"Die allgemeine Mannheimsche Kurve," Monatsh. für Math. u. Phys. **23** (1912), 289-296.

The general Mannheim curve

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Almost simultaneously, **H. Wieleitner** (1) and myself (2) generalized the concept of the Mannheim curve [as is known (3), that refers to the locus of curvature centers for the respective contact points of a curve Γ that rolls along a line] by allowing Γ to roll along a circle (4).

A closely-related consideration would be curves M that are described by the curvature centers m of the respective contact points of a curve Γ while it rolls along an arbitrary curve K.

Let Γ be referred to as the *base curve*, while K is called the *curvilinear axis*. Since the name of *generalized **Mannheim** curve* was employed by **Wieleitner** for the case in which Γ rolls along a circle, M will be called the *generalized **Mannheim** curve of* Γ *relative to K.*

I.

One obtains the point m of the general Mannheim curve M when one measures out the arc length s of the curve Γ from a fixed starting point a on K and measures out the corresponding curvature radius ρ of Γ along the associated normal to K. A point p of Γ with the natural coordinates ρ, s always corresponds to a point m on M then for which ρ means the perpendicular distance from the curvilinear axis and s means the arc length from a fixed starting point a on it to the foot k of the perpendicular. For the sake of uniqueness, a positive sense will be established for the arc length measurement. The positive directions of the normals can then be determined in such a way that the positive directions of the tangent and normals at the curve point considered coincide with the positions of the positive half-axes of a rectangular coordinate system.

These two givens, namely, the arc length s and the distance to the associated curve normals, which might be denoted by n, can be regarded as the coordinates of the point m. According to **G. Loria** (5), their introduction as coordinates goes back to **G. Mannheim**

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(1) "Über eine Verallgemeinerung des Begriffes der **Mannheim**schen Kurve," Math.-nat. Mitt. Württemberg (2) **9** (1907), 1-9.

(2) "Ein Analogon zur Mannheimschen Kurve," this journal **18** (1907), 315/6.

(3) **E. Wölffing**, "Über Pseudotrochoiden," Zeit. Math. Phys. **44** (1899).

(4) Cf., **Wieleitner**, *Spezielle ebene Kurven*, (Sammlung Schubert 56), Göschen, Leipzig, 1908, pp. 320, *et seq.*

(5) *Spezielle algebraische und transzendentale ebene Kurven*, (Sammlung Teubner 5), Teubner, Leipzig, 1902, pp. 606.

([1]). This coordinate system, which represents, as **G. de Longchamps** ([2]) has remarked already, merely a special case of the curvilinear coordinates that **Aoust** ([3]) treated, was addressed by **M. Petrovich** ([4]). In connection with that work, **Longchamps** published a construction of the tangent to a given curve in **Mannfred** or *inverse normal coordinates*, like the coordinates that were called s, n by **F. Ondraček** ([5]).

That then implies the theorem:

If one replaces the natural coordinate ρ, s *in the natural equation* $f(\rho, s) = 0$ *for a curve* Γ *with the inverse normal coordinates* n, s *relative to a curve* K *as their axis then the equation* $f(n, s) = 0$ *will represent the general* ***Mannheim*** *curve* M *of* Γ *relative to* K.

If K is a line, in particular, then the inverse normal coordinates will clearly go over to the Cartesian coordinates, and M will become the usual **Mannheim** curve of Γ. If K is a circle then that will yield the generalized **Mannheim** curve. On the other hand, when Γ is a circle, one will get the parallel curve of K. If K and Γ are congruent for corresponding positions of the curvature radius then M will become the evolute of K.

Naturally, one can also conversely replace the normal coordinates of a curve with the natural coordinates. One will then get the natural equation of that curve whose general **Mannheim** curve is the given one. If one switches the inverse normal coordinates with the Cartesian ones then the so-called *image curve* ([6]) to the given one will arise, which is the usual **Mannheim** curve to that curve Γ whose general **Mannheim** curve is the given one.

Let B be the usual **Mannheim** curve of Γ, let M be its general **Mannheim** curve, and let K be the curvilinear axes. One will then get M from B by a process that is similar to the one that **Varignon** ([7]) used in order to convert a curve $f(x, y) = 0$ into the curve $f(\rho, l\omega) = 0$. If one chooses the tangent at a point A of K to be the axis of the abscissa, the normal to be axis of the ordinate, and if the abscissa pq points to a point p of the curve B and makes the parallel curve K' to K that goes through q intersect the curve normal that is erected at the endpoint of the arc length $s = \overline{pq}$ that is measured from a then, as one can see immediately, the point of intersection p will be a point of M ([8]).

A spatial consideration will allow one to recognize a simple connection between B and M. If a point p runs along the curve K, while its plane moves parallel to itself in the perpendicular direction in such a way that its displacement is equal to the arc length of the curve that is described then p will describe a *general helix* G; i.e., a geodetic line on the cylinder that is projected over K. Any point q that is rigidly coupled to p by a curve normal will describe a path under this motion that projects onto its plane like a curve that is parallel to K. If one draws the normal plane ν to the curve K through the starting point

([1]) *De constructione aequationum differentialum primi gradus*, Bononiae, 1707.
([2]) "Le courbes images et les courbes symétriques," Nouv. Ann. math. (3) **14**, 373-278.
([3]) *Analyse infinitésimale des courbes planes*, Gauthier-Villars, Paris, 1873.
([4]) *Sur une système de coordonées sémi-curvilignes*, Prager, Berlin, 1898.
([5]) *Analytische Geometrie ebener Kurven in Büschelkoordinaten*, Vienna, 1903.
([6]) Petrovich, *loc. cit.*
([7]) "Nouvelles formation de spirales, etc.," Mém. de Paris Année 1704, Paris, 1722. Cf., **Loria**, *loc. cit.*, pp. 595.
([8]) The drawing of the very simple figure shall be left to the reader.

a for the measurement of arc length, and q makes a screwing motion in the plane ν then a point q_ν will come about whose Cartesian coordinates in n are clearly $x = n$, $z = -s$. If one performs this generalized screwing motion with an entire curve $M \equiv f(n, s) = 0$ then a *generalized helicoid* Φ will arise that intersects n along the curve $f(x, -z) = 0$, and thus, along the curve that is symmetric to the image curve of M with respect to the x-axis ([1]). If one draws the normal plane μ through a point b of K that is at a distance from a of s_0 along the curve then one will get the same curve of intersection with Φ as in ν, except that it is displaced parallel to itself along the z-direction through a distance of s_0 . If one refers to the intersections with μ, ν, … as the *meridians* that correspond to the helicoid and the intersections with normal planes to the axis of the cylinder as *normal sections* then that will yield:

The generalized helicoid Φ *that has the general Mannheim curve* M *to* Γ *as its normal section will be intersected by the meridian planes along the usual Mannheim curve* B *of* Γ.

All meridian sections of Φ *are congruent. The normal sections of* Φ *are different from each other, but they can be regarded as general Mannheim curves to the same curve* Γ *relative to* K *when the starting point of the rolling motion is thought of as being located at the respective point of intersection of* G *with the normal section plane in question.*

II.

The Cartesian equation of the general **Mannheim** curve shall now be presented. Let the equation for the curvilinear axis:

K be $y = f(x)$,

and let that of the rolling curve:

Γ be $\rho = \varphi(s)$.

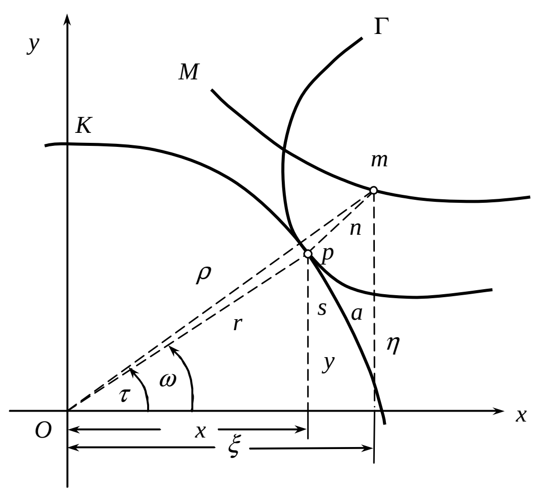

Figure 1

([1]) I propose to treat these surfaces thoroughly next from the descriptive-geometric standpoint.

If M has the running coordinates ξ, η then one will get n from (Fig. 1) the distance formula:

$$n^2 = (\eta - y)^2 + (\xi - x)^2, \tag{1}$$

and since m and k lie on a normal to K:

$$\frac{\eta - y}{\xi - x} = -\frac{1}{y'} . \tag{2}$$

One calculates the desired coordinates from that:

$$\boxed{\xi = x - \frac{n}{y'\sqrt{1+y'^2}}, \quad \eta = y + \frac{n}{\sqrt{1+y'^2}},} \tag{3}$$

from which $n = \varphi(s)$ can be eliminated by way of ([1]):

$$s = \int_{x_0}^{x} \sqrt{1+y'^2}\, dx .$$

It is just as easy to represent the general Mannheim curve in polar coordinates. If:

$$K\ldots \quad r = \psi(\omega),$$

$$\Gamma\ldots \quad \rho = \varphi(s)$$

are the given coordinates then m ($\mathfrak{r}$, r) will follow from the cosine law:

$$\boxed{n^2 = r^2 + \mathfrak{r}^2 - 2r\mathfrak{r}\cos(\tau - \omega).} \tag{5}$$

One further has:

$$\boxed{\frac{1}{\mathfrak{r}} - \frac{1}{r}\cos(\tau - \omega) - \frac{1}{r'}\sin(\tau - \omega) = 0.} \tag{6}$$

If one then uses:

$$s = \int_{\omega_0}^{\omega} \sqrt{r^2 + r'^2}\, d\omega \tag{7}$$

then the problem that was posed can be solved by one quadrature and one elimination.

([1]) In the cited paper by **Petrovich**, the formulas in question should correctly read:

$$y_1 = \varphi(t) = \frac{f'(t)\eta(\lambda)}{\sqrt{f'(t)^2 + \varphi'(t)^2}}, \qquad \text{instead of} \qquad y_1 = f'(t) + \frac{\varphi'(t)\eta(\lambda)}{\sqrt{f'(t)^2 + \varphi'(t)^2}} .$$

The results are in harmony with the ones in the paper of **Petrovich** that was mentioned, where one can also find the conditions for the curves to be rational and unicursal in inverse normal coordinates.

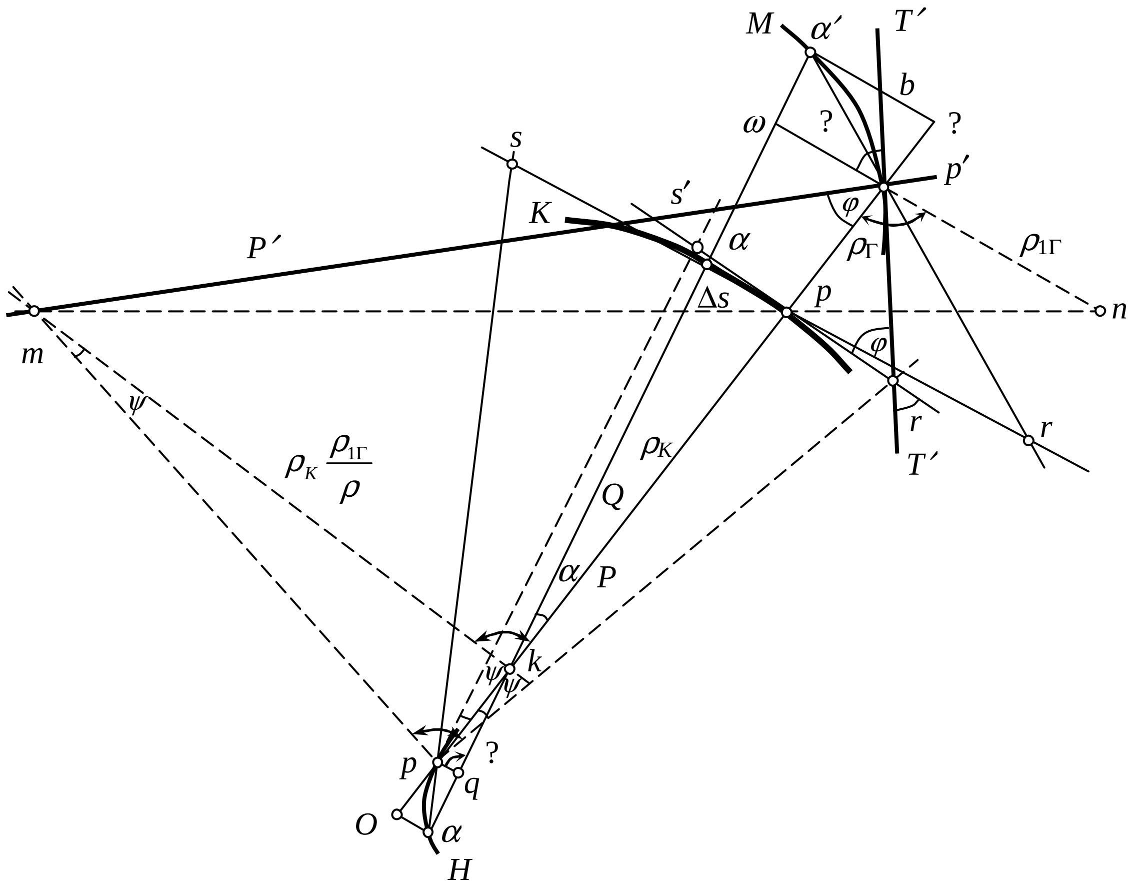

Figure 2.

Finally, the natural equation of the general Mannheim curve can be derived in parameter form. As Fig. 2 shows, ΔS is given by ([1]):

$$\Delta S = \sqrt{\left(\frac{\rho_K + \rho_\Gamma}{\rho_K}\right)^2 \Delta s^2 + \Delta\rho_\Gamma^2} \tag{8}$$

or in the limit by:

$$dS = ds\sqrt{\left(\frac{\rho_K + \rho_\Gamma}{\rho_K}\right)^2 + \left(\frac{d\rho_\Gamma}{ds}\right)^2}\ , \tag{8′}$$

([1]) If the rolling motion takes place around the inner circumference then ρ_Γ must be taken to be negative. (Cf., pp. 1)

$$\tan\varphi = \frac{d\rho_\Gamma}{ds\left(\dfrac{\rho_K + \rho_\Gamma}{\rho_K}\right)}. \tag{9}$$

or since:

$$\rho\frac{d\rho}{ds} = \rho_1 , \tag{10}$$

by:

$$\tan\varphi = \frac{\rho_{1\Gamma}\,\rho_K}{\rho_\Gamma\,(\rho_K + \rho_\Gamma)}. \tag{9$'$}$$

Now, the contingency angle of M is clearly:

$$d\mu = d\varphi + da \tag{11}$$

so

$$R = \frac{dS}{da + d\varphi}, \tag{12}$$

or, from (8′):

$$\boxed{R = \frac{\sqrt{\left(\dfrac{\rho_\kappa + \rho_\Gamma}{\rho_\kappa}\right)^2 + \left(\dfrac{d\rho_\Gamma}{ds}\right)^2}}{\dfrac{1}{\rho_\Gamma} + \dfrac{d}{ds}\arctan\dfrac{\rho_{1\Gamma}\rho_\kappa}{\rho_\Gamma(\rho_\kappa + \rho_\Gamma)}}.} \tag{12$'$}$$

(12′) and:

$$\boxed{S = \int ds\sqrt{\left(\frac{\rho_\kappa + \rho_\Gamma}{\rho_\kappa}\right)^2 + \left(\frac{d\rho_\Gamma}{ds}\right)^2}} \tag{12*}$$

define the natural equation for μ in parameter form.

III.

If one observes that the angle φ between corresponding tangents at M and K is equal to the angle between the associated normals then the use of equation (9′) will yield the following simple normal construction ([1]) for the general Mannheim curve.

One raises a perpendicular at p' on M to the normal P at the corresponding point of K, sets $p_n = \rho_{1\Gamma}$, and makes *the second curvature radius of K intersect the connecting line of n with p. The connecting line between the point of intersection m that one obtains and p' will be the desired normal.*

([1]) I would like to thank Herrn Dr. **L. Braude** in Bierstadt for communicating this construction to me in a letter.

Naturally, as one can see in the figure, it defines an angle with the normal to K whose tangent is equal to tan φ.

One can also draw the tangents to the general Mannheim curve in a way that is similar to how **Longchamps** ([1]) could construct tangents to the curve when it is expressed in inverse normal coordinates, assuming that the tangent can be constructed at K and the curvature center of K and second curvature radius of Γ is known.

Let (cf., Fig. 2) p and q be neighboring points on K, while p' and q', resp., are the corresponding points of Γ, P and Q are the associated normals to ϑ, and k is their point of intersection. If one lays out the line segment $\overline{k\,p^0} = \overline{p\,p'}$ along P in opposite directions, and likewise $\overline{k\,q^0} = \overline{q\,q'}$ along Q then the lines $p'\,q'$ and $p^0\,q^0$ the line $p\,q$ at the points r and s in such a way that $\overline{r\,p} = \overline{s\,q}$ ([2]). Now, if k is associated with the angle α and p^0 is associated with the angle ψ then the triangle $k\,p^0\,q^0$ will yield:

$$\frac{\overline{k\,p^0}}{\sin(\psi-\alpha)} = \frac{\overline{k\,q^0}}{\sin\psi} = \frac{\overline{k\,q^0} - \overline{k\,p^0}}{\sin\psi - \sin(\psi-\alpha)}, \tag{13}$$

or

$$\frac{f(s)}{\sin(\psi-\alpha)} = \frac{f(s+ds)}{\sin\psi} = \frac{\dfrac{f(s+ds)-f(s)}{ds}\,\dfrac{ds}{\alpha}}{\dfrac{\sin\alpha/2}{\alpha/2}\cos\left(\psi-\dfrac{\alpha}{2}\right)}. \tag{13'}$$

If one now shifts p towards q such that pq will be tangent then k will be the curvature center of K, lim $\alpha = 0$, $ds / \alpha = \rho_\kappa$, and (13′) will imply that:

$$\frac{f(s)}{\sin\psi} = \rho_\kappa \frac{f'(s)}{\cos\psi} \tag{13*}$$

or

$$\cot\,\psi = \rho_\kappa \frac{f'(s)}{f(s)}, \tag{13**}$$

which implies that:

([1]) *Loc. cit.*

([2]) An entirely elementary consideration will show that this implies: Let $p^0\,t \parallel q'\,u \parallel p\,q \parallel q^0\,v$, $\overline{p^0\,t} = \mathfrak{a}$, $\overline{q'u} = \mathfrak{b}$, so since $\overline{k\,q^0} = \overline{q\,q'}$, one also have $\overline{k\,v} = \overline{p\,u}$ and $\overline{v\,p^0} = \overline{q'u}$. $\overline{p\,r} = \dfrac{b\cdot\overline{p\,p'}}{\overline{p'u}}$, $\overline{q\,s} = \dfrac{a\cdot\overline{q^0\,q}}{\overline{q^0\,t}}$, $\mathfrak{a} = \dfrac{b\cdot\overline{t\,k}}{\overline{k\,q'}}$, so $\overline{q\,s} = \dfrac{b\cdot\overline{t\,k}\cdot\overline{q^0\,q}}{\overline{q^0\,t}\cdot\overline{k\,q'}}$. Since $\overline{q^0\,q} = \overline{k\,q'}$ and $\dfrac{\overline{t\,k}}{\overline{q^0\,t}} = \dfrac{\overline{p^0\,k}}{\overline{p^0\,v}}$, one will have $\overline{q\,s} = \dfrac{b\cdot\overline{p^0\,k}}{\overline{p^0\,v}} = \dfrac{b\cdot\overline{p\,p'}}{\overline{p'u}}$; hence, $\boxed{\overline{p\,r} = \overline{q\,s}.}$.

$$\boxed{\cot\psi = \rho_\kappa \cdot \frac{\rho_{1\Gamma}}{\rho_\Gamma^2}.} \tag{13***}$$

Moreover, the tangent construction can be performed as follows:

One proceeds until one has ascertained the point m in precisely the same way that one does with the construction of the normal (pp. 6), $\sphericalangle\, m\, p_0\, k = 90 - \psi$. *One now sets* $\sphericalangle\, s_1\, p_0\, p = \psi$ *in order to ascertain the point* r_1 *that is symmetric to* s_1 *relative to* p*, and to connect it with* p'. $r_1\, p'$ *will then be the desired tangent.*

However, it is simpler to find r_1 when one raises the perpendicular to $m\, p_0$ and intersects it with T, which is immediately clear on the grounds of symmetry.

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