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Introduction to the relativistic mechanics of continua with scalar structure

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Introduction

Apart from the fact that it realizes the natural context for the description of mechanical and electromagnetic phenomena, even in the purely-mechanical domain, special relativity constitutes a significant step in the process of conceptual unification that represents the highest aspiration of scientific thought. In the dynamics of particles with scalar structure, that process begins with the double unification of *mass* with (kinetic) *energy* and *mechanical* with *thermal action* (cf., e.g., [**1**], Chap. IX.2). That can be extended to continuum mechanics in a natural way that will emerge from this brief introduction, which starts from the classical situation and proceeds along a route that makes the adjustments that are indispensible for the relativistic theory. What will result is a profoundly modified picture, not only in the relativistic corrections to the traditional physical ingredients (mechanical stress, power of the internal force, internal energy, etc.), but also for the difference in its structural invariance. Nonetheless, the classical approach is more illuminating, especially in regard to the First Law of Thermodynamics (no. **6**), which will essentially constitute the correction of order c^{-2} (c is the speed of light *in vacuo*) to the principle of the conservation of mass (cf., Pham Mau Quan [**2**], no. 22).

For the sake of brevity, we shall examine, from the general viewpoint, only the case of *non-polar continua*, which are ordinarily characterized by the condition that the proper mechanical stress tensor should be symmetric. (The elimination of that hypothesis cannot be separated from the introduction of stress-moments and mass-moments; cf., G. Grioli [**3**], and from the geometrico-kinematical viewpoint, by the extension of the scalar structure by means of directors: cf., [**4**], Chap. III.) In that case, the thermal flux is not independent of the velocity, and is combined into a "thermal inertia" that naturally accompanies the "inertia of pure matter," which is what one has for fluids (cf., A. Lichnerowicz [**5**], Chap. IV, and Pham Mau Quan [**2**]).

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1. General equations of classical continuum mechanics

Let us briefly give an outline of the *general equations* of classical continuum mechanics in the context of Galilean reference frames ([1]). Let: S be the Galilean frame, which is referred to arbitrarily fixed internal coordinates of Cartesian type. Briefly, that means that $T \equiv \{0, \boldsymbol{c}_1, \boldsymbol{c}_2, \boldsymbol{c}_3\}$ is the *reference* triad. Let C be the *present configuration* (in S) of the continuous system, let x^i (i = 1, 2, 3) be the coordinates of the generic point $P \in C$, let $\boldsymbol{n}\, d\sigma$ be an arbitrary oriented surface element at P, and let $\boldsymbol{\phi}_n\, d\sigma$ be the *relative mechanical stress.*

Above all, the principle of conservation of mass (in S), when translated into the *Eulerian picture*, gives rise to the *continuity equation:*

$$\partial_t \mu + \partial_i (\mu v^i) = 0, \tag{1}$$

in which μ (x, t) is the *mass density* and $\boldsymbol{v}$ (x, t) is the *velocity*. In addition, the main equation of mechanics, when adapted to the generic part of the domain C, will ultimately give rise to three local conditions by a limited path. They are:

I) *Cauchy's theorem*, which specifies the dependency of the stress on $\boldsymbol{n}$:

$$\boldsymbol{\phi}_n = n_i\, \boldsymbol{\phi}^i \qquad \Rightarrow \qquad \boldsymbol{\phi}^i = (\boldsymbol{\phi}_n)_{n=\boldsymbol{c}_i}\,. \tag{2}$$

II) The *first indefinite equation:*

$$\mu \dot{\boldsymbol{v}} = \mu \boldsymbol{F} - \partial_i\, \boldsymbol{\phi}^i, \tag{3}$$

in which $\boldsymbol{F}$ is the *specific volume force*, and $()^{\bullet}$ is the *substantial derivative:*

$$()^{\bullet} \equiv \partial_t\, () + \partial_i\, ()\, v^i\,. \tag{4}$$

III) The *second indefinite equation*, or the reciprocity relation for the stresses:

$$\boldsymbol{\phi}_n \cdot \boldsymbol{n}' = \boldsymbol{\phi}_{n'} \cdot \boldsymbol{n}, \qquad \forall\, \boldsymbol{n}, \boldsymbol{n}'. \tag{5}$$

Finally, the energy theorem essentially translates (cf., [**6**], Chap. II.3) into the:

IV) *First Law of Thermodynamics:*

$$\dot{\varepsilon} = q - \frac{1}{\mu} w^{(i)}, \tag{6}$$

([1]) The passage to an arbitrary basis for reference is realized by adding the specific force of convection and that of Coriolis to the force of inertia in the equation of motion.

in which ε is the *specific internal energy* (i.e., per unit mass), q is the *specific thermal power* ([2]), and $w^{(i)}$ is the *specific power due to the contact force:*

(7) $$w^{(i)} = X^{ik}\,k_{ik}\,,$$

in which k_{ik} is the *rate of deformation* tensor:

(8) $$k_{ik} \equiv \tfrac{1}{2}(\partial_i\, v_k + \partial_i\, v_k)\,.$$

In the context of finite deformations, the Lagrangian viewpoint takes priority over the Eulerian one, and at its forefront, both classically and relativistically, there is an expression for the power dissipated or produced by the internal stresses, which is useful for the purpose of specifying the internal variables; however, we shall not address that here. We shall confine ourselves to the observation that in the classical domain:

a) There are *various dynamical ingredients that are, in fact, separate*: for instance, mass and internal energy, along with external volume forces and thermal action (through the power term q).

b) *The equations are invariant in both form and substance* with respect to the choice of Galilean reference frame; i.e., they obey the Galilean principle of relativity in its strong sense.

How does one modify this picture in the relativistic situation?

2. Relativistic extension.

Consider the viewpoint of special relativity ([3]), and take Minkowski space-time M_4 with the signature $- + + +$. Suppose that M_4 is oriented and also endowed with one of the two light semi-cones, which shall be called $\mathcal{C}^+$; i.e., it is endowed with only *Cartesian bases* $\{\boldsymbol{c}_\alpha\}$ ($\alpha = 0, 1, 2, 3$) *that obey* :

(9) $$\boldsymbol{c}_\alpha \cdot \boldsymbol{c}_\beta = m_{\alpha\beta} \quad (m_{00} = -1,\ m_{0i} = 0,\ m_{ik} = \delta_{ik}),$$

and have temporal axes that belong to $\mathcal{C}^+$.

In intrinsic terms, that is equivalent to considering only the class in M_4 of ∞^3 *Galilean reference frames* $\{S\}$ *that are equi-oriented in both time and space.*

In what follows, one can also take $\boldsymbol{c}_0 = \boldsymbol{\gamma}$ ($\boldsymbol{\gamma} \cdot \boldsymbol{\gamma} = -1$) in order to underscore the fact that $\boldsymbol{c}_0$ *plays a different role from the spatial vectors* $\boldsymbol{c}_i$ with respect to the Galilean

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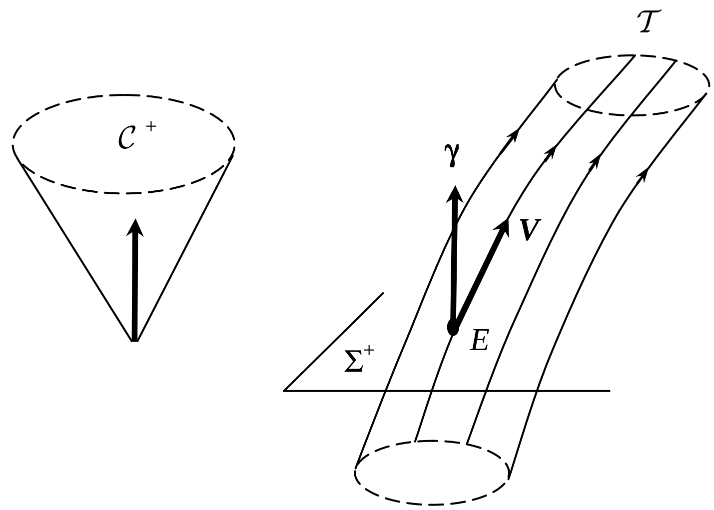
([2]) It is generally comprised of two terms, one of which relates to *radiation* and the other of which relates to *conduction:* $q = r - 1 / \mu \operatorname{div} \boldsymbol{q}$, in which $\boldsymbol{q}$ is the *thermal conduction vector* (cf., e.g., [**7**], pp. 295).

([3]) The passage to general relativity (i.e., to a curved space-time V_4) is automatic by way of the rule of transcription.

reference frame S that is associated with the Cartesian basis $\{\boldsymbol{c}_\alpha\}$. That is to say that S characterized by the unit vector $\boldsymbol{c}_0 \in \mathcal{C}^+$. By contrast, the unit vectors $\boldsymbol{c}_i$ are defined in the oriented 3-plane Σ^+ that is normal to $\boldsymbol{\gamma}$, up to an arbitrary spatial rotation.

In the relativistic situation, one also needs to distinguish the *absolute formulation* (which is invariant under space-time translations and rotations: viz., Lorentz transformations) *from the one that is relative to an arbitrary Galilean reference frame* (which is invariant under only time translations, and spatial translations and rotations). The latter, which is more significant from the physical standpoint, is *invariant in form* with respect to the choice of Galilean reference from (viz., the *principle of relativity*), *but not in substance*, in the sense that one must appeal to *relative magnitudes*, for which one must specify the way that they vary under a change of Galilean reference frame.



Let us start by considering the *absolute viewpoint:* The continuum is represented by a congruence of ∞^3 *time-like, oriented lines* that fill up a world-tube $\mathcal{T} \subseteq M_4$ [one and only one line of the congruence passes through each point $E \in \mathcal{T}$ ([4])] Let $\boldsymbol{V}(E)$ be the local 4-velocity of the continuum, and let $\boldsymbol{A}(E)$ be the vector that is its derivative with respect to proper time for the particles (viz., the 4-acceleration). Along with the 4-vectors $\boldsymbol{V}$ and $\boldsymbol{A}$ (one of which is time-like and normalized to $-c^2$, while the other one is space-like), two *fundamental scalars are defined in $\mathcal{T}$ that are both positive*: namely, $\mathcal{D}_0(E)$ and $\mu_0(E)$, which have the meanings of *proper number density* of the particles and the *proper density of proper mass.* From the relative viewpoint, those scalars come from an arbitrary Galilean reference frame $\boldsymbol{\gamma}$ ([5]). Indeed, along with the relationship:

(10) $$\boldsymbol{V} = \eta\,(\boldsymbol{v} + c\,\boldsymbol{\gamma}) = \text{inv.}, \qquad \eta \equiv 1/\sqrt{1 - v^2/c^2}\,,$$

([4]) The regularity that is assumed for the motion excludes any type of jump or laceration.
([5]) The passage to a more general reference frame for the fluid presents no difficulties (cf., [**8**])

that will give rise to the *local invariance relations* (with respect to the choice of reference frame, and in which $\mathcal{D}$ is the Jacobian of the x^i with respect to the Lagrangian coordinates):

(11) $$\eta\,\mathcal{D} = \text{inv.} = \mathcal{D}_0\,; \qquad \mu / \eta^2 = \text{inv.} = \mu_0\,,$$

which specify the significance of the scalars $\mathcal{D}_0$ and μ_0 in relation to the proper reference frame.

As far as the *relative stresses* $\boldsymbol{\phi}_n$ ($\boldsymbol{n} \in S$) *are concerned,* they will be replaced with the *4-stresses* $\boldsymbol{T_N}$, with $\boldsymbol{N} \in M_4$, given that the latter must have meaning in any Galilean reference system, and therefore in any spatial section Σ for $E \in \mathcal{T}$. Naturally, one assumes that the $\boldsymbol{T_N}$ satisfy the following properties, which are analogous to the classical ones (2) and (3):

(12) $$\boldsymbol{T_N} = N_\alpha \boldsymbol{T}^{\alpha}, \quad \boldsymbol{T_N} \cdot \boldsymbol{N}' = \boldsymbol{T_{N'}} \cdot \boldsymbol{N}\ , \qquad \forall\ \boldsymbol{N}, \boldsymbol{N}'.$$

We shall now address the *problem of how to relativistically extend* the Cauchy equation, which we must naturally postulate. The most natural extension is further suggested by the classical equations (1) – (3) if one interprets them as the temporal and spatial components, respectively, of the same 4-vector equation. More precisely, by virtue of (1) and (4), one will have:

$$\mu\,\dot{\boldsymbol{v}} \equiv (\mu\,\boldsymbol{v})^{\boldsymbol{\cdot}} - \dot{\mu}\,\boldsymbol{v} = \partial_t\,(\mu\,\boldsymbol{v}) + \partial_i\,(\mu\,\boldsymbol{v})\,v^i + \mu\,\boldsymbol{v}\,\partial_i\,v^i = \partial_t\,(\mu\,\boldsymbol{v}) + \partial_i\,(\mu\,v^i\,\boldsymbol{v}),$$

and the system (1) – (3) can be written in the form:

$$\partial_t\,\mu + \partial_i\,(\mu v^i) = 0, \qquad \partial_t\,(\mu\,\boldsymbol{v}) + \partial_i\,(\mu\ v^i\,\boldsymbol{v}) + \partial_i\,\boldsymbol{\phi}^{\,i} = \mu\,\boldsymbol{F}\ .$$

Hence, if one takes $(10)_1$ and $(11)_2$ into account then that will be equivalent (at least, in $\Sigma \oplus \Theta$) to the equation:

$$\partial_\alpha(\mu_0\,V^\alpha \boldsymbol{V}) + \partial_i\,\boldsymbol{\phi}^{\,i} = \mu_0\,\boldsymbol{f} \qquad\qquad (\boldsymbol{f} \equiv \eta^{\,2}\,\boldsymbol{F})\ ;$$

and therefore the most natural *relativistic extension* is:

(13) $$\partial_\alpha(\mu_0\,V^\alpha \boldsymbol{V} + \boldsymbol{T}^\alpha) = \mu_0\,\boldsymbol{f}\ .$$

3. Proper mechanical stresses and proper thermal energy (of conduction).

One can give a more expressive form to (13) by splitting each of the 4-stresses $\boldsymbol{T}^{\,\alpha}$ for all $\alpha = 0, 1, 2, 3$ into two parts that are *parallel* and *normal* to the 4-velocity $\boldsymbol{V}$ (*proper mechanical stresses* and *thermal conduction stresses*, resp.):

(14) $$\boldsymbol{T}^\alpha = \boldsymbol{\varphi}^\alpha + Q^\alpha \boldsymbol{V},$$

with

(15) $$\boldsymbol{\varphi}^\alpha \cdot \boldsymbol{V} = 0 .$$

Naturally, when one is dealing with an ordinary (i.e., non-polar) continuum, *one will assume that the proper mechanical stresses* $\boldsymbol{\varphi}_N$ *will also satisfy the axioms* (12):

(16) $$\boldsymbol{\varphi}_N = N_\alpha \boldsymbol{\varphi}^\alpha, \quad \boldsymbol{\varphi}_N \cdot \boldsymbol{N}' = \boldsymbol{\varphi}_{N'} \cdot \boldsymbol{N}, \qquad \forall\, \boldsymbol{N}, \boldsymbol{N}';$$

(15) will then become:

(17) $$\boldsymbol{\varphi}_V \equiv V_\alpha \boldsymbol{\varphi}^\alpha = 0 .$$

Having said that, if one scalar multiplies (14) by $\boldsymbol{V}$ and takes (15) into account then one will have, in turn:

$$- c^2 Q^\alpha = \boldsymbol{T}^\alpha \cdot \boldsymbol{V} = \boldsymbol{T}_V \cdot \boldsymbol{c}^\alpha = \boldsymbol{\varphi}_V \cdot \boldsymbol{c}^\alpha + V_\beta Q^\beta V^\alpha ;$$

i.e., as in (17):

(18) $$\boxed{Q^\alpha = \frac{1}{c^2} \varepsilon_{0,c} V^\alpha,}$$

in which $\varepsilon_{0,c}$ is the *proper thermal energy* (of conduction):

(19) $$\boxed{\varepsilon_{0,c} = -\boldsymbol{Q} \cdot \boldsymbol{V}.}$$

Naturally, one also introduces the *proper density of thermal conduction:*

(20) $$\mu_{0,c} \equiv \varepsilon_{0,c} / c^2 ;$$

(18) will then become, more simply:

(18′) $$Q^\alpha = \mu_{0,c} V^\alpha .$$

By definition, with the intervention of the proper mechanical stresses for the ones in (14), the relativistic Cauchy equation (13) will assume the following form:

(21) $$\boxed{\partial_\alpha (\tilde{\mu}_0 V^\alpha \boldsymbol{V} + \boldsymbol{\varphi}^\alpha) = \mu_0 \boldsymbol{f},}$$

in which $\tilde{\mu}_0$ is defined by:

(22) $$\tilde{\mu}_0 \equiv \mu_0 + \mu_{0,c} ,$$

and is the *total proper density*, which is the sum of the density of pure matter and thermal conduction.

One sees explicitly that *the property of the thermal conduction 4-vector* $\boldsymbol{Q}$ *that it is parallel to the 4-velocity* $\boldsymbol{V}$ *is a direct consequence of the hypothesis that the proper mechanical stresses are symmetric* that was referred to in (16). One is dealing with *a characteristic property of ordinary* (i.e., non-polar) *continua*. (18) will not persist for the continua that are characterized by *asymmetric* stresses (i.e., the ones that lead to an enlargement of the model from both the geometric and the dynamical viewpoints), and from the energetic viewpoint, the situation is presumably analogous to the one in the electromagnetic context.

In any case, *in Eulerian terms,* which introduce the components V^α of the 4-velocity, as well as the *stress characteristics* $X^{\alpha\beta}$, which have the decomposition:

(23) $$\boldsymbol{\varphi}^\alpha = X^{\alpha\beta}\,\boldsymbol{c}_\beta\,,$$

(21) will assume the typical aspect of *equations of conservation with sources;* indeed, they will have the type:

(21′) $$\partial_\alpha T^{\alpha\beta} = \mu_0 f^\beta \qquad (\beta = 0, 1, 2, 3),$$

in which:

(24) $$T^{\alpha\beta} \equiv \tilde{\mu}_0 V^\alpha V^\beta + X^{\alpha\beta},$$

as well as:

(25) $$V^\alpha V_\alpha = -c^2, \quad X^{\alpha\beta} = X^{\beta\alpha}, \qquad X^{\alpha\beta} V_\beta = 0\,.$$

Naturally, the *symmetric tensor field* $T^{\alpha\beta}$, with basis $\mathcal{T}$, *embodies the material schema* of the continuum, and is subject to only the condition that it must admit a time-like eigenvector ($\boldsymbol{V}$) and a corresponding negative eigenvalue ($\tilde{\mu}_0 > 0$). That says that knowing $T^{\alpha\beta}$ is equivalent to knowing $\tilde{\mu}_0$, V^α, and $X^{\alpha\beta}$, when the last two are consistent with (25).

4. Relative formulation.

The vector equation (21), or its scalar equivalent (21′), is completely satisfactory from the absolute viewpoint, given its invariant character with respect to the choice of Cartesian coordinates x^α ($\alpha = 0, 1, 2, 3$).

Nonetheless, it physical content in relation to an arbitrary Galilean reference frame S is not immediately evident. In order to specify that fundamental aspect of the equation, one must:

i) Fix the temporal direction $\boldsymbol{\gamma}$ that characterizes the reference frame and locally decompose all of the tensorial magnitudes that are in play along $\boldsymbol{\gamma}$ and the space Σ that is normal to it (viz., the *natural decomposition*).

ii) Correctly define ([6]) the relative magnitudes.

iii) Obtain the law of variance of all relative magnitudes under an arbitrary change of reference frame.

As far as mechanical stresses $\boldsymbol{\varphi}^{\alpha}$ are concerned ($\boldsymbol{\varphi}_V \equiv \boldsymbol{V}_{\alpha}\, \boldsymbol{\varphi}^{\alpha} = 0$), one will have, above all:

(26) $$\boxed{\boldsymbol{\varphi}^0 = \frac{1}{c} v_i\, \boldsymbol{\varphi}^i}$$

if one takes the decomposition (10) into account; that amounts to expressing $\boldsymbol{\varphi}^0$ in terms of the three vectors $\boldsymbol{\varphi}^i$. On the other hand, the natural decomposition of $\boldsymbol{\varphi}^i$: $\boldsymbol{\varphi}^i = \boldsymbol{\phi}^i - \boldsymbol{\varphi}^i \cdot \boldsymbol{\gamma\gamma}$, with:

(27) $$\boldsymbol{\phi}^i \equiv X^{ik}\, \boldsymbol{c}_k \in \Sigma \qquad (i = 1, 2, 3),$$

in which $\boldsymbol{\varphi}^i \cdot \boldsymbol{V} = 0 \Leftrightarrow \boldsymbol{\varphi}^i \cdot \boldsymbol{v} + c\, \boldsymbol{\varphi}^i \cdot \boldsymbol{\gamma} = 0$, is no different from:

(28) $$\boxed{\boldsymbol{\varphi}^i = \boldsymbol{\phi}^i + \frac{1}{c} \boldsymbol{\phi}^i \cdot \boldsymbol{v}\boldsymbol{\gamma}.}$$

Meanwhile: *The mechanical stresses* $\boldsymbol{\varphi}^{\alpha}$ *are well-defined functions of the relative stresses* $\boldsymbol{\phi}^i$ (in addition to $\boldsymbol{v}$ and $\boldsymbol{\gamma}$).

If one now projects (21) onto Σ and $\boldsymbol{\gamma}$ then one can proceed with the decomposition of $\partial_{\alpha}(\tilde{\mu}_0 V^{\alpha} \boldsymbol{V})$. If one takes (10) into account and sets:

(29) $$\tilde{\mu} = \tilde{\mu}_0\, \eta^2$$

then one will have, in turn:

$$\partial_{\alpha}(\tilde{\mu}_0 V^{\alpha} \boldsymbol{V}) = \partial_i[\tilde{\mu} v^i (\boldsymbol{v} + c\boldsymbol{\gamma})] + \frac{1}{c} \partial_t[\tilde{\mu}\, c\, (\boldsymbol{v} + c\boldsymbol{\gamma})] =$$

$$= \partial_i(\tilde{\mu} v^i \boldsymbol{v}) + \partial_t(\tilde{\mu} \boldsymbol{v}) + [\partial_i(\tilde{\mu} v^i) + \partial_t \tilde{\mu}]\, c\, \boldsymbol{\gamma}.$$

If one specifies the spatial derivatives and uses (4), as well as the kinematical identity (cf., e.g., [**1**], pp. 514):

([6]) Mathematically, as well as physically.

(30) $$\partial_i v^i \equiv \operatorname{div} \boldsymbol{v} = \frac{1}{\mathcal{D}}\dot{\mathcal{D}},$$

then the previous equation will imply the decomposition:

$$\partial_\alpha (\tilde{\mu}_0 V^\alpha \boldsymbol{V}) = \frac{1}{\mathcal{D}}[(\tilde{\mu}\dot{\mathcal{D}}\boldsymbol{v})^{\cdot} + (\tilde{\mu}\mathcal{D})^{\cdot} c\,\boldsymbol{\gamma}]\ .$$

Analogously, one has, as in (28) and (26):

$$\partial_\alpha \boldsymbol{\varphi}^\alpha = \partial_i \boldsymbol{\phi}^i + \frac{1}{c}\partial_i (\boldsymbol{\phi}^i \cdot \boldsymbol{v})\,\boldsymbol{\gamma} + \frac{1}{c^2}\partial_t (v_i \boldsymbol{\phi}^i) + \frac{1}{c^2}\partial_t (v_i \boldsymbol{\phi}^i \cdot \boldsymbol{v})\,\boldsymbol{\gamma}\,,$$

because if one introduces the *relative volume force:*

(31) $$\mu \boldsymbol{F} \equiv \mu_0 \boldsymbol{f}_\Sigma = \mu_0 (\boldsymbol{f} + \boldsymbol{f}\cdot\boldsymbol{\gamma}\,\boldsymbol{\gamma})$$

then (21) will give rise to the *quantity of motion theorem:*

(32) $$\boxed{\frac{1}{\mathcal{D}}(\tilde{\mu}\mathcal{D}\boldsymbol{v})^{\cdot} = \mu \boldsymbol{F}_{\text{tot}} \equiv \mu \boldsymbol{F} - \partial_i \boldsymbol{\phi}^i - \frac{1}{c^2}\partial_i (v_i \boldsymbol{\phi}^i)}$$

and the energy theorem:

$$\frac{1}{\mathcal{D}}(\tilde{\mu}\mathcal{D}c^2)^{\cdot} = -\,\mu_0\, c\, \boldsymbol{f}\cdot\boldsymbol{\gamma} - \partial_i (\boldsymbol{\phi}^i \cdot \boldsymbol{v}) - \frac{1}{c^2}\partial_i (v_i \boldsymbol{\phi}^i \cdot \boldsymbol{v})\,,$$

respectively.

Unlike (32), this latter equation cannot be presented in a more expressive form; nonetheless, it can be easily transformed by taking (10) into account: $-\,c\,\boldsymbol{\gamma} = \boldsymbol{v} - \boldsymbol{V}/\eta$, and developing the spatial derivatives. More precisely, if one introduces the following *specific relative magnitudes:*

(33) $$\begin{cases} \mu\varepsilon \equiv \tilde{\mu} c^2 & (\textit{internal energy}) \\ \mu q \equiv -\mu_0 \boldsymbol{f}\cdot \boldsymbol{V}/\eta & (\textit{thermal power}) \\ w^{(i)} \equiv \boldsymbol{\phi}^i \cdot (\partial_i \boldsymbol{v} + \dfrac{1}{c^2} v_i\, \partial_i \boldsymbol{v}) & (\textit{power due to the internal stresses}) \end{cases}$$

then one will get the *energy theorem* in the typical form:

(34) $$\frac{1}{\mathcal{D}}(\tilde{\mu}\mathcal{D}c^2)^{\cdot} = \boldsymbol{F}_{\text{tot}}\cdot\boldsymbol{v} + q - \frac{1}{\mu} w^{(i)}\,.$$

5. Laws of variance of the fundamental relative magnitudes.

The general equations (32) *and* (34) *satisfy the relativity principle*, insofar as they are invariant in form with respect to the choice of Galilean reference system S ; nonetheless, they are not invariant in substance (and cannot be otherwise in the relativistic context), given the relative character of the quantities in play.

In any case, *all of the relative quantities that were introduced* (in particular, ε, q, and $w^{(i)}$) *have a real physical content*; i.e., they do not vanish (like the forces that appear in relative motion), in the sense that they cannot disappear with a simple change of reference frame. More precisely, if one affixes the index 0 to the proper magnitudes then the following *local invariance properties* (both in form and substance) will be true:

$$\boxed{\begin{aligned} &\varepsilon = c^2 \tilde{\mu}_0 / \mu_0 = \text{inv.} = \varepsilon_0 , \\ &\eta^3 q = -\boldsymbol{f} \cdot \boldsymbol{V} = \text{inv.} = q_0 , \\ &\eta\, w^{(i)} = \boldsymbol{\varphi}^\alpha \cdot \partial_\alpha \boldsymbol{V} = \text{inv.} = w_0^{(i)} , \end{aligned}} \tag{35}$$

so if one takes the general law (cf., [**1**], pp. 570):

$$\eta' / \eta = \sigma / \alpha \tag{36}$$

into account, with:

$$\alpha \equiv \sqrt{1 - u^2 / c^2}\,, \qquad \sigma \equiv 1 - \boldsymbol{u} \cdot \boldsymbol{v} / c^2, \tag{37}$$

then that will imply the laws of variance for q and $w^{(i)}$ under the passage from S to S':

$$\boxed{w'^{(i)} = w^{(i)} \alpha / \sigma, \quad q' = q (\alpha / \sigma)^2 .} \tag{38}$$

Naturally, in the classical situation ($c \to \infty$), (38) will reduce to other kinds of invariant relations.

However, as far as the *volume forces* that were referred to in (31) are concerned, the relevant law of transformation is inferred directly in the case of a point with internal scalar structure, except for the subsequent adaptation to the presence of the density $\mu = \mu_0 \eta^2$. However, one has (cf., [**1**], pp. 571):

$$\boxed{\boldsymbol{F}' = [\alpha \boldsymbol{F} + (\boldsymbol{F} \cdot \boldsymbol{w} - q)\, \boldsymbol{u} / c^2]\, \alpha / \sigma^2 ,} \tag{39}$$

with the intervention of the thermal potential q that was referred to in $(33)_2$ and the vector $\boldsymbol{w}$:

$$\boldsymbol{w} \equiv \frac{1}{1+\alpha} \boldsymbol{u} - \boldsymbol{v} . \tag{40}$$

Finally, all that remains to be done is to find the *law of variance of the* (relative) *mechanical stresses* $\boldsymbol{\phi}^i \equiv X^{ik}\,\boldsymbol{c}_k$, or $\boldsymbol{\phi}_n = n_i\,\boldsymbol{\phi}^i$. In order to this, one must:

i) Start with the proper mechanical stresses $\boldsymbol{\varphi}^\alpha$:

$$\boldsymbol{\varphi}^\alpha \equiv X^{\alpha\beta}\,\boldsymbol{c}_\beta \tag{41}$$

or the associated tensor $X^{\alpha\beta}$.

ii) Take into account the Lorentz transformation (in x^1-standard coordinates):

$$x'^0 = (x^0 - \beta x^1)\,/\,\alpha, \quad x'^1 = (x^1 - \beta x^0)\,/\,\alpha, \quad x'^{2,3} = x^{2,3}, \qquad \beta \equiv u\,/\,c^2\,; \tag{42}$$

iii) Specify the components $X'^{ik} = A'^{\,i}_\alpha\,A'^{\,k}_\beta\,X^{\alpha\beta}$, with:

$$A'^{\,i}_\alpha \equiv \begin{Vmatrix} -\beta/\alpha & 1/\alpha & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{Vmatrix}.$$

Ultimately, one has the direct relationships:

$$\begin{cases} X'^{11} = (\beta^2 X^{00} - 2\beta X^{10} + X^{11})/\alpha^2, \\ X'^{12} = (X^{12} - \beta X^{02})/\alpha, \\ X'^{13} = (X^{13} - \beta X^{03})/\alpha, \\ X'^{22} = X^{22}, \quad X'^{23} = X^{23}, \quad X'^{33} = X^{33}, \end{cases}$$

and therefore, if one takes into account that the $X^{\alpha\beta}$ are expressed in terms of only X^i by means of (26) and (28):

$$X^{0k} = \frac{1}{c}\,v_i\,X^{ik}, \qquad X^{00} = \frac{1}{c^2}\,v_i\,v_k\,X^{ik}$$

then one will get the vectors $\boldsymbol{\phi}'^i \equiv X'^{ik}\,\boldsymbol{c}_k$ as functions of the vectors $\boldsymbol{\phi}^i$:

$$\boxed{\boldsymbol{\phi}'^i = \left(\delta^i_k + \frac{1}{\alpha c^2}\,u^i w_k\right)\left(\boldsymbol{\phi}^k + \frac{1}{\alpha c^2}\,\boldsymbol{\phi}^k \cdot \boldsymbol{w}\,\boldsymbol{u}\right).} \tag{43}$$

6. The energy theorem and the First Law of Thermodynamics.

The general equation (34) confirms the interpretation of *the first law of thermodynamics as the principle that substitutes for the energy theorem* (cf., [**6**], Chap. II.3). One eliminates the mechanical power $\mu\,\boldsymbol{F}_{\text{tot}}\cdot\boldsymbol{v}$ in (34) by using the relativistic Cauchy equation (32). More precisely, if one scalar multiplies by $\boldsymbol{v}$, while taking into account that $v^2 \equiv c^2\,(1 - 1/\eta^2)$, then one will get, in turn:

$$\mu\,\boldsymbol{F}_{\text{tot}}\cdot\boldsymbol{v} = \frac{1}{\mathcal{D}}(\tilde{\mu}\mathcal{D})^{\boldsymbol{\cdot}}v^2 + \frac{1}{2}\tilde{\mu}\,(v^2)^{\boldsymbol{\cdot}} = \frac{1}{\mathcal{D}}(\tilde{\mu}\,c^2\mathcal{D})^{\boldsymbol{\cdot}} - \frac{1}{\mathcal{D}\eta^2}(\tilde{\mu}\,c^2\mathcal{D})^{\boldsymbol{\cdot}} + \frac{1}{\eta^3}\tilde{\mu}\,c^2\dot{\eta}\ .$$

Meanwhile, (34) will become:

$$\frac{1}{\mathcal{D}\eta^2}(\tilde{\mu}\,c^2\mathcal{D})^{\boldsymbol{\cdot}} - \frac{1}{\eta^3}\tilde{\mu}\,c^2\dot{\eta} = \mu\,q - w^{(i)}\,,$$

or, by virtue of $(35)_{2,3}$ and $(11)_1$:

$$\frac{\eta^2}{\mathcal{D}_0}(\tilde{\mu}\,c^2\mathcal{D})^{\boldsymbol{\cdot}} - \tilde{\mu}\,c^2\dot{\eta} = \mu\,q_0 - \eta^2 w_0^{(i)}\ .$$

When one considers (29), this, in turn, will not differ from $(\tilde{\mu}_0\,c^2\mathcal{D}_0)^{\boldsymbol{\cdot}}\eta/\mathcal{D}_0 = \mu_0\,q_0 - w_0^{(i)}$, and meanwhile, in conformity with $(35)_1$, it will translate into the *First Law of Thermodynamics:*

$$\boxed{\eta\,(\mu_0\mathcal{D}_0\varepsilon)^{\boldsymbol{\cdot}}/\mu_0\,\mathcal{D}_0 = q_0 - w_0^{(i)}/\mu_0\,.} \tag{44}$$

One is obviously dealing with the energy theorem (34), as it would look in the (local) moving Galilean reference that is incipient to the generic element of the continuum. From that viewpoint, if:

$$\eta\ ()^{\boldsymbol{\cdot}} \equiv d/d\tau = V^{\alpha}\partial_\alpha \tag{45}$$

then the kinematical identity:

$$\frac{1}{\mathcal{D}_0}\frac{d\mathcal{D}_0}{d\tau} = \partial_\alpha V^{\alpha} \tag{30′}$$

will allow one to transform (44) into the "continuity equation" for the proper internal energy:

$$\partial_\alpha(\,\mu_0\,\varepsilon\,V^{\alpha}) = \mu_0\,q_0 - w_0^{(i)}\,; \tag{44′}$$

i.e., it will result directly from (21) upon scalar multiplication by $\boldsymbol{V}$.

7. Continua with no internal material structure.

(44) can ultimately be transformed by taking into account the fact that, as in (22), $\mu_0\,\varepsilon \equiv \tilde{\mu}_0\,c^2$ *combines the proper energy of pure matter with the proper thermal energy:*

$$\mu_0\,\varepsilon = \mu_0\,c^2 + \varepsilon_{0,c}\,. \tag{46}$$

More precisely, it can be written in the form:

$$\frac{\eta}{\mu_0\mathcal{D}_0}(\mu_0\,\mathcal{D}_0)^{\cdot} = \frac{1}{c^2}\left[q_0 - \frac{1}{\mu_0}w_0^{(i)} - \frac{\eta}{\mu_0\mathcal{D}_0}(\mathcal{D}_0\,\varepsilon_{0,c})^{\cdot}\right], \tag{47}$$

or also from (44′):

$$\partial_\alpha(\mu_0\,V^\alpha) = \frac{1}{c^2}[\mu_0\,q_0 - w_0^{(i)} - \partial_\alpha(\mu_0\,V^\alpha)]\,; \tag{47′}$$

that amount to specifying the *total sources for just the material energy.*

In any case, in the context of non-polar continua, (32) and (47) agree with the dynamical equations of a particle with variable proper mass (cf., [**1**], pp. 570). Meanwhile, they constitute the *general equations of relativistic mechanics of* (non-polar) *continua with internal material structure.* One has *four scalar equations in ten unknowns:* viz., μ, v^i, and $X^{ik} = X^{ki}$ (i, k = 1, 2, 3), if, as is natural, one intends to *assign the laws of all the sources:* viz., $\boldsymbol{F}$, q, and $\varepsilon_{0,c}$; the latter will be true directly or indirectly [through the thermal conduction vector that was referred to in (14), as in (19)] by means of the heat "equation."

Along with the equalities, it is enough (at least in the mechanical schema) to add six constitutive equations.

(47) – (47′) make it quite clear that *there is less of a connection to thermodynamics in the classical situation,* and one has the usual equation for the conservation of mass. That has the result that the first law of thermodynamics must be postulated *a priori*.

However, in the relativistic situation, special importance is placed upon the *continua with no material structure,* which are characterized by the *internal constraint:*

$$\mu_0\,\mathcal{D}_0 = \text{const.}, \qquad \forall \text{ elements of the continuum.} \tag{48}$$

For those continua, (47) translates into the following *limitation on the sources:*

$$\frac{\eta}{\mu_0\mathcal{D}_0}(\mathcal{D}_0\,\varepsilon_{0,c})^{\cdot} = q_0 - \frac{1}{\mu_0}w_0^{(i)}\;; \tag{49}$$

that limitation more properly corresponds to the classical content of the first law of thermodynamics (6).

(49), which is understood to be true for all transformations of the system, opens the door to reversible continua, in both the absolute and relative senses of the term; however, that would go beyond the scope of this brief introduction.

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