ON ELASTIC CURVES

Leonhard Euler

Additamentum I to

"Methodus inveniendi lineas curvas maximi minimive proprietate gaudentes"

Lausanne and Geneva 1744

Annotated by H. Linsenbarth

Translated by D. H. Delphenich

Excerpted from:

Abhandlungen über das Gleichgewicht und die Schwingungen der ebenen elastischen Kurven

by

JACOB BERNOULLI (1691, 1694, 1695) and LEON. EULER (1744)

Translated and edited by H. Linsenbarth

OSTWALD'S KLASSIKER DER EXACTEN WISSENSCHAFTEN
NO. 175

LEIPZIG
WILHELM ENGELMANN
1910

PREFACE TO THE ENGLISH TRANSLATION

Although this treatise dates back to 1744, nonetheless, its subject is sufficiently fundamental to other things that the applications of its methodology continue to have their utility to this day. The subject in question is that of the shapes and vibratory modes that can be assumed by elastic curves or *elastica*, which are also the basis for the method of spline interpolations in computer graphics.

More precisely, one should say *lamina elastica*, or elastic bands, as opposed to *corda elastica*, or elastic strings. Hence, although the following fact is mentioned specifically only once in all that follows, the objects that will be bending and vibrating will have a non-zero, but otherwise negligible, thickness and a width that will usually be assumed to be constant, except in one section on non-uniform elasticity. Other than that, one might as well be talking about the deformation of curves, such as the longitudinal curve through the band that intersects the center of mass of each cross-section. In that sense, the discussion that follows can be seen as extension of the usual theory of flexible, inextensible curves and vibrating strings to a higher order of differential equations, namely, four. Although we say "inextensible," in a few cases, such as the vibration of a band that is fixed in a wall at both ends, the band will be assumed to maintain its original length.

Although this translator usually avoids the tendency to alter mathematical symbols and equations in any way, except to incorporate errata and correct obvious errors in typesetting, in the present case, some effort was made to make the logic of Euler's argument more transparent by "modernizing" some of the instances where he employed a technique in calculus that is currently frowned upon, namely, treating differentials as if they were capable of being multiplied and divided, and not just added, subtracted, and multiplied by scalars. For instance, one will typically find that expressions of the form:

$$\frac{dx^2}{d^2y} \quad \text{and} \quad \frac{ds^2}{dx}$$

will be converted into the more modern forms:

$$\left(\frac{d^2y}{dx^2}\right)^{-1} \qquad \text{and} \qquad \frac{dx}{ds}\,ds\,,$$

respectively.

One should also notice that since this treatise seems to have predated the introduction of the term "curvature" (in the Frenet-Serret sense), one must intuit its role in many cases where one is dealing with that very concept, but in the form of the radius of curvature. For instance, the basic variational problem can be posed as one of finding a planar curve that minimizes an action functional that amounts to simply the integral of the square of that curvature along the curve from among the planar curves that have the same length.

D. H. Delphenich, 2018

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PREFACE TO THE GERMAN TRANSLATION

For a practical understanding of the older ground-breaking works on *elastica*, a knowledge of the connection between the Ansätze that they contain and the methods of stereo-mechanics and continuum mechanics is prerequisite.

The following introductory remarks shall also make it possible for those readers who have not studied general mechanics in detail to make a critical evaluation of the most important original works on that topic.

a) One imagines that the original straight elastic wire (viz., *lamina*) has been replaced with a chain (i.e., series) of infinitely-small, rigid material elements that are coupled to each other with pivot joints whose axes all remain perpendicular to a fixed plane. The forces – $\mathbf{r}$ and – $\mathbf{r}'$ are applied to the joints C and C', resp., of an arbitrary material element $\mathfrak{K}$, and those forces are transferred to the previous and following elements, resp. Along with those isolated forces at C and C', moments – $\mathbf{R}$ and $\mathbf{R}'$ will also be applied when the links exhibit resistance to rotation of the terms about the pivot axis. Let the resultant of the applied (external) forces on $\mathfrak{K}$ be $d\,\mathbf{k}$, and let its moment relative to the rotational point C of the link $\mathfrak{K}$ be $d\,\mathbf{M}$. The reaction moments $\mathbf{R}$ and $\mathbf{R}'$ can be referred to an arbitrary pole O in the plane.

If one sets $\overline{OC} = \mathbf{c}$, $\overline{OC'} = \mathbf{c}'$, $\overline{c'-c} = d\,\mathbf{c}'$, $\overline{r'-r} = d\,\mathbf{r}$, $\overline{R'-R} = d\,\mathbf{R}$ then the basic rules of elementary statics will give the following conditions for equilibrium of the forces on the material element $\mathfrak{K}$:

$$d\,\mathbf{r} + d\,\mathbf{k} = 0, \qquad d\,\mathbf{R} + d\mathbf{c}\cdot\mathbf{r} + d\,\mathbf{M} = 0 \qquad \text{(pole } C\text{)}.$$

With the introduction of the arc element ds of the equilibrium curve (axis of the flexible, inextensible wire), which couples to the pivot (C), one can employ specific quantities $\mathbf{k} = d\,\mathbf{k} / ds$ and $\mathbf{m} = d\,\mathbf{M} / ds$, which refer to a unit length of the wire axis, in place of the absolute quantities $d\,\mathbf{k}$ and $d\,\mathbf{M}$, resp. If one then sets $d\mathbf{c}/ds = \boldsymbol{\sigma}$ then the static Ansatz will assume the form:

$$\text{(1)} \qquad \frac{d\mathbf{r}}{ds} + \boldsymbol{\kappa} = 0, \qquad\qquad \text{(2)} \qquad \frac{d\mathbf{R}}{ds} + \boldsymbol{\sigma}\,\mathbf{r} + \mathbf{m} = 0.$$

These equations appear many times in recent literature, while they are absent from the older authors (**Jac. Bernoulli**, **Euler**) in that explicit form. **Euler** knew of the corresponding form for chains whose links have finite dimensions (confer the discussion by **Routh** in his *Dynamik*, German edition, v. 2, pp. 71), but he did not pass to the limit. Confer also **Euler**'s Ansatz in no. **57** (pp. 43) of the following.

Equations (1) and (2) were presented by **Clebsch** in his *Elastizität fester Körper*, Leipzig, 1862, pp. 204-222, and were used as the basis for **Kirchhoff**'s theory of wires. One also finds them in **Thomson** and **Tait**, *Natural Philosophy*, Part 2, 1st ed., Oxford, 1867, 2nd ed., Cambridge, 1895, pp. 152-155, in **Love**, *Theory of Elasticity*, 2nd ed., Cambridge, 1906, pp. 370-372, and their direct relationship to the theory of chains of objects with finite links is discussed in **K. Heun** in the Zeit. Math. Phys. **56** (1908), pp. 68, *et seq*. Those equations are treated thoroughly from a more general viewpoint by **E.** and **F. Cosserat**, *Théorie des corps déformables*, Paris, 1909, pp. 5-65.

b) Let the position of the axis element *d* **c** relative to the *x*-axis of a fixed axis-cross *Oxy* be determined by the angle ϑ. Let the magnitude and direction of the contingency angle of the elastica be denoted by $d\boldsymbol{\vartheta}$. With those preliminaries, $d\boldsymbol{\vartheta} / ds$ will be the specific rotation of the axis element *d* **c**. Along with the direction of the tangent ($\boldsymbol{\sigma}$), one introduces the direction of the normal to the curve $\boldsymbol{\nu}$. Hence, $\boldsymbol{\eta} = \boldsymbol{\sigma}\,\boldsymbol{\nu}$ is the segment of the altitude (of unit length) to the plane of the curve, such that:

$$\frac{d\boldsymbol{\vartheta}}{ds} = \mathfrak{w} = \frac{1}{a_1}\boldsymbol{\eta},$$

in which a_1 is the radius of curvature of the elastica.

From **Daniel Bernoulli**'s hypothesis, one has $\mathbf{R} = \mathsf{P}\,\mathfrak{w}$, if P means a constant that depends upon the dimensions of the cross-section and the elastic coefficients.

Furthermore, let:

$$\kappa_x = \frac{\partial u}{\partial c_x}, \qquad \kappa_y = \frac{\partial u}{\partial c_y}, \qquad \kappa_z = \frac{\partial u}{\partial \vartheta}.$$

The function *u* can be referred to as the potential of the applied forces. It will now follow from the basic equations of statics (1) and (2) that:

$$\frac{d\mathbf{r}}{ds}\frac{d\mathbf{c}}{ds} + \frac{d\mathbf{R}}{ds}\frac{d\boldsymbol{\vartheta}}{ds} + \boldsymbol{\sigma}\,\mathbf{r}\frac{d\boldsymbol{\vartheta}}{ds} + \frac{du}{ds} = 0$$

or

$$\frac{d\mathbf{r}}{ds}\boldsymbol{\sigma} + \boldsymbol{\sigma}\,\mathbf{r}\,\mathfrak{w} + \mathsf{P}\mathfrak{w}\frac{d\mathfrak{w}}{ds} + \frac{du}{ds} = 0.$$

If one then sets:

$$\mathbf{r} = r_\sigma\,\boldsymbol{\sigma} + r_\nu\,\boldsymbol{\nu}$$

then one will have:

$$\frac{d\mathbf{r}}{ds} = \frac{dr_\sigma}{ds}\boldsymbol{\sigma} + \frac{dr_\sigma}{ds}\boldsymbol{\nu} + \mathfrak{w}\,r_\sigma\boldsymbol{\nu} + r_\nu\frac{d\boldsymbol{\nu}}{ds}$$

and

$$\boldsymbol{\sigma}\frac{d\mathbf{r}}{ds} = \frac{dr_\sigma}{ds} + r_\nu\,\boldsymbol{\sigma}\frac{d\boldsymbol{\nu}}{ds}.$$

However, one has:

$$\frac{d\boldsymbol{\nu}}{ds} = -\,\mathfrak{w}\;\boldsymbol{\sigma}.$$

It will then follow that:

$$\boldsymbol{\sigma}\frac{d\mathbf{r}}{ds} = -\,\mathfrak{w}\,r_\nu + \frac{dr_\sigma}{ds}, \qquad \boldsymbol{\sigma}\,\mathbf{r} = r_\nu\,\boldsymbol{\eta}\,.$$

One sees from this that the equation:

$$\frac{dr_\sigma}{ds} + \mathsf{P}\,\mathfrak{w}\frac{d\mathfrak{w}}{ds} + \frac{du}{ds} = 0$$

is integrable, such that:

$$(3) \qquad \tfrac{1}{2}\mathsf{P}\,\mathfrak{w}^2 + r_\sigma + u = h^0.$$

That equation exhibits a certain analogy with the principle of *vis viva* in kinetics.

c) One can set $u = 0$ for an elastica with no applied forces. Equation (3) will then assume the simplified form:

$$(3') \qquad \tfrac{1}{2}\mathsf{P}\,\mathfrak{w}^2 + r_\sigma = h^0.$$

The virtual work due to bending is $R\,\delta\vartheta$. One then defines:

$$\frac{d}{ds}(R\,\delta\vartheta) = \frac{dR}{ds}\delta\vartheta + R\frac{d\,\delta\vartheta}{ds},$$

or with the use of equation (2):

$$\frac{d}{ds}(R\,\delta\vartheta) = -\,r_\nu\,\delta\vartheta + R\,\delta\mathfrak{w}.$$

If one denotes the endpoints of the elastica by A and B then an integration along the axis of the wire will yield:

$$[R\,\delta\vartheta]_A^B = \int_A^B (R\,\delta\mathfrak{w} - r_\nu\,\delta\vartheta)\,ds\,.$$

If one sets the virtual rotation $\delta\vartheta$ equal to zero at the limits A and B then one will have:

$$\int_A^B (R\,\delta\mathfrak{w} - r_\nu\,\delta\vartheta)\,ds = 0,$$

or, since:

$$r_\nu\,\delta\vartheta = \delta r_\sigma,$$

one will have:

$$\delta\int_A^B (\tfrac{1}{2}\mathsf{P}\,\mathfrak{w}^2 - r_\sigma)\,ds = 0.$$

However, from equation (3′), one will have:

$$-r_\sigma = \tfrac{1}{2}\mathsf{P}\,\mathfrak{w}^2 - h^0.$$

As a result:

(4) $$\delta\left[\int_A^B \mathsf{P}\,\mathfrak{w}^2 ds - h^0 \int_A^B ds\right] = 0.$$

In **Euler**'s way of looking at things, the integral $\int_A^B \mathsf{P}\,\mathfrak{w}^2 ds$ will then be a maximum-minimum with the isoperimetric condition $\int_A^B ds = l$ (const.).

It follows from equation (1) that $\mathbf{r} = \mathbf{r}^0$. One will then have:

$$r_\sigma = r_x^0 \cos\vartheta + r_y^0 \sin\vartheta \qquad \text{and} \qquad r_\nu = -r_x^0 \sin\vartheta + r_y^0 \cos\vartheta.$$

Ordinarily, one chooses the axes O_x , O_y in such a way that one will have $r_y^0 = 0$.

d) One can regard the quantity $\frac{1}{2}\mathsf{P}\,\mathfrak{w}^2 = e$ in equation (3) as an energy. The sum $u + r_\sigma = u'$ can be considered to be a modified potential energy. We then set $e - u' = f$ and let f denote the *Lagrangian function* for statics (in analogy to stereo-kinetics). The static analogue to the *Lagrangian* equation of kinetics will have the form:

$$\frac{d}{d\tau}\frac{df}{d\mathfrak{w}} - \frac{df}{d\vartheta} = 0,$$

and will be identical to the equation:

$$\frac{dR}{ds} + r_\nu + m = 0$$

in the present case.

With that, the **Kirchhoff** analogy is proved. One can find further details on that analogy in **Love**, *Elasticity*, 2nd ed., pp. 382, and **W. Hess**, Math. Ann. **25** (1885).

e) **Euler** gives instructions on how to treat the isoperimetric problem in Chapter 5 of *Methodus inveniendi…*, whose German version by **P. Staeckel** is included in this collection as volume **46**.

K. Heun

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Appendix I to: "Method of finding curves that possess a property of having the highest or lowest degree." (1744)

Introduction

1. – For some time, some brilliant mathematicians have recognized that the methods that are proposed in this book are of use, not only in analysis, but also in the solution of physical problems. Namely, since the plan of the entire universe is complete and was established by the wisest Creator, nothing will happen in the world that is not based upon some relationship to the maximum or minimum. For that reason, no further doubt can exist that all phenomena in the world can be just as well determined from the final causes with the help of the method of maxima and minima as they can from the causes that that are in effect. So many excellent examples of that fact exist already that further examples would not be necessary in order to state that truth. Moreover, in every type of question of natural science, one must then strive to determine a quantity that assumes a largest or smallest value. That problem seems to belong more to philosophy than mathematics. Since a double path is then given for exploring the phenomena of nature, once in terms of the initial causes, which one would prefer to call the "direct method," and secondly, from the final causes, the mathematician might appeal to either of them with the same effect. Namely, when the initial causes lie hidden, but the final causes are clear, the problem can be solved by the indirect method. In contrast, the direct method can be applied whenever the effects can be defined by the initial causes. However, one must see, in particular, how to make the solution tractable in both cases. Not only will the one serve to confirm the other, but the agreement between them will give us the greatest satisfaction. The curvature of a cable or a hanging chain can be ascertained in two ways: First, *a priori*, from the effects of gravity, and then by the method of maxima and minima, since it is clear that such a cable must assume a curvature such that the center of gravity will lie as deeply as possible. In the same way, the curvature of the rays that go through a transparent medium of varying density can be determined *a priori*, as well as from the fact that they must arrive at a given location in the shortest time.

Very many similar examples were furnished by the esteemed **Bernoulli** and others, whom we can thank for significantly completing the *a priori* method of solution and our understanding of the initial causes. Thus, although the great number of clever examples means that no doubt remains that the property of being a maximum or minimum will appear for all curves that arise by solving the problems of mathematical physics, nonetheless, that maximum or minimum is often very difficult to recognize, even though one can recognize it from the *a priori* solution. Thus, the form that a curved elastic band assumes has been known for some time, but up to now no one has remarked how that curve could be explored by the method of maxima and minima; i.e., from final causes. Now, the most highly honored, as well as most incisive in regard to that way of exploring nature, **Daniel Bernoulli** has communicated to me that the total force that a curved elastic band includes, which he called the *potential force*, can be summarized in a formula and that this expression must be a minimum for the elastic curve ([1]). Since a

([1]) **Daniel Bernoulli** pointed out the potential force to **Euler** in a letter on 20 October 1742. (Letter 26, vol. 2, footnote v.: *Correspondance mathématiques et physique*, Petersburg, 1843). At the end of the letter, he said: "Since no one has mastered the isoperimetric method (i.e., the variational calculus that **Euler** had founded as a special branch of analysis) as completely as you, you will solve this problem, in which one requires that $\int ds / R^2$ must be a minimum." **Dan. Bernoulli** had already known about **Euler**'s method of finding curves before the Appendix on elastic curves had appeared, as he spoke about it with great interest in his letter to **Euler** in 1743. See no. **63** and the remark on page 63 below.

new and wonderful light shines upon my method of maxima and minima because of that discovery, and its very far-reaching applications clearly emerge, I have believed that this very desirable situation cannot be passed over, without, at the same time, my making the application of my method clear by publishing that exceptional property of the elastic curve that the renowned **Bernoulli** had discovered. That property also implies second-order differentials that are still not present in the solution of the isoperimetric problem.

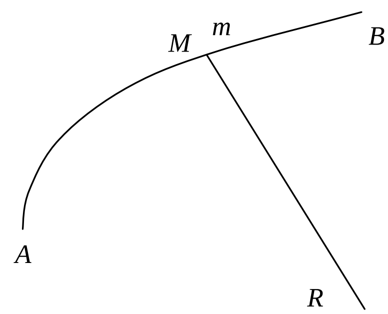

Figure 1.

On the curvature of uniform elastic bands

2. – Let *AB* (Fig. 1) be an arbitrarily-curved elastic band. The arc length *s* will be denoted by *s* and the radius of curvature at *M* by *R*. According to **Bernoulli**, the force potential that is included in the segment *AM* will be expressed by the formula $\int ds / R^2$, when the band is equally thick and wide and elastic everywhere, and in the natural state it is straight ([1]). Therefore, let it be a property of the curve *AM* that this expression is a minimum for it. However, since second-order differentials enter into the expression for that radius of curvature *R*, one will need four conditions to determine a curve that is endowed with that property, and that will correspond precisely to the nature of our problem. Namely, since infinitely many elastic bands of the same length can be drawn through two given endpoints *A* and *B*, the solution can be unique only when two more points are given, in addition to the two points *A* and *B*, or (what amounts to the same thing) the positions of the tangents at the endpoints *A* and *B*. The given elastic band, which is longer than the rectilinear distance *AB*, can be bent in such a way that it not only goes through the endpoints, but also in such a way that the tangents to it at those points will possess given directions. Hence, the problem of finding the curvature of elastic bands can be posed as follows:

Among all curves of the same length that go through the points A and B and contact lines at those points whose positions are given, determine the one for which the expression $\int ds / R^2$ *is a minimum.*

3. – Since the solution shall refer to rectangular coordinates, an arbitrary line *AD* will be taken to be an axis. For it, let *AP* = *x*, and let the ordinate *PM* be equal to *y*. As the

([1]) For the special formulas that are employed here, confer Chapter II and V of **Euler**'s *Methodus inveniendi lineas curvas*, etc. (Volume **46** of this collection, published by **P. Staeckel**.)

method that was found prescribes, one sets $dy = p\,dx$, $dp = q\,dx$. Let the curve element Mm be:

$$Mm = ds = dx\sqrt{1+p^2}\,.$$

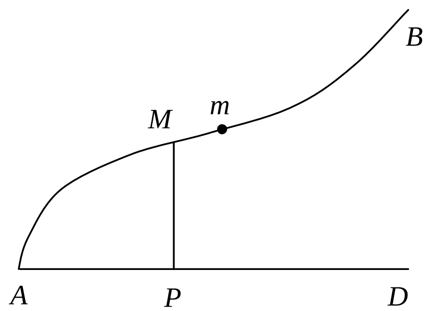

Figure 2.

Since the curves from which the desired one is to be ascertained should be isoperimetric, one must first consider the expression:

$$\int dx\sqrt{1+p^2}\,.$$

When one compares this to the general $\int Z\,dx$ and differentiates it, that will give $\frac{d}{dx}\frac{p}{\sqrt{1+p^2}}$. Secondly, since the radius of curvature is equal to:

$$\left(\frac{dp}{dx}\right)^{-1}(1+p^2)^{3/2} = \frac{(1+p^2)^{3/2}}{q} = R,$$

the formula $\int ds / R^2$, which should be a minimum, will become $\int \frac{q^2 dx}{(1+p^2)^{5/2}}$.

When this is compared with the general formula $\int Z\,dx$, that will give:

$$Z = \frac{q^2}{(1+p^2)^{5/2}}\,.$$

When one sets $dZ = M\,dx + N\,dy + P\,dp + Q\,dq$, that will give:

$$M = 0, \qquad N = 0, \qquad P = -\frac{2pq^2}{(1+p^2)^{7/2}}, \qquad Q = \frac{2q}{(1+p^2)^{5/2}}\,.$$

The differential expression that is derived from the formula $\int \frac{q^2 dx}{(1+p^2)^{5/2}}$ is then:

$$-\frac{dP}{dx} + \frac{d^2Q}{dx^2}\,.$$

One will then have the following equation for the desired curve:

$$\alpha \frac{d}{dx} \frac{p}{\sqrt{1+p^2}} = \frac{dP}{dx} + \frac{d^2Q}{dx^2}.$$

If one multiplies this by dx and integrates then one will get:

$$\frac{\alpha p}{\sqrt{1+p^2}} + \beta = P - \frac{dQ}{dx}.$$

This equation in then multiplied by $q\,dx = dp$, such that what will emerge is:

$$\frac{\alpha p\,dp}{\sqrt{1+p^2}} + \beta\,dp = P\,dp - q\,dQ.$$

However, due to the fact that $M = 0$, $N = 0$:

$$dZ = P\,dp + Q\,dq, \qquad \text{so} \qquad P\,dp = dZ - Q\,dq.$$

If one substitutes that value then what will arise is:

$$\frac{\alpha p\,dp}{\sqrt{1+p^2}} + \beta\,dp = dZ - Q\,dq - q\,dQ.$$

After another integration, what will follow is:

$$\alpha\sqrt{1+p^2} + \beta p + \gamma = dZ - Q\,dq - q\,dQ.$$

However, since $Z = \dfrac{q^2}{(1+p^2)^{5/2}}$ and $Q = \dfrac{2q}{(1+p^2)^{5/2}}$, one will have:

$$\alpha\sqrt{1+p^2} + \beta p + \gamma = \frac{-q^2}{(1+p^2)^{5/2}}.$$

If one takes the arbitrary constants to be negative then one will get:

$$q = (1+p^2)^{5/4}\sqrt{\alpha\sqrt{1+p^2} + \beta p + \gamma} = \frac{dp}{dx}.$$

It will then follow from this that:

$$dx = \frac{dp}{(1+p^2)^{5/4}\sqrt{\alpha\sqrt{1+p^2}+\beta p+\gamma}}.$$

Since $dy = p\,dx$, one also has:

$$dy = \frac{p\,dp}{(1+p^2)^{5/4}\sqrt{\alpha\sqrt{1+p^2}+\beta p+\gamma}}.$$

These two equations suffice to find the curve by quadratures.

4. – This formula, which was derived in a completely general way, is hardly integrable. However, it can be constrained in such a way that the constraint allows it to be integrated. Namely, since:

$$d\,\frac{\sqrt{\alpha\sqrt{1+p^2}+\beta p+\gamma}}{\sqrt[4]{1+p^2}} = \frac{dp\,(\beta-\gamma p)}{(1+p^2)^{5/4}\sqrt{\alpha\sqrt{1+p^2}+\beta p+\gamma}},$$

one will have:

$$\frac{2\sqrt{\alpha\sqrt{1+p^2}+\beta p+\gamma}}{(1+p^2)^{1/4}} = \beta x - \gamma y + \delta.$$

Since the position of the origin along the axis is arbitrary, one can drop δ. One can change the axis in such a way that ([1]):

([1]) Let it be pointed out for this coordinate transformation that the new axes will again be rectilinear. The new x-axis defines an angle of φ with the old one that is determined by $\tan\varphi = \gamma / \beta$. When $P = dY / dX$, that will become:

$$p = \frac{\beta P-\gamma}{\beta+\gamma P},$$

and therefore:

$$1 + p^2 = \frac{(\beta^2+\gamma^2)(1+P^2)}{(\beta+\gamma P)^2}.$$

One substitutes this value in **Euler**'s last equation and obtains:

$$\frac{2\sqrt{\alpha\sqrt{1+P^2}+P\sqrt{\beta^2+\gamma^2}}}{(1+P^2)^{1/4}} = X\sqrt{\beta^2+\gamma^2}.$$

Let $\sqrt{\beta^2+\gamma^2} = \beta_1$. If one introduces the lower-case symbols, in place of the upper-case ones, then one will get:

$$2\sqrt{\alpha\sqrt{1+p^2}+\beta_1\gamma} = \beta_1\,x\,(1+p^2)^{1/4},$$

as in the text.

$$X = \frac{\beta x - \gamma y}{\sqrt{\beta^2 + \gamma^2}} \qquad \text{and} \qquad Y = \frac{\gamma x + \beta y}{\sqrt{\beta^2 + \gamma^2}}.$$

One can certainly set γ equal to zero in this, since nothing stands in the way of once more denoting the new abscissa by x. One will then get the equation for the elastic curve:

$$2\sqrt{\alpha\sqrt{1+p^2} + \beta p} = \beta x \sqrt[4]{1+p^2}.$$

Squaring this will yield:

$$4\alpha\sqrt{1+p^2} + 4\beta p = \beta^2 x^2 \sqrt{1+p^2}.$$

In order to make this homogeneous, let:

$$\alpha = \frac{4m}{a^2} \qquad \text{and} \qquad \beta = \frac{4n}{a^2},$$

which will make:

$$n\,a^2\,p = (n^2x^2 - ma^2)\sqrt{1+p^2},$$

so

$$p = \frac{n^2x^2 - ma^2}{\sqrt{n^2a^4 - (n^2x^2 - ma^2)^2}} = \frac{dy}{dx}.$$

When one changes the constants and either increases or diminishes the abscissa x by a given constant ([1]), one will get the following equation for the general elastic curve:

$$dy = \frac{(\alpha + \beta x + \gamma x^2)\,dx}{\sqrt{a^4 - (\alpha + \beta x + \gamma x^2)^2}},$$

from which it will follow that:

$$ds = \frac{a^2\,dx}{\sqrt{a^4 - (\alpha + \beta x + \gamma x^2)^2}}.$$

([1]) One sets:

$$n = \gamma, \qquad x = x_1 + \frac{\beta}{2\gamma}, \qquad m = \frac{1}{a^2}\left(\frac{\beta^2}{4} - \alpha\gamma\right).$$

Naturally, the quantities α, β, γ that are introduced here are different from the quantities α, β, γ that were introduced at the beginning of this section. One will then have:

$$n^2\,x^2 - m\,a^2 = \gamma(\alpha + \beta x_1 + \gamma x_1^2),$$

so:

$$dy = \frac{dx_1\,(\alpha + \beta x_1 + \gamma x_1^2)}{\sqrt{a^4 - (\alpha + \beta x_1 + \gamma x_1^2)^2}}.$$

If one drops the index on x then one will get the penultimate equation of no. **4** in the text.

The agreement of the curve that is found in that way with the elastic curve that was ascertained before will emerge from that equation.

5. – In order to make this agreement emerge more clearly, I will also examine the nature of elastic curves directly. Although the very learned **Jacob Bernoulli** has already done this excellently, I will nonetheless take this opportunity to add a few observations about the properties of elastic curves, such as their various types and figures, that I believe have either been overlooked or only touched upon.

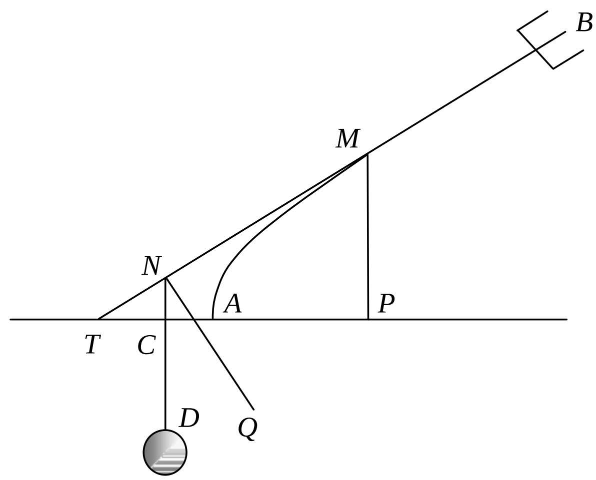

Figure 3.

Let the elastic band *AB* be fixed to a wall or floor (Fig. 3) in such a way that the end *B* will not only remains fixed, but the position of the tangent at *B* will also be determined. Let the band be constrained at *A* by the rigid rod *AC*, to which the force *CD* = *P* is applied perpendicularly. In that way, one will find the band to be in the curved configuration *BMA*. Take *AC* to be the axis, and let *AC* equal *c*, *AP* = *x*, *MP* = *y*. Now, if the band suddenly loses its elasticity at *M* and is completely flexible then it will be bent by a force whose moment is equal to *P* (*c* + *x*). In order for no motion to follow from that bending, the elasticity of the band at *M* must be in equilibrium with the moment of the applied for *P* (*c* + *x*). However, elasticity depends, first of all, upon the material of the band, which I shall assume to be the same everywhere, but then the same thing will be true of the curvature of the band at *M*, such that it will be inversely proportional to the radius of curvature at *M*. Let that be $R = -\left(\frac{dx}{ds}\frac{d^2y}{ds^2}\right)^{-1}$, $ds = \sqrt{dx^2 + dy^2}$ in this, and *dx* is constant, so $\mathrm{E}\,k^2 / R$ will express the elastic force in the band at *M* that brings about equilibrium with the moment of the applied force *P* (*c* + *x*). The equation then exists:

$$P\,(c + x) = \frac{\mathrm{E}\,k^2}{R} = -\,\mathrm{E}\,k^2\left(\frac{dx}{ds}\frac{d^2y}{ds^2}\right)^{-1}.$$

When that equation is multiplied by *dx*, it will be integrable. That integral will then be:

$$P\left(\tfrac{1}{2}x^2 + cx + f\right) = \frac{-\mathrm{E}\,k^2 dy}{\sqrt{dx^2 + dy^2}},$$

so:

$$dy = \frac{-P\,dx\left(\tfrac{1}{2}x^2 + cx + f\right)}{\sqrt{\mathrm{E}^2 k^4 - P^2\left(\tfrac{1}{2}x^2 + cx + f\right)^2}}.$$

That equation coincides with the one that I derived by the method of maxima and minima from **Bernoulli**'s principle.

6. – One can derive the force that is required in order to produce the given curvature of the band from a comparison of that equation with the one that was found before. The elastic band might be represented by the curve *AMB*, whose equation is:

$$dy = \frac{(\alpha + \beta x + \gamma x^2)\,dx}{\sqrt{a^4 - (\alpha + \beta x + \gamma x^2)^2}}.$$

$\mathrm{E}\,k^2$ might express the absolute elasticity of that band, such that when $\mathrm{E}\,k^2$ is divided by the radius of curvature at an arbitrary location, that might yield the true elastic force.

In order to perform the comparison, the numerator and denominator is multiplied by $\mathrm{E}\,k^2 / a^2$, such that one will have:

$$dy = \frac{\mathrm{E}\,k^2 / a^2 (\alpha + \beta x + \gamma x^2)\,dx}{\sqrt{\mathrm{E}^2 k^4 - \dfrac{\mathrm{E}^2 k^4}{a^4}(\alpha + \beta x + \gamma x^2)^2}}.$$

One will then have:

$$-\tfrac{1}{2}P = \frac{\mathrm{E}\,k^2\gamma}{a^2}, \qquad -P\,c = \frac{\mathrm{E}\,k^2\gamma}{a^2}, \qquad -P\,f = \frac{\mathrm{E}\,k^2\gamma}{a^2},$$

so the applied force will be $CD = -\,2\,\mathrm{E}\,k^2\,\gamma / a^2$, the length will be $AC = c = \beta / 2\gamma$ and the constant f will be equal to $\alpha / 2\gamma$.

7. – Hence, in order for the elastic band *AB*, one end of which *B* is fixed to the wall, to be bent into the form *AMB*, that band must be subjected to a force of $CD = -\,2\,\mathrm{E}\,k^2\,\gamma / a^2$ in the direction *CD* that is perpendicular to the axis *AP*. As the figure shows, that force will act in the opposite direction when γ is a positive quantity. Since $\mathrm{E}\,k^2 / R$ represents the moment of the driving force, $\mathrm{E}\,k^2 / a^2$ will be equivalent to the a pure force, and that force will be determined from the elasticity of the band; let it be *F*. The bending force *CD* will then have the same ratio to the force *F* that $-\,2\gamma$ has to 1, where γ is a pure number.

8. – One can now further determine the force that is necessary in order for the segment *BM* of the band to keep its position when the segment *AM* is cut. If the segment *AM* is cut then the elastic band will end at the rigid bar *MT* (Fig. 3), which is coupled to the band in such a way that it will always give the direction of the tangent at the point *M*, which will also bend the band. With those conventions, it will be clear from the foregoing that in order to maintain the curvature *BM* of the bar *MT* at the point *N* in the direction *ND*, one must apply a force of $-2\,\mathsf{E}\,k^2\,\gamma/a^2$. *ND* is normal to the axis *AP* and $AC = \beta/2\gamma$. The distance *MN* will be:

$$\frac{ds}{dx}CP = \frac{ds}{dx}\cdot\frac{\beta+2\gamma x}{2\gamma} = \frac{(\beta+2\gamma x)\,ds}{2\gamma\,dx},$$

so:

$$\frac{ds}{dx} = \frac{a^2}{\sqrt{a^4-(\alpha+\delta x+\gamma x^2)^2}}.$$

If the force *ND* is decomposed into two components, namely, *NQ*, which is perpendicular to the tangent *MY*, and *NT*, which is in the direction of the tangent, then one will have:

$$NQ = \frac{-2\mathsf{E}\,k^2\gamma}{a^2}\frac{dx}{ds}$$

and

$$NT = \frac{-2\mathsf{E}\,k^2\gamma}{a^2}\frac{dy}{ds}.$$

9. – If one cuts out the segment *BM* then *AM* will be subjected to the force $-2\,\mathsf{E}\,k^2\,\gamma/a^2$ in the direction *CD*, as before. In order to maintain the curvature of the segment *AM*, the endpoint *M*, which is regarded as fixed to the rigid rod *MN* in the direction of the tangent, must driven by a force $-2\,\mathsf{E}\,k^2\,\gamma/a^2$ at *N*, but in the opposite direction to the one that was found in the previous case. Namely, one continually removes that force, which must be applied at the two endpoints. It must then be equal and have the opposite direction. The forces that are established in a segment of the cut band in order for the existing curvature to remain the same can then be determined very easily.

10. – Let *AM* be a curved elastic band that is fixed to the rigid fixed rods *AB*, *MN* at *A* and *M*. The equal forces *DE* and *NR* might be applied to those rods in opposite directions in order to produce equilibrium with the curvature of the band *AM*, and the equation for that shall be presented. The line *AP* that goes through *A* and is perpendicular to the directions of the previous two forces will be taken to be the axis. The absolute elasticity of the band will be set equal to $\mathsf{E}\,k^2$. Let $CAD = m$, $\cos CAD = n$, where *CAD* is the angle that the tangent at *A* makes with the axis. One will then have $m^2 + n^2 = 1$. Furthermore, let $AC = c$ and let the bending force be $DE = NR = P$, and finally let $AP = x$ and $PM = y$. The curve will then be expressed by the equation:

$$dy = \frac{-P\left(\frac{1}{2}x^2 + cx + f\right)dx}{\sqrt{\mathsf{E}^2k^4 - P^2\left(\frac{1}{2}x^2 + cx + f\right)^2}}.$$

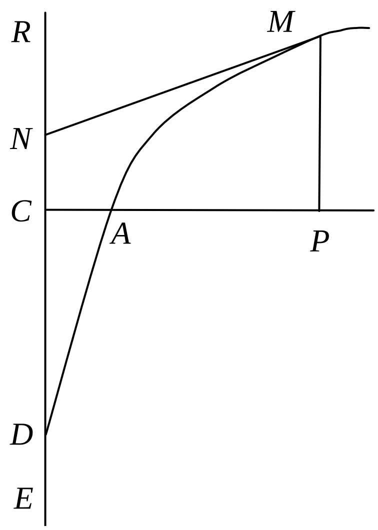

Figure 4.

Since the direction of the tangent at A is given, one must have:

$$\frac{dy}{dx} = \frac{m}{n} \qquad \text{for } x = 0\,;$$

hence:

$$m = -\frac{Pf}{\mathsf{E}k^2}.$$

The constant f can be determined from this; it is:

$$f = \frac{-m\mathsf{E}k^2}{P}.$$

The entire curve is determined now.

11. – In order to bend the band AM into the curve that expressed by the equation above, the force $DE = P$ must be applied to the tangent AB at the point D, where $AB = c / n$. Let its direction be parallel to the ordinate PM.

The force DE must be decomposed into two components Dd and Df that are perpendicular to each other; one will then have $Dd = Pn$ and $Df = Pm$. In order for the consideration of the line AD to be unnecessary in the calculation, two forces can be substituted for the force Dd at the given points A and B, where $AB = h$, namely, $Aa = p$ and $Bb = q$, which are likewise perpendicular to the rod AB. In order to do that, one must assume that $p \cdot h = Pn \cdot BD = Pn\,(c / n - h)$ and that $q = p + n\,P$. However, since it does not matter at which point of the rod AD one applies the tangential force $Df = Pm$, it will be applied to the point *A* precisely, and one will set $AF = mP$. Let $AF = r$, such that the band *AM* will be under the influence of three forces $Aa = p$, $Bb = q$, and $AF = r$. We would like to investigate the influence that those three forces might have on curvature.

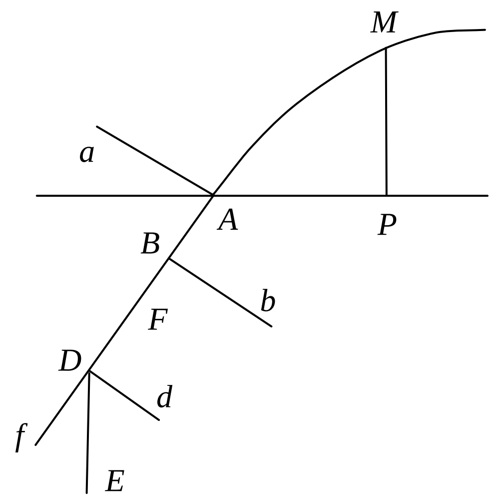

Figure 5.

12. – One has $m\,P = r$, so $P = r / m$. When substituted in the previous equations, that will give:

$$ph = \frac{cr}{m} - \frac{nhr}{m} \qquad \text{and} \qquad q = p + \frac{n\,r}{m},$$

so:

$$\frac{n}{m} = \frac{q-p}{r},$$

and the position of the axis *AB* will be known from that equation, which will be:

$$\tan CAD = \frac{m}{n} = \frac{r}{q-p},$$

so

$$m = \frac{r}{\sqrt{r^2 + (q-p)^2}} \qquad \text{and} \qquad n = \frac{q-p}{\sqrt{r^2 + (q-p)^2}}.$$

It will follow from the equation:

$$hp = \frac{cr}{m} - \frac{nhr}{m} = \frac{cr}{m} - hq + hp$$

that:

$$c = \frac{mhq}{r} \qquad \text{or} \qquad c = \frac{h\,q}{\sqrt{r^2 + (q-p)^2}}$$

and

$$P = \sqrt{r^2 + (q-p)^2}\,.$$

Now, since:

$$f = \frac{-m\mathsf{E}k^2}{P} = \frac{-m\mathsf{E}k^2}{r^2 + (q-p)^2}\,,$$

one will have:

$$\tfrac{1}{2}x^2 + c\,x + f = \tfrac{1}{2}x^2 + \frac{hqx}{\sqrt{r^2 + (q-p)^2}} - \frac{\mathsf{E}k^2 r}{r^2 + (q-p)^2}\,.$$

One will get the following equation for the desired curve:

$$dy = \frac{dx\left[\dfrac{\mathsf{E}k^2 r}{\sqrt{r^2+(q-p)^2}} - hqx - \tfrac{1}{2}x^2\sqrt{r^2+(q-p)^2}\right]}{\sqrt{\mathsf{E}k^2 r - \left[\dfrac{\mathsf{E}k^2 r}{\sqrt{r^2+(q-p)^2}} - hqx - \tfrac{1}{2}x^2\sqrt{r^2+(q-p)^2}\right]^2}}\,.$$

That equation is very convenient for the investigation of the most common kind of the bending of a band when one grabs it with either a pair of pliers or two fingers. The one finger pushes in the direction *Aa*, while other pushes in the direction *Bb*, and the band can still be bent in the direction *AF*, in addition.

13. – If the tangential force $AF = r$ vanishes then the axis *AP* will fall along the direction of the tangent *AF*. One will then have:

$$dy = \frac{-dx\left[hqx - \tfrac{1}{2}(q-p)\,x^2\right]}{\sqrt{\mathsf{E}^2 k^4 - \left[hqx + \tfrac{1}{2}(q-p)\,x^2\right]^2}}\,.$$

If the forces *p* and *q* are equal to each other then the axis *AP* will be perpendicular to the tangent *AF*, since $n = 0$. The equation of the curve will then be:

$$dy = \frac{dx\left[\mathsf{E}k^2 - hqx - \tfrac{1}{2}r\,x^2\right]}{\sqrt{2\,\mathsf{E}k^2(hqx + \tfrac{1}{2}r\,x^2) - (hqx + \tfrac{1}{2}r\,x^2)^2}}\,.$$

When one sets $r = 0$ in this, such that the band is subject to equal, but oppositely-directed, forces at *A* and *B*, the expression for the associated curve will be:

$$dy = \frac{dx\left[\mathsf{E}k^2 - hqx\right]}{\sqrt{h\,q\,(2\,\mathsf{E}k^2 x - hq\,x^2)}}\,.$$

When that equation is integrated, that will give:

$$y = \sqrt{\frac{2\mathsf{E}k^2x - hqx^2}{hq}},$$

which is then the equation for a circle. The band is then bent into a circle in this case, and the radius of that circle will be $\mathsf{E}\,k^2 / hq$.

The new types of elastic curves

14. – Since we have seen that not just the circle, but an infinite manifold of curves are found among the elastic curves, it might be worth the effort to attempt an enumeration of all types of curves that belong to that genre. In that way, not only the nature of those curves will be seen more precisely, but in any sort of case in question, one will be able to evaluate which type a curve belongs to from its form alone. We would like to establish the difference between the types in such a way that the types can be enumerated in the same way that the types of algebraic curves of a given order can be enumerated ([1]).

15. – The general equation of the elastic curve:

$$dy = \frac{(\alpha + \beta x + \gamma x^2)\,dx}{\sqrt{a^4 - (\alpha + \beta x + \gamma x^2)^2}}$$

will assume a simpler form when the coordinate origin is shifted along the axis by $\beta / 2\gamma$, and a^2 is written for a^2 / γ (or when one sets $\gamma = 1$). The equation will, in fact, become:

$$dy = \frac{(\alpha + x^2)\,dx}{\sqrt{a^4 - (\alpha + x^2)^2}}.$$

Now, $a^4 - (\alpha + x^2)^2 = (a^2 - \alpha - x^2)(a^2 + \alpha + x^2)$, so one sets $a^2 - \alpha = c^2$ or $\alpha = a^2 - c^2$. The equation will go to:

$$dy = \frac{(a^2 - c^2 + x^2)\,dx}{\sqrt{(c^2 - x^2)(2a^2 - c^2 + x^2)^2}}.$$

Since $\beta = 0$ (see no. **6**), the direction of the force at A that bends the band will be perpendicular to the axis. AD will then represent the direction of the applied force whose magnitude is $2\,\mathsf{E}\,k^2 / a^2$, if $\mathsf{E}\,k^2$ expresses the absolute elasticity (see Fig. 6).

([1]) Here, **Euler** was thinking of **Newton**'s celebrated classification of third-order curves. In the following discussion, AP (see Fig. 6) will always be the direction of the positive x-axis, and AB will be that of the positive y-axis. The direction of the applied force in no. **5** is parallel to the negative y-axis.

16. – Let $x = 0$, so one will have $\frac{dy}{dx} = \frac{a^2 - c^2}{c\sqrt{2a^2 - c^2}}$. That expression represents the tangent to the angle that the curve *AM* makes with the axis *AP* at *A*, so the sine of that angle will then be $(a^2 - c^2) / a^2$.

When one then has $a = \infty$ and the bending force $2\,\mathrm{E}\,k^2 / a^2$ then vanishes, the band will be perpendicular to the axis *AP* at *A* and will have no curvature. For $a = \infty$, the band will take the form of a straight line. The first type of elastic curves will then be represented by lines that extend to infinity in both directions.

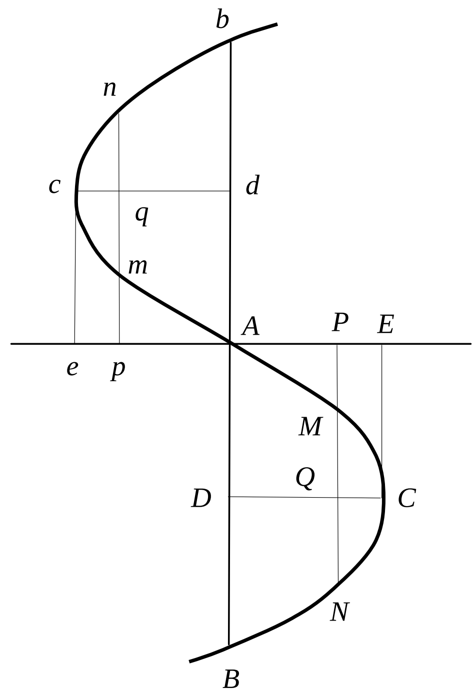

Figure 6.

17. – Before we enumerate the remaining types, it would be good to first make a few general remarks on the figure of the elastic curve. When *a* decreases, the angle *PAM* (Fig. 6) that the curve makes with the axis at *A* will decrease; i.e., as the bending force $2\mathrm{E}k^2/a^2$ increases. If $a^2 = c^2$ then that axis *AP* will contact the curve at *A*. If $a^2 < c^2$ then the curve *AM*, which has moved downwards up to now (as in Fig. 6), will now turn upwards until $a^2 = \frac{1}{2}c^2$. However, when $a^2 < \frac{1}{2}c^2$, the angle will become imaginary, and as a result, no segment of the curve will exist at *A*. Those different cases give rise to the different types.

18. – Since the equation will not change form when *x* and *y* are both taken to be negative, it will further emerge that the curve through *A* will possess similar and equal branches *AMC* and *Amc* that lie alternately; *A* will then be an inflection point. Therefore, when the segment *AMC* is known, if one takes $Ap = AP$ then one will have $pm = PM$. If *x* increases then the curve will move further away on both sides of the axis until the abscissa $AE = c$ is attained. The ordinate EC will contact the curve because $dy / dx = \infty$ for $x = c$. It is clear that beyond $AE= c$, the abscissa x cannot increase any more, since dy / dx would then become imaginary. The entire curve would then lie between the limiting ordinates $SC = ec$. The curve cannot exceed those limits. Up to now, we have established two curve branches AC and Ac that extend to the limits on both sides of A.

19. – How does the curve behave beyond C and c ? One takes the line CD that is parallel to AE to be the axis and sets the new coordinates to $CQ = t$, $QM = u$. One will then have $t + x = AE = CD = c$ and $y + u = CE = AD = b$; hence $x = c - t$, $y = b - u$, and $dy = - du$, $dx = - dt$. If one substitutes those values then that will imply the equation of the curve in the new coordinates:

$$dy = \frac{(a^2 - 2ct + t^2)\,dt}{\sqrt{t\,(2c - t)(2a^2 - 2ct + t^2)}}\,.$$

It will next follow from this that when t is taken to be infinitely small:

$$du = \frac{a^2\,dt}{2a\sqrt{ct}}\,,$$

so

$$u = a\sqrt{\frac{t}{c}}\,.$$

That equation shows that beyond C, the curve will advance to N in a manner that is similar to the way that C goes on to M ([1]). The double-valuedness of the $\sqrt{\ }$ sign in the denominator of the first equation is enough to make it clear that the ordinate u can be taken to be negative and positive in the same way. It is then clear that CD is a diameter of the curve and the arc length CNB is similar and equal to the arc length CMA.

20. – Likewise, the line cd, which runs through c parallel to the other side of the axis AE, will also be a diameter, so the branch Axb will then be equal and similar to the branch ACB. The bending at the points B and b will be precisely opposite to the bending at A; the curve will then go beyond them in the same way. The curve will then possess

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([1]) The form of the curve in the vicinity of C can also be derived from $u = a\sqrt{t / c}$ in such a way that $u^2 = a^2\, t / c$ will represent a parabola. If one sets x to be very small in the original equation then one will have $y = (a^2 - c^2)\, dx / \sqrt{2a^2c^2}$, so $y = a^2 - c^2\, x / (ac\sqrt{2})$; i.e., the curve will have the form of a straight line in the neighborhood of A. That will also follow from the fact that A is an inflection point of the curve; one will then have that

$$\frac{d^2y}{dx^2} = 2x\frac{a^4}{\sqrt{(c^2 - a^2)^3(2a^2 - c^2 + x^2)^3}}$$

will vanish for $x = 0$. The curve will not possess inflection points for other values of x.

infinitely many diameters *CD*, *cd*, etc., that are parallel to each other and have the same separation distance *Dd*. The curve will then consist of infinitely many mutually-similar and equal parts. Therefore, the entire curve will be known when *AMC* is known sufficiently.

21. – Since *A* is an inflection point, the radius of curvature will be infinitely large there. That is also implied by the nature of the curve itself. Namely, the force $2\,E\,k^2 / a^2$ is applied to *A* in the direction *AD*. From the basic property of elasticity, it will be $2E\,k^2 x / a^2$ or $E\,k^2 / R$ (no. **5**), at an arbitrary point, if *R* means the radius of curvature at that point, so $R = a^2 / 2x$. The curvature is greatest at those points, which lie as far from the line *BAb* as possible ([1]).

22. – Although the abscissa $AE = c$ is determined for the point *C*, *EC* can only be found by integrating the equation:

$$dy = \frac{(a^2 - c^2 + x^2)\,dx}{\sqrt{(c^2 - x^2)(2a^2 - c^2 + x^2)}}.$$

If one sets $x = c$ after integrating then the value of the associated *y* will yield the distance *CE*, and when that is doubled, it will yield *AB* or the interval *Dd*, which is equal to *AB* and lies between the diameters. Integration is likewise necessary in order to determine the length *AC* of the bent band. When the arc length is set to $AM = s$, one will have:

$$ds = \frac{a^2\,dx}{\sqrt{(c^2 - x^2)(2a^2 - c^2 + x^2)}}.$$

The integration of that will yield the length of the curve *AC* when one sets $x = c$ ([2]).

([1]) The elastic curve was treated at various places in **Euler**'s "Methodus inveniendi lineas curvas…," but not very thoroughly, either (v. 46 of this collection, pp. 110, 111, 127, 131). In Chapter 5, § **46**, **Euler** proved the important property that among all curves of the same length that go through the same two points, the elastic curve is the one that will generate the body of largest volume when it is rotated around an axis. He also mentioned the relationship $R = a^2 / 2x$ there; i.e., that the radius of curvature is inversely proportional to the abscissa.

([2]) If one sets $x / c = u$ and $c^2 / (c^2 - 2a^2) = k^2$ then one will have:

$$s = \frac{a^2}{\sqrt{2a^2 - c^2}} \int \frac{du}{\sqrt{(1-u^2)(1-k^2u^2)}};$$

i.e., in **Legendre**'s terminology, *s* will be an "elliptic function of the first kind." With the same substitution, *y* will go to:

$$\sqrt{2a^2 - c^2} \int \frac{\sqrt{1-k^2u^2}\,du}{\sqrt{1-u^2}} - \frac{a^2}{\sqrt{2a^2 - c^2}} \int \frac{du}{\sqrt{(1-u^2)(1-k^2u^2)}}.$$

The first term is an elliptic integral of the second kind, while the second term is once more an elliptic integral of the first kind. The integrations for *s* and *y* cannot be carried in closed form then.

23. – Since that formula cannot be integrated, we will try to conveniently express the values of the interval *AD* and the curve segment *AC* by approximation. If we then set $z = \sqrt{c^2 - x^2}$ then:

$$PM = y = \int \frac{(a^2 - z^2)\,dx}{z\sqrt{2a^2 - z^2}}$$

and

$$AM = s = \int \frac{a^2\,dx}{z\sqrt{2a^2 - z^2}} .$$

However, from a series development, one has:

$$\frac{1}{\sqrt{2a^2 - z^2}} = \frac{1}{a\sqrt{2}}\left(1 + \frac{1}{4}\frac{z^2}{a^2} + \frac{1\cdot 3}{4\cdot 8}\frac{z^4}{a^4} + \frac{1\cdot 3\cdot 5}{4\cdot 8\cdot 12}\frac{z^6}{a^6} + \cdots\right),$$

so one will have:

$$s = \frac{1}{\sqrt{2}}\int dx\left(\frac{a}{z} + \frac{1}{4}\frac{z}{a} + \frac{1\cdot 3}{4\cdot 8}\frac{z^3}{a^3} + \frac{1\cdot 3\cdot 5}{4\cdot 8\cdot 12}\frac{z^5}{a^5} + \cdots\right)$$

and

$$s - y = \frac{1}{\sqrt{2}}\int dx\left(\frac{z}{a} + \frac{1}{4}\frac{z^3}{a^3} + \frac{1\cdot 3}{4\cdot 8}\frac{z^5}{a^5} + \frac{1\cdot 3\cdot 5}{4\cdot 8\cdot 12}\frac{z^7}{a^7} + \cdots\right).$$

24. – We shall consider this integral mainly for the case $x = c$, so $z = 0$, and the integral can be expressed conveniently with the help of the circular periphery. If the ratio of the diameter to the periphery is equal to $1 / \pi$ then:

$$\int \frac{dx}{z} = \int_0^c \frac{dx}{\sqrt{c^2 - a^2}} = \frac{\pi}{2} .$$

The following integral will be determined in the same way ([1]):

$$\int_0^c z\,dx = \frac{1}{2}\cdot\frac{\pi}{2}c^2, \qquad \int_0^c z^3\,dx = \frac{1\cdot 3}{2\cdot 4}\cdot\frac{\pi}{2}c^4,$$

$$\int_0^c z^5\,dx = \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\cdot\frac{\pi}{2}c^6, \qquad \int_0^c z^7\,dx = \frac{1\cdot 3\cdot 5\cdot 7}{2\cdot 4\cdot 6\cdot 8}\cdot\frac{\pi}{2}c^8.$$

With the help of this integral, it will follow that:

([1]) The reference to the limits of the definite integral has been added for the sake of brevity, but it is not found in **Euler**, since it was first introduced by **Fourier** (1882) in his *Traité analytique de la chaleur*.

$$AC = \frac{\pi a}{2\sqrt{2}}\left(1 + \frac{1^2}{2^2}\frac{c^2}{2a^2} + \frac{1^2\cdot 3^2}{2^2\cdot 4^2}\frac{c^4}{4a^4} + \frac{1^2\cdot 3^2\cdot 5^2}{2^2\cdot 4^2\cdot 6^2}\frac{c^6}{8a^6} + \cdots\right),$$

$$AD = \frac{\pi a}{2\sqrt{2}}\left(1 - \frac{1^2}{2^2}\cdot\frac{3}{1}\cdot\frac{c^2}{2a^2} - \frac{1^2\cdot 3^2}{2^2\cdot 4^2}\cdot\frac{5}{3}\cdot\frac{c^4}{4a^4} - \frac{1^2\cdot 3^2\cdot 5^2}{2^2\cdot 4^2\cdot 6^2}\cdot\frac{7}{5}\cdot\frac{c^6}{8a^6} - \cdots\right).$$

When $AE = c$ and $AD = b$ are given, one can then find the segment a and the curve length AC from those equations. Conversely, one can also determine the lines AD and CD from the given curve length AC and the segment a, since the latter determine the bending force.

25. – As is known, we have already established the first type of curve in such a way that we would have $c = 0$ or $a / c = \infty$ in the general equation:

$$dy = \frac{(a^2 - c^2 + x^2)\,dx}{\sqrt{(c^2 - x^2)(2a^2 - c^2 + x^2)}}.$$

That case corresponds to the straight line. It represents the natural state of the elastic band. However, the cases for which the quantity c is so small that they can be neglected in comparison to a should also be counted among the first kind. However, since x cannot dominate the quantity c, x can also be neglected in comparison to a, and one will get the equation:

$$dy = \frac{a\,dx}{\sqrt{2(c^2 - x^2)}},$$

whose integral is $y = \frac{a}{\sqrt{2}}\arcsin\frac{x}{c}$. That is the equation of a trochoid that extends to infinity ([1]). AD will be equal to $\frac{\pi a}{2\sqrt{2}}$. The length of the curve AC will deviate infinitely little from that, since the angle DAM is infinitely small. Let the length of the band be $ACB = 2f$, and let its absolute elasticity be equal to $\mathsf{E}\,k^2$. Since $f = \frac{\pi a}{2\sqrt{2}}$, the force that this infinitely-small curvature of the band provokes will have a finite magnitude, and

([1]) $y = \frac{a}{\sqrt{2}}\arcsin\frac{x}{c}$ will imply that $x = c \sin\frac{y\sqrt{2}}{a}$. With the current terminology, one calls the curve that is represented by that equation a *sinusoid*. One now understands a trochoid to mean an extended or truncated cycloid. In fact, the sinusoid can be regarded as a special case of a truncated cycloid.

indeed, it will be $\frac{\mathsf{E}k^2}{f^2}\frac{\pi^2}{4}$; i.e., when the ends *A* and *B* are linked together with a filament, that filament will be tensed with a force of $\frac{\mathsf{E}k^2}{f^2}\frac{\pi^2}{4}$.

26. – The second kind of curve is defined by the cases in which $c > 0$, but $c < a$; i.e., *c* lies between the limits 0 and *a*. The angle *DAM* will then be smaller than a right angle. One will then have $\sin PAM = \cos DAM = (a^2 - c^2) / a^2$. In that case, the form of the curve is the one that it is represented in Fig. 6. Since $c < a$, one will have $\frac{c^2}{2a^2} < \frac{1}{2}$, so $AC = f > \frac{\pi a}{2\sqrt{2}}$. Therefore, $a^2 < \frac{8f^2}{\pi^2}$, so the force that draws the ends *A* and *B* of the band towards each other with the help of the filament *AB* will be greater than in the foregoing case, namely $> \frac{\mathsf{E}k^2}{f^2}\cdot\frac{\pi^2}{4}$.

27. – I take the case of $c = a$ to be the third kind of curve. Since the axis *AP* contacts the curve at *A* in that case, that kind of curve will take the special name of a *rectangular* elastic curve. One has:

$$dy = \frac{x^2\,dx}{\sqrt{a^4 - x^4}} \qquad \text{and} \qquad ds = \frac{a^2\,dx}{\sqrt{a^4 - x^4}}$$

for them. The values of *AD* and *AC* in that case are given by:

$$AC = f = \frac{\pi a}{2\sqrt{2}}\left(1 + \frac{1^2}{2^2}\cdot\frac{1}{2} + \frac{1^2\cdot 3^2}{2^2\cdot 4^2}\cdot\frac{1}{4} + \frac{1^2\cdot 3^2\cdot 5^2}{2^2\cdot 4^2\cdot 6^2}\cdot\frac{1}{8} + \cdots\right),$$

$$AD = b = \frac{\pi a}{2\sqrt{2}}\left(1 - \frac{1^2}{2^2}\cdot\frac{3}{1\cdot 2} - \frac{1^2\cdot 3^2}{2^2\cdot 4^2}\cdot\frac{5}{3\cdot 4} - \frac{1^2\cdot 3^2\cdot 5^2}{2^2\cdot 4^2\cdot 6^2}\cdot\frac{7}{5\cdot 8} - \cdots\right).$$

Although neither *b* nor *a* can be expressed in closed form, I have proved elsewhere that a remarkable relationship exists between those quantities. Namely, I have shown that $4bf = \pi a^2$ ([1]); the right angle between *AD* and *AC* is equal to the area of the circle

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([1]) The Euler relation $4bf = \pi a^2$ can be easily derived with the help of the **Legendre** relation:

$$K\,\mathsf{E}' + K'\mathsf{E} - KK' = \frac{\pi}{2}.$$

(For the formulas on elliptic integrals that are needed for this, one can confer, e.g., **E. Pascal**: *Repertorium der höheren Mathematik, deutsche Ausgabe von* **A. Schepp**, pp. 156.) One will then have:

whose diameter is *AE*. By carrying out the calculation, one will find, approximately, that $f = \frac{5a}{6}\cdot\frac{\pi}{2}$ or $a = \frac{12f}{5\pi}$. Therefore, the force that draws the ends *A* and *B* of the band together will be $\frac{\mathsf{E}k^2}{f^2}\cdot\frac{25}{72}\pi^2$. More precisely, one will find that $f = \frac{\pi a}{2\sqrt{2}}\cdot 1.1803206$. Hence:

$$b = \frac{\pi a^2}{4f} = \frac{a}{\sqrt{2}\cdot 1.803206}.$$

In terms of mere numbers, it will then follow that ([1]) :

$$\frac{f}{a} = 1.311006 \qquad \text{and} \qquad \frac{b}{a} = 0.59896.$$

28. – When $c > a$, the fourth kind of curve will arise, although for them one should have $AD = b > 0$. That will imply a second limit for *c*, namely, from the equation:

$$f = \int_0^a \frac{a^2\,dx}{\sqrt{(a^2-x^2)(a^2+x^2)}}.$$

If one sets $x = a\cos\varphi$ then one will have:

$$f = \frac{a}{\sqrt{2}}\int_0^{\pi/2}\frac{d\varphi}{\sqrt{1-\frac{1}{2}\sin^2\varphi}} = \frac{a}{\sqrt{2}}\cdot K.$$

Moreover, one has:

$$b = \int_0^a \frac{x^2\,dx}{\sqrt{(a^2-x^2)(a^2+x^2)}} = \int_0^a \frac{dx\sqrt{a^2+x^2}}{\sqrt{a^2-x^2}} - \int_0^a \frac{a^2\,dx}{\sqrt{(a^2-x^2)(a^2+x^2)}}.$$

If one again sets $x = a\cos\varphi$ then one will have:

$$b = a\sqrt{2}\int_0^{\pi/2} d\varphi\sqrt{1-\sin^2\varphi} - \frac{a}{\sqrt{2}}K = \frac{a}{\sqrt{2}}(2\,\mathsf{E} - K).$$

As the formulas show, the complete integrals *K* and *E* belong to the modulus $k^2 = 1/2$ in this case, so $1 - k^2$ will also be 1/2 . However, if one replaces k^2 with E and *K* in $1 - k^2$ then it will go to E' and K', resp. One will then have $K = K'$ and $\mathsf{E} = \mathsf{E}'$ here. The **Legendre** relation above will the give the equation:

$$K(2\,\mathsf{E} - K) = \frac{\pi}{2}.$$

However, one now has $b\,f = (a^2 / 2)\,K\,(2\,\mathrm{E} - K)$; hence, $4\,b\,f = a^2\pi$. One will find another proof in **Todhunter**: *A History of the Theory of Elasticity*, Cambridge, 1886. vol. 1, pp. 36.

([1]) One will find a computational error in these numerical calculations. **Euler** has set:

$$b = \frac{a}{\sqrt{2}}\cdot 1.1803206, \qquad \text{instead of} \qquad b = \frac{a}{\sqrt{2}\cdot 1.1803206}.$$

(This was corrected in the text.) It will then follow that $b / a = 0.59896$; i.e., approximately 0.6. That will yield the formula $f = \frac{5a}{6}\cdot\frac{\pi}{2}$. In the text, **Euler** falsely said that $b / a = 0.834612$.

$$1 = \frac{1^2}{2^2}\cdot\frac{3}{1}\cdot\frac{c^2}{2a^2} + \frac{1^2\cdot 3^2}{2^2\cdot 4^2}\cdot\frac{5}{3}\cdot\frac{c^4}{4a^4} + \frac{1^2\cdot 3^2\cdot 5^2}{2^2\cdot 4^2\cdot 6^2}\cdot\frac{7}{5}\cdot\frac{c^5}{8a^6} + \ldots$$

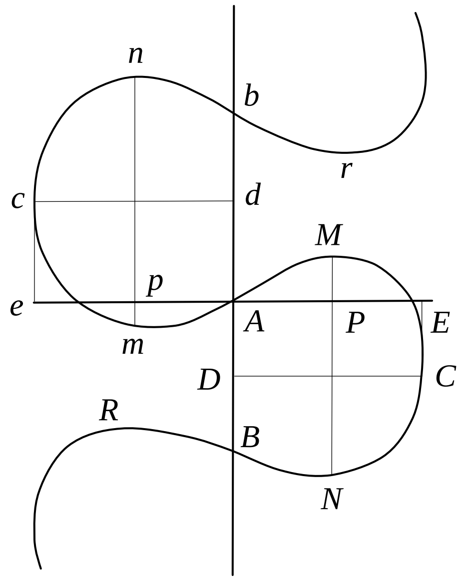

Figure 7.

For the fourth kind of curve, since $c > a$, the curve will climb above the axis *AE* at *A*. It will then define an angle *PAM* whose sine is equal to $(c^2 - a^2) / a^2$; however, we will soon see that this angle is smaller than $40^o\ 41'$. The distance *AD* will vanish when that angle assumes that value; I shall treat that case as one of the fifth kind. The curves of the fourth kind include the cases for which c^2 / a^2 are found between the limits 1 and 1.6511868. The form of those curves can be see in the figure, where one must remark that the closer that c^2 / a^2 approaches the limit 1.6511868, the smaller that the distance *AD* will become, and the closer that the endpoints *A* and *B* of the band will come to each other. It can happen then that the nodes *m* and *R*, as well as *M* and *r*, will not only touch each other, but even intersect. Finally, it can happen that all diameters *DC*, *dc* might coincide with each other and the axis *AE*.

29. – When that happens, the fifth kind of curve will come about, which is expressed by the equation:

$$dy = \frac{(a^2 - c^2 + x^2)\,dx}{\sqrt{(c^2 - x^2)(2a^2 - c^2 + x^2)}},$$

in which the relationship exists between a and c that $AD = b = 0$. One sets $c^2 / 2a^2 = v$. v will then be determined from the following equation:

$$1 = \frac{1\cdot 3}{2\cdot 2}v + \frac{1\cdot 1\cdot 3\cdot 5}{2\cdot 2\cdot 4\cdot 4}v^2 + \frac{1\cdot 1\cdot 3\cdot 3\cdot 5\cdot 7}{2\cdot 2\cdot 4\cdot 4\cdot 6\cdot 6}v^3 + \ldots$$

When one seeks the limits between which the true value of *v* lies, whether by some familiar method or mere trial and error, one will find that $v = 0.824$ and $v = 0.828$.

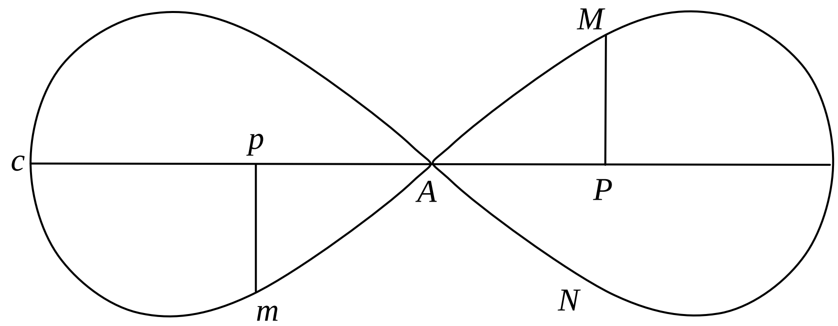

Figure 8.

If the two values are substituted in the equation then one can conclude from the discrepancy that will arise in both cases that one must have $v = 0.825934$, so $c^2 / a^2 = 1.651868$ and $(c^2 - a^2) / a^2 = 0.651868$, and that value is the sine of the angle *PAM*. From a Table, one will find that this angle is $40^{o}\ 41'$, so the angle *MAN*, which is twice as large, will be $81^{o}\ 22'$. Therefore, when the endpoints of the elastic band are close enough that they will contact, the curve *AMCNA* (Fig. 8) will be determined, and the two ends will meet each other at $81^{o}\ 22'$ ([1]).

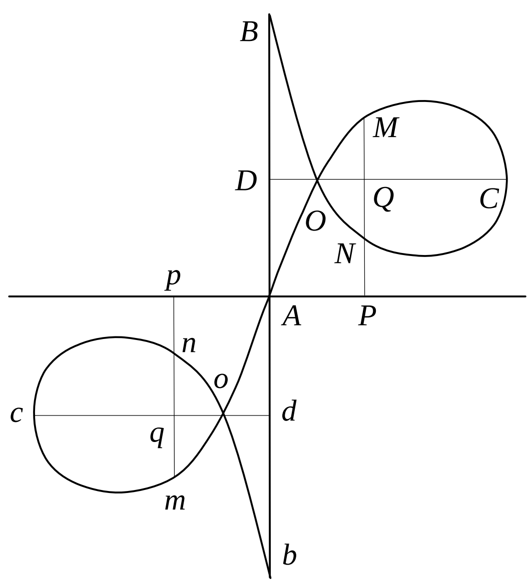

Figure 9.

([1]) **W. Hess** treated the problem of the elastic line in analogy with the motion of a pendulum and gave a series of figures for their possible forms [Mat. Ann. **25** (1885), 1-38]. For **Euler**, the force direction for the **Euler** curve of the fifth kind – i.e., the altitude to *AP* at *A* (Fig. 8) – defines an angle of $90^{o} + 40^{o}\ 41' = 130^{o}\ 41'$ with the curve. Hess gave 129.3^{o} for that angle. He obtained that angle from **Legendre**'s tables of elliptic integrals. One must find the value of the modulus k^2 for which $2\,\mathsf{E} - K$ vanishes. As the editor has suggested, that will give $130^{o}\ 41'$, and in addition, a check of **Euler**'s equation for *v* will show that **Euler** had calculated in correctly, since one would get 0.8261 for *v*, instead of **Euler**'s value 0.8259.

30. – When the two ends *A* and *B* of the band first get close enough to touch and are then bent in opposite directions with increasing force, a curve of the form *AMCNB* (Fig. 9) will result, which defines the sixth kind. For the curves of this kind, one has $c^2 / 2a^2 >$ 0.825934, but $c^2 / 2a^2 < 1$. For $c^2 = 2a^2$, one will have the seventh kind, which we shall explain immediately.

For those curves, the angle *PAM* that the curve makes with the axis at *A* will be greater than $40^{\circ}\, 41'$, but smaller than 90°. If $\sin PAM = (c^2 - a^2) / a^2$ and since $c^2 < 2a^2$, that value will be smaller than 1, and it will be equal to 1 only when $c^2 = 2a^2$.

31. – Let $c^2 = 2a^2$, so our seventh kind of curve will be in question. In that case, the equation reads:

$$dy = \frac{(a^2 - x^2)\, dx}{x\sqrt{2a^2 - x^2}} .$$

From it, one sees that the branches *A* and *B* of the curve extend to infinity in such a way that *AB* will be an asymptote of the curve. Therefore, each branch *AMC* and *BNC* will go to infinity, as one can see from the series that was found for the arc length *AC* before:

$$AC = \frac{\pi a}{2\sqrt{2}}\left(1 + \frac{1^2}{2^2} + \frac{1^2 \cdot 3^2}{2^2 \cdot 4^2} + \frac{1^2 \cdot 3^2 \cdot 5^2}{2^2 \cdot 4^2 \cdot 6^2} + \cdots\right) .$$

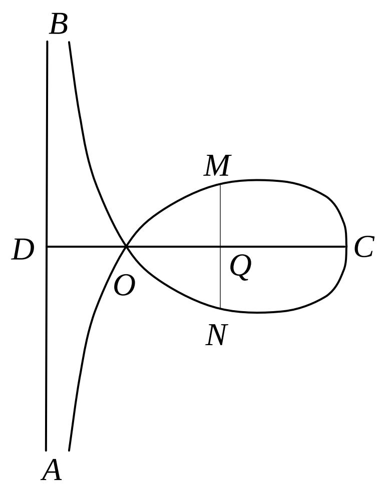

Figure 10.

The sum of this series in is infinite. If the length of the band $AC = f$ is to be finite then one must have $a = 0$, so one will also have $CD = c = 0$. Hence, after the band has bent into a knot, it will again extend rectilinearly. In order to achieve that, one will need an infinitely-large force. However, when the band is infinitely long, it will define a curve with a node that goes to the asymptote *AB*, so one has $CD = c$. (Fig. 10) The equation of this curve can be integrated with the help of logarithms, and one will get:

$$y = \sqrt{c^2 - x^2} - \frac{c}{2}\log\frac{c+\sqrt{c^2-x^2}}{x};$$

The abscissa x is taken to be the diameter DC in this. One has $DQ = x$ and $QM = y$, so for $x = CD = c$, one will have $y = 0$. y will likewise be zero at the node O. In order to find O, one must set:

$$\frac{2\sqrt{c^2-x^2}}{c} = \log\frac{c+\sqrt{c^2-x^2}}{x}.$$

Let $\cos\vartheta = x / c$, so $\sin\vartheta = 2\sqrt{c^2-x^2}/c$, and the equation will become:

$$2\sin\vartheta = \log\tan\left(45^\circ + \frac{\vartheta}{2}\right).$$

The logarithm is taken to be the natural logarithm. For want of a table of such things, one might look up the tangent of the angle $45^\circ + \vartheta / 2$ in the usual table of logarithms, when the 10 is removed from the reference number. What will remain is ω; one will then have $2\sin\vartheta = \omega \cdot 2.30258509$ ([1]). If one again takes natural logarithms then one will have $\log 2 + \log\sin\vartheta = \log\omega + 0.3622156886$, or $\log\sin\vartheta = \log\omega + 0.0611856930$. Upon applying this artifice, one will soon find an approximate value for ϑ, and from that, one can find the true value of ϑ by the *regula falsi* ("false position") method, which will determine DO. One will then find that $\vartheta = 73^\circ\ 14'\ 12''$, from which, it will follow that:

$$\frac{x}{c} = 0.2884191 \qquad \text{and} \qquad \frac{\sqrt{c^2-x^2}}{c} = 0.9575042.$$

The angle QOM is $2\vartheta - 90^\circ = 56^\circ\ 28'\ 24''$, so the angle MON will become $112^\circ\ 56'\ 48''$. For the fifth kind, the angle at the node will be $81^\circ\ 22'$, so for the sixth kind, the angle at the node will be between $81^\circ\ 22'$ and $112^\circ\ 56'\ 48''$. When a node appears in the fourth kind, the angle will be less than $81^\circ\ 22'$.

32. – Finally, let $c^2 > 2a^2$, so one sets $c^2 = 2a^2 + g^2$. The equation of the curve reads:

([1]) Here, the well-known relation $\ln n = \log_{10} n \cdot \ln 10$ has been applied. $\ln 10$ is 2.302585. At the conclusion of this section, one will set $\angle QOM = 2\vartheta = 90^\circ$, so:

$$\tan\varphi = \frac{dy}{dx} = \frac{(c^2/2 - x^2)}{x\sqrt{c^2-x^2}}.$$

One again sets $x = c\cos\vartheta$, which will yield:

$$\tan\varphi = \frac{1 - 2\cos^2\vartheta}{2\sin\vartheta\cos\vartheta} = -\cot 2\vartheta.$$

Hence, $\tan\varphi = \cos(180^\circ - 2\varphi) = \tan(2\varphi - 90^\circ)$, so $\varphi = 180^\circ - 2\varphi$.

$$dy = \frac{(x^2 - \frac{1}{2}c^2 - \frac{1}{2}g^2)\,dx}{\sqrt{(c^2 - x^2)(x^2 - g^2)}}.$$

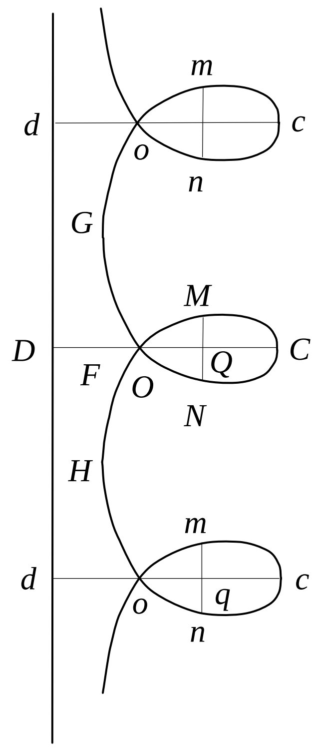

Figure 11.

This equation represents the eighth kind of curve. The line *dDd* might be the direction of the applied force, so $x = DQ$, $y = QM$. First, it is clear that y can be real only when $x > g$, but then x cannot go beyond the segment $DC = c$. Thus, when one sets $DF = g$, the entire curve will lie between two lines that are parallel to *dd* and go through *C* and *F*, and simultaneously contact the curve (Fig. 11). It makes no difference which of the two segments c or g is larger. As long as both of them are unequal, the equation will not change when c and g are exchanged with each other. Furthermore, that curve has infinitely many, mutually-parallel diameters *DC*, *dc*, *dc*, etc., and in addition the line that is drawn through *G* and *H* perpendicular to *dDd* will be the diameter ([1]). Nowhere along the entire curve will one find an inflection point, so the curvature will go uniformly to infinity in both directions, as is shown in the figure. The angles *MON* at the nodes are greater than $112^{\circ}\,56'\,48''$.

33. – Since these (eight) kinds of curve include not only the cases for which $g^2 < c^2$, but also the ones for which $g^2 > c^2$, all that remains is the case for which $c = g$. The entire

([1]) The curve will remain unchanged when c and g are exchanged. Therefore, it must have the same form in the neighborhood of *G* that is has in the neighborhood of *C*, only the curvature at *C* will be greater than it is at *G*. The altitude to *Dd* at *G* will then be a diameter of the curve, as well, like the altitude to *Dd* at *C*.

curve will vanish in that case, because $CF = 0$. Now, when c and g are both fixed to be infinite, but in such a way that their difference remains finite, the curve will possess a definite form. In order to find it, one sets $g = c - 2h$ and $x = c - h + t$. Since c is very large, but h and t are finite, $\frac{1}{2}c^2 + \frac{1}{2}g^2 = c^2 - 2ch$; $x^2 - \frac{1}{2}c^2 - \frac{1}{2}g^2 = 2\,ct$. However, one will then have $c^2 - x^2 = 2c\,(h - t)$ and $x^2 - g^2 = 2c\,(h + t)$. The following equation will emerge from that then:

$$dy = \frac{t\,dt}{\sqrt{h^2 - t^2}},$$

which will give a circle. In that case, the elastic band will be curved into a circle, as was mentioned before. The circle defines the ninth and final kind.

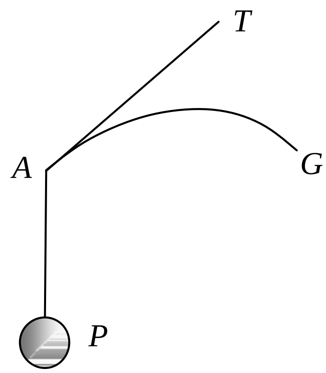

Figure 12.

34. – From the classification of these curves, it is easy to find which kind a given curve belongs to in a particular case. Let the elastic band be fixed to a wall at *G* (Fig. 12), and hang a weight *P* from the end *A*, so that the band assumes the form *GA*. If one draws the tangent *AT* then making a decision will be possible only by way of the angle *TAP*. When it is acute, the curve will be one of the second kind, and when it is a right angle, it will be one of the third kind, and the elastic curve will be rectangular.

If the angle is obtuse, but smaller than $130^\circ\,41'$, then the curve will be one of the fourth kind. When the angle $TAP = 130^\circ\,41'$, it will be one of the fifth kind. If *TAP* is greater than that then the curve will be one of the sixth kind. It will be one of the seventh kind when that angle is equal to two right angles, which cannot happen in reality, though. That kind of curve, along with the last two, cannot be produced by hanging a weight on the band directly.

35. – In order to explain how the latter kinds of curve can be produced by the curvature of the band, let a weight be hung from *C* that points in the direction *CD* (Fig. 3 in no. **5**), not directly from the band that is fixed at *B*, but along the rigid rod *AC* that is coupled solidly with the end *A* of the band.

Let the distance *AC* be *h*, let the absolute elasticity of the band be $\mathsf{E}\,k^2$, and let the sine of the angle *MAP* that the band makes with the horizontal at *A* be *m*. Furthermore, let $AP = t$ and $PM = y$, so the equation of that curve will be found to be:

I. $$dy = \frac{dt\,(m\mathsf{E}k^2 - Pht - \tfrac{1}{2}Pt^2)}{\sqrt{\mathsf{E}^2 k^4 - (m\mathsf{E}k^2 - Pht - \tfrac{1}{2}Pt^2)^2}}.$$

In order to bring this equation into the form that we appealed to in our classification of kinds, we will set $CP = x = h + t$. We will get ([1]):

II. $$dy = \frac{dx\,(m\mathsf{E}k^2 - \tfrac{1}{2}Ph^2 - \tfrac{1}{2}Px^2)}{\sqrt{\mathsf{E}^2 k^4 - (m\mathsf{E}k^2 - \tfrac{1}{2}Ph^2 - \tfrac{1}{2}Px^2)^2}}.$$

A comparison of that equation with:

III. $$dy = \frac{-dx\,(a^2 - c^2 + x^2)}{\sqrt{a^4 - (a^2 - c^2 + x^2)^2}}$$

will yield $\tfrac{1}{2}Pa^2 = \mathsf{E}\,k^2$, or $a^2 = 2\,\mathsf{E}\,k^2 / P$, and $\tfrac{1}{2}Pc^2 - \tfrac{1}{2}Pa^2 = m\,\mathsf{E}\,k^2 + \tfrac{1}{2}Ph^2$, so:

$$c^2 = \frac{2(1+m)\mathsf{E}k^2}{P} + h^2.$$

36. – The curve will then be one of the second kind when one has $2m\,\mathsf{E}\,k^2 / P + h^2 < 0$ or $P < -\,2m\,\mathsf{E}\,k^2 / h^2$. If the angle *PAM* is not negative then the force P must be negative, and the rod at C must point upwards. The curve will be one of the third kind when $P = -\,2m\,\mathsf{E}\,k^2 / h^2$. It will be one of the fourth kind when $2m\,\mathsf{E}\,k^2 + P\,h^2 > 0$, but at the same time, $2m\,\mathsf{E}\,k^2 + P\,h^2 < 2\alpha\,\mathsf{E}\,k^2$, where $\alpha = 0.651868$. However, if $P = 2(\alpha - m)\,\mathsf{E}\,k^2 / h^2$ then the curve will be one of the fifth kind. If one has $P\,h^2 > 2\,(\alpha - m)\,\mathsf{E}\,k^2$, but at the same time $P\,h^2 < 2\,(1 - m)\,\mathsf{E}\,k^2$, then one will be dealing with a curve of the sixth kind. One will get the seventh kind when $P\,h^2 = 2\,(1 - m)\,\mathsf{E}\,k^2$, and one will get the eighth kind when $P\,h^2 > 2\,(1 - m)\,\mathsf{E}\,k^2$. When the angle *PAM* is a right angle, one will have $1 - m = 0$, and the curve will always be one of the eighth kind. Finally, the ninth kind will arise when $h = \infty$, as I pointed our before.

([1]) Equation I will become the equation at the end of no. **5** when one replaces x with t, c with h, and sets $m\mathsf{E}k^2 = -Pf$. That relation is derived at the end of no. **10**. If one shifts the coordinate origin from A to C then I will go to II. The normal form equation (III) of the elastic curve will not change when the x-axis is displaced parallel to itself, since only dy enters into it, but not y. The origin will then become an arbitrary point along the line AB (Fig. 6). The applied force acts at it in the direction AB, which is also the case at the point C for II. The minus sign in front of dx in III can be explained by the fact that in Fig. 3, no. **5**, the force acts in the direction of the negative y-axis, but in Fig. 6, it acts in the direction of the positive y-axis (see the remark on page 13). Hence, since II and III are now referred to the same coordinate axes, those equations can be made to coincide.

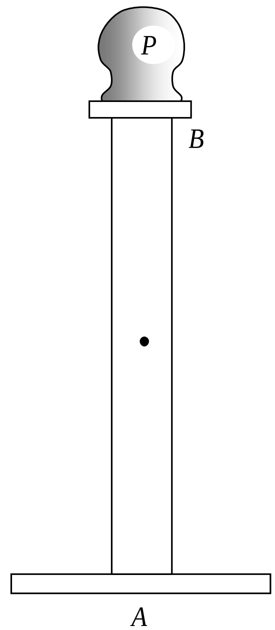

Figure 13.

On the load capacity of columns

37. – As was remarked before in regard to the first kind of curve, they can serve to determine the load capacity of columns. Let *AB* (Fig. 13) be a vertical over the base *A* that the column lies on; it carries the weight *P*. Let the column be chosen so that the weight cannot slide. If the weight *P* is not too large then one might expect at most a bending of the column. In that case, the column can be regarded as also being endowed with elasticity. Let the absolute elasticity of the column be $\mathsf{E}\,k^2$, and let its height be $2f = a = AB$. In no. **25**, we saw that the force that is required in order to bend the column by a very small amount is:

$$\frac{\pi^2 \mathsf{E} k^2}{4 f^2} = \frac{\pi^2}{a^2} \mathsf{E} k^2 .$$

Therefore, when the load *P* is not greater than:

$$\frac{\mathsf{E}\, \pi^2 k^2}{a^2},$$

one should not expect any bending. However, if *P* is larger then the column then the column cannot resist bending, but if the elasticity of the column, and therefore its density, as well, remains unchanged then the load *P* that it can safely carry will behave conversely like the square of the height. A column that is twice as high can carry only one-fourth as much load. That can be especially useful in regard to wooden columns, which are quite subject to bending.

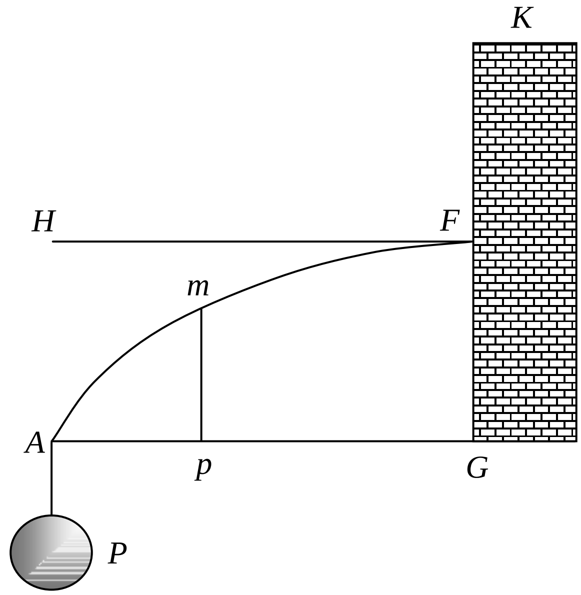

Figure 14.

Determining the absolute elasticity by experiments

38. – In order to be able to determine the curvature of any elastic band *a priori*, the absolute elasticity that we have expressed by $\mathsf{E}\, k^2$ will be known. That can be achieved conveniently by a single experiment. Let the uniform elastic band *FH* whose absolute elasticity is to be found (Fig. 14) be fixed at one end *F* to a solid wall *GK*, such that it will assume the horizontal position *FH*; one can, in fact, neglect its own weight in this. An arbitrarily-chosen weight *P* is hung at the end *H*, which will curve the band into the position *AF*. Let the length of the band be $AF = HF = f$, let the length of the horizontal grade be $AG = g$, and let the length of the vertical be $GF = h$. All of those values can be obtained from measurements. One can compare with the curve *AF* whose general equation is expressed by:

$$dy = \frac{(c^2 - a^2 - x^2)\,dx}{\sqrt{(c^2 - x^2)(2a^2 - c^2 + x^2)}}.$$

In it, *a* and *c* are to be determined from *f*, *g*, *h*. The bending force will be:

$$P = \frac{2\mathsf{E}\, k^2}{a^2}, \quad \text{so} \qquad \mathsf{E}\, k^2 = \tfrac{1}{2} P\, a^2.$$

39. – Since the tangent is horizontal at *F*, one will have $dy / dx = 0$, so $x = \sqrt{c^2 - a^2}$. $AG = g$ will then become $\sqrt{c^2 - a^2}$ and $a^2 = c^2 - g^2$. Hence, when one sets $x = g$ in:

$$dy = \frac{(c^2 - g^2)\,dx}{\sqrt{(c^2 - x^2)(c^2 - 2g^2 + x^2)}},$$

that must imply that $y = GF = h$ and $s = AF = f$. However, one has:

$$ds = \frac{(c^2 - g^2)\,dx}{\sqrt{(c^2 - x^2)(c^2 - 2g^2 + x^2)}}.$$

The weight P might be chosen to be small enough that the band is bent only slight, so a, and therefore c, as well, will be very large. One will then have:

$$(c^4 - 2\,c^2\,g^2 + 2\,g^2\,x^2 - x^4)^{-1/4} = \frac{1}{c^2} + \frac{g^2}{c^4} - \frac{g^2 x^2}{c^6} + \frac{x^4}{2c^6},$$

approximately. Integration will then yield:

$$s = \frac{(c^2 - g^2)\,x}{c^2} + \frac{(c^2 - g^2)\,g^2 x}{c^4} - \frac{(c^2 - g^2)\,g^2 x^3}{3c^6} + \frac{(c^2 - g^2)\,x^5}{10\,c^6},$$

approximately, and:

$$y = \frac{g^2\,x}{c^2} + \frac{g^4 x}{c^4} - \frac{g^4 x^3}{3c^6} + \frac{g^2\,x^5}{10\,c^6} - \frac{x^3}{3c^2} - \frac{g^2 x^3}{3c^2} + \frac{g^2\,x^5}{5\,c^6} - \frac{x^7}{14\,c^6}.$$

If one sets $x = g$ then one will have ([1]):

$$f = g - \frac{37\,g^5}{30\,c^4}$$

and

$$h = \frac{2g^3}{3c^2} + \frac{2g^5}{3c^4}.$$

In order to find c, one employs the value of h and also neglects the second term. One will then get:

([1]) The value of f is not correct. From the figure, it is clear that $f > g$, with no further assumptions. If one continues the series development of $(c^4 - 2c^2\,g^2 + 2\,g^2\,x^2 - x^4)^{-1/2}$ further then one will get $+\ \frac{3}{8} \cdot \frac{1}{c^{10}}\,[4\,c^4\,g^4 + 4\,c^2\,g^2\,x^4 - 8\,c^2\,g^4\,x^2 + \ldots]$. If one integrates and multiplies by $(c^2 - g^2)$ and then sets $x = g$ then the first term in brackets will give the term $\frac{3}{8} \cdot \frac{4}{c^4}\,g^5 = \frac{3}{2} \cdot \frac{g^5}{c^4}$. **Euler** seems to have overlooked this. As a result, one will have $f = g + \frac{8}{30}\,\frac{g^5}{c^4}$.

$$c^2 = \frac{2g^3}{3h},$$

so

$$a^2 = \frac{g^2(2g-3h)}{3h},$$

and therefore:

$$\mathsf{E}\,k^2 = \tfrac{1}{2}Pa^2 = \frac{Pg^2(2g-3h)}{6h}.$$

That value will differ only slightly from the true value as long as one does not let the curvature of the band get too large.

40. – The absolute elasticity $\mathsf{E}\,k^2$ depends, first of all, upon the nature of the material of which the band is composed. Secondly, it depends upon the width of the band, such that when everything else stays the same, the expression $\mathsf{E}\,k^2$ will be proportional to the width of the band. Thirdly, however, the thickness of the band will play a significant role in the determination of the value of $\mathsf{E}\,k^2$. $\mathsf{E}\,k^2$ then seems to be proportional to the square of the thickness. The expression $\mathsf{E}\,k^2$ will be a term that relates to the elastic material and includes the first power of the width of the band and the second power of the thickness. As a result, one can determine the elasticities of all materials and compare them to each other by experiments in which one measures their lengths and thicknesses.

On the curvature of non-uniform elastic bands

41. – Up to now, I have assumed that the absolute elasticity $\mathsf{E}\,k^2$ of a band whose curvature I determined is constant along its entire length. However, the solution can result from the same method when $\mathsf{E}\,k^2$ is assumed to vary arbitrarily. Namely, let the absolute elasticity be an arbitrary function of the arc length *AM* = *s* (Fig. 3). That function will be called *S*. Let *R* be the radius of curvature at *M*. The curve *AM* that the band assumes will be arranged so that among all other curves of the same length, $\int S\,ds/R^2$ will be a minimum. That case will be solved by the second general formula ([1]).

Let $dy = p\,dx$, $dp = q\,dx$, and $dS = T\,ds$. The problem will then come down to finding the curve for which:

$$\int \frac{S\,q^2 dx}{(1+p^2)^{5/2}}$$

([1]) Here, **Euler** is referring to the formulas that were given in "Methodus inveniendi lineas curvas, Chapter IV, no. 7, II (pp. 132 of that book). They are not included in v. **46** of this collection.

is a minimum from among all curves for which $\int dx\sqrt{1+p^2}$ has the same value. The first formula $\int dx\sqrt{1+p^2}$ will then give the differential expression:

$$\frac{1}{dx}d\frac{p}{\sqrt{1+p^2}}.$$

When the second formula:

$$\int\frac{S\,q^2dx}{(1+p^2)^{5/2}}$$

is compared with $\int Z\,dx$ that will give:

$$Z=\frac{S\,q^2}{(1+p^2)^{5/2}}.$$

Now, one has set:

$$dZ = L\,d\Pi + M\,dx + N\,dy + P\,dp + Q\,dq, \qquad \Pi=\int[Z]\overline{dx},$$

$$d\,[Z] = [M]\overline{dx}+[N]\overline{dy}+[P]\overline{dp}, \qquad L\,\overline{d\Pi} = \frac{q^2T\,\overline{ds}}{(1+p^2)^{5/2}},$$

so:

$$L=\frac{q^2T}{(1+p^2)^{5/2}};$$

but

$$\overline{d\Pi} = \overline{ds} = \overline{dx}\sqrt{1+p^2},$$

so

$$[Z] = \sqrt{1+p^2}, \qquad [M] = 0, \qquad [N] = 0, \qquad [P] = \frac{p}{\sqrt{1+p^2}}.$$

Furthermore, $M = 0$, $N = 0$, so:

$$P=-\frac{5S\,q^2p}{(1+p^2)^{7/2}} \quad\text{and}\quad Q=\frac{2S\,q}{(1+p^2)^{5/2}}.$$

One will then have:

$$dZ=\frac{q^2dS}{(1+p^2)^{5/2}}+P\,dp+Q\,dq.$$

42. – One defines:

$$\int L\,dx = \int\frac{q^2T\,dx}{(1+p^2)^{5/2}} = \int\frac{q^2dS}{(1+p^2)^3}.$$

Let the value of that integral when one sets $x = a$ be H. The constant a will soon emerge once more from calculation. One will then have:

$$V = H - \int \frac{q^2 dS}{(1+p^2)^3}.$$

The differential expression will then be:

$$-\frac{dP}{dx} - \left(\frac{d}{dx}[P]\right) \cdot V + \frac{d^2 Q}{dx^2}.$$

The equation of the desired curve follows from the two differential expressions:

$$\alpha \frac{d}{dx}\left(\frac{p}{\sqrt{1+p^2}}\right) = +\frac{dP}{dx} + \left(\frac{d}{dx}[P]\right) \cdot V - \frac{d^2 Q}{dx^2},$$

or

$$\frac{\alpha\, p}{\sqrt{1+p^2}} + \beta = \frac{H\, p}{\sqrt{1+p^2}} - \frac{p}{\sqrt{1+p^2}} \int \frac{q^2 dS}{(1+p^2)^3} + P - \frac{dQ}{dx}.$$

The constant H can be combined with the constant α, and the constant a will emerge from the equation. One will then get:

$$\frac{\alpha\, p}{\sqrt{1+p^2}} + \beta = P - \frac{dQ}{dx} - \frac{p}{\sqrt{1+p^2}} \int \frac{q^2 dS}{(1+p^2)^3}.$$

43. – Multiply that equation by $dp = q\, dx$. From the last equation of number **41**, $P\, dp$ can then be replaced with:

$$dZ - Q\, dq - \frac{q^2 dS}{(1+p^2)^{5/2}},$$

and the following integrable equation will arise:

$$\frac{\alpha\, p\, dp}{\sqrt{1+p^2}} + \beta\, dp = dZ - q\, dQ - Q\, dq - \frac{q^2 dS}{(1+p^2)^{5/2}} - \frac{p\, dp}{\sqrt{1+p^2}} \int \frac{q^2 dS}{(1+p^2)^3}.$$

Its integral is:

$$\alpha\sqrt{1+p^2} + \beta\, p + \gamma = Z - qQ - \sqrt{1+p^2} \int \frac{q^2 dS}{(1+p^2)^3},$$

or

$$\alpha\sqrt{1+p^2}+\beta p+\gamma=\frac{-S\,q^2}{(1+p^2)^{5/2}}-\sqrt{1+p^2}\int\frac{q^2dS}{(1+p^2)^3}.$$

Divided that equation by $\sqrt{1+p^2}$ and differentiate once more:

$$\frac{\beta\,dp}{(1+p^2)^{3/2}}-\frac{\gamma p\,dp}{(1+p^2)^{3/2}}+\frac{2q^2\,dS}{(1+p^2)^3}+\frac{2S\,q\,dq}{(1+p^2)^3}-\frac{6S\,pq^2\,dq}{(1+p^2)^4}=0.$$

If one multiplies this by $(1 + p^2)^{3/2} / 2q$ then that will imply that:

$$\frac{\beta\,dp}{2q}-\frac{\gamma p\,dp}{2q}+\frac{q\,dS+S\,dq}{(1+p^2)^{3/2}}-\frac{3S\,pq\,dq}{(1+p^2)^{5/2}}=0.$$

However, since $dp = q\,dx$ and $dy = p\,dx$, the integral of the latter equation will be:

$$\alpha+\tfrac{1}{2}\beta x-\tfrac{1}{2}\gamma y+\frac{S\,q}{(1+p^2)^{3/2}}=0.$$

However, the radius of curvature is $R = -(1 + p^2)^{3/2} / q$, so when one doubles the constants *g* and *b*, the following equation will arise:

$$\frac{S}{R}=\alpha+\beta x-\gamma y.$$

This equation agrees splendidly with the direct method. Namely, it expresses $\alpha+\beta x-\gamma y$ as the moment of the bending force when an arbitrary line is assumed to be the axis ([1]). That moment must be equal to the absolute elasticity divided by the radius of curvature. With that, not only is the property of the elastic curve that was observed by the very distinguished **Bernoulli** explained completely, but the application of my complicated formulas to this example are also confirmed excellently.

([1]) That expression can be established more rigorously as follows: Let *CP* be the arbitrary line whose equation (Fig. 3) is $Ax + By + C_1 = 0$, and let it be the axis. The moment of the force P about the point *M* (corresponding to the developments in no. **5**) will be $\mathsf{P}\cdot CP$. For the moment, the point *M* will have the coordinates ξ and η, so the equation of *MP* will be given by $A(y-\eta)-B(x-\xi)=0$. The point *C* has the coordinates $x = k$, $y = l$, so one will have $Ak + Bl + C_1 \equiv 0$, and the length of the altitude to *MP* at *C* – i.e., *CP* – is given by:

$$\frac{A(l-\eta)-B(k-\xi)}{\sqrt{A^2+B^2}}.$$

The moment is $\mathsf{P}\cdot CP$, so when one again replaces ξ and η with *x* and *y*, respectively, that moment will be:

$$\frac{\mathsf{P}(Al-Bk)}{\sqrt{A^2+B^2}}+\frac{B\mathsf{P}}{\sqrt{A^2+B^2}}\,x-\frac{A\mathsf{P}}{\sqrt{A^2+B^2}}\,y,$$

which corresponds to $\alpha+\beta x-\gamma y$ in the text.

44. – Therefore, when the curve is given that the nonuniform elastic band will define under the action of the force P (Fig. 3), the absolute elasticity of the band at an arbitrary location can be derived from it. The line CP that is perpendicular to the direction of the applied force is taken to be the axis. Let $CP = x$ and $PM = y$, let the arc length $AM = s$, and let the radius of curvature at M be equal to R. Since the moment of the force P at the point M is equal to Px, one will have $S / R = Px$, and therefore $S = PRx$, and as a result, since R can be assumed to be known at the individual points of the given curve, the absolute elasticity will be known at each location. Hence, when the substance that the band is made of and its thickness remains the same everywhere, but the width does change, then since the absolute elasticity is proportional to the width, the width at the individual locations will be found.

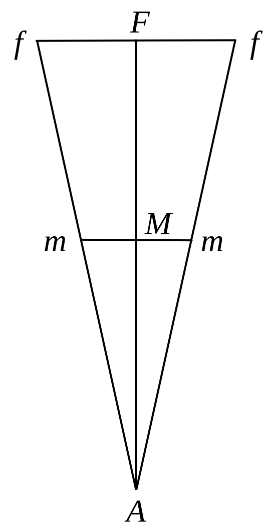

Figure 15.

45. – Let the triangular wedge *f A f* (Fig. 15) be cut out of the elastic band, which has the same density everywhere. The width mm is proportional to the length AM at any arbitrary location M. If one sets $AM = s$ then the absolute elasticity at M will be proportional to s. Let it be $E\,ks$. Let the end ff of the band be horizontal and fixed in a wall, and hang a weight P at the vertex A, which will curve the midline AF into a curve FmA (Fig. 14), whose nature will be examined. In the horizontal axis, let $Ap = x$, $pm = y$, and arc length $Am = s$, so $Px = E\,ks / R$. One has:

$$R = -\left(\frac{dx}{ds}\frac{d^2y}{ds^2}\right)^{-1}.$$

When one multiplies this equation by dx that will yield:

$$P\,x\,dx = -\,E\,ks\left(\frac{dx}{ds}\frac{d^2y}{ds^2}\right)^{-1},$$

or

$$\frac{Px}{\mathrm{E}k}dx + s\left(\frac{dx}{ds}\frac{d^2y}{ds^2}\right)^{-1}dx = 0.$$

However, one has:

$$d\left(s\frac{dy}{ds}\right) = s\frac{dy}{ds}dy - s\frac{dy}{ds}\frac{ds}{ds}ds + dy.$$

If one assumes that dx is constant then:

$$d^2s = \frac{dy}{ds}d^2y\,,$$

and therefore:

$$d\left(s\frac{dy}{ds}\right) = s\frac{dx}{ds}\frac{d^2y}{ds^2}dx + dy.$$

$$\int s\frac{dx}{ds}\frac{d^2y}{ds^2}dx = s\frac{dy}{ds} - y\,.$$

Upon integration, that will yield the original equation:

$$\frac{Px^2}{2\mathrm{E}k} + a = -s\frac{dy}{ds} + y.$$

46. – Let $dy = p\,dx$, so $ds = dx\sqrt{1+p^2}$. One sets $2\,\mathrm{E}\,k / P = c$, so the equation above will become:

$$a + \frac{x^2}{c} = y - \frac{s\,p}{\sqrt{1+p^2}}\,,$$

or

$$\frac{a\sqrt{1+p^2}}{p} + \frac{x^2\sqrt{1+p^2}}{cp} = \frac{y\sqrt{1+p^2}}{p} - s.$$

When differentiated, that will yield:

$$\frac{-a\,dp}{p^2\sqrt{1+p^2}} + \frac{2x\,dx\sqrt{1+p^2}}{cp} - \frac{x^2\,dp}{cp^2\sqrt{1+p^2}} = \frac{dy\sqrt{1+p^2}}{p} - \frac{y\,dp}{p^2\sqrt{1+p^2}} - dx\sqrt{1+p^2}\,.$$

Since $dy = p\,dx$, the right-hand side will reduce to $-\,y\,dp\,/\,(p^2\sqrt{1+p^2})$. Hence:

$$a - y = \frac{2px(1+p^2)}{c}\left(\frac{dp}{dx}\right)^{-1} - \frac{x^2}{c}.$$

dp is assumed to be constant, so upon differentiating, one will get:

$$-p\,dx = \frac{2px\,(1+p^2)}{c}d\left(\frac{dp}{dx}\right)^{-1} + \frac{2p(1+p^2)}{c}\left(\frac{dp}{dx}\right)^{-1}dx + \frac{2x\;dx(1+3p^2)}{c} - \frac{2x\,dx}{c},$$

or

$$0 = c\,dx\,dp + 2x\,d^2x\,(1+p^2) + 2\,dx^2\,(1+p^2) + 6p\,x\,dx.$$

Solving that equation any further is not possible. The simplest curve equation (from no. **45**, conclusion) is:

$$\frac{y\,ds - s\,dy}{ds} = \frac{Px^2}{2\mathsf{E}k}.$$

When one sets $x = 0$, y and s must also vanish, so the constant a must be zero.

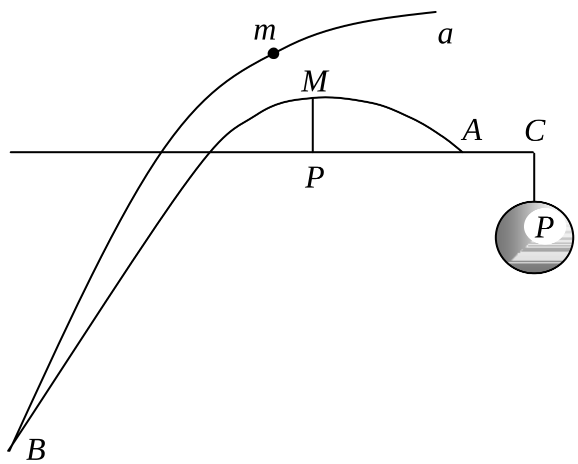

Figure 16.

On the curvature of elastic bands that are not rectilinear in their natural state

47. – In that way, the curvature of the uniform elastic band, just like the non-uniform one, is determined when a force is applied, and the band that is being evaluated is rectilinear in its natural state. However, when the band is already curved in its natural state, it will assume a different curvature as a result of the applied force. In order to find the latter, one must know the elasticity of that natural figure in addition to the applied force. Therefore, let the curve *Bma* represent the natural form of the band (Fig. 16) whose elasticity is $\mathsf{E}\,k^2$ everywhere. That natural curve will go over to the form *BMA* as a result of the applied force *P*. The line *CAP* is drawn through *A* perpendicular to the direction of the force, and that line is taken to be the axis. Let $AC = c$, $AP = x$, $PM = y$, so the moment of the applied force at the point *M* will be $P\,(c + x)$.

48. – Furthermore, let the radius of curvature of the desired curve at *M* be equal to *R*, and let the arc length *am* of the natural curve be equal to *s*, so: arc length *am* = arc length *AM* = *s*. Let *r* be the radius of curvature at *m*, which will be given by the arc length *s* of the known curve *amB*. Since the curvature at *M* is larger, $R < r$. The overshoot of the elementary angle at *M* over the one at *m* is $\frac{ds}{R} - \frac{ds}{r}$. However, that overshoot is due to the applied force. One then has:

$$P\,(c + x) = \mathsf{E}\,k^2 \left(\frac{1}{R} - \frac{1}{r}\right).$$

That is the equation of the desired curve, since *r* is given in terms of *s* ; i.e., it is a function of *x* and *y*. However, that curve cannot be reduced to one of the kinds that were considered before.

49. – We would like to assume that the band *amB* has the form of circle in its natural state, so we then set $r = a$, and: $P\,(c + x) = \mathsf{E}\,k^2 \left(\frac{1}{R} - \frac{1}{a}\right)$. When that is multiplied with *dx* and integrated (see no. **5**, towards the end), that will give:

$$\frac{P}{\mathsf{E}k^2}\left(\tfrac{1}{2}x^2 + cx + f\right) = \frac{-dy}{ds} - \frac{x}{a}.$$

Write $c - \frac{\mathsf{E}k^2}{Pa}$ for *c*. That equation will then become:

$$\frac{P}{\mathsf{E}k^2}\left(\tfrac{1}{2}x^2 + cx + f\right) = \frac{-dy}{ds}.$$

That is the same equation that we found before (no. **5**) for the band that is rectilinear in its natural state. The bands that are circular in their natural state will then be bent into the same curve that the naturally-rectilinear curves will assume, but the point at which the force is applied, and so the segment $AC = c$, as well, must generally be different for each of the two cases. The same nine kinds of curves that we enumerated before will then yield figures that the naturally-circular bands can assume. When *AC* is assumed to be infinite, the circular band can be extended into a straight line. When yet another arbitrary force acts, as well, the same effect can arise as when it alone is applied to a naturally-rectilinear band.

50. – We would like to assume that the point *C* is infinitely distant, and that assumption is entirely independent of the natural form of the band. The moment of the applied force will then be the same everywhere, so when it is divided by $\mathsf{E}\,k^2$, let that equal 1 / *b*, and we will now have:

$$\frac{1}{b}=\frac{1}{R}-\frac{1}{r} \qquad \text{and} \qquad \frac{1}{R}=\frac{1}{b}+\frac{1}{r},$$

$$\int\frac{ds}{R}=\frac{s}{b}+\int\frac{ds}{r}.$$

$\int ds / R$ will be referred to as the *amplitude* of the arc *AM*, and likewise $\int ds / r$ will be the *amplitude* of arc *am*. The most distinguished **Joh. Bernoulli** cared to apply the expression "amplitude" in his excellent treatise "de motu reptorio" ([1]). Let $\frac{s}{b}+\int\frac{ds}{r}$ be an arc of a circle with radius 1. Since *r* is given in terms of *s*, the circular arc will also be known in terms of *s*. The rectangular coordinates *x* and *y* of a point on *BMA* will be found from that, namely:

$$x=\int ds\sin\left(\frac{s}{b}+\int\frac{ds}{r}\right), \qquad y=\int ds\cos\left(\frac{s}{b}+\int\frac{ds}{r}\right).$$

The desired curve can then be found with the help of quadratures.

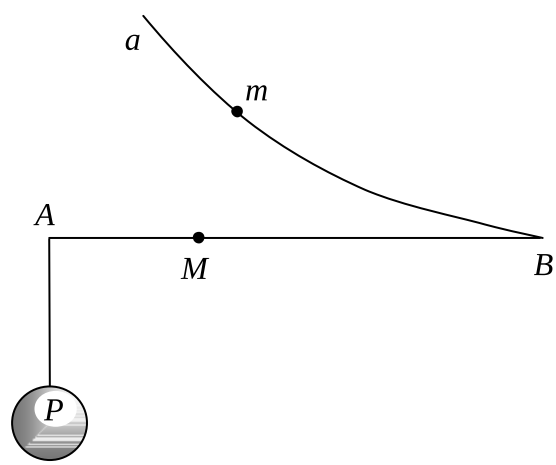

Figure 17.

51. – The form *amB* that the band must have in the natural position in order for it to extend into the straight line *AMB* in the direction *AP* of the applied force (Fig. 17) can be determined from this. Let *AM* be assumed to be equal to *s*, so *Ps* will be the moment of the applied force at the point *M*, and the radius of the curvature circle at *M* is infinite, by assumption, so 1 / *R* = 0. Furthermore, the length of the arc *am* = *r*, so *r*, viz., the radius of curvature at *m*, must be assumed to be negative here, since the curve is convex to the

([1]) One can find **Joh. Bernoulli**'s treatise "De motu reptorio" in *Actis Erudit.*, Aug. 1705 (Werke I, pp. 408).

axis. Therefore, $Ps = \mathrm{E}\, k^2 / r$, or $rs = a^2$. That equation will embody the essence of the curve *amB*.

52. – Since $\frac{1}{r} = \frac{s}{a^2}$, one will have $\int \frac{ds}{r} = \frac{s^2}{2a^2}$. The amplitude of the arc *am* will then behave like the square of the arc length. The rectangular coordinates of that curve are then given by:

$$x = \int ds \sin \frac{s^2}{2a^2}, \qquad y = \int ds \cos \frac{s^2}{2a^2}.$$

The arc $s^2 / 2a^2$ must then be cut from a circular of radius 1, and its sines and cosines will serve to determine the coordinates. It will then follow immediately from the equation that with increasing s the radius of curvature will decrease continually, so it will be obvious that the curve cannot extend to infinity, even when s is infinite. The curve will then belong to the same kind as the spirals that converge to a certain point, such as a center, after winding about it infinitely-many times. It seems to be very difficult to find that point by construction. It is certain that analysis would be required essentially whenever anyone discovers a method by means of which the values of the integrals $\int ds \sin \frac{s^2}{2a^2}$ and $\int ds \cos \frac{s^2}{2a^2}$ can be given for the case of $s = \infty$, at least approximately. That problem seems to have enough merit that the mathematicians should direct their efforts towards it ([1]).

53. – Let $2a^2 = b^2$, so:

$$\sin \frac{s^2}{b^2} = \frac{s^2}{b^2} - \frac{1}{3!}\frac{s^6}{b^6} + \frac{1}{5!}\frac{s^{10}}{b^{10}} - \frac{1}{7!}\frac{s^{14}}{b^{14}} + \ldots - \ldots,$$

$$\cos \frac{s^2}{b^2} = 1 - \frac{1}{2!}\frac{s^4}{b^4} + \frac{1}{4!}\frac{s^8}{b^8} - \frac{1}{6!}\frac{s^{12}}{b^{12}} + \ldots - \ldots$$

([1]) **Euler** himself has further directed his own efforts to the determination of those two integrals, as he said on pp. 339 of the fourth volume of the *Institutiones calculi integralis* (St. Petersburg, 1794): "Recently, I have found by a happy accident with the help of an entirely singular method that (see also no. **54**):

$$\int_0^\infty \frac{d\vartheta \cos \vartheta}{\sqrt{\vartheta}} = \frac{\pi}{2} \qquad \text{and also} \qquad \int_0^\infty \frac{d\vartheta \sin \vartheta}{\sqrt{\vartheta}} = \frac{\pi}{2}."$$

Hence, the coordinates of the desired point are $x = y = a\sqrt{\pi}/2$. The curve that is being investigated here is the one whose natural equation reads $rs = a^2$. It was called the *clothoid* by **Cesaro**. One can find more details and an illustration, which **Euler** also found in the aforementioned place, in **Loria**, *Spezielle algebraische und transzendente Kurven der Ebene*, German trans. by **F. Schütte**, Leipzig, 1902, pp. 458.

The coordinates x and y of the desired curve can then be expressed conveniently as infinite series. One will then get:

$$x = \frac{1}{1\cdot 3}\frac{s^3}{b^2} - \frac{1}{3!\,7}\frac{s^7}{b^6} + \frac{1}{5!\,11}\frac{s^{11}}{b^{10}} - \frac{1}{7!\,15}\frac{s^{15}}{b^{14}} + \ldots - \ldots$$

$$y = s - \frac{1}{2!\,5}\frac{s^5}{b^4} + \frac{1}{4!\,9}\frac{s^9}{b^8} - \frac{1}{6!\,13}\frac{s^{13}}{b^{12}} + \ldots - \ldots$$

The values of the coordinates x and y can be determined precisely from these series, which converge strongly for values of the arc length s that are not too large. However, the values that x and y will assume when the arc length s is set to infinitely large cannot by any means be excluded from these series.

54. – Since setting $s = \infty$ will raise very great difficulties, that disadvantage can be remedied in the following way: Let $s^2 / b^2 = v$, so $s = b\sqrt{v}$ and $ds = \frac{b\,dv}{2\sqrt{v}}$; hence:

$$x = \frac{b}{2}\int \frac{dv}{\sqrt{v}}\sin v \qquad \text{and} \qquad y = \frac{b}{2}\int \frac{dv}{\sqrt{v}}\cos v\,.$$

However, I now assert that the desired values of x and y when $s = \infty$, can be found from the following integral formulas ([1]):

$$x = \frac{b}{2}\int dv\left(\frac{1}{\sqrt{v}} - \frac{1}{\sqrt{\pi+v}} + \frac{1}{\sqrt{2\pi+v}} - \frac{1}{\sqrt{3\pi+v}} + \cdots - \cdots\right)\sin v,$$

$$y = \frac{b}{2}\int dv\left(\frac{1}{\sqrt{v}} - \frac{1}{\sqrt{\pi+v}} + \frac{1}{\sqrt{2\pi+v}} - \frac{1}{\sqrt{3\pi+v}} + \cdots - \cdots\right)\cos v.$$

([1]) Namely, one subdivides the interval from zero to infinity in the following way: 1: from 0 to π, 2: from π to 2π, 3: from 2π to 3π, etc. One will then have:

$$x = \frac{b}{2}\left[\int_0^{\pi}\frac{dv\sin v}{\sqrt{v}} + \int_{\pi}^{2\pi}\frac{dv\sin v}{\sqrt{v}} + \int_{2\pi}^{3\pi}\frac{dv\sin v}{\sqrt{v}} + \cdots\right].$$

One sets $v = v_1 + \pi$ in the second integral, $v = v_2 + 2\pi$ in the third, etc., and then gets:

$$x = \frac{b}{2}\left[\int_0^{\pi}\frac{dv\sin v}{\sqrt{v}} - \int_0^{\pi}\frac{dv_1\sin v_1}{\sqrt{v_1+\pi}} + \int_0^{\pi}\frac{dv_2\sin v_2}{\sqrt{v_2+2\pi}} + \cdots\right].$$

Since the notation for the variables in the definite integrals is irrelevant, one can set $v = v_1 = v_2 = \ldots$ That will give the value in the text. The derivation for y is similar.

After integration, one sets $v = \pi$, where π means the arc length that belongs to an angle of two right angles. In that way, one can indeed avoid setting anything to infinity, but in order to do that, the infinite series:

$$\frac{1}{\sqrt{v}} - \frac{1}{\sqrt{\pi+v}} + \frac{1}{\sqrt{2\pi+v}} - \dots + \dots$$

will be introduced into the calculation. Since its sum is unknown up to now, seeking the answer to this question will raise the greatest complications at the present time.

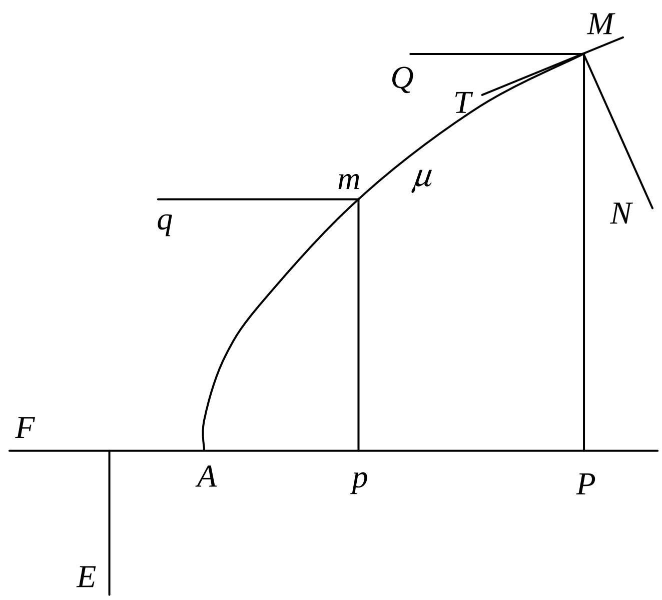

Figure 18.

On the curvature of an elastic band with arbitrary forces acting at individual points

55. – Once the method has been analyzed for the problem of finding the curvature of an arbitrary elastic band when one force is applied at one location, one might further investigate the problem of finding the curvature when the band is stressed by several forces, if not an infinitude of them. However, since it has still not been established up to now what sort of expression is a maximum or minimum in those cases, I will apply the direct method in order to perhaps ascertain the property of being a maximum or minimum from the solution. Let the elastic band that is rectilinear in the natural state go to the form *AmM*, initially from the finite forces *P* and *Q*, which are applied in the mutually-perpendicular directions *CE* and *CF* (Fig. 18), but then from infinitely-small forces that are applied to the individual elements of the band $m\mu$ in the directions *mp* and *mq*, which are parallel to the directions *CF* and *CE*. With those conventions, one seeks the nature of the curve *AmM*.

56. – The line *FCA* will be taken to be an axis, so let $AC = c$, $AP = x$, $PM = y$, let the arc length of the curve be $AM = s$, and let the radius of the curvature circle at *M* be *R*. Let the constant absolute elasticity of the band be $\mathsf{E}\, k^2$, and the sum of the moments that arise from the of the applied forces relative to *M* must equal $\mathsf{E}\, k^2 / R$. First, the moment $P\,(c + x)$ will be produced by the finite force *P* that points in the direction *CE*. It acts in such a way that the elastic force will be in equilibrium. The moment that is produced by the other force *Q* – namely, *Qy* – will stress it in the other direction. Thus, the moment $P\,(c + x) - Qy$ will be produced by the two finite forces *P* and *Q*. Now, an arbitrary intermediate element $m\mu$ of the band will be considered whose abscissa *Ap* is equal to ζ and whose ordinate is set equal to η. Let the force that the element $m\mu$ exerts in the direction *mp* be *dp*, and let the force that it exerts in the direction *mq* equal *dq*. The moment that is produced by those forces at the point *M* will be $(x - \xi)\, dp - (y - \eta)\, dq$.

57. – In order to find the sum of all those moments, the point *M*, and therefore *x* and *y*, as well, must temporarily be considered to be constants, such that only the coordinates ζ and η, along with the forces *dp* and *dq*, will be regarded as variable. The sum of the moments that are produced by the forces that are applied to the arc *Am* will then be: $xp - \int \zeta\, dp - yp + \int \eta\, dp$. In that expression, *p* expresses the sum of all forces that are applied to the arc *AM* in directions that are parallel to *pm*, and *q* denotes the sum of all forces that are applied to the arc *AM* in directions *Ap* that are parallel to the axis. However, one has:

$$\int \zeta\, dp = \zeta\, p - \int p\, d\zeta \qquad \text{and} \qquad \int \eta\, dp = \eta\, q - \int q\, d\eta\,.$$

The sum above will then be:

$$(x - \zeta)\, p + \int p\, d\zeta - (y - \eta)\, q - \int q\, d\eta\,.$$

The point *m* will now be displaced, and then one will have $\zeta = x$, $\eta = y$, and $d\zeta = dx$, $d\eta = dy$. Thus, the sum of all moments taken along the entire arc *AM* will be $\int p\, dx - \int q\, dy$.

One will then get the following equation for the desired curve:

$$\frac{\mathsf{E}\, k^2}{R} = P\,(c + x) - Qy + \int p\, dx - \int q\, dy\,.$$

58. – When the integrals $\int p\, dx$ and $\int q\, dy$ cannot be performed, the equation that is found by differentiation must be satisfied by the integrals. One will then get:

$$\frac{-\mathrm{E}\,k^2\,dR}{R^2} = P\,dx - Q\,dy + p\,dx - q\,dy.$$

However, when neither p nor q can be given by closed-form expressions, but only represented as the sum of infinitely many, infinitely-small forces, the values p and q must be arranged by further differentiation such that dp and dq then appear along with the second-order differentials d^2p and d^2q.

After a first differentiation (when the previous equation is divided by dx), one will get:

$$-\mathrm{E}\,k^2\,d\left(\frac{1}{R^2}\frac{dR}{dx}\right) = dp - (Q+q)\cdot d\,\frac{dy}{dx} - \frac{dy}{dx}\,dq\,.$$

Let $dy / dx = \omega$, so from repeated differentiation (when the equation that was just obtained is divided by $d\omega$), one will get:

$$-\mathrm{E}\,k^2\,d\left[\frac{d}{d\omega}\left(\frac{1}{R^2}\frac{dR}{dx}\right)\right] = d\,\frac{dp}{d\omega} - 2\,dq - \omega\,d\,\frac{dq}{d\omega}.$$

That equation will involve fourth-order differentials.

59. – Instead of vertical and horizontal forces p and q, resp., two forces might be applied to the individual points, one of which points in the direction of the normal $MN = dv$ and the other of which points in the direction of the tangent $MT = dt$ (Fig. 18). One will then have:

$$dp = \frac{dx}{ds}dv + \frac{dy}{ds}dt \qquad \text{and} \qquad dq = \frac{dx}{ds}dt - \frac{dy}{ds}dv\,.$$

Since $dy = \omega\,dx$ and $ds = dx\sqrt{1+\omega^2}$, one will have:

$$dp = \frac{dv}{\sqrt{1+\omega^2}} + \frac{\omega\,dt}{\sqrt{1+\omega^2}} \qquad \text{and} \qquad dq = \frac{dt}{\sqrt{1+\omega^2}} - \frac{\omega\,dv}{\sqrt{1+\omega^2}}\,.$$

If one substitutes this in the latter equation then that will give:

$$-\mathrm{E}\,k^2\,d\left[\frac{d}{dv}\left(\frac{1}{R^2}\frac{dR}{dx}\right)\right] = \frac{-dt}{\sqrt{1+\omega^2}} + \frac{2\omega\,dv}{\sqrt{1+\omega^2}} + \sqrt{1+\omega^2}\,d\,\frac{dv}{d\omega}\,.$$

When this equation is multiplied by $\sqrt{1+\omega^2}$, it will be integrable. For the sake of brevity, let $z = \frac{1}{R^2}\frac{dR}{dx}$. Integration will then yield:

$$A - t + \frac{dv(1+\omega^2)}{d\omega} = -\mathrm{E}\,k^2\left[\frac{dz\cdot\sqrt{1+\omega^2}}{d\omega} - \frac{\omega z}{\sqrt{1+\omega^2}} + \frac{1}{2R^2}\right]$$

$$= -\mathrm{E}\,k^2\left[(1+\omega^2)\frac{d}{d\omega}\left(\frac{1}{R^2}\frac{dR}{dx\sqrt{1+\omega^2}}\right) + \frac{1}{2R^2}\right].$$

Now, since:

$$R = -\,(1+\omega^2)^{3/2}\,\frac{dx}{d\omega}dx,$$

one will have:

$$d\omega = -\,\frac{(1+\omega^2)^{3/2}}{R}\,dx.$$

If one substitutes this value for $d\omega$ then, since $dx\sqrt{1+\omega^2} = ds$, one will have:

$$A - t - R\,\frac{dv}{ds} = -\mathrm{E}\,k^2\left[\frac{1}{2R^2} - R\frac{d}{ds}\left(\frac{1}{R^2}\frac{dR}{ds}\right)\right],$$

or, with some rearrangement:

$$t + R\,\frac{dv}{ds} - A = \mathrm{E}\,k^2\left[\frac{1}{2R^2} - R\frac{d}{ds}\left(\frac{1}{R^2}\frac{dR}{ds}\right)\right].$$

60. – It is now clear that the band will turn into a completely flexible string with the elastic force $\mathrm{E}\,k^2$ vanishes. The previous equations will then include all curves that can be defined by a completely-flexible string that is subject to arbitrary forces. If the string is pulled down by its own weight then one would have $q = 0$, p would be equal to the weight of the string *AM*, and $P = 0$; hence, from the first equation in no. **58**, one would get $p\,(dx\,/\,dy) = Q$ = constant. That is the general equation for a catenary of any kind. The individual points of the completely-flexible string will be subject to forces whose directions (Fig. 18) are themselves normal to the curve; e.g., the string might be subjected to the force dv in the direction *MN* at *M*. Since $t = 0$, (from no. **59**) one will have the equation $R\,(dv\,/\,ds) = A$ = constant. That is the general property for the lintearia (*Muldenkurve*) and all similar ones that come about in that way.

On the curvature of an elastic band that is produced by its own weight

61. – I shall return to the elastic curves for which special attention is required for the problem of finding the form that an elastic band will assume when it is curved by its own weight. Let *AmM* be the desired curve (Fig. 18). Since only the vertical forces that originate in gravity will come into question, $P = 0$, $Q = 0$, $q = 0$, and p expresses the weight of the band *AM*. Let *F* be the weight of the band of length *a*, so $p = Fs / a$, since the band was assumed to be uniform. The nature of the curve will then be expressed by the following equation (from no. **58**):

$$\frac{-\mathsf{E}k^2}{R^2}dR = \frac{Fs}{a}dx.$$

Let the amplitude of the curve (i.e., $\int ds / R$) be *u*, so $R = ds / du$ and $dx = ds \cdot \sin u$, and therefore when the element *ds* is assumed to be constant, one will find the equation:

$$s\, ds \sin u - \frac{\mathsf{E}ak^2}{F} \cdot \frac{du}{ds} du = 0.$$

However, as mere inspection will show, that equation cannot be reduced any further.

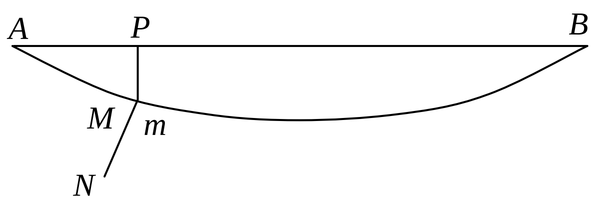

Figure 19.

62. – Special emphasis is warranted by the curves that are assumed by a fluid at rest that are a kind of infinitely-extended elastic band. Let *AMB* (Fig. 19) be the desired form of the curve, and let $AP = x$, $PM = y$, $AM = s$. The element *Mm* will be pushed in the normal direction *MN* by a force that is proportional to *ds*. Hence, $dv = n\, ds$, $dt = 0$. One derives the vertical force $dp = b\, dx$ and the horizontal force $dq = -n\, dy$ from this. One will then have $p = nx$ and $q = -ny$. The equation of no. **57** will then become:

$$\frac{\mathsf{E}k^2}{R} = P\,(c + x) - Qy + \tfrac{1}{2}n\,x^2 + \tfrac{1}{2}n\,y^2.$$

The coordinates *x* and *y* can be increased or decreased by constant quantities in such a way that the equation of the curve will assume the form:

$$x^2 + y^2 = A + \frac{B}{R}.$$

One multiplies that equation by $x\,dx + y\,dy$ in order to make it integrable; namely:

$$\int \frac{x\,dx + y\,dy}{R} = -\int \frac{x + y\,\omega}{(1+\omega^2)^{3/2}}\,d\omega$$

[when one sets $dy = \omega dx$]

$$= \frac{y-\omega x}{\sqrt{1+\omega^2}} = y\frac{dx}{ds} - x\frac{dy}{ds}.$$

Integration will then imply that:

$$(x^2+y^2)^2 = A\,(x^2+y^2) + B\left(y\frac{dx}{ds} - x\frac{dy}{ds}\right) + C.$$

Let:

$$z = \sqrt{x^2+y^2} \quad \text{and} \quad y = u\,z, \quad \text{so} \quad x = z\sqrt{1-u^2}\,,$$

so

$$y\,dx - x\,dy = -\frac{z^2}{\sqrt{1-u^2}}\,du\,; \qquad ds = \sqrt{dz^2 + \frac{z^2\,du^2}{1-u^2}}\,.$$

If one then sets:

$$\frac{1}{\sqrt{1-u^2}}\,du = dr \qquad \text{then one will have:} \qquad z^4 - A\,z^2 - C = -\frac{Bz^2}{\sqrt{dz^2+z^2dr^2}}\,dr\,,$$

so

$$dr = \frac{z^4 - Az^2 - C}{z\sqrt{B^2z^2-(z^4-Az^2-C)^2}}\,dz\,.$$

When A and C are equal to zero, the associated curve will be algebraic. One will then have the equation:

$$dr = \frac{1}{\sqrt{1-u^2}}\,du = \frac{z^2}{\sqrt{B^2-z^6}}\,dz = \frac{3z^2}{3\sqrt{a^6-z^6}}\,dz\,.$$

Integration will yield:

$$\arcsin u = \tfrac{1}{3}\arcsin\frac{z^3}{a^3} \quad \text{or} \quad \frac{z^3}{a^3} = 3u - 4u^3 = \frac{3y}{z} - \frac{4y^3}{z^3}\,.$$

What will ultimately follow is the equation $z^6 = 3\,a^3\,y\,z^2 - 4\,a^3\,y^3$, or :

$$(x^2+y^2)^3 = 3\,a^3\,x^2\,y - a^3\,y^3\,.$$

On the oscillatory motion of elastic bands

63. – The oscillatory motion of the elastic band and things that are prepared to move in an arbitrary way can be derived from the foregoing. The renowned **Daniel Bernoulli** was the first to address that truly interesting topic, and he had already suggested the problem to me of determining elastic oscillations of a band with one end that is fixed in a solid wall some years ago. I gave the solution in Comment. Petropol, vol. VII (1740).

Since that time, however, I succeeded in treating that problem more simply, and through my communications with the esteemed **Bernoulli**, I arrived at many questions and viewpoints whose clarification I would like to address here due to its relationship to the current topic. If the oscillatory motion is sufficiently fast, a sound will be produced by the oscillating band whose height and relationship to other sounds can be determined with the help of a study of sound on the basis of these principles. Since the nature of sound is easily accessible to experiments, one can study the agreement between calculation and reality in that way, and thus confirm the theory. Our knowledge of the essence of elastic bodies will be extended appreciably by that.

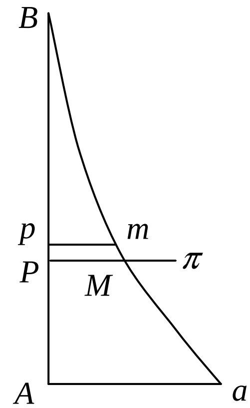

Figure 20.

64. – However, one might first object that the problem here has only been treated for very small oscillations, such that the interval that the band moves within during its oscillations is likewise very small. However, that restriction does not diminish the utility and applications in any way. Namely, not only would the oscillations be free from isochronism if they were to cover great distances, but the formation of different sounds of the kind that we have mainly ignored here would also require very small oscillations. Therefore, I shall consider a uniform elastic band here that is rectilinear in its natural state, whose one end is fitted into an immobile wall, such that the band would have the form of the straight line *AB* (Fig. 20) if it were left to itself. Let its length be $AB = a$, and let its absolute elasticity at the individual locations be $\mathsf{E}\, k^2$; its weight will be considered, but we shall assume that it has been arranged in such a way that its state cannot be perturbed by the force of gravity.

The oscillations of the elastic bands with one end fixed to a wall

65. – When this band is driven by an arbitrary force, it might carry out very small oscillations, during which, it will sweep out the very small interval *Aa* about both sides of its natural state *AB*. Let *BMa* be an arbitrary state that the band assumes during its oscillations. Since that state will be infinitely close to the natural state *BPA*, the line *MP*, like *Aa*, will represent the path that the points *M* and *a* on the band will pass through, or more precisely, that path will have a ratio with the true path that differs only slightly from a ratio of unity. In order to determine the oscillatory motion, it is absolutely necessary to known the nature of the curve *BMa* that the side assumes under oscillation. Let $AP = x$, $PM = y$, arc $aM = s$, let the radius of curvature at M equal R, and let the very small interval Aa equal b. On the basis of our condition, the arc length s will be very close to equal to the abscissa x, so ds can be assumed to be dx, namely, dy will vanish in comparison to dx. With dx taken to be constant, the radius of curvature $\left(\frac{dx}{ds}\frac{d^2y}{dx^2}\right)^{-1}$ will be $R = \left(\frac{d^2y}{dx^2}\right)^{-1}$, in the present case, so the convexity of the curve *BMa* will point to the axis *BA*. Since the band is fixed to a solid wall at *B*, *AB* will be a tangent to the curve at *B*.

66. – With those conventions, let f be the length of a simple, isochronous pendulum in order to determine the curve *BMa* and the oscillatory motion. The fact that very small oscillations are isochronous can be explained by the nature of things, and in addition, the calculations that will be done will show that. The acceleration by which the point M of the band moves to P will be $PM / f = y / f$. If one sets the mass of the entire band equal to M, by which, its weight will be expressed, then the element $Mm = ds = dx$ will take on the mass $(M / a)\, dx$. The force that drives the element in the direction MP will then be $(M\, y / a\, f)\, dx$. Therefore, the forces that excite the individual parts of the band will be known, on the one hand, from the curve *BMa*, and on the other hand, from the length f of the simple, isochoronous pendulum. However, since the motion of the band is, in reality, driven by the elastic force, the nature of the curve *BMa* and the length of the simple, isochronous pendulum will be determined when one knows that force.

67. – Since the band then moves as if forces equal to $(M\, y / a\, f)\, dx$ acted upon the individual elements Mm in the direction MP, it will follow that the band will be in equilibrium in the state *BMa* when equal forces $(M\, y / a\, f)\, dx$ are applied to the individual elements in the opposite direction $M\pi$. As a result, the band will assume the curvature under oscillation that it assumes in the rest state when its individual points M are subjected to forces $(M\, y / a\, f)\, dx$ in the direction $M\pi$. From rule that was found the above (nos. **56** and **57**), all of those forces that are applied along the arc aM can be combined, and that will give the sum $(M / a\, f) \int y\, dx$, which must be substituted for p. Since the remaining forces P, Q, and q happen to vanish there, the curve will have the equation:

$$\frac{\mathrm{E}k^2}{R} = \int p\,dx \qquad \text{or} \qquad \frac{\mathrm{E}k^2}{R} = \frac{M}{a\,f}\int dx \int y\,dx\ .$$

Since $R = 1/\frac{d^2y}{dx^2}$, one will have $\mathrm{E}k^2\frac{d^2y}{dx^2} = \frac{M}{a\,f}\int dx\int y\,dx$. Differentiation will yield: $\mathrm{E}k^2\frac{d^3y}{dx^3} = \frac{M}{a\,f}\int y\,dx$. With another differentiation, one will get the fourth-order differential equation:

$$\mathrm{E}k^2\frac{d^4y}{dx^4} = \frac{My}{af}.$$

68. –The nature of the curve *BMa* can be expressed by that equation, and the length *f* can be determined from it when it is adapted to the present case. If *f* is known then the oscillatory motion will also be known. However, above all, that equation must be integrated. Since it belongs to the type of higher-order differential equation whose general integration I have pointed out in *Misc. Berol.*, v. VII, one will arrive at the following equation for the integral, when one sets $\mathrm{E}\;k^2\;a\;f\;/\;M\;=c^4$, for the same of brevity:

$$y = Ae^{x/c} + Be^{-x/c} + C\sin\frac{x}{c} + D\cos\frac{x}{c}.$$

e denotes the number whose hyperbolic logarithm equals 1, and sin (x / c) and cos (x / c) denote the sine and cosine of the arc x / c of a circle of radius 1. *A*, *B*, *C*, *D* are four constants that are introduced by fourfold integration, which one must determine in order to adapt the calculations to the present case.

69. – The determination of the constants happens in the following way: First, one sets $x = 0$, so one must have $y = b$, and that will then imply the first equation: $b = A + B + D$. Secondly, since one has:

$$c^4\frac{d^2y}{dx^2} = \int dx\int y\,dx,$$

one must have $\frac{d^2y}{dx^2} = 0$ for $x = 0$, since $\int p\,dx$ vanishes for $x = 0$. One will then get the second equation:

$$A + B - D = 0.$$

Thirdly: Since $c^4\frac{d^3y}{dx^3} = \int y\,dx$, one will also have $\frac{d^3y}{dx^3} = 0$ for $x = 0$; i.e., that will imply the third equation $0 = A - B - C$. Fourth: y will vanish for $x = a$, so that will yield the fourth equation:

$$0 = Ae^{a/c} + Be^{-a/c} + C\sin\frac{a}{c} + D\cos\frac{a}{c}.$$

Fifth: Since *AB* contacts the curve at *B*, dy / dx must be zero for $x = a$. Therefore, the fifth equation will emerge:

$$0 = Ae^{a/c} - Be^{-a/c} + C\cos\frac{a}{c} - D\sin\frac{a}{c}.$$

The four constants *A*, *B*, *C*, *D* will first be determined from these five equations, and then (and this is the main result) the value of $c = \sqrt[4]{\frac{\mathrm{E}k^2 af}{M}}$ will be found. The length *f* of the simple, isochronous pendulum can be derived from that, and in that way the period of the oscillations will also be known.

70. – It follows from the second and third equation that:

$$C = A - B \qquad \text{and} \qquad D = A + B.$$

When those values are substituted in the fourth and fifth equation, that will yield:

$$0 = Ae^{a/c} + Be^{-a/c} + (A-B)\sin\frac{a}{c} + (A+B)\cos\frac{a}{c},$$

$$0 = Ae^{a/c} - Be^{-a/c} + (A-B)\cos\frac{a}{c} - (A+B)\sin\frac{a}{c},$$

resp.

It follows from this that:

$$\frac{A}{B} = -\frac{e^{-a/c} + \sin\frac{a}{c} - \cos\frac{a}{c}}{e^{a/c} + \sin\frac{a}{c} + \cos\frac{a}{c}} = \frac{e^{-a/c} + \cos\frac{a}{c} + \sin\frac{a}{c}}{e^{a/c} + \cos\frac{a}{c} - \sin\frac{a}{c}}.$$

One will then get the equation:

$$2 + (e^{a/c} + e^{-a/c})\cos\frac{a}{c} = 0,$$

or also:

$$e^{2a/c}\cos\frac{a}{c} + 2\,e^{a/c} + \cos\frac{a}{c} = 0.$$

Therefore:

$$e^{a/c} = -\frac{1 \pm \sin\frac{a}{c}}{\cos\frac{a}{c}}.$$

When $e^{a/c}$ is a positive quantity, cos (a / c) must always be negative; i.e., the angle a / c must be greater than a right angle.

71. – One sees that the last equation will yield infinitely many angles a / c that satisfy it. Infinitely many types of oscillations of that band will arise from that. Namely, the curve can cut the axis AB at one or more points before it contacts the axis at B. In that way, many (indeed, infinitely many) types of oscillation are possible. Here, we would like to consider mainly the case in which B is the first point that the side has in common with the axis. That case is satisfied by the smallest angle a / c that can appear as a solution of the equation that was found. Since it is greater than one right angle, it might be set equal to $\pi / 2 + \varphi$, where φ is smaller than one right angle. Since sin (a / c) = cos φ and cos (a / c) = – sin φ, that will imply the double equation:

$$e^{a/c} = \frac{1 \mp \cos\varphi}{\sin\varphi};$$

it will follow from this that:

$$e^{a/c} = \tan\tfrac{1}{2}\varphi \qquad \text{or} \qquad e^{a/c} = \cot\tfrac{1}{2}\varphi.$$

That latter equation yields a smaller value for φ; it will then correspond to the conditions that were imposed.

72. – The further possible types of oscillations will be found when the angle a / c is set to something greater than $2R$, but smaller than $3R$. Let $a / c = \frac{3}{2}\pi - \varphi$, so sin ($a$ / c) = – cos φ and cos (a / c) = – sin φ. That will yield:

$$e^{a/c} = \frac{1 \pm \cos\varphi}{\sin\varphi}.$$

Hence, $e^{a/c} = \tan(\varphi / 2)$ or $e^{a/c} = \cot(\varphi / 2)$. Other types of oscillations will be found in a similar way when one sets:

$$\frac{a}{c} = \tfrac{5}{2}\pi + \varphi, \quad \frac{a}{c} = \tfrac{7}{2}\pi - \varphi, \quad \text{etc.}$$

When one employs natural logarithms, those assumptions will imply the following equations:

I.	$\frac{1}{2}\pi + \varphi = \log\cot\frac{1}{2}\varphi$,	II.	$\frac{1}{2}\pi + \varphi = \log\tan\frac{1}{2}\varphi$,
III.	$\frac{3}{2}\pi - \varphi = \log\cot\frac{1}{2}\varphi$,	IV.	$\frac{3}{2}\pi - \varphi = \log\tan\frac{1}{2}\varphi$,
V.	$\frac{5}{2}\pi + \varphi = \log\cot\frac{1}{2}\varphi$,	VI.	$\frac{5}{2}\pi + \varphi = \log\tan\frac{1}{2}\varphi$,

$$\text{VII.}\quad \tfrac{7}{2}\pi - \varphi = \log\cot\tfrac{1}{2}\varphi, \qquad \text{VIII.}\quad \tfrac{7}{2}\pi - \varphi = \log\tan\tfrac{1}{2}\varphi,$$

etc.

The third of these equations coincides with the second one; namely, if one sets $\tfrac{1}{2}\varphi = \tfrac{1}{2}\pi - \tfrac{1}{2}\vartheta$ then one will have cot $\tfrac{1}{2}\varphi$ = tan$\tfrac{1}{2}\vartheta$. The third equation will then go to $\tfrac{1}{2}\pi + \vartheta$ = log tan $\tfrac{1}{2}\vartheta$; however, that is the second equation. Similarly, the fourth one coincides with the first one, the fifth one with the eight one, and the sixth one with the seventh one. Therefore, the only distinct equations are the following ones:

$$\text{I.}\quad \tfrac{1}{2}\pi + \varphi = \log\cot\tfrac{1}{2}\varphi, \qquad \text{II.}\quad \tfrac{1}{2}\pi + \varphi = \log\tan\tfrac{1}{2}\varphi,$$

$$\text{III.}\quad \tfrac{5}{2}\pi + \varphi = \log\cot\tfrac{1}{2}\varphi, \qquad \text{IV.}\quad \tfrac{5}{2}\pi + \varphi = \log\tan\tfrac{1}{2}\varphi,$$

$$\text{V.}\quad \tfrac{9}{2}\pi + \varphi = \log\cot\tfrac{1}{2}\varphi, \qquad \text{VI.}\quad \tfrac{9}{2}\pi + \varphi = \log\tan\tfrac{1}{2}\varphi,$$

etc.

73. – Let u be the natural logarithm of the tangent or cotangent of the angle $\tfrac{1}{2}\varphi$. Refer to a table of common logarithms for that tangent or cotangent, and let the value be v. It is then known that $u = 2.302585092994 \times v$, so if one again takes ordinary logarithms then:

$$\log u = \log v + 0.3622156886.$$

Now, $u = n\pi / 2 + \varphi$, so log u = log $(n\pi / 2 + \varphi)$. In order to evaluate that, φ must be expressed as a fraction of the radius, while $\pi / 2$ is 1.57079632679. One converts φ into seconds and extracts the number 5.3244252332 ([1]) as the common logarithm of that number, so one will then get log φ, and the appropriate value of φ by exponentiation. For every arbitrary type of oscillation, one will always have: $a / c = u = n\pi / 2 + \varphi$.

([1]) The start of this number is somewhat truncated, which can be explained by the use of tables, similarly to what was said in no. **31**. It yields φ in terms of the number β, which is φ converted into seconds. One then has, in arc units:

$$\varphi = \beta\,\frac{\pi}{180\cdot 60^2}, \quad \text{so} \qquad \log\varphi = \log\beta - \log\frac{180\cdot 60^2}{\pi}.$$

The latter logarithm is 5.3144… That will explain the appearance of that number in nos. **73**, **74**, **78**, and **85**, and the table calculation is truncated in the last of those sections. In order to get ln cot $\varphi / 2$, **Euler** used the known formula:

$$\ln\cot\frac{\varphi}{2} = \frac{1}{M}\log\cot\frac{\varphi}{2}.$$

In order to perform the multiplication on the right-hand side, he again employed ordinary logarithms, and log $1 / M$ is 0.362215… The appearance of this number in the second part of the table calculation becomes understandable in that way. At the conclusion of the calculation, **Euler** applied the *regula falsi* method, as he also did in no. **85**.

74. – When one observes those preliminaries in regard to the calculation, the value of the angle φ will not be difficult to ascertain by approximation for any type of oscillation. If one assigns values to φ at will and then determines $n\pi / 2 + \varphi$ and log tan $\varphi / 2$ or log cot $\varphi / 2$ then one will rapidly arrive at a reasonably precise value for φ. If the angle φ initially lies between relatively wide limits then one will soon find narrower ones, and from them, the true value of φ. I have then ascertained the following limits for the first equation $a / c = \pi / 2 + \varphi = \log\cot(\varphi / 2)$: namely, $17^o\ 26'$ and $17^o\ 27'$. Those will imply the true value of φ by the following calculation:

$\varphi = 17^o\ 26'\ 0''$, 62760″ in seconds log = 4.7976829349 subtract: 5.3144251332	$\varphi = 17^o\ 27'\ 0''$, 62820″ in seconds 4.7980979321 5.3144251332
log φ = 0.4832578017 – 1 φ = 0.3042690662 $\frac{1}{2}\pi$ = 1.5707963268	0.4836727989 – 1 0.3045599545 1.5707963268
$\frac{1}{2}\pi + \varphi$ = 1.8750653930	1.8753562813
$\frac{1}{2}\varphi = 8^o\ 43'\ 0''$ $v = \log\cot\frac{1}{2}\varphi$ = 0.8144034109 log v = 0.9108395839 – 1 add: 0.3622156886	$\frac{1}{2}\varphi = 8^o\ 43'\ 30''$ 0.81339819342 0.9106147660 0.3622156886
log u = 0.2730552725 u = 1.8752331540	0.2728304546 1.8742626675
Difference: + 1677610	– 10936138

From the deviation between both limiting values, one then concludes that $\varphi = 17^o\ 26'\ 7.98''$, so:

$$\tfrac{1}{2}\pi + \varphi = \frac{a}{c} = 107^o\ 26'\ 7.98''.$$

In seconds, φ = 62967.98, so: log φ =	4.7977381525
–	5.3144251332
	0.4833130193
in arc units, φ =	0.3043077545
adding $\frac{1}{2}\pi$ =	1.5707963268
a / c =	1.8751040813

Once that is found, one will have ([1]):

$$\frac{A}{B} = \tan \tfrac{1}{2}\varphi = 0.1533390624.$$

One then finds the relationship between C and D and A and B from the known relationship between the constants A and B.

75. – The first equation $b = A + B + D$ still remains. Since $D = A + B$, it will then follow that $A + B = \tfrac{1}{2}b$. Now, $A = B \tan \tfrac{1}{2}\varphi$, so:

$$B = \frac{b}{2(1+\tan\tfrac{1}{2}\varphi)}.$$

However, $\tan \tfrac{1}{2}\varphi = 0.1533390624$, so the constants can be determined as follows:

$$\frac{A}{b} = \frac{\tan\tfrac{1}{2}\varphi}{2(1+\tan\tfrac{1}{2}\varphi)} = \frac{0.1533390624}{2.3066781248},$$

$$\frac{B}{b} = \frac{1}{2(1+\tan\tfrac{1}{2}\varphi)} = \frac{1.0000000000}{2.3066781248},$$

$$\frac{C}{b} = \frac{-1+\tan\tfrac{1}{2}\varphi}{2(1+\tan\tfrac{1}{2}\varphi)} = \frac{-0.8466609376}{2.3066781248},$$

$$\frac{D}{b} = \frac{1+\tan\tfrac{1}{2}\varphi}{2(1+\tan\tfrac{1}{2}\varphi)} = \frac{1.1533390624}{2.3066781248}.$$

([1]) One easily finds the formula $A / B = \tan \tfrac{1}{2}\varphi$, which has not been derived up to now as follows: Add the first two formulas in no. **70**, which contain only A and B; it will then follow that:

$$2A\, e^{a/c} - 2B \sin(a/c) + 2A \cos(a/c) = 0,$$

or, since $\sin(a/c) = \cos\varphi$, $\cos(a/c) = -\sin\varphi$, and $e^{a/c} = \cot\varphi/2$, one will have:

$$\cos\varphi = \frac{A}{B}\left(\cot\frac{\varphi}{2} - \sin\varphi\right),$$

$$\cos\varphi = \frac{A}{B}\cot\frac{\varphi}{2}\left(1-2\sin^2\frac{\varphi}{2}\right) = \frac{A}{B}\cot\frac{\varphi}{2}\cos\varphi,$$

so

$$\frac{A}{B} = \tan\frac{\varphi}{2}.$$

Once the constants have been found, the nature of the curve that the band assumes during its oscillation will be expressed by this equation:

$$\frac{y}{b} = \frac{A}{b}e^{x/c} + \frac{B}{b}e^{-x/c} + \frac{C}{b}\sin\frac{x}{c} + \frac{D}{b}\cos\frac{x}{c}.$$

76. – Most of what is worth knowing about the rapidity of the oscillations can be recognized from the equation a / c = 1.8751040813. For the sake of brevity, let n = 1.875…, so $a = nc$. Now, $c^4 = \mathsf{E}\, k^2\, a\, f / M$, where M / a expresses the specific weight of the band (i.e., the mass per unit length). One will then have $a^4 = n^4\, \mathsf{E}\, k^2\, a\, f / M$, so $f = \frac{a^4}{n^4}\cdot\frac{1}{\mathsf{E}k^2}\cdot\frac{M}{a}$; i.e., the length of the simple isochronous pendulum is proportional to the fourth power of the length of the band and the specific weight and inversely proportional to the absolute elasticity. Let g be the length of the simple second pendulum, so g = 3.16625 Rhenish feet. Since the period of oscillation is proportional to the square root of the length of the pendulum, the period of an oscillation that our elastic band will complete will be:

$$\frac{\sqrt{f}}{\sqrt{g}}\text{ seconds} = \frac{a^2}{n^2}\sqrt{\frac{1}{g}\cdot\frac{1}{\mathsf{E}k^2}\frac{M}{a}}.$$

The number of oscillations that are completed in one second will then be:

$$\frac{n^2}{a^2}\sqrt{g\mathsf{E}k^2\cdot\frac{a}{M}}.$$

That number expresses the height of the sound that excites the band.

That sound that is produced by various bands with one end attached to a solid wall will then behave like the square root of the absolute elasticity, and inversely like the square root of the specific weight, and inversely like the square of the length. Hence, when two elastic bands differ only in their lengths, the associated sound will behave like the square of the length; i.e., a band that is twice as long will give a sound that is two octaves lower. However, a tensed string will give a sound that is only one octave lower. It then becomes clear from this that the sounds of elastic bands behave quite differently from the sounds of tensed strings ([1]).

77. – As far as the behavior of a curve beyond its ends a and B is concerned, it will first be clear that the curve beyond a will advance in such a way that it will be continually separated from the axis AB. Namely, if x is taken to be negative then:

([1]) Here, **Euler** is referring to the difference between the oscillations of bodies that are elastic due to stress – viz., tensed strings or *corda elastica* – and the ones that are elastic due to rigidity – viz., elastic bands or *lamina elastica.*

$$y = Be^{x/c} + Ae^{-x/c} - C\sin\frac{x}{c} + D\cos\frac{x}{c}.$$

All terms in this are positive, since only the coefficient C has a negative value (no. **75**). When x increases, so does y, since B is greater than A, and therefore outweighs the term $Be^{x/c}$. However, if x / c has attained only a middle value then the term $Be^{x/c}$ will have already decreased so much that the remaining terms will vanish along with it. Since the radius of curvature is not infinite at B, namely, one has:

$$\frac{\mathsf{E}k^2}{R} = \frac{M}{af}\int dx \int y\,dx,$$

the curve will have no inflection point at B; it will then advance further on the same side of the axis. When the abscissa x increases beyond $AB = a$, the first term $A\,e^{x/c}$ will soon become so large that the other ones will appear to be very small in comparison.

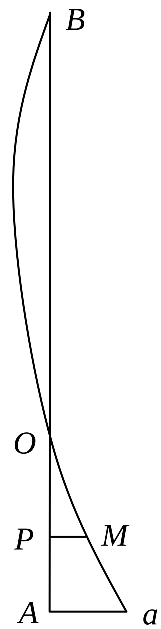

Figure 21.

78. – Up to now, the first kind of oscillations were treated as ones among the infinitely-many oscillations that the same band can accommodate. The second kind, which is represented in the figure (Fig. 21), in which the band that is fixed at B cuts the axis AB at a point O, will be derived from the equation:

$$\frac{a}{c} = \tfrac{1}{2}\pi + \varphi = \log\tan\tfrac{1}{2}\varphi,$$

or

$$\tfrac{3}{2}\pi - \varphi = \log \cot \tfrac{1}{2}\varphi = \frac{a}{c}.$$

Here, I have found, from some experiments, that the angle φ is included within the limits $1^{\circ}\ 2'\ 40''$ and $1^{\circ}\ 3'\ 0''$. The true value of that φ can be ascertained from them, as before.

	$\varphi = 1^{\circ}\ 2'\ 40''$, in seconds: 3760″	$\varphi = 1^{\circ}\ 3'\ 0''$, in seconds: 3780″
log	3.5751878450	3.5774917998
minus	5.3144251332	5.3144251332
$\log \varphi =$	0.2607627118 – 2	0.2630666666 – 2
$\varphi =$	0.0182289944	0.0183259571
$\tfrac{3}{2}\pi =$	4.7123889804	4.7123889804
$\tfrac{3}{2}\pi - \varphi = \frac{a}{c} =$	4.6951599860	4.6940630233
$\tfrac{1}{2}\varphi =$	31′ 20″	= 31′ 30″
$\log \cot \tfrac{1}{2}\varphi =$	2.0402552577	2.0379511745
$\log v =$	0.3096845055	0.3091937748
add	0.3622156886	0.3622156886
$\log u =$	0.6719001941	0.6714094634
$u =$	4.6978613391	4.6925559924
$\frac{a}{c} =$	4.6941599860	4.6940630233
Deviation:	37013531	– 15070309

The true value of the angle φ is ascertained to be $1^{\circ}\ 2'\ 54.213''$ from those deviations, and $a/c = 268^{\circ}\ 57'\ 5.787''$. That will yield $a/c = 4.6940910795$ in arc units.

The pitch of the oscillating band of the previous type will relate to the pitch of this band as the square of the number 1.8751040813 relates to the square of 4.6940910795;

i.e., like 1 to 6.266891, or, in smaller numbers, like 4 : 25 or like 1 : 6 $\frac{4}{15}$. The latter pitch will be two octaves plus a fifth plus the next half-tone from the former one ([1]).

79. – The angle φ will be much smaller for the following types of oscillations of the same elastic band under which the band cuts the axis *AB* in two or more points. One will then have the following equation:

$$\tfrac{5}{2}\pi + \varphi = \log \cot \tfrac{1}{2}\varphi = \frac{a}{c}, \quad \text{so} \qquad e^{5\pi/2+\varphi} = \cot \tfrac{1}{2}\varphi$$

for the third type. Due to the very small value of φ, $e^{5\pi/2+\varphi}$ can be developed into:

$$e^{5\pi/2}\left(1+\varphi+\tfrac{1}{2}\varphi^2+\tfrac{1}{6}\varphi^3+\cdots\right),$$

$$\cot \tfrac{1}{2}\varphi = \frac{\cos\frac{1}{2}\varphi}{\sin\frac{1}{2}\varphi} = \frac{1-\frac{1}{8}\varphi^2+\cdots-\cdots}{\frac{1}{2}\varphi-\frac{1}{48}\varphi^3+\cdots-\cdots} = \frac{2}{\varphi}-\frac{\varphi}{6}.$$

One will then have:

$$e^{5\pi/2} = \frac{2}{\varphi}, \quad \text{so} \qquad \varphi = 2\,e^{-5\pi/2},$$

approximately, or more precisely ([2]):

$$\varphi = \frac{1}{1+\frac{1}{2}e^{5\pi/2}}.$$

Hence:

$$\frac{a}{c} = \tfrac{5}{2}\pi + \frac{2}{e^{5\pi/2}+2};$$

([1]) If the lower tone is C then the higher one will not be as low as G sharp. If C has the oscillation number N then G sharp will have $\frac{25}{4}N$ (i.e., $6\frac{4}{16}N$, instead of $6\frac{4}{15}N$, as in **Euler**'s calculation). See the remark in the next section.

([2]) Namely, one has $e^{5\pi/2}(1+\varphi) = 2/\varphi$, approximately, so $e^{5\pi/2} + \varphi\cdot e^{5\pi/2} = 2/\varphi$. However, from the first approximation in the text, $\varphi\cdot e^{5\pi/2} = 2$, so $2/\varphi = e^{5\pi/2}+2$; i.e.:

$$\varphi = \frac{1}{1+\frac{1}{2}e^{5\pi/2}}.$$

If one sets $\nu = \frac{1}{a^2}\sqrt{g\mathsf{E}k^2\frac{a}{M}}$ then the various types of oscillations will correspond to tones with the oscillation numbers:

$$1.815^2\nu, \qquad 4.69^2\nu, \qquad \tfrac{25}{4}\pi^2\nu, \qquad \tfrac{49}{4}\pi^2\nu, \qquad \ldots$$

All of those tones were found experimentally by **Chladni** (**Chladni**, *Akustik*, Leipzig 1802, pp. 94-103). They are in the best agreement with **Euler**'s results, as **Chladni** also found for the following cases (see the remark on page 63).

the last term is very small. Similarly, one will get:

$$\frac{a}{c} = \tfrac{7}{2}\pi - 2e^{-7\pi/2}, \qquad \text{etc.,}$$

approximately, for the fourth type of oscillation. Since the second term will always be smaller, a / c will assume the values $\tfrac{9}{2}\pi$, $\tfrac{11}{2}\pi$, etc., which will deviate from reality less and less as one advances in that series.

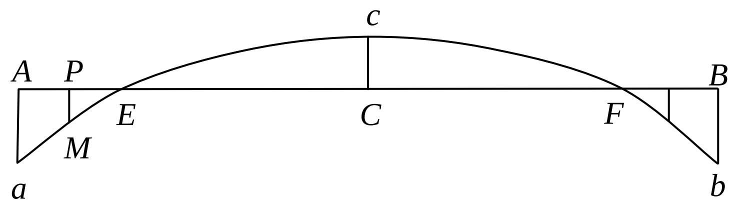

Figure 22.

The oscillations of a free elastic band

80. – We shall now consider an elastic band that is nowhere fixed and either lies freely in a very smooth plane or is found to be weightless in empty space. It is quite clear that such a band can assume an oscillatory motion, namely, when the band *acb* (Fig. 22) moves with alternating curvature, on one side and the other of its rest position *AB*. That oscillatory motion can be determined in a similar manner to the foregoing case, except that the associated calculations in this case must be adapted suitably. Hence, let *acb* be one form of the band that will appear during its oscillation, and let *ACB* be the position of that band in its equilibrium state, through which it will go during any oscillation. As before, the length of the band will be $AB = a$, its absolute elasticity = $\mathsf{E}\, k^2$, and its weight or mass will be set to *M*. Furthermore, let $AP = x$, $PM = y$, arc $aM = s$, which coincides with the abscissa *x*, such that one can set $ds = dx$. The radius of curvature at *M* then proves to be $R = 1 / (d^2y / dx^2)$. Furthermore, let the first ordinate by $Aa = b$. From the conventions, one can pose the same argument as before (nos. **66** and **67**), and arrive at the same equation:

$$\frac{\mathsf{E}k^2}{R} = \frac{M}{af}\int dx \int y\,dx = \mathsf{E}k^2 \frac{d^2y}{dx^2}.$$

81. – One sets $c^4 = \mathsf{E}\, k^2\, a\, f / M$, in which *f* expresses the length of the simple isochronous pendulum, as before. Upon integration, one will get the following equation for the curve:

$$y = Ae^{x/c} + Be^{-x/c} + C\sin\frac{x}{c} + D\cos\frac{x}{c}.$$

That equation is adapted to the present case as follows: If one sets $x = 0$ then one must have $y = b$, so:

$$b = A + B + D.$$

Secondly, since:

$$c^4 \frac{d^2 y}{dx^2} = \int dx \int y\, dx\,, \quad \text{one must have } \frac{d^2 y}{dx^2} = 0 \text{ for } x = 0\,;$$

therefore:

$$0 = A + B - D.$$

Thirdly, since:

$$c^4 \frac{d^3 y}{dx^3} = \int y\, dx\,, \qquad \text{one must have } \frac{d^2 y}{dx^2} = 0 \text{ for } x = 0\,;$$

one will then have:

$$0 = A - B - C.$$

Fourthly, when one sets $x = a$, $\int y\, dx$ must vanish, since $\int y\, dx$ expresses the sum of all forces that pull the band in a direction that is perpendicular to the axis. When that sum is not zero, the band will be subject to a local motion, contrary to assumption. On that basis, $d^3 y / dx^3$ will then be equal to zero; i.e.:

$$0 = A e^{a/c} - B e^{-a/c} - C \sin\frac{a}{c} + D \cos\frac{a}{c}.$$

Fifth, since the band is free at the end B, it cannot have any curvature there, so one will also have $d^2 y / dx^2 = 0$ for $x = a$; hence:

$$0 = A e^{a/c} + B e^{-a/c} - C \sin\frac{a}{c} - D \cos\frac{a}{c}.$$

In regard to these five conditions, not just the four constants A, B, C, D will be determined, but also the value of the fraction a / c, with which, the length of the simple isochronous pendulum can then be known.

82. – It follows from the second and third equation that:

$$D = A + B, \qquad C = A - B.$$

One substitutes these values in the following ones and then finds that:

$$\frac{A}{B} = \frac{e^{-a/c} - \cos\frac{a}{c} - \sin\frac{a}{c}}{e^{a/c} - \cos\frac{a}{c} + \sin\frac{a}{c}} = \frac{-e^{-a/c} - \sin\frac{a}{c} + \cos\frac{a}{c}}{e^{a/c} - \sin\frac{a}{c} - \cos\frac{a}{c}}.$$

That equality will imply the equation:

$$0 = 2 - e^{a/c}\cos\frac{a}{c} - e^{-a/c}\cos\frac{a}{c} \qquad \text{or} \qquad e^{a/c} = \frac{1 \pm \sin\frac{a}{c}}{\cos\frac{a}{c}}.$$

However, the following equations can be derived from this:

I. $$\frac{a}{c} = \tfrac{1}{2}\pi - \varphi = \log\tan\tfrac{1}{2}\varphi.$$

That will yield $a / c = 0$; i.e., the band will keep its natural position ([1]).

II. $$\frac{a}{c} = \tfrac{1}{2}\pi - \varphi = \log\cot\tfrac{1}{2}\varphi,$$

III. $$\frac{a}{c} = \tfrac{3}{2}\pi + \varphi = \log\cot\tfrac{1}{2}\varphi,$$

IV. $$\frac{a}{c} = \tfrac{5}{2}\pi - \varphi = \log\cot\tfrac{1}{2}\varphi,$$

V. $$\frac{a}{c} = \tfrac{7}{2}\pi + \varphi = \log\cot\tfrac{1}{2}\varphi,$$

VI. $$\frac{a}{c} = \tfrac{9}{2}\pi - \varphi = \log\cot\tfrac{1}{2}\varphi,$$

VII. $$\frac{a}{c} = \tfrac{11}{2}\pi + \varphi = \log\cot\tfrac{1}{2}\varphi,$$

etc.

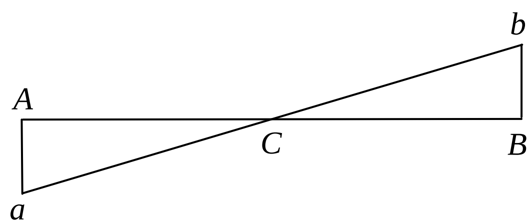

Figure 23.

83. – These equations again yield infinitely-many kinds of oscillations. For the second equation, the band *AB* will cut the axis *AB* just once, for the third one, twice, for the fourth one, three times, for the fifth one, four times, etc. From this, it is clear that the second, fourth, and sixth kinds are not appropriate for the present problem. Namely, since the number of intersection points is odd for those types, on the second case, the band will have a position as is represented in Fig. 23 during its oscillation, and for which, although the sum of the forces that act upon the entire band will vanish, nonetheless, they will cause the band to execute a rotational motion about the midpoint *C*, since the forces that are applied to the two halves *aC* and *bC* would combine to produce that rotation of the band. On that basis, since the rotational motion must be excluded entirely, the form of the band under oscillation must be arranged in such a way that not only must the sum

([1]) The case $a / c = 0$, which will also occur more frequently in what follows, is dealt with thus: Since a is not zero, one must have $c = \infty$; i.e., since $c^4 = E\,k^2\,a\,f / M$, one must have $f = \infty$. The associated isochronous pendulum would be infinitely long, so its period of oscillation would also be infinite. The band would need an infinitely-long time in order to complete its oscillation; i.e., it would remain at rest.

of the forces that are applied to the entire band be zero, but the sum of their moments must also vanish. That will demand that the curve must possess a diameter *cC* at the midpoint *c* (Fig. 22). However, that will appear when the curve cuts the axis *AB* in either two or four, or more generally, an even number of points. Hence, only the third, fifth, seventh, etc., equations will yield suitable solutions ([1]).

84. – That restriction on the solutions is based in the problem itself when we allow only those curves that have the line *Cc* for their diameter; i.e., ones for which the same value of *y* will result when one replaces *x* with $a - x$. If we set $a - x$ in place of *x* in the general equation then we will get:

$$y = A e^{a/c} e^{-x/c} + B e^{-a/c} e^{x/c} + C \sin\frac{a}{c}\cos\frac{x}{c} - C\cos\frac{a}{c}\sin\frac{x}{c} + D\cos\frac{a}{c}\cos\frac{x}{c} + D\sin\frac{a}{c}\sin\frac{x}{c}.$$

That equation must coincide with:

$$y = A e^{x/c} + B e^{-x/c} + C\sin\frac{x}{c} + D\cos\frac{x}{c}.$$

One will then have:

$$A\, e^{a/c} = B, \qquad C\left(1 + \cos\frac{a}{c}\right) = D\sin\frac{a}{c}, \qquad C\sin\frac{a}{c} = D\left(1 - \cos\frac{a}{c}\right).$$

The last two equations amount to the same thing. Since one then has:

$$\frac{A}{B} = e^{-a/c},$$

a comparison of this value with the previous one (no. **82**) will give:

([1]) Only the case that is represented in Fig. 23 of oscillatory motion with one node must be rejected for free elastic bands, but not the other ones. **Dan. Bernoulli** expressed his amazement at **Euler**'s error in a letter on 4 Sept. 1743 (Letter 20 in *Fuß*'s Correspondance math. et physique):

"These motions proceed freely, and I have calculated various properties of them and performed very many beautiful experiments on the position of the nodes and the height of the sound that agreed with the theory beautifully. I have arrived at the conclusion that the few words that you said about that in the *Supplemento* should be deleted."

In the Actis Acad. Petrop. (1779), Part 1, page 103, **Euler** once more carried out his examination of oscillating bands under other viewpoints and allowed the oscillations with an odd number of nodes with no further restrictions. He also treated six types of oscillations in regard to the ends of the band (whether free, fixed to the supports, or embedded in a wall), whereas here, he only treated four. **Lord Rayleigh** gave a thorough presentation of the transverse oscillations of elastic bands in Chap. VIII of his *Theory of Sound* (German edition by **Fr. Neesen**, Braunschweig, 1879), where he also gave, e.g., a figure for the free oscillation with three nodes. See also **Strehlke**, Poggendorfs Annalen, Bd. **27** and **A. Seebeck**, Abhandl. d. Kgl. Sächs Ges. d. Wiss, 1852.

$$e^{-a/c} - \cos\frac{a}{c} - \sin\frac{a}{c} = 1 - e^{-a/c}\cos\frac{a}{c} + e^{-a/c}\sin\frac{a}{c}.$$

or

$$e^{-a/c} = \frac{1+\cos\frac{a}{c}+\sin\frac{a}{c}}{1+\cos\frac{a}{c}-\sin\frac{a}{c}} = \frac{1+\sin\frac{a}{c}}{\cos\frac{a}{c}} = \frac{\cos\frac{a}{c}}{1-\sin\frac{a}{c}}.$$

85. – We will then have:

$$e^{a/c} = \frac{1-\sin\frac{a}{c}}{\cos\frac{a}{c}}.$$

Previously (no. **82**), we found the equation:

$$e^{a/c} = \frac{1\pm\sin\frac{a}{c}}{\cos\frac{a}{c}}.$$

That implies that we now have only half of the cases of solutions to the problems that were derived from this equation above (no. **82**, end), and indeed only the cases that were indexed by odd numbers. Since the first equation represents the rest state of the band, all types of oscillations will be represented by the following equations:

I. $$\frac{a}{c} = \tfrac{3}{2}\pi + \varphi = \log\tan\tfrac{1}{2}\varphi,$$

II. $$\frac{a}{c} = \tfrac{7}{2}\pi + \varphi = \log\tan\tfrac{1}{2}\varphi,$$

III. $$\frac{a}{c} = \tfrac{11}{2}\pi + \varphi = \log\tan\tfrac{1}{2}\varphi,$$

etc.

The first of these equations represents the main type of oscillation, for which the value of the angle φ can be found approximately in a manner that is similar to what was done before. The limits to the angle φ will soon prove to be $1^\circ\,0'\,40''$ and $1^\circ\,1'\,10''$. The true value of φ can be ascertained in that way by the following calculation [which is shortened and analogous to the ones in nos. **74** and **78**]:

$\varphi =$	$1^\circ\,0'\,40'' = 3640''$	$1^\circ\,1'\,0'' = 3660''$
$\varphi =$	0.2466762504 – 2	0.2490559522 – 2
$\varphi =$	0.0176472180	0.0177441807

$\frac{a}{c} = \varphi + \frac{3}{2}\pi =$	4.7300361984	4.7301331611
$\frac{1}{2}\varphi =$	30′ 20″	30′ 30″
log v =	0.3126728553	0.3121694510
log u =	0.6748885339	0.6743851396
u =	4.7302983543	4.7248186037
Deviation:	+ 6336341	+ 53145574 636341
	Difference:	52509233

From this, one will see that the true value of that φ does not lie within those limits, but is somewhat smaller ([1]). Nevertheless, it will be implied by the deviations. Let $\varphi = 1^{\circ}\ 0'\ 40'' - n$, so one will have the proportion:

$$20'' : 52509233 = n'' : 636341.$$

One finds that n = 2423 / 10000, so:

$$\varphi = 1^{\circ}\ 0'\ 39.7556'' = 3639.7556'',$$

or, in arc units, 0.0176460438, so:

$$\frac{a}{c} = \frac{3}{2}\pi + \varphi = 4.7300350232 \qquad \text{[correct value = 4.7300408].}$$

86. – Let the last number be m, so since:

$$c^4 = \frac{\mathrm{E}k^2 af}{M}, \quad \text{it will follow that} \quad a^4 = \frac{m^4\,\mathrm{E}k^2 af}{M} \quad \text{and} \quad f = \frac{a^4}{m^4} \cdot \frac{1}{\mathrm{E}k^2} \cdot \frac{M}{a}.$$

In the same way (as in no. **76**), that will imply the number of oscillations that this band completes in one second:

([1]) The deviation in the table on the left is incorrect. It must read – 2621559, namely, 4.73003… – 4.73029… The angle φ will then lie between $1^{\circ}\ 0'\ 40''$ and $1^{\circ}\ 1'\ 0''$. If one sets $\varphi = 1^{\circ}\ 0'\ 40'' + n$ then the *regula falsi* method will give:

$$\frac{n}{20''} = \frac{2621559}{53145574 + 2621559}.$$

One finds that n = 0.94, so $\varphi = 1^{\circ}\ 0'\ 40.94''$. That value was found by, e.g., **Rayleigh** in *Theorie des Schalles*, v. 1, pp. 298. One will then set a / c equal to 4.7300408. Since the error first shows up in the fifth decimal place, the other numerical values will remain correct up to that point. The correct value will be used in the following sections.

$$\frac{m^2}{a^2}\sqrt{g\,\mathrm{E}\,k^2\,\frac{a}{M}},$$

in which g = 3.16625 in Rhenish feet. Hence, when that band is arranged in such a way that, in one case, it has one end fixed to a wall and in the other case, it is free to produce a tone, the two tones will relate to each other like $n^2 : m^2$; i.e., like:

$$1.8751040813^2 : 4.7300408^2,$$

which is like 1 : 6.36324 or approximately 11 : 70. The interval between those sounds will then be defined by two octaves, a fifth, and the next half-tone. However, when the latter free band is assumed to be twice as long as the former, fixed, band, the interval between the tones will be almost a minor sixth:

$$\left[\text{i.e., } \frac{8}{5} = \frac{72}{45}, \text{ instead of } \frac{70}{44}\right].$$

87. – The equation of the curve can be determined more closely once that value for the fraction a / c has been found. Namely, one has:

$$e^{a/c} = \frac{1-\sin\frac{a}{c}}{\cos\frac{a}{c}},$$

and $A\,e^{a/c} = B$, so:

$$B = \frac{1-\sin\frac{a}{c}}{\cos\frac{a}{c}}A.$$

$$C = A - B = A\left(\cos\frac{a}{c}+\sin\frac{a}{c}-1\right) : \cos\frac{a}{c},$$

$$D = A + B = A\left(\cos\frac{a}{c}-\sin\frac{a}{c}+1\right) : \cos\frac{a}{c}.$$

Now:

$$b = A + B + D = 2D = 2A\left(\cos\frac{a}{c}-\sin\frac{a}{c}+1\right) : \cos\frac{a}{c}.$$

One will then have:

$$A = \frac{b\cos\frac{a}{c}}{2\left(\cos\frac{a}{c}-\sin\frac{a}{c}+1\right)} = \frac{b\left(+1+\sin\frac{a}{c}-\cos\frac{a}{c}\right)}{4\sin\frac{a}{c}},$$

$$B = \frac{b\left(1-\sin\frac{a}{c}\right)}{2\left(\cos\frac{a}{c}-\sin\frac{a}{c}+1\right)} = \frac{b\left(-1+\sin\frac{a}{c}+\cos\frac{a}{c}\right)}{4\sin\frac{a}{c}},$$

$$C = \frac{b\left(-1+\sin\frac{a}{c}+\cos\frac{a}{c}\right)}{2\left(\cos\frac{a}{c}-\sin\frac{a}{c}+1\right)} = \frac{b\left(1-\cos\frac{a}{c}\right)}{2\sin\frac{a}{c}},$$

$$D = \frac{b}{2} = \frac{b\sin\frac{a}{c}}{2\sin\frac{a}{c}}.$$

If one substitutes those values then that will imply the equation:

$$\frac{y}{b} = \frac{e^{x/c}\cos\frac{a}{c}+e^{-x/c}\left(1-\sin\frac{a}{c}\right)}{2\left(1-\sin\frac{a}{c}+\cos\frac{a}{c}\right)} + \frac{\left(1-\cos\frac{a}{c}\right)\sin\frac{x}{c}+\sin\frac{a}{c}\cos\frac{x}{c}}{2\sin\frac{a}{c}}.$$

88. – However, since the line cC is a diameter to the curve, the computed abscissa of the midpoint C will be $CP = z$, so $x = \frac{1}{2}a - z$. One will then have:

$$e^{x/c} = e^{a/2c}\cdot e^{-z/c} = e^{-z/c}\cdot\sqrt{\frac{1-\sin\frac{a}{c}}{\cos\frac{a}{c}}}$$

and

$$e^{-x/c} = e^{z/c}\cdot\sqrt{\frac{\cos\frac{a}{c}}{1-\sin\frac{a}{c}}};$$

one will then have:

$$\frac{A e^{x/c}+B e^{-x/c}}{b} = \frac{\left(e^{z/c}+e^{-z/c}\right)\sqrt{\cos\frac{a}{c}\left(1-\sin\frac{a}{c}\right)}}{2\left(1-\sin\frac{a}{c}+\cos\frac{a}{c}\right)} = \frac{e^{z/c}+e^{-z/c}}{2\left(e^{a/2c}+e^{-a/2c}\right)}.$$

Furthermore:

$$\left(1-\cos\frac{a}{c}\right)\sin\frac{x}{c}+\sin\frac{a}{c}\cos\frac{x}{c}=\sin\frac{x}{c}+\sin\frac{a-x}{c}=\sin\left(\frac{a}{2c}-\frac{z}{c}\right)+\sin\left(\frac{a}{2c}+\frac{z}{c}\right)$$

$$=2\sin\frac{a}{2c}\cos\frac{z}{c}.$$

If one substitutes everything into the equation of the curve then one will get:

$$\frac{2y}{b}=\frac{e^{z/c}+e^{-z/c}}{e^{a/2c}+e^{-a/2c}}+\frac{\cos\frac{z}{c}}{\cos\frac{a}{2c}}.$$

That is the simplest form for the equation of the curve *aM cb*. It is obvious that regardless of whether *z* is taken to be positive or negative, that will yield the same value of *y* ([1]).

$$e^{a/2c}+e^{-a/2c}\quad\text{is also equal to}\quad\frac{2\cos\frac{a}{2c}}{\sqrt{\cos\frac{a}{c}}},$$

([1]) This formula must read:

$$e^{a/2c}+e^{-a/2c}=-\frac{2\cos\frac{a}{2c}}{\sqrt{\cos\frac{a}{c}}}.$$

Since $a / c = 270^\circ\ 0'\ 40.94''$, $\cos(a / 2c) = \cos 135^\circ\ 30'\ 20.47''$ must necessarily be negative, so the right-hand side of the formula will have a positive value, which is as it should be, since $e^{a/2c}+e^{-a/2c}$ is positive. The formula can be derived as follows: One has:

$$e^{a/2c}+e^{-a/2c}=\sqrt{e^{a/c}}+\sqrt{e^{-a/c}}=\frac{\left(1-\sin\frac{a}{c}\right)+\cos\frac{a}{c}}{\sqrt{\cos\frac{a}{c}\left(1-\sin\frac{a}{c}\right)}}.$$

That value was used before in order to calculate $\frac{1}{b}\left(Ae^{x/c}+Be^{-x/c}\right)$. If one introduces half-angles then:

$$e^{a/2c}+e^{-a/2c}=\frac{2\cos^2\frac{a}{2c}-2\cos\frac{a}{2c}\sin\frac{a}{2c}}{\sqrt{\cos\frac{a}{c}}\cdot\sqrt{\sin^2\frac{a}{2c}+\cos^2\frac{a}{2c}-2\sin\frac{a}{2c}\cos\frac{a}{2c}}}=\frac{2\cos\frac{a}{2c}\left(\cos\frac{a}{2c}-\sin\frac{a}{2c}\right)}{\sqrt{\cos\frac{a}{c}}\cdot\left(\sin\frac{a}{2c}-\cos\frac{a}{2c}\right)}=-\frac{2\cos\frac{a}{2c}}{\sqrt{\cos\frac{a}{c}}}.$$

One sets the second square root in the denominator equal to $\sin\frac{a}{2c}-\cos\frac{a}{2c}$, since that value is positive.

The formulas of no. **89** are the ones that correspond to the correct values that were derived here, as opposed to the ones that **Euler**'s published. The last formula is correct in the text, and likewise, the numerical values are also correct. *Cc* / *Aa* proves to be negative, since those ordinates have different directions in Fig. **22**.

and we have found that the angle $a / c = 271^{\circ}\ 0'\ 40.94''$.

89. – If one sets $z = 0$ in the equation thus-obtained then y will assume the value of Cc, namely:

$$\frac{2Cc}{b} = -\frac{2\sqrt{\cos\frac{a}{c}}}{2\cos\frac{a}{2c}} + \frac{1}{\cos\frac{a}{2c}}$$

or

$$\frac{Cc}{Aa} = \frac{1-\sqrt{\cos\frac{a}{c}}}{2\cos\frac{a}{2c}}.$$

Now, $\cos a / c = \sin 1^{\circ}\ 0'\ 40.94''$ and $\cos a / 2c = -\sin 45^{\circ}\ 30'\ 20.5''$. One finds from this that $Cc / Aa = -0.60784$ (Fig. 22).

The points E and F at which the curve intersects the axis will be found when one sets $y = 0$. That will imply:

$$e^{z/c} + e^{-z/c} = -\frac{\cos\frac{z}{c}}{\cos\frac{a}{2c}}\left(e^{a/2c} + e^{-a/2c}\right) = \frac{2\cos\frac{z}{c}}{\sqrt{\cos\frac{a}{c}}}.$$

One finds that $CE / CA = 0.551685$ and $AE / AC = 0.448315$ by approximation. Those points E and F will remain immobile while the band performs its oscillations. Therefore, that oscillatory motion, which can be hard to produce by a direct impact, will still be easy to produce. If the band were held fixed at the points E and F that were just determined, it would continue to oscillate as if it were completely free.

90. – When the second of the equations that were found above – namely, $\frac{a}{c} = \frac{7}{2}\pi + \varphi = \log\cot\frac{1}{2}\varphi$ – is treated in the same way, one will find $\varphi = 0$, approximately, for that case, and the second type, for which the free end can oscillate, will emerge from that, namely, when the axis AB is cut at four points. Hence, the band will further oscillate as if it were fixed at those four points. Thus, when the band is fixed at those four points or only at two of them, it will likewise oscillate as if it were free. However, it will have a much higher pitch. Its oscillation number will relate to that of the sound that the previous type of oscillation produced almost like 7^2 to 3^2. Both of those tones are separated by an interval of two octaves, plus a fourth, and one-half of the next half-tone. For the third type of oscillation, one will have the equation $\frac{a}{c} = \frac{11}{2}\pi + \varphi = \log\cot\frac{1}{2}\varphi$. The curve acb has six points of intersection with the axis AB. It creates a pitch that is one octave and a minor third higher then the previous one ($121 / 49 = 2 \cdot \frac{6}{5}$, approximately). The band will produce that pitch when it is fixed at two of those points. It is clear from this how different sounds can be created by the same band, according to whether it is fixed at two points in different ways. When the two fixed points do not coincide with the intersection points of the first, second, or third kind, the oscillations will continue from any of the following kinds up to infinity. It will then create a sound that is so high that it can no longer be heard, or (what amounts to the same thing) the band will not exhibit any oscillatory motion at all, but it will generate an indeterminate sound of the kind that is produced by an oscillating string that is plucked at a point where the parts do not possess a rational ratio.

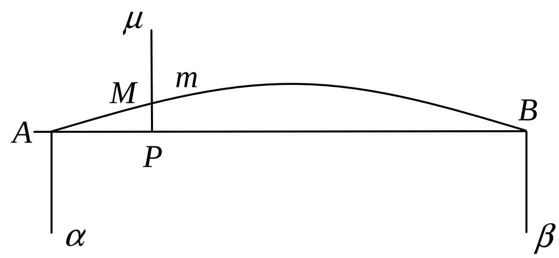

Figure 24.

The oscillations of an elastic band that is fixed at both ends

91. – The elastic band is now fixed at both endpoints *A* and *B* (Fig. 24), but in such a way that the tangents to the curve at those points are indeterminate. In order to realize that case in an experiment, let two very thin knife-edges $A\alpha$, $B\beta$ be rigidly coupled with the band, which embed the band in a wall at the endpoints *A* and *B* and make it immobile there. In order to deduce the oscillatory motion of that elastic band, as before, set: The absolute elasticity = $\mathsf{E}\,k^2$, the length $AB = a$, its weight = M, the length of the simple, isochronous pendulum = f. Let *AMB* be the curvilinear form that the band assumes under oscillation. Furthermore, set $AP = AM = x$, $PM = y$, and set the radius of curvature at M equal to R. One lets P denote the force that the knife-edge $A\alpha$ must support in the direction $A\alpha$. Since the force to which the element *Mm* must be subjected in the direction $M\mu$ in order for the band to keep its position is equal to $M\, y\, dx / a\, f$, one will obtain the following equation for the curve from the rules that were given above (nos. **57**, **66**, **67**):

$$\frac{\mathsf{E}\,k^2}{R} = Px - \frac{M}{a\,f}\int dx \int y\,dx\,.$$

Since the curve is concave to the axis, R will be $-\,1/d^2y/dx^2$ here; one will then have:

$$\mathsf{E}\,k^2\,\frac{d^2y}{dx^2} = \frac{M}{a\,f}\int dx \int y\,dx - Px\,.$$

For $x = 0$, that will imply $R = \infty$; i.e., one will also have $d^2y = 0$.

92. – If this equation is differentiated twice then the same equation will follow that we found before in the previous cases:

$$\mathrm{E}\, k^2\, d^4 y = \frac{M}{a\, f}\; y\, dx^4 .$$

If one then sets $\mathrm{E}\, k^2\, a\, f / M = c^4$ then one will get the integral of the equation:

$$y = A e^{x/c} + B e^{-x/c} + C \sin\frac{x}{c} + D \cos\frac{x}{c} .$$

For the further determination, one sets $x = 0$, so one will also have $y = 0$, and $0 = A + B + D$. Secondly, one sets $x = a$, so y will again be 0, and one then has:

$$0 = A e^{a/c} + B e^{-a/c} + C \sin\frac{a}{c} + D \cos\frac{a}{c} .$$

Thirdly, $d^2 y / dx^2$ must vanish for $x = 0$ and $x = a$. One will then have:

$$A + B - D = 0 \qquad \text{and} \qquad A e^{a/c} + B e^{-a/c} - C \sin\frac{a}{c} - D \cos\frac{a}{c} = 0.$$

The equations $A + B - D = 0$ and $A + B + D = 0$ imply:

$$D = 0 \quad \text{and} \quad B = -A.$$

If one substitutes these values in the other two equations then one will get:

$$0 = A\left(e^{a/c} - e^{-a/c}\right) + C \sin\frac{a}{c}$$

and

$$0 = A\left(e^{a/c} - e^{-a/c}\right) - C \sin\frac{a}{c} .$$

Those equations can be satisfied only when $A = 0$, so $e^{a/c} - e^{-a/c}$ can vanish only for the case of $a / c = 0$ (see the remark on pp. 62). However, one must then have $C \sin (a / c) = 0$. One cannot set $C = 0$ in that, since otherwise no oscillatory motion would exist anymore, since all constants would then be zero. One must then have $\sin (a / c) = 0$, so one will either have $a / c = \pi$ or $a / c = 2\pi$, etc. That will once more yield infinitely-many different kinds of oscillations, according to whether the curve *AMB* cuts the axis nowhere except for the endpoints *A* and *B* or at one or two or more points. That follows from the equation $y = C \sin (x / c)$. However, no matter how many points of intersection arise, they will have equal distances between them.

93. – For the first and most important kind of oscillation, $a / c = \pi$, so:

$$a^4 = \pi^4 c^4 = \pi^4 \cdot \mathsf{E}\, k^2 \cdot \frac{a}{M} \cdot f\,.$$

Hence:

$$f = \frac{a^4}{\pi^4} \cdot \frac{1}{\mathsf{E}\, k^2} \cdot \frac{M}{a}\,.$$

The pitches will then (no. **76**), in turn, be inversely proportional to the square of the length of the band. The pitch that this band will produce for $a / c = \pi$ will relate to the pitch of the same band when the end B is fixed as π^2 relates to the square of 1.8751040813; i.e., as 2.807041 : 1 or, in smallest numbers, like 160 to 57. The interval between them is one octave plus almost a third half-tone. If the oscillations occur as the second kind, for which $a / c = 2\pi$, then the pitch will be two octaves higher, and when $a / c = 3\pi$, it will be three octaves plus the next whole tone higher than in the case $a / c = \pi$, etc. ([1]). In order to be able to easily test this experimentally, it should be remarked that the oscillation must be made to be as small as possible, such that no essential elongation of the band will arise. Therefore, in order for the rigidity of the band that opposes a very small extension of the band, but without any oscillations taking place, to not be harmful here, the points must be arranged such that such a small extension is possible. That will happen when they lie on an entirely smooth plane. It will be such that the elastic band AB that is equipped with the points $A\alpha$ and $B\beta$ at A and B, resp., will produce a sound that corresponds to the calculation when the points are placed on a mirror.

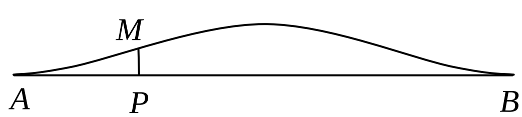

Figure 25.

The oscillations of an elastic band that is fixed to a wall at both ends

94. – Now that we have dealt with the previous cases, our treatise on elastic bands might conclude with the oscillatory motion that an elastic band exhibits when both of its ends A and B are attached to a wall (Fig. 25) such that not only the points A and B will remain immobile under the oscillation, but the line AB will continually contact the curve AMB at A and B. However, one must once more be careful that the bolts that fix the endpoints A and B are not rigid, but admit a small extension, which the curvature would necessitate. Therefore, one will arrive at the sort of forces one that one would also need

([1]) On the tonal intervals that occur in this section, let it be remarked: In the C-major scale, the first interval 160 / 57 is in the interval from the root tone C to F-sharp of the next-higher octave (2.78, instead of 2.81). The second interval 4 : 1 is the one from the root tone C to the C that is two octaves higher, the third interval 9 : 1 reaches from the root tone C to the D tone that lies three octaves higher. On the realization of those oscillations in practice, which **Euler** thought would be complicated, see **Chladni**, *Akustik*, pp. 99.

in order to fix the band at the endpoints A and B by using the following fourth-order differential equation:

$$\mathsf{E}\, k^2\, d^4 y = \frac{M}{a\,f}\, y\, dx^4\,.$$

Let:

$$\frac{\mathsf{E}\, k^2 af}{M} = c^4.$$

As above, one will get the integral:

$$y = Ae^{x/c} + Be^{-x/c} + C\sin\frac{x}{c} + D\cos\frac{x}{c}\,.$$

95. – The constants A, B, and D must be determined in such a way that not only y, but also dy, will vanish for $x = 0$, since the curve must contact the axis AB at A. The same thing must take place when one sets $x = a$. That will yield the following four equations:

I. $0 = A + B + D,$

II. $0 = A - B + C,$

III. $0 = Ae^{a/c} + Be^{-a/c} + C\sin\frac{a}{c} + D\cos\frac{a}{c}\,,$

IV. $0 = Ae^{a/c} - Be^{-a/c} + C\cos\frac{a}{c} - D\sin\frac{a}{c}\,.$

It follows from the first and second one that:

$$C = -A + B, \quad D = -A - B.$$

One substitutes these values in the remaining two equations. It will follow that:

$$0 = Ae^{a/c} + Be^{-a/c} - (A-B)\sin\frac{a}{c} - (A+B)\cos\frac{a}{c}\,,$$

$$0 = Ae^{a/c} - Be^{-a/c} - (A-B)\cos\frac{a}{c} + (A+B)\sin\frac{a}{c}\,.$$

One takes the sum of both equations and gets:

$$\frac{A}{B} = \frac{\sin\frac{a}{c}}{\cos\frac{a}{c} - e^{a/c}}\,.$$

It follows from the difference between the two equations that:

$$\frac{A}{B} = \frac{e^{-a/c} - \cos\frac{a}{c}}{\sin\frac{a}{c}},$$

so:

$$2 = \left(e^{a/c} + e^{-a/c}\right)\cos\frac{a}{c} \qquad \text{or} \qquad e^{a/c} = \frac{1 \pm \sin\frac{a}{c}}{\cos\frac{a}{c}}.$$

That equation agrees with the one that was found in no. **82**. The following infinitude of solutions will satisfy it:

I. $\frac{a}{c} = \frac{1}{2}\pi - \varphi = \log\cot\frac{1}{2}\varphi$, II. $\frac{a}{c} = \frac{3}{2}\pi + \varphi = \log\cot\frac{1}{2}\varphi$,

III. $\frac{a}{c} = \frac{5}{2}\pi - \varphi = \log\cot\frac{1}{2}\varphi$, IV. $\frac{a}{c} = \frac{7}{2}\pi + \varphi = \log\cot\frac{1}{2}\varphi$.

96. – The first of these can be satisfied only when $\varphi = 90^{\circ}$, so one will then have a / c (see the remark on pp. 62). The first type of oscillation is derived from the equation $\frac{a}{c} = \frac{3}{2}\pi + \varphi = \log\cot\frac{1}{2}\varphi$. That was dealt with before (no. **85**) and will give $a / c = 4.7300408$. Therefore, the elastic band that is embedded in a fixed wall at both ends will carry out its oscillations just as if it were entirely free. However, that agreement relates to only the first kind of oscillation ([1]). Namely, the second kind, for which $\frac{a}{c} = \frac{5}{2}\pi - \varphi = \log\cot\frac{1}{2}\varphi$, and for which the band cuts the axis at one point during oscillation, does have its equivalent for the free band. The third type of oscillation of the band that is fixed at both ends coincides with the second type for the free band, and so forth.

97. – The last two genres of oscillations (nos. **91** and **94**) cannot be rigorously tested by experiments, on the cited grounds. However, the first genre (no. **65**) is not only very suited to the demands of experiments, but can also be converted in such a way that the absolute elasticity of any band, which we have denoted by $\mathsf{E}\,k^2$, can be ascertained. Namely, when the sound that a band that is fixed at one end in a wall creates is heard, and one produces the same sound with a string, the number of oscillations in one second will

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([1]) That agreement will be found for all types of oscillation. **Euler**'s misleading statement is based upon the fact that he rejected oscillations with an odd number of nodes for the free band; see the remark on pp. 63.

be known. If that were set equal to the expression $\frac{n^2}{a^2}\sqrt{g \cdot \mathsf{E}\, k^2 \cdot \frac{a}{M}}$ then one would find the value of the expression $\mathsf{E}\, k^2$, since the number n would be known and the quantities g, a, and M could be obtained by measurement. One would then know the absolute elasticity. It can then be compared with the one that is found from the curvature (no. **35**) ([1]).

([1]) These experiments, which are very important to engineering, were carried out quite extensively. Admittedly, the formulas that were given here did not lead to any useful results, since they did not bring the cross-section of the elastic band under consideration; see, e.g., **Kupffer**, "Recherches expérimentales sur l'élasticité des métaux," St. Petersburg, 1860.