak{a}_{\mu,m+k}) = 0 \qquad (k = 1, \ldots, n-m).$$

Now, since it is clear from the outset that the totality of all of those pencils is invariantly coupled with the point $x_1, \ldots, x_n$ and the bundle B then that will imply:

Theorem 6:

The Pfaff system:

$$\text{(1)} \qquad dx_{m+k} - \sum_{\mu=1}^{m} \mathfrak{a}_{\mu,m+k}(x_1, \ldots, x_n)\, dx_\mu = 0 \qquad (k = 1, \ldots, n-m)$$

in invariantly coupled with the system of equations:

$$
(30)\qquad \begin{cases} dx_{m+k} - \sum_{\mu=1}^{m} \mathfrak{a}_{\mu,m+k}\, dx_\mu = 0, \\ \delta x_{m+k} - \sum_{\mu=1}^{m} \mathfrak{a}_{\mu,m+k}\, \delta x_\mu = 0, \\ \sum_{\mu,\nu=1}^{m} (dx_\mu\, \delta x_\nu - dx_\nu\, \delta x_\mu)(\mathfrak{A}_\mu\, \mathfrak{a}_{\nu,m+k} - \mathfrak{A}_\nu\, \mathfrak{a}_{\mu,m+k}) = 0 \quad (k = 1, \ldots, n-m). \end{cases}
$$

If $m = n - 1$ then that theorem will imply the known bilinear covariant of a **Pfaff** equation. One would strongly suspect that such a bilinear structure must also appear for *systems* of **Pfaff** equations; for that reason, Theorem **6** is not really very remarkable. Theorem **5**, which naturally cannot be stated in the special case of $m = n - 1$, is all the more remarkable since it will be meaningless in exactly that case.