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Geometrization of the Dirac theory of electrons

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The Dirac equations are written in a general, invariant form with the help of the concept of the parallel translation of a semi-vector. The energy tensor is constructed, and the macroscopic, as well as the quantum-mechanical, equations of motion are presented. The former have the usual form – viz., the divergence of the energy tensor equals the Lorentz force – and the latter are essentially identical with those of the geodetic line. The appearance of the four-potential φ_l, along with the Ricci coefficients γ_{ikl}, in the formula for parallel translation gives, on the one hand, a simple geometric basis for the appearance of the expression $p_l - e / c\ \varphi_l$ in the wave equation and, on the other hand, shows that the potentials φ_l, deviating from Einstein's view, have an autonomous place in the geometric universe, and must not be, perhaps, functions of the γ_{ikl}.

In a paper of D. Ivanenko and the author (*), the opinion was expressed that the Dirac matrices have a purely geometric meaning. In another paper (**) by this author, the notion of the parallel translation of a semi-vector (i.e., a quadruple of quantities that transform like Dirac's ψ-functions) was presented.

In a subsequent notice (***), the author further applied this concept to the presentation of the general-relativistic wave equation for the electron and derived the macroscopic equations of motion in Einsteinian form.

The present paper is a summary and completion of the arguments that were given in the aforementioned notices.

1. The transformation properties of the Dirac ψ-functions were studied thoroughly by F. Möglich (****) and J. v. Neumann (†). The transformation law takes on a particularly simple form when one chooses the following expressions for the first three Dirac α-matrices:

$$\alpha_1 = \sigma_1, \qquad \alpha_2 = \rho_3\sigma_1, \qquad \alpha_3 = \sigma_3, \tag{1}$$

(*) V. Fock and D. Ivanenko, "Über eine mögliche geometrische Deutung der relativistischen Quantentheorie," Zeit. f. Phys. **54** (1929), 798.

(**) *Ibidem*, "Géométrie quantique linéaire et déplacement paralléle," C. R. Acad. Sci. Paris **188** (1929), 1470. This paper was presented on 20 May 1929 at the physics conference in Kharkov.

(***) V. Fock, "Sur les équations de Dirac dans la théorie de relativité génerale," C. R. Acad. Sci. Paris **189** (1929), 25.

(****) F. Möglich, "Zur Quantentheorie des rotierienden Elektrons," Zeit. f. Phys. **48** (1928), 852.

(†) J. v. Neumann, "Einige Bemerkungen zur Diracschen Theorie des relativistischen Drehelektrons," *ibidem*, pp. 868.

and one of the following matrices for the fourth one (†):

$$\alpha_4 = \rho_2\,\sigma_2, \qquad \alpha_1 = \rho_1\,\sigma_2\,, \tag{1*}$$

where ρ_1, ρ_2, ρ_3 ; $\sigma_1, \sigma_2, \sigma_3$ are the four-rowed matrices that Dirac introduced.

Then, in fact, a general Lorentz transformation corresponds to the following transformation of the ψ-functions:

$$\left.\begin{aligned} \psi_1' &= \alpha\psi_1 + \beta\psi_2, \qquad \psi_3' = \bar{\alpha}\psi_3 + \bar{\beta}\psi_4, \\ \psi_2' &= \gamma\psi_1 + \delta\psi_2, \qquad \psi_4' = \bar{\gamma}\psi_3 + \bar{\delta}\psi_4. \end{aligned}\right\} \tag{2}$$

The complex quantities $\alpha, \beta, \gamma, \delta$ satisfy the condition:

$$\alpha\delta - \beta\gamma = 1, \tag{3}$$

and in the case of a purely spatial rotation they go over to the usual parameters of Cayley and Klein.

If one lets α_0 denote the identity matrix then the quantities:

$$A_i = \bar{\psi}\,\alpha_i\,\psi \qquad\qquad (i = 0, 1, 2, 3) \tag{4}$$

define the components of a four-vector, and the quantities:

$$A_4 = \bar{\psi}\,\alpha_4\,\psi\,, \qquad\qquad A_5 = \bar{\psi}\,\alpha_5\,\psi \tag{4*}$$

are invariants. This fact can be expressed in formulas as follows: If one denotes the transformation (2) by S:

$$\psi' = S\,\psi, \qquad \bar{\psi}' = \bar{\psi}S^+, \tag{5}$$

where S^+ refers to the adjoint matrix to S (i.e., the transposed conjugate), then one has the equations:

$$S^+\alpha_i\,S = \sum_{k=0}^{3} a_{ik}\alpha_k\,, \qquad S^+\alpha_4\,S = \alpha_4\,; \quad S^+\alpha_5\,S = \alpha_5\,, \tag{6}$$

where a_{ik} are the coefficients of a general Lorentz transformation. Since:

$$\bar{\psi}'\,\alpha_i\,\psi' = \bar{\psi}\;S^+\alpha_i\,S\;\psi$$

the quantities (4) and (4*) then transform according to the formulas:

(†) See V. Fock, "Über den Begriff der Geschwindigkeit in der Diracschen Theorie des Elektrons," Appendix. *ibidem* **55** (1929), 127.

$$A'_i = \sum_{k=0}^{3} a_{ik} A_k \,; \qquad A'_4 = A_4 \,; \qquad A'_5 = A_5 \,; \tag{7}$$

i.e., like a four-vector (like invariants, resp.). Since the quantities Ai (i = 0, 1, 2, 3), which are quadratic in ψ, define a four-vector, we would like to refer to the quantities ψ with the transformation properties (2) as "semi-vectors" (††).

The explicit expressions for the quantities A_i (i = 0, 1, 2, 3, 4, 5) read:

$$\begin{aligned}
A_0 &= \bar\psi_1\psi_1 + \bar\psi_2\psi_2 + \bar\psi_3\psi_3 + \bar\psi_4\psi_4, \\
A_1 &= \bar\psi_1\psi_2 + \bar\psi_2\psi_1 + \bar\psi_3\psi_4 + \bar\psi_4\psi_3, \\
A_2 &= -i\bar\psi_1\psi_2 + i\bar\psi_2\psi_1 + i\bar\psi_3\psi_4 - i\bar\psi_4\psi_3 \\
A_3 &= \bar\psi_1\psi_1 - \bar\psi_2\psi_2 + \bar\psi_3\psi_3 - \bar\psi_4\psi_4, \\
A_4 &= -\bar\psi_1\psi_4 + \bar\psi_2\psi_3 + \bar\psi_3\psi_2 + \bar\psi_4\psi_1, \\
A_5 &= -i\bar\psi_1\psi_4 + i\bar\psi_2\psi_3 - i\bar\psi_3\psi_2 + i\bar\psi_4\psi_1.
\end{aligned}$$

On the basis of these expressions, one confirms the following identity between the quantities A_i :

$$A_1^2 + A_2^2 + A_3^2 + A_4^2 + A_5^2 = A_0^2 \,. \tag{8}$$

2. We have considered the transformation properties of the ψ-functions under a Lorentz transformation in the space of special relativity. If we take the standpoint of general relativity then in order to be able to introduce the concept of semi-vector, we must have an orthogonal (more precisely, *pseudo-orthogonal*) reference system at every spacetime point. To that end, we introduce a net of four orthogonal congruences of curves and, with Einstein, refer to the directions of these congruences as "beins." The considerations of the previous paragraph then also remain valid for the case of general relativity if we understand A_i to mean the components of a vector relative to the beins.

We enumerate the beins with Latin indices and the coordinates with Greek ones, which all range through the values 0, 1, 2, 3; for the summation over the Latin indices, the summation will be given explicitly, while for the summation over Greek indices, it will be suppressed. We denote the parameters of the congruence of curves by h_k^α and the moments by $h_{k,\alpha}$. Since we are dealing with an indefinite metric, with Eisenhart (*), we introduce the quantities $e_1 = e_2 = e_3 = -1$; $e_0 = +1$. The components of a vector relative to the coordinate direction (A_σ) and relative to the beins (**) are then expressed in one way or the other as follows:

(††) This term was introduced by L. Landau.

(*) L. Eisenhart, *Riemannian Geometry*, Princeton, 1926. – Also see the splendid summary of the most important formulas and facts in the paper of T. Levi-Civita: "Vereinfachte Herstellung der Einsteinschen einheitlichen Feldgleichungen," Berl. Ber. (1929), 3.

(**) In the sequel, the bein and coordinate components will often be denoted by one and the same notation; therefore, in order to avoid confusion, the former will be provided with a prime.

$$A'_k = A_\sigma h_k^\sigma ; \quad A_\sigma = \sum_k e_k A'_k \, h_{k,\sigma} \, . \tag{9}$$

If we denote the bein components of an infinitesimal displacement by ds_k then from the formulas:

$$\delta A_\alpha = \Gamma^\beta_{\alpha\sigma} A_\beta dx^\sigma ; \qquad \Gamma^\beta_{\alpha\sigma} = \begin{Bmatrix} \alpha\,\sigma \\ \beta \end{Bmatrix} \tag{10}$$

for the variation of the components of a vector under parallel translation, one gets the following expression for the variation of its bein components:

$$\delta A'_i = \sum e_k e_l \gamma_{ikl} A'_k ds_l \, , \tag{11}$$

where the γ_{ikl} are the rotation coefficients that Ricci introduced:

$$\gamma_{ikl} = (\nabla_\sigma h_i^\beta) \, h_{k,\beta} h_l^\sigma = (\nabla_\sigma h_{i,\beta}) \, h_k^\beta h_l^\sigma \, . \tag{12}$$

In this, ∇_σ denotes the covariant derivative of x_σ.

3. We would now like to consider the variation of the components of a semi-vector ψ under infinitesimal parallel displacement. For this variation, we make the Ansatz:

$$\delta\psi = \sum_l e_l \, C_l \, ds_l \, \psi \, . \tag{13}$$

The C_l are matrices with the elements $(C_l)_{mn}$, and we understand the $C_l \psi$ to mean four functions whose m^{th} one is given by the formula:

$$(C_l \, \psi)_m = \sum_{n=1}^{4} (C_l)_{mn} \psi_n \, .$$

The equation that is complex-conjugate to (13) reads:

$$\delta\bar\psi = \bar\psi \sum_l e_l \, C_l^+ \, ds_l \, , \tag{13*}$$

where C_l^+ denotes the adjoint matrix. Now, the parallel displacement of a semi-vector is already determined from that of a vector by the law (13); namely, here we must have:

$$\delta A'_i = \delta(\bar\psi \alpha_i \psi) = \delta\bar\psi \, \alpha_i \, \psi + \bar\psi \, \alpha_i \, \delta\psi = \bar\psi \sum_l e_l (C_l^+ \alpha_i + \alpha_i C_l) \, ds_l \, \psi \, . \tag{14}$$

Should this variation coincide with the one that is given by (11), then the C_l must satisfy the conditions:

$$C_l^+ \alpha_i + \alpha_i C_l = \sum_k e_k \alpha_k \gamma_{ikl} . \tag{15}$$

Since $A_4' = \bar{\psi}\alpha_4\psi$ and $A_5' = \bar{\psi}\alpha_5\psi$ are invariants, moreover, one must have:

$$\delta A_4' = \bar{\psi}\sum_l e_l (C_l^+ \alpha_4 + \alpha_4 C_l)\, ds_l\, \psi = 0, \tag{16}$$

and likewise for $\delta A_5'$, from which the additional equations follow:

$$C_l^+ \alpha_4 + \alpha_4 C_l = 0; \qquad C_l^+ \alpha_5 + \alpha_5 C_l = 0. \tag{17}$$

One immediately convinces oneself that the general solution of equations (15) and (17) is given by the formula:

$$C_l = \tfrac{1}{4}\sum_{mk} \alpha_m \alpha_k e_k \gamma_{mkl} + i\Phi_l' , \tag{18}$$

where Φ_l' are Hermitian matrices that must commute with all of the α_i , as well as α_4 and α_5 . If one remains in the realm of four-rowed matrices then it follows from the commutability with all α-matrices that such a matrix must be proportional to the identity matrix. By contrast, if one considers matrices with more than four rows (*) then one does not exclude the case in which the Φ_l' are not proportional to the identity matrix. We would like to remain in the domain of four-rowed matrices and consider the Φ_l' as real numbers.

One should note that the C_l do not include the matrices α_4 and α_5 , such that they transform the first two ψ-functions among themselves and the last two among themselves.

4. Now that we have presented the concept of the parallel displacement of a semi-vector, we can define that of the covariant derivative $D_l'\psi$ of a semi-vector ψ along a bein direction *l* by the formula:

$$D_l'\psi = \frac{\partial\psi}{\partial s_l} - C_l\,\psi, \tag{19}$$

where $\dfrac{\partial\psi}{\partial s_l} = h_l^\sigma \cdot \dfrac{\partial\psi}{\partial x^\sigma}$ denotes the derivative in the direction of the l^{th} bein. We denote the covariant derivative of a semi-vector relative to the coordinate x^σ by:

$$D_\sigma\,\psi = \frac{\partial\psi}{\partial x^\sigma} - \Gamma_\sigma\,\psi, \tag{19*}$$

(*) Such matrices can possibly arise in certain generalizations of the Dirac equation; e.g., to the two-body problem.

where, to abbreviate, we have set:

$$\Gamma_\sigma = \sum_k e_k h_{k,\sigma} C_k \ . \tag{20}$$

If one considers space to be pseudo-Euclidian, for the moment, and sets γ_{ikl} equal to zero then the expression (20) for D'_l will be equal to:

$$D'_l \psi = D'_l \psi = \frac{\partial \psi}{\partial s_l} - i\,\Phi'_l\,\psi \,.$$

This is, however, precisely the expression that arises in the Dirac equation when one understands the Φ'_l to mean the quantities:

$$\Phi'_l = \frac{2\pi e}{hc}\varphi'_l\,, \tag{21}$$

where φ'_l denote the bein components of the vector potential. In the sequel, we would like to assume this physical interpretation for the geometric quantities Φ'_l. We have thus achieved a geometric interpretation for the appearance of the vector potential in the Dirac equation, and indeed this interpretation is such that the potential can also be different from zero when the gravitational terms that include the quantities γ_{ikl} vanish.

If we now turn to the formula (13) for $\delta\psi$ then we see that it is precisely Weyl's linear differential form that enters into it:

$$\sum_l e_l\,\varphi'_l\,ds = \varphi_\sigma\,dx^\sigma,$$

in agreement with the conjecture that Weyl's expressed (*). The appearance of Weyl's differential form in the law of parallel translation of a semi-vector is closely related with the fact that was pointed out by the author (**) and Weyl (*loc. cit.*) that the addition of a gradient to the four-potential corresponds to the multiplication of the ψ-function by a factor with an absolute value of 1. This fact was referred to as the "principle of gauge invariance" by Weyl.

5. The concept of covariant derivative of a semi-vector makes it possible to present the Dirac wave equation for the electron in the general theory of relativity. To that end, we consider the operator:

$$F\psi = \frac{h}{2\pi i}\sum_k e_k \alpha_k \left(\frac{\partial \psi}{\partial s_k} - C_k \psi\right) - mc\ \alpha_4\ \psi\,. \tag{22}$$

(*) H. Weyl, *Gruppentheorie und Quantenmechanick*, § 19, pp. 88. Leipzig, 1928.

(**) V. Fock, "Über die invariante Form der Wellen- und der Bewegungsgleichungen für einen geladenen Massenpunkt," Zeit. f. Phys. **39** (1926), 226.

We would like to show that it is self-adjoint (*). In order to see this, we go over from the coordinates to the beins, and introduce the matrices:

$$\gamma^\rho = \sum_k e_k \alpha_k h_k^\sigma \tag{23}$$

and the matrices Γ_σ that are defined by (20). Analogous relations for the matrices that were just introduced follow from equations (15):

$$\Gamma_\alpha^+ \gamma^\sigma + \gamma^\rho \Gamma_\alpha = -\nabla_\alpha \gamma^\rho . \tag{24}$$

This formula can be proved easily when one goes back to the definition (12) of the γ_{ikl}.

When expressed in terms of coordinates, the operator F reads:

$$F\psi = \frac{h}{2\pi i} \gamma^\sigma \left(\frac{\partial \psi}{\partial x^\sigma} - \Gamma_\sigma \psi \right) - mc\, a_4\, \psi . \tag{25}$$

While observing (24), one easily proves the identity:

$$\bar{\psi} F \psi - \overline{(F\psi)} \psi = \frac{h}{2\pi i} \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^\sigma} (\bar{\psi} \sqrt{g}\, \gamma^\sigma \psi) , \tag{26}$$

where g denotes the absolute value of the determinant $\| g_{\rho\sigma} \|$. This identity expresses the fact that the operator F is self-adjoint. This fact allows one to make the following Ansatz for the Dirac equation in the general theory of relativity:

$$F\ \psi = 0. \tag{27}$$

If ψ satisfies this equation then it follows from the identity (26) that the divergence of the current vectors:

$$S^\rho = \bar{\psi} \gamma^\rho \psi , \tag{28}$$

which is obviously real, due to the Hermitian character of the γ^ρ matrices, vanishes:

$$\frac{1}{\sqrt{g}} \frac{\partial}{\partial x^\sigma} (\sqrt{g} S^\sigma) = 0. \tag{29}$$

(*) Here, we understand the word "self-adjoint" to have a somewhat extended meaning. Namely, we intend that to mean that the expression $\bar{\psi} F \psi - \overline{(F\psi)} \psi$ can be written in the form of a (generally four-dimensional) divergence.

It is easy to prove (*) that the Ansätze (25) and (27) for the Dirac equation is invariant (more precisely, *covariant*) not only in relation to the choice of coordinates, but also in relation to the choice of orthogonal congruence of curves.

To prove that, we first remark that the Γ_σ can be defined uniquely, in agreement with the previous definitions (18), (20), (21), by the equations:

$$\left.\begin{aligned} \Gamma_\alpha^+ \gamma^\sigma + \gamma^\sigma \Gamma_\alpha &= -\nabla_\alpha \gamma^\sigma, \\ \tfrac{1}{4}\,\mathrm{Spur}\,\Gamma_\alpha &= \frac{2\pi ie}{hc}\varphi_\sigma. \end{aligned}\right\} \tag{30}$$

If one now introduces any net of congruences of curves and one denotes the quantities that are referred to this net by an asterisk then the new Γ_σ^*-solutions of the analogous equations are:

$$\left.\begin{aligned} \Gamma_\alpha^{*+} \gamma^\sigma + \gamma^{*\sigma} \Gamma_\sigma^* &= -\nabla_\alpha \gamma^{*\sigma}, \\ \tfrac{1}{4}\,\mathrm{Spur}\,\Gamma_\sigma^* &= \frac{2\pi ie}{hc}\varphi_\sigma. \end{aligned}\right\} \tag{30*}$$

However, at every spacetime point, the transition to new bein-directions has the character of a local Lorentz transformation. As a result, the new components of the semi-vector ψ^* and the new matrices $\gamma^{*\sigma}$ are linked with the old ψ and γ^σ by relations of the form:

$$\psi^* = S\,\psi; \qquad \gamma^\sigma = S^+ \gamma^{*\sigma} S, \tag{31}$$

[cf., formulas (5) and (6)], where S denotes a matrix of the form:

$$S = \begin{Bmatrix} \alpha & \beta & 0 & 0 \\ \gamma & \delta & 0 & 0 \\ 0 & 0 & \bar\alpha & \bar\beta \\ 0 & 0 & \bar\gamma & \bar\delta \end{Bmatrix}, \qquad\qquad \alpha\delta - \beta\gamma = 1$$

with variable elements.

However, the transformation law for the coefficients Γ_σ of parallel translation reads:

$$\Gamma_\sigma^* = S\,\Gamma_\sigma S^{-1} + \frac{\partial S}{\partial x^\sigma} S^{-1}, \tag{32}$$

so this expression is the solution of (30*) (**).

One further has the relation:

(*) This passage (up to the end of § 5) was added by the editor.

(**) One has: Spur $\dfrac{\partial S}{\partial x^\sigma} S^{-1} = 0$.

$$\frac{\partial \psi^*}{\partial x^\sigma} - \Gamma_\sigma \psi^* = S\left(\frac{\partial \psi}{\partial x^\sigma} - \Gamma_\sigma \psi\right). \tag{33}$$

If one lets $F^* \psi^*$ denote the expression that is analogous to (25) that one gets when one furnishes the γ_σ, Γ_σ, and ψ in it with an asterisk then it follows from (31) and (33) that:

$$F\ \psi = S^+ F^* \psi^*. \tag{34}$$

The equation $F\psi = 0$ is then equivalent to $F^* \psi^* = 0$, which was to be proved.

6. In this paragraph, we would like to represent the operator F in another form, in which we calculate the sum $\sum_k e_k \alpha_k C_k$ that enters into formula (22).

In order to be able to represent the result in a clear form, we proceed as follows. We introduce the quantities ε_{ijkl}, which should vanish when two equal indices appear in the indices $ijkl$, and in the case of differing indices, it should equal + 1 or – 1, according to whether the number sequence $ijkl$ emerges from 0123 by an even or odd permutation, resp. With the help of these quantities, we define the "bein-vector":

$$f_i = \frac{1}{2}\sum_{jkl} e_j e_k e_l\, \varepsilon_{ijkl}\, \gamma_{jkl} \tag{35}$$

with the components:

$$\left.\begin{aligned} f_0 &= -e_0(\gamma_{123} + \gamma_{231} + \gamma_{312}), \\ f_1 &= -e_1(\gamma_{203} + \gamma_{032} + \gamma_{320}), \\ f_2 &= -e_2(\gamma_{301} + \gamma_{013} + \gamma_{130}), \\ f_3 &= -e_3(\gamma_{102} + \gamma_{021} + \gamma_{210}). \end{aligned}\right\} \tag{35*}$$

If we observe the identities:

$$\left.\begin{aligned} \alpha_1\alpha_2\alpha_3 &= i\rho_3\alpha_0, \\ \alpha_2\alpha_3 &= i\rho_3\alpha_1, \\ \alpha_3\alpha_1 &= i\rho_3\alpha_2, \\ \alpha_1\alpha_2 &= i\rho_3\alpha_3, \end{aligned}\right\} \tag{1**}$$

which emerge from the definition (1) of the matrices α_i , then we can write the sum $\sum_l e_l \alpha_l C_l$ in the form:

$$\sum_l e_l \alpha_l C_l = \sum_l e_l \alpha_l \left(i\Phi'_l - \frac{1}{2}\sum_l e_l \gamma_{jlj} - \frac{i}{2}\rho_3 f_l \right). \tag{*}$$

We denote:

$$k_l = -\sum_j e_j \gamma_{jij} = \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^\sigma}\left(\sqrt{g}\, h_i^\sigma\right) \tag{36}$$

and introduce the expression (*) into (22). We then obtain:

$$F\psi = \sum_j e_j\alpha_j\left(\frac{h}{2\pi i}\frac{\partial\psi}{\partial s_j} - \frac{e}{c}\varphi_j\psi + \frac{h}{4\pi i}k_j\psi\right) + \frac{h}{4\pi}\rho_3\sum_j e_j\alpha_j f_j - mc\,\alpha_4\psi\,. \qquad (22^*)$$

We remark that the first and second sums in this expression are individually self-adjoint operators.

In the event that all of the congruences are normal congruences, the "bein-vector" f_i vanishes, since every Ricci symbol γ_{ikl} with three different indices vanishes identically; Furthermore, we can then choose the hypersurfaces whose normals give the curve congruences to be coordinate surfaces. We then have:

$$ds^2 = \sum_j e_j H_j^2 dx_j^2\,; \qquad \sqrt{g} = H_0\,H_1\,H_2\,H_3\,; \qquad (37)$$

$$h_i^i = \frac{1}{H_i}; \qquad h_{i,i} = e_i\,H_i\,; \qquad f_i = 0, \qquad (37^*)$$

while all parameters $h_{i,\sigma}$ and h_i^σ with differing indices vanish. The expression for the operator F then reads:

$$F\psi = \sum_j e_j\alpha_j\frac{1}{H_j}\left(\frac{h}{2\pi i}\frac{\partial\psi}{\partial x_j} - \frac{e}{c}\varphi_j\psi + \frac{h}{4\pi i}\frac{\partial}{\partial x_j}\left(\ln\frac{\sqrt{g}}{H_j}\right)\psi\right) - mc\,\alpha_4\psi\,. \qquad (38)$$

This formula allows one to write down the Dirac equation in arbitrary curvilinear, orthogonal coordinates immediately. One must observe in the following that if one writes equation (38) – e.g., in the case of an ordinary Euclidian space – once in Cartesian coordinates and once more in curvilinear coordinates then the ψ-functions that enter into (38) in both cases are not identical, but are coupled to each other by a transformation of the form (2) with variable coefficients α, β, γ, δ. One must keep this situation in mind for the presentation of the uniqueness requirements for the ψ-functions.

To conclude this paragraph, let it be remarked here that, as is known, it is not always possible to choose all curve congruences to be normal congruences in a general Riemannian space. That is possible in any event for the important special case of a static gravitational field with central and axial symmetry, as the solutions of the Einstein equations that were found by *Schwarzschild* and *Levi-Civita* have shown.

7. We would now like to attempt to find the energy tensor. To that end, we consider the tensor:

$$A^{\sigma\cdot}_{\cdot\alpha} = \bar{\psi}\,\gamma^\sigma\left(\frac{\partial\psi}{\partial x^\alpha} - \Gamma_\alpha\psi\right) = \bar{\psi}\,\gamma^\sigma D_\alpha\psi \qquad (39)$$

and calculate its divergence (*).

We write the Dirac equation, along with its complex conjugate, in the form:

$$\gamma^\sigma \left(\frac{\partial \psi}{\partial x^\sigma} - \Gamma_\sigma \psi \right) - \frac{2\pi i}{h} mc\, \alpha_4 \psi = 0, \tag{40}$$

$$\left(\frac{\partial \bar{\psi}}{\partial x^\sigma} - \bar{\psi}\, \Gamma_\sigma^+ \right) \gamma^\sigma + \frac{2\pi i}{h} mc\, \bar{\psi}\, \alpha_4 = 0. \tag{41}$$

We differentiate (40) with respect to x^α and multiply on the left by $\bar{\psi}$; we multiply equation (40*) on the right by $\partial \psi / \partial x^\alpha$ and add the results. If we observe the formula that follows from (24), namely:

$$\Gamma_\sigma^+ \gamma^\sigma + \gamma^\sigma \Gamma_\sigma = - \frac{1}{\sqrt{g}} \frac{\partial \sqrt{g}\, \gamma^\sigma}{\partial x^\sigma}, \tag{41}$$

then we can write the sum in the form:

$$\frac{1}{\sqrt{g}} \frac{\partial}{\partial x^\sigma} \left(\bar{\psi} \sqrt{g}\, \gamma^\sigma \frac{\partial \psi}{\partial x^\alpha} \right) + \bar{\psi} \frac{\partial \gamma^\sigma}{\partial x^\alpha} D_\alpha \psi - \bar{\psi} \gamma^\sigma \frac{\partial \Gamma_\sigma}{\partial x^\alpha} \psi = 0. \tag{42}$$

We further multiply (40) on the left by $-\bar{\psi}\, \Gamma_\alpha^+$ and (40*) on the right by $\Gamma_\alpha \psi$ and add them; we replace $\Gamma_\alpha^+ \gamma^\sigma$ and $\Gamma_\sigma^+ \gamma^\sigma$ with their expressions in (24) and (41) in the sum. In this way, we get:

$$\frac{1}{\sqrt{g}} \frac{\partial}{\partial x^\sigma} \left(\bar{\psi} \sqrt{g}\, \gamma^\sigma \Gamma_\alpha \psi \right) + \bar{\psi} (\nabla_\alpha \gamma^\sigma) D_\alpha \psi - \bar{\psi} \gamma^\sigma \frac{\partial \Gamma_\sigma}{\partial x^\alpha} \psi + \bar{\psi} \gamma^\sigma (\Gamma_\sigma \Gamma_\alpha - \Gamma_\alpha \Gamma_\sigma) \psi = 0. \tag{43}$$

If we replace $\nabla_\alpha \gamma^\sigma$ here by:

$$\nabla_\alpha \gamma^\sigma = \frac{\partial \gamma^\sigma}{\partial x^\alpha} + \Gamma^\sigma_{\alpha\rho} \gamma^\rho$$

then subtraction of (43) from (42) yields:

$$\frac{1}{\sqrt{g}} \frac{\partial}{\partial x^\sigma} (\sqrt{g}\, A^{\sigma\cdot}_{\cdot\alpha}) - \Gamma^\sigma_{\alpha\rho} A^{\rho\cdot}_{\cdot\sigma} = \psi\, \gamma^\sigma D_{\sigma\alpha}\, \psi, \tag{44}$$

where, to abbreviate, we have set:

(*) One can also derive the result of this paragraph in an elegant way by considering an infinitesimal coordinate transformation (cf., H. Tetrode, "Allgemein-relativistische Quantentheorie des Elektrons," Zeit. f. Phys. **50** (1928), 336.) However, we prefer to proceed in a more elementary way.

$$D_{\sigma\alpha} = \frac{\partial \Gamma_\sigma}{\partial x^\alpha} - \frac{\partial \Gamma_\alpha}{\partial x^\sigma} + \Gamma_\sigma \Gamma_\alpha - \Gamma_\alpha \Gamma_\sigma . \tag{45}$$

We must now calculate the matrix $D_{\sigma\alpha}$. We have:

$$D_{\sigma\alpha} = D_\sigma D_\alpha - D_\alpha D_\sigma = \sum_{kl} e_k e_l h_{k,\sigma} h_{l,\alpha} D'_{kl} , \tag{46}$$

where we have set:

$$D'_{kl} = D'_k D'_l - D'_l D'_k + \sum_m e_k (\gamma_{mlk} - \gamma_{mkl}) D'_m . \tag{47}$$

The operator (47) is equal to:

$$D'_{kl} = \frac{1}{4} \sum_{ij} \alpha_i \alpha_j e_j \gamma_{ijkl} + \frac{2\pi i e}{hc} M'_{kl} , \tag{48}$$

where γ_{ijkl} denotes the bein-components of the Riemann tensor:

$$\gamma_{ijkl} = \frac{\partial \gamma_{ijk}}{\partial s_l} - \frac{\partial \gamma_{ijl}}{\partial s_k} + \sum_m e_m [\gamma_{ijm} (\gamma_{mkl} - \gamma_{mlk}) + \gamma_{mil} \gamma_{mjk} - \gamma_{mik} \gamma_{mjl}], \tag{49}$$

and the skew-symmetric tensor M'_{kl}:

$$M'_{kl} = \frac{\partial \varphi'_k}{\partial s_l} - \frac{\partial \varphi'_l}{\partial s_k} + \sum_m e_m (\gamma_{mkl} - \gamma_{mlk}) \varphi'_m , \tag{50}$$

represents the electromagnetic field.

We next express the matrix $\gamma^\sigma D_{\sigma\alpha}$ in terms of D'_{kl}:

$$\gamma^\sigma D_{\sigma\alpha} = \sum_{kl} e_k e_l \alpha_k h_{l\alpha} D'_{kl} . \tag{51}$$

The sum $\sum_k e_k \alpha_k D'_{kl}$ that appears here can be calculated with the help of (48), in which one should observe the cyclic symmetry of the Riemann tensor. One gets:

$$\sum_k e_k \alpha_k D'_{kl} = \sum_k e_k \alpha_k \left(-\frac{1}{2} R'_{kl} + \frac{2\pi i e}{hc} M'_{kl} \right), \tag{52}$$

where:

$$R'_{kl} = - \sum_i e_i \gamma_{ikil} \tag{53}$$

denote the bein-components of the contracted Riemann tensor. If one substitutes (52) in (51) then one gets:

$$\gamma^{\sigma} D_{\sigma\alpha} = \gamma^{\rho}\left(-\frac{1}{2}R_{\rho\alpha} + \frac{2\pi ie}{hc}M_{\rho\alpha}\right). \tag{51*}$$

We thus have the following expression for the divergence of the tensor $A^{\sigma\cdot}_{\cdot\alpha}$:

$$\frac{1}{\sqrt{g}}\frac{\partial}{\partial x^{\sigma}}(\sqrt{g}A^{\sigma\cdot}_{\cdot\alpha}) - \Gamma^{\sigma}_{\alpha\rho}A^{\rho\cdot}_{\cdot\sigma} = S^{\rho}\left(-\frac{1}{2}R_{\rho\alpha} + \frac{2\pi ie}{hc}M_{\rho\alpha}\right). \tag{54}$$

If we set:

$$\frac{ch}{2\pi i}A^{\sigma\cdot}_{\cdot\alpha} = W^{\sigma\cdot}_{\cdot\alpha} = T^{\sigma\cdot}_{\cdot\alpha} + iU^{\sigma\cdot}_{\cdot\alpha}, \tag{55}$$

where $T^{\sigma\cdot}_{\cdot\alpha}$ and $U^{\sigma\cdot}_{\cdot\alpha}$ denote the real and imaginary parts of the complex tensor $W^{\sigma\cdot}_{\cdot\alpha}$, then formula (54) can be written in the form:

$$\nabla_{\sigma}W^{\sigma\cdot}_{\cdot\alpha} = S^{\rho}\left(eM_{\rho\alpha} - \frac{hc}{4\pi i}R_{\rho\alpha}\right), \tag{56}$$

or, when one separates the real and imaginary parts:

$$\left.\begin{aligned} \nabla_{\sigma}T^{\sigma\cdot}_{\cdot\alpha} &= eS^{\rho}M_{\rho\alpha}, \\ \nabla_{\sigma}U^{\sigma\cdot}_{\cdot\alpha} &= \frac{hc}{4\pi}S^{\rho}R_{\rho\alpha}. \end{aligned}\right\} \tag{57}$$

The second of these equations is an identity that is easy to prove, if the tensor $U^{\sigma\cdot}_{\cdot\alpha}$ is equal to:

$$U^{\sigma\cdot}_{\cdot\alpha} = -\frac{hc}{4\pi}\nabla_{\alpha}S^{\sigma}, \tag{58}$$

and the divergence of the vector S^{σ} vanishes from (29).

Equation (57) says that the divergence of the tensor $T^{\sigma\cdot}_{\cdot\alpha}$ is equal to the Lorentz force. For that reason, we can interpret $T^{\sigma\cdot}_{\cdot\alpha}$ as the energy tensor. Equations (57) are then the equations of motion for the general theory of relativity. Perhaps it would be more consistent to interpret the total complex tensor $W^{\sigma\cdot}_{\cdot\alpha}$ as the energy tensor, instead of the real part $T^{\sigma\cdot}_{\cdot\alpha}$; we shall not go into the question here of which interpretation is preferable.

It is remarkable that the electromagnetic tensor $M_{\rho\alpha}$ appears here along with the Riemann tensor $R_{\rho\alpha}$ in the form of a Hermitian matrix:

$$\left\| R_{\rho\alpha} - \frac{4\pi ie}{hc}M_{\rho\alpha} \right\|.$$

8. In order to derive the quantum-mechanical equations of motion that correspond to those of a mass point (geodetic line) from the results that we obtained, we proceed as follows:

We choose a complete system of functions in the domain of the spatial variables x_1, x_2, x_3 :

$$\psi_n(x_0, x_1, x_2, x_3; \zeta) \qquad (\zeta = 1, 2, 3, 4), \tag{59}$$

each of which satisfies the Dirac equation (*) and is normalized by the requirement that:

$$\iiint \bar{\psi}_n \gamma^0 \psi_n \sqrt{g}\, dx_1 dx_2 dx_3 = 1. \tag{60}$$

Due to (26) and (27), it follows from the validity of this equation for a special value of x_0 that it is valid for any value of x_0 . We define the matrix element for an operator L by the formula:

$$L_{mn} = \iiint \bar{\psi}_m L \psi_n \sqrt{g}\, dx_1 dx_2 dx_3 . \tag{61}$$

We observe that what we did in the previous paragraphs, and especially equation (54), remains unchanged when we replace ψ with ψ_n and $\bar{\psi}$ with $\bar{\psi}_m$ in $A^{\sigma\cdot}_{\cdot\alpha}$ and S^ρ, and thus, replace them with two different solutions of the Dirac equation. We now write equation (54) in the form:

$$\frac{1}{\sqrt{g}} \frac{\partial}{\partial x^\sigma} (\bar{\psi}_m \sqrt{g}\, \gamma^\sigma D_\alpha \psi_n) = \Gamma^\sigma_{\alpha\rho}\, \bar{\psi}_m\, \gamma^\rho D_\alpha \psi_n \to \bar{\psi}_m\, \gamma^\rho \left(-\frac{1}{2} R_{\rho\alpha} + \frac{2\pi ie}{hc} M_{\rho\alpha} \right) \psi_n . \tag{62}$$

If we multiply (62) by $\sqrt{g}\, dx_1\ dx_2\ dx_3$ and integrate over all of space then only one term remains in the sum on the left-hand side of (62), and we obtain:

$$\frac{d}{dx^0} \left\{ \iiint \bar{\psi}_m \gamma^0 D_\alpha \psi_n \sqrt{g}\, dx_1 dx_2 dx_3 \right\}$$

$$= \iiint \bar{\psi}_m \left[\Gamma^\sigma_{\alpha\rho} \gamma^\rho D_\sigma + \gamma^\rho \left(-\frac{1}{2} R_{\rho\alpha} + \frac{2\pi ie}{hc} M_{\rho\alpha} \right) \right] \psi_n \cdot \sqrt{g}\, dx_1 dx_2 dx_3 , \tag{63}$$

an equation that we can also write symbolically in the form:

$$\frac{d}{dx^0} (\gamma^0 D_\alpha) = \Gamma^\sigma_{\alpha\rho} \gamma^\rho D_\sigma + \gamma^\rho \left(-\frac{1}{2} R_{\rho\alpha} + \frac{2\pi ie}{hc} M_{\rho\alpha} \right), \tag{64}$$

or also, if we set:

$$P_a = \frac{h}{2\pi i} D_\alpha , \tag{65}$$

(*) Cf., on this, V. Fock, Zeit. f. Phys. **49** (1928), as well as **55** (1929), 127.

in the form:

$$\frac{d}{dx^0}(\gamma^0 P_\alpha) = \Gamma^\sigma_{\alpha\rho}\gamma^\rho P_\sigma + \gamma^\rho\left(-\frac{e}{c}M_{\rho\alpha} + \frac{hc}{4\pi i}R_{\rho\alpha}\right). \tag{66}$$

We can now interpret the operators γ^ρ as representatives of the classical velocity dx^ρ / dx^0 and P_σ as those of the covariant quantities of motion $m\, g_{\sigma\rho}\, dx^\rho / dx^0$. This interpretation makes the transition to the classical theory possible. If we do this, and consequently neglect the term on the right-hand side that is provided with h, then we obtain precisely the classical equations of motion for a charged mass point in a gravitational field, and in particular – when no electromagnetic field is present – the differential equation for the geodetic lines.

9. The pure covariant tensor:

$$W_{\sigma\alpha} = g_{\sigma\rho}\; W^{\rho\cdot}_{\cdot\alpha} = c\bar{\psi}\gamma_\sigma P_\alpha \psi \tag{55*}$$

is not symmetric in its indices. Due to the meaning of the operators $c\ \gamma_\sigma$ and P_α (viz., velocity and quantity of motion), the quantum-mechanical quantities $W_{\sigma\alpha}$ correspond to the classical quantities $\rho_0\, u_\sigma\, u_\alpha$:

$$W_{\sigma\alpha} \rightarrow \rho_0\, u_\sigma\, u_\alpha, \tag{67}$$

where u_σ denote the classical covariant components of the four-velocity and ρ_0 denotes the rest density of the matter. However, the quantity $\rho_0\, u_\sigma\, u_\alpha$ is symmetric in its indices.

The Dirac equation (27) can be derived from a variational principle, which can be formulated with the help of the energy tensor as follows:

$$\delta\iiint\int (W^{\sigma\cdot}_{\cdot\sigma} - mc^2\bar{\psi}\alpha_4\psi)\sqrt{g}\; dx_0\, dx_1\, dx_2\, dx_3 = 0. \tag{68}$$

This equation provides a simple physical interpretation of the invariant $m\bar{\psi}\alpha_4\psi$ as the rest density of matter.

10. We would now like to summarize the results of our investigation.

The concept of parallel translation of a semi-vector served as the starting point for us. With the help of this concept, the appearance of the potentials φ_α, along with the impulses p_α in the Dirac equation could be interpreted in a purely geometric way: The purely formal conversion of the expression $p_\alpha - e/c\varphi_\alpha$ from classical mechanics to quantum mechanics was therefore superfluous. Furthermore, the aforementioned concept allowed us an informal association of the potential in the geometric schema of the general theory of relativity, which can be of use for the presentation of a unified theory of electricity and gravitation.

Moreover, the Dirac equations were present in the general theory of relativity, which are invariant in regard to the choice of coordinates and of "beins." As a closely-related result, this yielded an explicit representation of the Dirac equation in curvilinear, orthogonal coordinates. A tensor was constructed whose divergence was equal to the

Lorentz force; this tensor was interpreted as the energy tensor, and the equation that it satisfied was interpreted as the macroscopic equation of motion. Furthermore, the quantum-mechanical equations of motion for the electron that correspond to the classical equations for a charged mass point or – in the absence of an electromagnetic field – those of a geodetic line were derived. In conclusion, the variational principle from which the Dirac equation can be derived was written down.

The goal that we have tried to achieve was the geometrization of the Dirac theory of electrons and its association with the general theory of relativity. Thus, the difficulties that adhere to the Dirac theory – such as the appearance of negative energy values and a non-vanishing probability of the charge reversal of an electron – were not involved at all. However, our considerations might perhaps contribute to the solution of these difficulties in an indirect way, in that they show how the original, unaltered Dirac theory can be derived.

Leningrad, Physical Institute of the University, May/June 1929.

Appendix

After the completion of this paper, I learned of a very interesting paper of H. Weyl (*). Weyl's basic mathematical idea is essentially identical with the concept of the parallel displacement of a semi-vector. The physical content of Weyl's paper is, however, completely different from that of my own paper.

We can summarize the essential features of Weyl as follows:

1. Weyl regarded the Dirac equation as a wave equation for the electron-proton system, not for the electron.

2. In the additional gravitational terms, Weyl believed he would find a substitute for the term $mc\alpha_4$, which was simply deleted.

In my opinion, both of these features can scarcely be justified when one encounters essential difficulties that I would like to point out here.

The quantum-mechanical equations of motion that follow from the Dirac equation are a complete analogue for the classical equations of motion for a charged mass point (and not, say, for a system of two bodies), as was already shown in my earlier paper (**).

The Dirac equation, and indeed with the term $mc\alpha_4$, is suitable entirely to the description of a force-free motion of an electron as a wave in the sense of the original de Broglie picture.

The decomposition that Weyl carried out of the current vector S into two summands $S^{(+)}$ and $S^{(-)}$, which were interpreted as currents of positive and negative electricity, resp., cannot be justified if these summands are null vectors, and only their sum $S = S^{(+)} + S^{(-)}$ is a timelike vector (***). The current is, however, a static-macroscopic quantity and, as

(*) H. Weyl, "Gravitation and the electron," Proc. Nat. Acad. Amer. **15** (1929), 323.

(**) V. Fock, "Über den Begriff der Geschwindigkeit, usw.," Zeit. f. Phys. **55** (1929), 127.

(***) Proof: The timelike character of S follows from the identity (8) (where S_i is to be used now, instead of A_i), since it yields:

$$S_0^2 - S_1^2 - S_2^2 - S_3^2 = S_4^2 + S_5^2 . \tag{*}$$

such, must have the same character as in the classical theory, so it must necessarily be timelike.

The Weyl equations should describe the electron-proton system; one may then demand that they correctly duplicate the energy levels of the hydrogen atom. Due to the omission of the term $mc\alpha_4$, this is, however, hardly possible, and unproved, in any event.

The gravitational terms [viz., the "bein-vector" f_i, in our formula (35)] that Weyl interpreted as a substitute for mass can be made to vanish as long as a system of normal congruences exists, and especially in the case of spherical symmetry, as well as in the static case of axial symmetry. However, one might expect a high degree of symmetry for the electron-proton system.

Finally, it remains entirely unclear how the constants m and M – viz., the masses of the electron and proton – should actually emerge from the gravitational terms.

Due to these difficulties, I cannot consider Weyl's attempt to tackle the quantum-mechanical problem of mass, as well as the two-body problem, as successful. By contrast, I heartily agree with Weyl's general idea that both problems are closely coupled to each other and to gravitation.

In conclusion, I would like to make some general remarks on the physical content of the Dirac equations and the quantum-mechanical two-body problem.

In my way of looking at things, only the electron is described quantum-mechanically by the Dirac equation, while the rest of the world (perhaps also the mass of the electron) is described macroscopically. The proton is also counted among the rest of the world. The solution of the two-body problem must consist of one finding a quantum-mechanical description of the electron, the proton, the electromagnetic field, and mass. On the contrary, for the macroscopic description of gravitation and electricity, the one-body problem can perform a useful service.

One obtains $S_i^{(+)}$ ($S_i^{(-)}$, resp.) from S_i if one sets ψ_3 and ψ_4 (ψ_1 and ψ_2, resp.) equal to zero; S_4 and S_5 vanish in both cases, and thus, also the left-hand side of (*), which was to be proved.